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Research Article

Generalized Hausdorff Operators on $\dot{K}_{\alpha,q}^{\beta,p}(\mathbb{R})$ and $H\dot{K}_{\alpha,q}^{\beta,p,N}(\mathbb{R})$ in the Dunkl Settings

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In the present paper, we obtain some new results, and we generalize some known results for the Hausdorff operators. We have studied the generalized Hausdorff operators $\mathscr{H}_{\alpha,\varphi}$ on the Dunkl-type homogeneous weighted Herz spaces $\dot{K}_{\alpha,q}^{\beta,p}(\mathbb{R})$ and Dunkl Herz-type Hardy spaces $H\dot{K}_{\alpha,q}^{\beta,p,N}(\mathbb{R})$. We have determined simple sufficient conditions for these operators to be bounded on these spaces. As applications, we provide necessary and sufficient conditions for generalized Cesàro operator to be bounded on $\dot{K}_{\alpha,q}^{\beta,p}(\mathbb{R})$ and Hardy inequality for $\dot{K}_{\alpha,q}^{\beta,p}(\mathbb{R})$.

1. Introduction and Preliminaries

We recall that the Fourier transform $\widehat{f}$ of a (complex-valued) function f in $L^1(\mathbb{R})$ is defined as

$$\widehat{f}(t) := \int_{\mathbb{R}} f(x)e^{-itx}dx, \quad t \in \mathbb{R}. \tag{1}$$

The Hausdorff operator $\mathscr{H}$ generated by a function φ in $L^1(\mathbb{R})$ as introduced in [1] can be defined both directly and via the Fourier transform. The latter reads as follows:

$$\left(\mathscr{H}_{\varphi}f\right)^{\wedge}(t) := \int_{\mathbb{R}} \widehat{f}(tx)\varphi(x)dx, \quad t \in \mathbb{R}, \tag{2}$$

where f is also in $L^1(\mathbb{R})$. The existence of such a function $\mathscr{H}f$ in $L^1(\mathbb{R})$ is established in [1]. The theory of Hausdorff operators, while dating in a sense back to Hurwitz and Silverman [2] in 1917 with summability of number series, now becomes a notable ingredient in modern harmonic analysis and has received an extensive attention in recent years. To save the length of this article, we refer the reader to the survey article [3] for its background and historical developments.

Dunkl's theory generalizes classical Fourier analysis on $\mathbb{R}^d$. This theory began twenty years ago with Dunkl's seminal work in [4]. It was later developed by many mathematicians. On the real line, the Dunkl operators $\mathscr{D}_\alpha$ are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. An important motivation to study Dunkl operators originates from their relevance for the analysis of quantum many-body systems of the Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics (see [5]).

Let μ_α be the measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) = |x|^{2\alpha+1}dx. \tag{3}$$

We denote by $L_\alpha^p(\mathbb{R})$, $1 \le p \le \infty$, the space of measurable functions on $\mathbb{R}$ such that

$$\|f\|_{L_\alpha^p} := \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, \quad \text{if } 1 \le p < \infty, \tag{4}$$

$$\|f\|_{L_\alpha^\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty. \tag{5}$$

The Dunkl-Hausdorff operator $\mathscr{H}_\alpha$ (see [6–9]) acting on $L^1_\alpha(\mathbb{R})$ generated by a function φ belonging to $L^1(\mathbb{R})$ is defined directly as:

$$\mathscr{H}_{\alpha,\varphi}f(x) := \int_0^\infty \frac{\varphi(t)}{t^{2\alpha+2}} f\left(\frac{x}{t}\right) dt, \quad f \in L^1_\alpha(\mathbb{R}), x \in \mathbb{R}, \tag{6}$$

and for all function f in $L^1_\alpha(\mathbb{R})$, the Dunkl-Hausdorff operator $\mathscr{H}_\alpha$ verifies

$$\mathscr{F}_\alpha(\mathscr{H}_\alpha f)(t) = \int_{\mathbb{R}} \mathscr{F}_\alpha(f)(tx)\varphi(x)dx, \quad t \in \mathbb{R}, \tag{7}$$

where $\mathscr{F}_\alpha$ is the Dunkl transform. When $\alpha = -1/2$, the operator $\mathscr{H}_{\alpha,\varphi}$ is the direct definition of the Hausdorff operator associated with Fourier transform defined in (2)

$$\mathscr{H}_\varphi f(x) = \int_0^\infty \frac{\varphi(t)}{t} f\left(\frac{x}{t}\right) dt, \tag{8}$$

from which several well-known operators can be deduced for suitable choices of φ, e.g., for $\varphi(t) = (1/t)\chi_{(1,\infty)}(t)$, the operator $\mathscr{H}_\varphi$ reduces to the standard Hardy averaging operator

$$\mathscr{H}f(x) = \frac{1}{x}\int_0^x f(t)\, dt, \tag{9}$$

while for $\varphi(t) = \chi_{[0,1]}(t)$, it reduces to the adjoint of Hardy averaging operator

$$\mathscr{H}^* f(x) = \int_x^\infty \frac{f(t)}{t}\, dt. \tag{10}$$

Chen et al. [10] established boundedness of the classical Hausdorff operators in Herz type spaces, which are a natural generalization of the Lebesgue spaces L^p. Gasmi et al. [11] introduced a new weighted Herz space associated with the Dunkl operators on $\mathbb{R}$. They also characterize the corresponding Herz-type Hardy spaces by atomic decomposition. Motivated by this result concerning Herz spaces (see also [12–14] and reference therein), this paper is aimed at extending these results to the context of Dunkl theory. We investigate the Dunkl-Hausdorff operators on the Dunkl-type homogeneous weighted Herz spaces $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ and Dunkl Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ in the spirit of those in [10]. As applications, we provide necessary and sufficient conditions for Dunkl-Cesàro operator and sufficient conditions for Dunkl-Hardy operator to be bounded on the homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$.

This paper is organized as follows: in Section 1, we have presented some definitions and fundamental results from Dunkl's analysis. In Sections 2 and 3, we have presented and proven our main results.

For $\alpha \geq -1/2$, the Dunkl differential-difference operator is defined as (see [4])

$$\mathscr{D}_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x} \cdot \frac{f(x) - f(-x)}{2}, \quad f \in C^1(\mathbb{R}). \tag{11}$$

For $\lambda \in \mathbb{C}$, the initial problem

$$\mathscr{D}_\alpha(f)(x) = \lambda f(x), f(0) = 1, \quad x \in \mathbb{R}, \tag{12}$$

has a unique solution $E_\alpha(\lambda .)$ (called the *Dunkl kernel*) given by:

$$E_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), \quad z \in \mathbb{C}, \tag{13}$$

where j_α is the normalized Bessel function of the first kind (with order α) defined on $\mathbb{C}$ by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^\infty (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(n+\alpha+1)}. \tag{14}$$

The integral representation of E_α is given by

$$E_\alpha(i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+(1/2))} \int_{-1}^1 (1-t)\left(1-t^2\right)^{\alpha-1/2} e^{-i\lambda x t} dt. \tag{15}$$

The Dunkl transform $\mathscr{F}_\alpha$ is defined for $f \in L^1_\alpha(\mathbb{R})$ by:

$$\mathscr{F}_\alpha(f)(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}} E_\alpha(-ixy) f(y) d\mu_\alpha(y), c_\alpha = 2^{\alpha+1}\Gamma(\alpha+1). \tag{16}$$

This transform satisfies the following properties:

(i) For all $f \in L^1_\alpha(\mathbb{R})$ such that $\mathscr{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$, we have the inversion formula

$$f(x) = \frac{1}{c_\alpha} \int_{\mathbb{R}} E_\alpha(i\lambda x) \mathscr{F}_\alpha(f)(\lambda) d\mu_\alpha(\lambda), a.e.,x \in \mathbb{R}. \tag{17}$$

(ii) For all $f \in \mathscr{S}(\mathbb{R})$ (the usual Schwartz space)

$$\mathscr{F}_\alpha(\mathscr{D}_\alpha f)(x) = ix\mathscr{F}_\alpha(f)(x). \tag{18}$$

For any $x, y, z \in \mathbb{R}$, we consider:

$$w_\alpha(x,y,z) = \frac{(\Gamma(\alpha+1))^2}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+(1/2))} \cdot \left(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\right) q_\alpha(|x|,|y|,|z|), \tag{19}$$

where

$$\sigma_{x,y,z} == \begin{cases} \frac{x^2+y^2-z^2}{2xy}, & \text{if } xy \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{20}$$

and for $x, y, z > 0$,

$$q_\alpha(x,y,z) = \frac{\left(\left[(x+y)^2 - z^2\right]\left[z^2 - (x-y)^2\right]\right)^{\alpha-1/2}}{(xyz)^{2\alpha}} \mathscr{X}_{[|x-y|,x+y]}(z). \tag{21}$$

We further have (see [15])

$$\int_0^{+\infty} q_\alpha(x,y,z) d\mu_\alpha(z) = 1 \text{ and} \int_{\mathbb{R}} |w_\alpha(x,y,z)| d\mu_\alpha(z) \le \sqrt{2}. \tag{22}$$

In the sequel, we consider the signed measure $\gamma_{x,y}$ on $\mathbb{R}$ given by

$$d\gamma_{x,y}(z) = \begin{cases} w_\alpha(x,y,z) d\mu_\alpha(z), & \text{if } x, y \in \mathbb{R}^*, \\ d\delta_y(z), & \text{if } x = 0, \\ d\delta_x(z), & \text{if } y = 0. \end{cases} \tag{23}$$

For $x, y \in \mathbb{R}$ and a continuous function f on $\mathbb{R}$, we put

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z), \tag{24}$$

which is called the Dunkl translation operator. The Dunkl translation operator has the following properties:

(i) For $x, y \in \mathbb{R}$ and a continuous function f on $\mathbb{R}$, we have

$$\tau_x(f)(y) = \tau_y(f)(x). \tag{25}$$

(ii) (Product formula) For all $x, y, z \in \mathbb{R}$

$$\tau_x(E_\alpha(iy))(z) = E_\alpha(ixy) E_\alpha(iyz). \tag{26}$$

(iii) For all $x, y \in \mathbb{R}$ and $f \in L^1_\alpha(\mathbb{R})$, we have

$$\mathscr{F}_\alpha(\tau_x(f))(y) = E_\alpha(ixy) \mathscr{F}_\alpha(f)(y). \tag{27}$$

The Dunkl convolution of two functions f, g on $\mathbb{R}$ is defined by the relation

$$f *_\alpha g(x) = \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y). \tag{28}$$

Let $f \in \mathscr{S}'(\mathbb{R})$ and $\phi \in \mathscr{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi(x) d\mu_\alpha(x) = 1$, we have

$$\lim_{t \longrightarrow 0} f *_\alpha \phi_t = f, \text{in} \mathscr{S}'(\mathbb{R}) \tag{29}$$

where ϕ_t is the dilation of ϕ given by

$$\phi_t(x) := t^{-2(\alpha+1)} \phi\left(\frac{x}{t}\right), \quad x \in \mathbb{R}. \tag{30}$$

For all $N \in \mathbb{N}$, we denote by F_N the subset of $\mathscr{S}(\mathbb{R})$ constituted by all those $\phi \in \mathscr{S}(\mathbb{R})$ such that supp $(\phi) \subset [-1, 1]$, and for all $m, n \in \mathbb{N}$ such that $m, n \le N$, we have

$$\rho_{m,n}(\phi) := \sup_{x \in \mathbb{R}} (1 + |x|)^m |\mathscr{D}^n_\alpha \phi(x)| \le 1. \tag{31}$$

Moreover, the system of seminorms $\{\rho_{m,n}\}_{m,n \in \mathrm{N}}$ generates the topology of $\mathscr{S}(\mathbb{R})$ (see [16]).

Let $f \in \mathscr{S}'(\mathbb{R})$ and $N \in \mathbb{N}$. The α-grand maximal function of N-order $G_{\alpha,N}(f)$ of f is defined by

$$G_{\alpha,N}(f)(x) := \sup_{t>0, \phi \in F_N} |\phi_t *_\alpha f(x)|, \quad x \in \mathbb{R}. \tag{32}$$

The α-grand maximal function $G_{\alpha,N}$ is a bounded continuous operator from $L^p_\alpha(\mathbb{R})$ into itself, for every $p \in]1, \infty]$, provided that$N > 2(\alpha + 1)$ (see [11]).

2. Boundedness of $\mathscr{H}_{\alpha,\varphi}$ on the Homogeneous Weighted Herz Space $\dot{K}^{\beta,p}_{\alpha,q}$

Let $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The homogeneous weighted Herz space $\dot{K}^{\beta,p}_{\alpha,q}$ is the space constituted by all the functions $f \in L^q_\alpha(\mathbb{R})_{loc}$, such that

$$\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} := \left(\sum_{k=-\infty}^{+\infty} 2^{2(\alpha+1)\beta k p} \|f \chi_k\|^p_{L^q_\alpha(\mathbb{R})}\right)^{1/p} < +\infty, \tag{33}$$

where χ_k is the characteristic function of the set

$$A_k = \left\{x \in \mathbb{R} \, ; 2^{k-1} \le |x| \le 2^k\right\}, \quad \text{for } k \in \mathbb{Z}, \tag{34}$$

and $L^q_\alpha(\mathbb{R})_{loc}$ is the space $L^q_{loc}(\mathbb{R}, |x|^{2\alpha+1} dx)$.

Note that $\dot{K}^{\beta,0}_{\alpha,q}(\mathbb{R}) = L^q_\alpha(\mathbb{R})$. The main result of this subsection is the following theorem.

Theorem 1. *Let $\alpha \ge (-1/2)$, $\beta \in \mathbb{R}$, $1 < p < +\infty$, $1 \le q < +\infty$, and φ a measurable function on $\mathbb{R}$ such that*

$$C_{q,\alpha,\beta,\varphi} = \int_0^\infty |\varphi(t)| t^{2(\alpha+1)(\beta-1+(1/q))} dt < \infty. \tag{35}$$

Then, the Dunkl-Hausdorff operator $\mathscr{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself, i.e.,

$$\|\mathscr{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} \lesssim C_{q,\alpha,\beta,\varphi}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}. \tag{36}$$

Proof. Note that for any $t>0$, there exists an integer number $l=l(t)$ satisfying $2^{l-1}<t\le 2^l$. The Minkowski inequality for $L^q_\alpha(\mathbb{R})$ guarantees that for any $k\in\mathbb{Z}$,

$$\|\mathscr{H}_{\alpha,\varphi}f\chi_k\|_{L^q_\alpha(\mathbb{R})} \le \int_0^\infty |\varphi(t)|t^{2(\alpha+1)(1/q-1)}\left(\|f\chi_{k-l}\|_{L^q_\alpha(\mathbb{R})}+\|f\chi_{k-l+1}\|_{L^q_\alpha(\mathbb{R})}\right)dt. \tag{37}$$

By applying the Minkowski inequality for l^p, we obtain

$$\begin{aligned}\|\mathscr{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})} &= \left(\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\|\mathscr{H}_{\alpha,\varphi}f\chi_k\|^p_{L^q_\alpha(\mathbb{R})}\right)^{1/p}\\ &\le \left(\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\left(\left(\int_0^\infty|\varphi(t)|t^{2(\alpha+1)(1/q-1)}\left(\|f\chi_{k-l}\|_{L^q_\alpha(\mathbb{R})}+\|f\chi_{k-l+1}\|_{L^q_\alpha(\mathbb{R})}\right)dt\right)^p\right)\right)^{1/p}\\ &\le \int_0^\infty|\varphi(t)|t^{2(\alpha+1)(1/q-1)}\left\{\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\left(\|f\chi_{k-l}\|_{L^q_\alpha(\mathbb{R})}\right)^{1/p}+\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\left(\|f\chi_{k-l+1}\|_{L^q_\alpha(\mathbb{R})}\right)^{1/p}\right\}dt.\end{aligned} \tag{38}$$

Since $2^{l-1}<t\le 2^l$ and the definition of the Herz space $\dot{K}^{\beta,p}_{\alpha,q}$, we estimate

$$\begin{aligned}&\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\left(\|f\chi_{k-l-1}\|_{L^q_\alpha(\mathbb{R})}\right)^{1/p}\\ &\quad+\sum_{k=-\infty}^{+\infty}2^{2(\alpha+1)\beta kp}\left(\|f\chi_{k-l}\|_{L^q_\alpha(\mathbb{R})}\right)^{1/p}\\ &\le\left(2^{2(l-1)(\alpha+1)\beta}+2^{2l(\alpha+1)\beta}\right)\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}\lesssim t^{2(\alpha+1)\beta}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}.\end{aligned} \tag{39}$$

Therefore, we obtain

$$\|\mathscr{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}\lesssim\int_0^\infty|\varphi(t)|t^{2(\alpha+1)(\beta-1+1/q)}dt\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}, \tag{40}$$

which implies that $\mathscr{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself. □

Remark 2. When $\alpha=-1/2$, Theorem 1 reduce to ([10], Theorem 2.4).

2.1. Hardy Inequality for $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$. If $\varphi(t)=(\chi_{(1,\infty)(t)})/t$, then (6) is of the following form:

$$\mathscr{H}_\alpha f(x)=\frac{1}{x^{2\alpha+1}}\int_0^x f(\xi)d\mu_\alpha(\xi). \tag{41}$$

In this case, $\mathscr{H}_{\alpha,\varphi}$ reduces to the Hardy-type averaging operator for which we deduce the following result.

Corollary 3. *Let $\alpha\ge(-1/2)$, $1<p<+\infty$, $1\le q<+\infty$, $0<\beta<1-(1/q)$, and $\varphi(t)=(\chi_{(1,\infty)(t)})/t$. Then, the Dunkl-Hardy operator $\mathscr{H}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself and we have*

$$\|\mathscr{H}_{\alpha,\varphi}f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}\le\frac{1}{2(\alpha+1)(\beta-1+1/q)}\|f\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}. \tag{42}$$

Proof. From Theorem 1, we have

$$\begin{aligned}C_{q,\alpha,\beta,\varphi}&=\int_0^\infty|\varphi(t)|\left|t\right|^{2(\alpha+1)(\beta-1+1/q)}dt,\\ &=\int_1^\infty t^{2(\alpha+1)(\beta-1+1/q)-1}dt,\\ &=\frac{1}{2(\alpha+1)(\beta-1+1/q)}.\end{aligned} \tag{43}$$

□

2.2. Generalized Cesàro Operator. If φ is supported in the interval $[0;1]$, then $\mathscr{H}_{\alpha,\varphi}$ reduces to the generalized Cesàro operator $\mathscr{C}_{\alpha,\varphi}$ defined by

$$\mathscr{C}_\varphi f(x):=\int_0^1\frac{\varphi(t)}{t^{2\alpha+2}}f\left(\frac{x}{t}\right)dt,\quad x\in\mathbb{R}, \tag{44}$$

(see [6, 17]).

Corollary 4. *Let $\alpha\ge(-1/2)$, $\beta\in\mathbb{R}^\times$, $1<p<+\infty$, $1\le q<+\infty$, and φ a nonnegative measurable function defined on $[0,1]$.*

Then, the generalized Cesàro operator $\mathscr{C}_{\alpha,\varphi}$ is bounded from $\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$ to itself if and only if

$$C_{q,\alpha,\varphi} = \int_0^1 \varphi(t) t^{2(\alpha+1)(\beta+1/q-1)} dt < \infty. \tag{45}$$

Proof. By Theorem 1, we need just to prove the necessity part. For any $\varepsilon \in (0,1)$, we set

$$f_\varepsilon(x) = \begin{cases} |x|^{-(2\beta(\alpha+1)+\varepsilon+2(\alpha+1)/q)}, & \text{if } |x| > 1, \\ 0 & \text{otherwise,} \end{cases} \tag{46}$$

then for $j = 0, -1, -2, \cdots, \|f_\varepsilon \chi_j\|_{L^q_\alpha(\mathbb{R})} = 0$, and for $j \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} \|f_\varepsilon \chi_j\|^q_{L^q_\alpha(\mathbb{R})} &= \int_{2^{j-1} \le |x| \le 2^j} |x|^{-(2(\alpha+1)\beta+\varepsilon+2(\alpha+1)/q)q} d\mu_\alpha(x) \\ &= 2\int_{2^{j-1}}^{2^j} x^{-(2(\alpha+1)\beta+\varepsilon)q-1} dx \\ &= C_\varepsilon\, 2^{-j(2(\alpha+1)\beta+\varepsilon)q}, \end{aligned} \tag{47}$$

where

$$C_\varepsilon = 2\frac{2^{(2(\alpha+1)\beta+\varepsilon)q} - 1}{(2(\alpha+1)\beta+\varepsilon)q}. \tag{48}$$

Hence, it yields

$$\begin{aligned} \left(\sum_{j=-\infty}^{+\infty} 2^{2(\alpha+1)\beta jp} \|f_\varepsilon \chi_j\|^p_{L^q_\alpha(\mathbb{R})}\right)^{1/p} &= \left(\sum_{j=1}^{+\infty} 2^{2(\alpha+1)\beta jp} \|f_\varepsilon \chi_j\|^p_{L^q_\alpha(\mathbb{R})}\right)^{1/p} \\ &= C_\varepsilon^{1/q} \left(\sum_{j=1}^{+\infty} 2^{-j\varepsilon p}\right)^{1/p} = C_\varepsilon^{1/q} \frac{2^{-\varepsilon}}{(1-2^{-p\varepsilon})^{1/p}}, \end{aligned} \tag{49}$$

thus $f_\varepsilon \in H^{\beta,k}_{p,q}$.

Now, it is easy to see that

$$\begin{aligned} \mathscr{H}_{\alpha,\varphi} f_\varepsilon(x) &= \int_0^{|x|} \left|\frac{x}{t}\right|^{-(2(\alpha+1)\beta+\varepsilon+2(\alpha+1)/q)} t^{-2(\alpha+1)} \varphi(t) dt \\ &= |x|^{-(2\beta(\alpha+1)+\varepsilon+2(\alpha+1)/q)} \int_0^{|x|} t^{2\beta(\alpha+1)+\varepsilon-2(\alpha+1)(1-1/q)} \varphi(t) dt. \\ \sum_{j=-\infty}^{+\infty} & 2^{2(\alpha+1)\beta jp} \|(\mathscr{H}_{\alpha,\varphi} f_\varepsilon)\chi_j\|^p_{L^q_\alpha(\mathbb{R})} \\ &= \sum_{j=-\infty}^{+\infty} 2^{2(\alpha+1)\beta jp} \left[\int_{A_j} \left(|x|^{-(2\beta(\alpha+1)+\varepsilon+2(\alpha+1)/q)} \int_0^{|x|} t^{2\beta(\alpha+1)+\varepsilon-2(\alpha+1)(1-1/q)} \varphi(t) dt\right)^q d\mu_\alpha(x)\right]^{p/q} \\ &\ge \left(\int_0^1 t^{2\beta(\alpha+1)+\varepsilon-2(\alpha+1)(1-1/q)} \varphi(t) dt\right)^p \sum_{j=1}^{+\infty} 2^{2(\alpha+1)\beta jp} \left(\int_{A_j} |x|^{-(2\beta(\alpha+1)+\varepsilon+2(\alpha+1)/q)q} d\mu_\alpha(x)\right)^{p/q}. \end{aligned} \tag{50}$$

It follows from (49) that

$$\begin{aligned} &\left(\sum_{j=-\infty}^{+\infty} 2^{2(\alpha+1)\beta jp} \|(\mathscr{H}_{\alpha,\varphi} f_\varepsilon)\chi_j\|^p_{L^q_\alpha(\mathbb{R})}\right)^{1/p} \\ &\ge C_\varepsilon^{1/q} \frac{2^{-\varepsilon}}{(1-2^{-q\varepsilon})^{1/p}} \left(\int_0^1 t^{2\beta(\alpha+1)+\varepsilon-2(\alpha+1)(1-1/q)} \varphi(t) dt\right), \end{aligned} \tag{51}$$

which implies when $\varepsilon \longrightarrow 0$,

$$\int_0^1 t^{2\beta(\alpha+1)-2(\alpha+1)(1-1/q)} \varphi(t) dt < \infty. \tag{52}$$

This completes the proof. □

3. Boundedness of $\mathscr{H}_{\alpha,\varphi}$ on the Dunkl Herz-Type Hardy Space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$

Definition 5. Let $\alpha \ge (-1/2)$, $N \in \mathbb{N}$, $\beta \in \mathbb{R}$, $0 < p < +\infty$, and $1 \le q < +\infty$. The Herz-type Hardy space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ is the space of distributions $f \in \mathscr{S}'(\mathbb{R})$ such that $G_{\alpha,N}(f) \in \dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})$. Moreover, we have

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \|G_{\alpha,N}(f)\|_{\dot{K}^{\beta,p}_{\alpha,q}(\mathbb{R})}. \tag{53}$$

In the sequel, we are interested in the spaces $H\dot{K}^{\beta,p,N}_{\alpha,q}$

$(\mathbb{R})$, when $\beta \geq 1 - 1/q$. Now, we turn to the atomic characterization of the space $H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$.

Definition 6. Let $\alpha \geq (-1/2)$, $1 \leq q \leq \infty$, and $\beta \geq 1 - 1/q$. A measurable function a on $\mathbb{R}$ is called a (central) $(\beta\,;q)$ -atom if it satisfies:

(1) supp $a \subset [-r, r]$, for a certain $r > 0$

(2) $\|a\|_{q,\alpha} \leq r^{-2(\alpha+1)\beta}$,

(3) $\int_{\mathbb{R}} a(x)x^k d\mu_\alpha(x) = 0$, $k = 0, 1, \cdots, 2s + 1$

where s is the integer part of $(\alpha + 1)(\beta - 1 + 1/q)$.

Theorem 7. *Let $\alpha \geq (-1/2)$, $0 < p < +\infty$, $1 \leq q < +\infty$, $\beta \geq 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s + 3 + \alpha)$. Then, $f \in H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$ if and only if there exist, for all $j \in \mathbb{N} \setminus \{0\}$, an $(\beta\,;q)$ -atom a_j and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ and $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover,*

$$\|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})} = \inf \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p}, \tag{54}$$

where the infimum is taken over all atomic decompositions of f.

The main result of this subsection is the following theorem.

Theorem 8. *Let $\alpha \geq (-1/2)$, $0 < p \leqslant 1 < q < \infty$, $\beta \geq 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s + 3 + \alpha)$.*

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_0^{\infty} t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)(1 + |\log_2|t||)^\sigma dt. \tag{55}$$

If for some $\sigma > ((1 - p)/p)$, $C_p \coloneqq C_{p,\sigma} < \infty$, then

$$\|\mathscr{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,p,N}_{\alpha,q}}. \tag{56}$$

(ii) For $p = 1$, let

$$C_1 = \int_0^{\infty} t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t) dt. \tag{57}$$

If $C_1 < \infty$, then

$$\|\mathscr{H}_{\alpha,\varphi}(f)\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}} \lesssim \|f\|_{H\dot{K}^{\beta,1,N}_{\alpha,q}}. \tag{58}$$

Proof. By the central atomic decomposition, for $f \in H\dot{K}^{\beta,p,N}_{\alpha,q}(\mathbb{R})$, we write

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \tag{59}$$

where

$$\sum_{j=1}^{\infty} |\lambda_j|^p \simeq \|f\|^p_{H\dot{K}^{\beta,p,N}_{\alpha,q}}. \tag{60}$$

Then, we have

$$\mathscr{H}_{\alpha,\varphi}(f) = \sum_{j=1}^{\infty} \lambda_j \mathscr{H}_{\alpha,\varphi}(a_j). \tag{61}$$

Let us show that

$$\mathscr{H}_{\alpha,\varphi}(a_k) = \sum_{j\in\mathbb{Z}} c_{k,j} a_{k,j}, \tag{62}$$

where each $a_{k,j}$ again is a central $(\beta\,;q)$-atom and

$$\sum_{k\in\mathbb{Z}} |c_{k,j}|^{p\circ} \infty. \tag{63}$$

We write

$$b_{k,j}(x) = \int_{2^j \leq t \leq 2^{j+1}} a_k\left(\frac{x}{t}\right) t^{-(2\alpha+2)} \varphi(t) dt. \tag{64}$$

So,

$$\mathscr{H}_{\alpha,\varphi}(a_k)(x) = \sum_{j\in\mathbb{Z}} b_{k,j}(x). \tag{65}$$

Now, we check that each $b_{k,j}$ satisfies the same cancellation condition as a_k.

For $i = 0, 1, \cdots, 2s + 1$, where s is the integer part of $(\alpha + 1)(\beta - 1 + 1/q)$, we have

$$\begin{aligned}\int_{\mathbb{R}} b_{k,j}(x)x^i d\mu_\alpha(x) &= \int_{\mathbb{R}} x^i \int_{2^j \leq t \leq 2^{j+1}} a_k\left(\frac{x}{t}\right) t^{-(2\alpha+2)} \varphi(t) dt \\ &= \int_{2^j \leq t \leq 2^{j+1}} t^{-(2\alpha+2)} \varphi(t) dt \int_{\mathbb{R}} a_k\left(\frac{x}{t}\right) x^i d\mu_\alpha(x) \\ &= \int_{2^j \leq t \leq 2^{j+1}} t^{-(2\alpha+2)} \varphi(t) dt \int_{\mathbb{R}} a_k\left(\frac{x}{t}\right) x^i d\mu_\alpha(x) \\ &= \int_{2^j \leq t \leq 2^{j+1}} t^i \varphi(t) dt \int_{\mathbb{R}} a_k(u) u^i d\mu_\alpha(u) = 0.\end{aligned} \tag{66}$$

Also, the size of $b_{k,j}$ is

$$\|b_{k,j}\|_{L^q_\alpha} \leq \int_{2^j \leq t \leq 2^{j+1}} \left\| a_k\left(\frac{\cdot}{t}\right) \right\|_{L^q_\alpha} t^{-2(\alpha+1)(1/q-1)} \varphi(t) dt, \tag{67}$$

then

$$\|b_{k,j}\|_{L^q_\alpha} \le r_k^{-2(\alpha+1)\beta} \int_{2^j \le t \le 2^{j+1}} t^{-2(\alpha+1)(1/q-1)} \varphi(t)dt. \quad (68)$$

If $|x| > 2^{j+1} r_k$, we have

$$\frac{|x|}{t} \ge 2^{-j-1}|x| > r_k, \quad (69)$$

which means $a_k(x/t) = 0$ for all $2^j \le t \le 2^{j+1}$. This tells us that

$$\operatorname{supp}\left(b_{k,j}\right) \subset B\left(0, 2^{j+1} r_k\right). \quad (70)$$

Now, we write

$$\mathscr{H}_{\alpha,\varphi}(a_k) = \sum_{k\in}^{\infty} c_{k,j} a_{k,j}, \quad (71)$$

where

$$c_{k,j} = 2^{2(j+1)(\alpha+1)\beta} \int_{2^j \le t \le 2^{j+1}} t^{-2(\alpha+1)(1/q-1)} \varphi(t)dt, \quad (72)$$

$$a_{k,j} = c_{k,j}^{-1} b_{k,j}. \quad (73)$$

It is easy to check that $a_{k,j}$ is a central (α, q) atom and we have

$$\begin{aligned} c_{k,j} &= 2^{2(\alpha+1)\beta} \int_{2^j \le t \le 2^{j+1}} t^{-2(\alpha+1)(1/q-1)} 2^{2j(\alpha+1)\beta} \varphi(t)dt \\ &\le 2^{2(\alpha+1)\beta} \int_{2^j \le t \le 2^{j+1}} t^{-2(\alpha+1)(1/q-1)} t^{2(\alpha+1)\beta} \varphi(t)dt \left(2^j \le |t|\right) \\ &= 2^{2(\alpha+1)\beta} \int_{2^j \le t \le 2^{j+1}} t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)dt. \end{aligned} \quad (74)$$

Let

$$c'_{k,j} = \int_{2^j \le t \le 2^{j+1}} t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)dt, \quad (75)$$

using Holder inequality yields the following

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \left(c'_{k,j}\right)^p &= \sum_{j\in\mathbb{Z}} \left(\left(c'_{k,j}\right)^p (1+|j|)^{\sigma p} (1+|j|)^{-\sigma p}\right) \\ &\le \left(\sum_{j\in\mathbb{Z}} c'_{k,j} (1+|j|)^{\sigma}\right)^p \left(\sum_{j\in\mathbb{Z}} (1+|j|)^{-\sigma p/1-p}\right)^{1-p} \\ &\le \left(\int_{\mathbb{R}} t^{2(\alpha+1)\left(\beta-1+\frac{1}{q}\right)} \varphi(t)(1+|\log_2 t|)^{\sigma} dt\right)^p \\ &\quad \cdot \left(\sum_{j\in\mathbb{Z}} (1+|j|)^{-\sigma p/1-p}\right)^{1-p}, \end{aligned} \quad (76)$$

since $\sigma > ((1-p)/p)$, then $\sum_{j\in\mathbb{Z}} (1+|j|)^{-\sigma(p/(1-p))} < \infty$. It follows from (75) that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} |c_{k,j}|^p \lesssim \sum_{j\in\mathbb{Z}} \left(c'_{k,j}\right)^p \\ &\lesssim \left(\int_0^\infty t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)(1+|\log_2|t||)^{\sigma} dt\right)^p = \left(C_{p,\sigma}\right)^p. \end{aligned} \quad (77)$$

This shows

$$\mathscr{H}_{\Phi}(f) = \sum_{k=1}^{\infty} \lambda_k \mathscr{H}_{\Phi}(a_k) = \sum_{k=1}^{\infty} \sum_{j\in\mathbb{Z}} \lambda_k c_{k,j} a_{k,j}. \quad (78)$$

By the atomic decomposition, we obtain

$$\begin{aligned} \left\|\mathscr{H}_{\varphi,\alpha}(f)\right\|_{H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\lesssim \left(\sum_{k=1}^{\infty} \sum_{j\in\mathbb{Z}} \left|\lambda_k c_{k,j}\right|^p\right)^{1/p} \lesssim \\ &\cdot \left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right)^{1/p} \lesssim \|f\|_{H\dot{K}_{\alpha,q}^{\beta,p,N}}, \end{aligned} \quad (79)$$

and this end the proof of (i). □

The argument of part (ii) can be proved in an analogous way.

Remark 9. When $\alpha = -1/2$, Theorem 8 reduce to ([10], Theorem 2.5).

We now return to the example of the generalized Cesàro operator $\mathscr{C}_{\alpha,\varphi}$.

Corollary 10. *Let $\alpha \ge (-1/2)$, $0 < p \leqslant 1 < q < \infty$, $\beta \ge 1 - 1/q$, and $N \in \mathbb{N}$; $N > 2(2s + 3 + \alpha)$.*

(i) For $0 < p < 1$, let

$$C_{p,\sigma} = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)(1+|\log_2|t||)^{\sigma} dt. \quad (80)$$

If for some $\sigma > ((1-p)/p)$, $C_p := C_{p,\sigma} < \infty$, *then*

$$\left\| \mathscr{C}_{\alpha,\varphi}(f) \right\|_{H\dot{K}_{\alpha,q}^{\beta,p,N}} \lesssim \|f\|_{H\dot{K}_{\alpha,q}^{\beta,p,N}}. \tag{81}$$

(ii) For $p = 1$, *let*

$$C_1 = \int_0^1 t^{2(\alpha+1)(\beta-1+1/q)} \varphi(t)\, dt. \tag{82}$$

If $C_1 < \infty$, *then*

$$\left\| \mathscr{C}_{\alpha,\varphi}(f) \right\|_{H\dot{K}_{\alpha,q}^{\beta,1,N}} \lesssim \|f\|_{H\dot{K}_{\alpha,q}^{\beta,1,N}}. \tag{83}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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