Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6610836, 8 pages
https://doi.org/10.1155/2021/6610836

Research Article

Inequalities for a Unified Integral Operator for Strongly (α, m)-Convex Function and Related Results in Fractional Calculus

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Received 6 December 2020; Accepted 11 February 2021; Published 16 March 2021

Academic Editor: Shuangping Tao

In this paper, we study integral inequalities which will provide refinements of bounds of unified integral operators established for convex and (α, m)-convex functions. A new definition of function, namely, strongly (α, m)-convex function is applied in different forms and an extended Mittag-Leffler function is utilized to get the required results. Moreover, the obtained results in special cases give refinements of fractional integral inequalities published in this decade.

1. Introduction

Fractional integral operators are very useful and are extensively utilized in mathematics, physics, engineering, and many other subjects. The researchers have introduced variety of fractional integral operators, most of them generalize classical Riemann-Liouville integrals. In recent study of mathematical inequalities, fractional integral operators are playing an important role. Many classical inequalities have been studied for fractional integral operators of different kinds. For example, one can see recent articles dealing with fractional inequalities in [1–15] and in the references therein. Our aim in this article is to study an integral operator for a newly defined strongly (α, m)-convex function, which has direct consequences in fractional integral operators. Next, we give some definitions of generalized fractional integral operators.

Definition 1 (see [16]). Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by

$$\begin{aligned} {}_{g}^{\mu}I_{a^+}f(x) &= \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1}g(t)f(t)dt, \quad x>a, \\ {}_{g}^{\mu}I_{b^-}f(x) &= \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1}g(t)f(t)dt, \quad x<b, \end{aligned} \tag{1}$$

where $\Gamma(.)$ is the gamma function.

A k-analogue of the above definition is defined as follows.

Definition 2 (see [17]). Let $f : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ ; $\Re(\mu$), $k > 0$ are defined by

$$
{}_{g}^{\mu}I_{a^+}^{k}f(x)=\frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{\mu/k-1}g(t)f(t)dt,\ x>a,
$$
$$
{}_{g}^{\mu}I_{b^-}^{k}f(x)=\frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{\mu/k-1}g(t)f(t)dt,\quad x<b,
\tag{2}
$$

where $\Gamma_k(.)$ is given by

$$
\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-t^k/k}dt,\quad \Re(x)>0. \tag{3}
$$

A well-known function named Mittag-Leffler function is defined by [18]

$$
E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{4}
$$

where $\alpha, z\in\mathbb{C}$ and $\Re(\alpha)>0$.

One can see the references [19–22] to study the Mittag-Leffler function and its generalizations.

Fractional integral operator containing an extended Mittag-Leffler function is defined as follows.

Definition 3 (see [1]). Let $\omega,\mu,\alpha',l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\alpha),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$ and $0<k\leq\delta+\Re(\mu)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then, the generalized fractional integral operators $\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha',l,\omega,a^+}f$ and $\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha',l,\omega,b^-}f$ are defined by

$$
\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha',l,\omega,a^+}f\right)(x;p)=\int_a^x (x-t)^{\alpha'-1}E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(\omega(x-t)^\mu;p)f(t)dt,
$$
$$
\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha',l,\omega,b^-}f\right)(x;p)=\int_x^b (t-x)^{\alpha'-1}E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(\omega(t-x)^\mu;p)f(t)dt,
\tag{5}
$$

where

$$
E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(t;p)=\sum_{n=0}^{\infty}\frac{\beta_p(\gamma+nk,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nk}}{\Gamma\left(\mu n+\alpha'\right)}\frac{t^n}{(l)_{n\delta}}, \tag{6}
$$

is the extended generalized Mittag-Leffler function.

Recently, Farid defined the following unified integral operator which unifies several kinds of fractional integrals. Also, new kinds of fractional integrals can be generated from it.

Definition 4 (see [23]). *Let $f,g:[a,b]\to\mathbb{R}$, $0<a<b$, be the functions such that f be positive and $f\in L_1[a,b]$, and g be differentiable and strictly increasing. Also let ϕ/x be an increasing function on $[a,\infty)$ and $\alpha',l,\gamma,c\in\mathbb{C}$, $\Re(\alpha'),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$, $p,\mu,\delta\geq 0$ and $0<k\leq\delta+\mu$. Then, for $x\in[a,b]$, the left and right integral operators are defined by*

$$
\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(x,\omega;p)=\int_a^x K_x^y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)f(y)d(g(y)), \tag{7}
$$

$$
\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(x,\omega;p)=\int_x^b K_y^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)f(y)d(g(y)), \tag{8}
$$

where $K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi)=(\phi(g(x)-g(y)))/(g(x)-g(y))\,E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(\omega(g(x)-g(y))^\mu;p)$.

The following property of the kernel used in the unified integral operator will be used in sequel.

P: Let g and ϕ/I be increasing functions. Then, for $m<t<n$, $m,n\in[a,b]$, the kernel $K_m^n(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi)$ satisfies the following inequality:

$$
K_t^m\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)g'(t)\leq K_n^m\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)g'(t). \tag{9}
$$

This can be obtained from following two straightforward inequalities:

$$
\frac{\phi(g(t)-g(m))}{g(t)-g(m)}g'(t)\leq\frac{\phi(g(n)-g(m))}{g(n)-g(m)}g'(t),
$$
$$
E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(\omega(g(t)-g(m))^\mu;p)\leq E^{\gamma,\delta,k,c}_{\mu,\alpha',l}(\omega(g(n)-g(m))^\mu;p).
\tag{10}
$$

The reverse of inequality (9) holds when g and ϕ/I are of opposite monotonicity. For suitable settings of functions ϕ, g and certain values of parameters included in the Mittag-Leffler function (6), many well-known fractional integral operators can be reproduced, see ([11], Remarks 6 and 7).

The objective of this paper is to obtain bounds of unified integral operators using strongly (α,m)-convexity. It is defined as follows [24].

Definition 5. A function $f:[0,b]\to\mathbb{R}$, $b>0$ is said to be strongly (α,m) -convex, where $(\alpha,m)\in[0,1]^2$ if

$$
\begin{aligned}
f(tx+m(1-t)y)\leq{}& t^\alpha f(x)+m(1-t^\alpha)f(y)\\
&-\lambda m t^\alpha(1-t^\alpha)|y-x|^2,
\end{aligned}
\tag{11}
$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$.

Remark 6. (i) If we put $\lambda=0$, then (11) gives the definition of (α,m) -convex functions

(ii) If we put $\alpha=1$, then (11) gives the definition of strongly m-convex functions

(iii) If we put $(\alpha,m)=(1,m)$, then (11) gives the definition of m-convex function

(iv) If we put $(\alpha, m) = (1, 1)$, then (11) gives the definition of convex function

(v) If we put $(\alpha, m) = (1, 0)$, then (11) gives the definition of star-shaped function

(vi) If we put $\lambda = 0$ and $(\alpha, m) = (1, 0)$, then (11) gives the definition of convex function

In the upcoming section, bounds of unified integral operators are established by using strongly (α, m)-convexity. These bounds provide general formulas to get bounds of many fractional integral operators along with described in [11], Remarks 6 and 7]. Many mathematicians worked on new types of Hadamard inequalities using convex functions, see [9, 25–27]. We also established general Hadamard type inequality by applying Lemma 10 which further produces various inequalities of Hadamard type for fractional integrals.

2. Main Results

Theorem 7. *Let $f : [a, b] \to \mathbb{R}$ be a positive integrable strongly (α, m) -convex function with $m \in (0, 1]$, $0 < a < mb$. Let $g : [a, b] \to \mathbb{R}$ be differentiable and strictly increasing function, also let ϕ/x be an increasing function on $[a, b]$. Then, unified integral operators (7) and (8) satisfy the following inequality:*

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(x,\omega;p)\\
&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\left(\left(mf\left(\frac{x}{m}\right)g(x) - f(a)g(a)\right)\right.\\
&\quad - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(mf\left(\frac{x}{m}\right) - f(a)\right)^\alpha I_{a^+}g(x)\\
&\quad \left.- \frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\left(\Gamma(\alpha+1)^a I_{a^+}g(x) - \frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right)\\
&\quad + K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g;\phi\right)\left(\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right.\\
&\quad - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b) - mf\left(\frac{x}{m}\right)\right)^\alpha I_{b^-}g(x)\\
&\quad \left.- \frac{\lambda(mb-x)^2}{m(b-x)^\alpha}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(x)}{(b-x)^\alpha} - \Gamma(\alpha+1)^\alpha I_{b^-}g(x)\right)\right),
\end{aligned} \tag{12}$$

where ${}^\alpha I_{a^+}g(x), {}^\alpha I_{b^-}g(x)$ are Riemann-Liouville fractional integrals.

Proof. According to property **P**, the following inequalities hold:

$$K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)g'(t) \le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)g'(t), t \in [a, x], \tag{13}$$

$$K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)g'(t) \le K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g;\phi\right)g'(t), t \in (x, b]. \tag{14}$$

Strongly (α, m)-convex function f satisfies the following inequalities:

$$\begin{aligned}
f(t) \le& \left(\frac{x-t}{x-a}\right)^\alpha f(a) + m\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)f\left(\frac{x}{m}\right)\\
&- \lambda m\left(\frac{x-t}{x-a}\right)^\alpha\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left(\frac{x-ma}{m}\right)^2,
\end{aligned} \tag{15}$$

$$\begin{aligned}
f(t) \le& \left(\frac{t-x}{b-x}\right)^\alpha f(b) + m\left(1 - \left(\frac{t-x}{b-x}\right)^\alpha\right)f\left(\frac{x}{m}\right)\\
&- \lambda m\left(\frac{t-x}{b-x}\right)^\alpha\left(1 - \left(\frac{t-x}{b-x}\right)^\alpha\right)\left(\frac{mb-x}{m}\right)^2.
\end{aligned} \tag{16}$$

Multiplying (13) with (15) and integrating over $[a, x]$, one can obtain

$$\begin{aligned}
&\int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)f(t)d(g(t))\\
&\le f(a)K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x\left(\frac{x-t}{x-a}\right)^\alpha d(g(t))\\
&\quad + mf\left(\frac{x}{m}\right)K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)d(g(t))\\
&\quad - \frac{\lambda}{m}(x-ma)^2K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x\left(\frac{x-t}{x-a}\right)^\alpha\\
&\quad \cdot\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)d(g(t)).
\end{aligned} \tag{17}$$

From the above inequality, one can obtain

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(x,\omega;p)\\
&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\left(\left(mf\left(\frac{x}{m}\right)g(x)\right.\right.\\
&\quad - f(a)g(a)\Big) - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(mf\left(\frac{x}{m}\right)\right.\\
&\quad - f(a)\Big)^\alpha I_{a^+}g(x) - \frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\\
&\quad \left.\cdot\left(\Gamma(\alpha+1)^\alpha I_{a^+}g(x) - \frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right).
\end{aligned} \tag{18}$$

Now, adopting the same procedure as we did for (13) and (15), the following inequality can be obtained from (14) and (16):

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(x,\omega;p)\\
&\le K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g;\phi\right)\left(\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right.\\
&\quad - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b) - mf\left(\frac{x}{m}\right)\right)^\alpha I_{b^-}g(x)\\
&\quad - \frac{\lambda(mb-x)^2}{m(b-x)^\alpha}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(x)}{(b-x)^\alpha}\right.\\
&\quad \left.\left.-\Gamma(\alpha+1)^\alpha I_{b^-}g(x)\right)\right).
\end{aligned} \tag{19}$$

From inequalities (18) and (19), inequality (12) can be obtained.

Corollary 8. *If we consider $\alpha = 1$ and $\lambda = 0$ in (12), then the following inequality holds for m-convex functions:*

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(x,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(x,\omega;p) \leq K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\left((g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right)\right) + K_b^x\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\left((g(b)-g(x))\left(f(b)+mf\left(\frac{x}{m}\right)\right)\right). \tag{20}$$

Remark 9. (i) If we consider $\lambda = 0$ in (12), ([28], Theorem 7) is obtained, otherwise the refinement is obtained

(ii) If we consider $(\alpha, m) = (1, 1)$ in (12), ([12], Theorem 4]) is obtained

(iii) If we consider $\phi(t) = t^{\alpha'}$ and $g(x) = x$, in (12), then ([24], Theorem 4]) is obtained

(iv) If we consider $\lambda = 0$, $(\alpha, m) = (1, 1)$ in (12), ([11], Theorem 8]) is obtained

(v) If we consider $\lambda = 0$, $\phi(t) = (\Gamma(\alpha')t^{\alpha'/k})/(k\Gamma_k(\alpha'))$ and $p = \omega = 0$ in (12), then ([10], Theorem 7]) can be obtained

(vi) If we consider $\alpha' = \beta$ in the result of (v), then ([10], Corollary 8]) can be obtained

(vii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\mu)t^{\mu}$, $p = \omega = 0$, and $(\alpha, m) = (1, 1)$ in (12), ([6], Theorem 7]) is obtained

(viii) If we consider $\alpha' = \beta$ in the result of (vi), ([6], Corollary 8]) is obtained

(ix) If we consider $\lambda = 0$, $\phi(t) = (\Gamma(\alpha')t^{\alpha'/k})/(k\Gamma_k(\alpha'))$, $(\alpha, m) = (1, 1)$, $g(x) = x$, and $p = \omega = 0$, then ([4], Theorem 7]) can be obtained

(x) If we consider $\alpha' = \beta$ in the result of (xi), then ([4], Corollary 8]) can be obtained

(xi) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha')t^{\alpha'}$, $g(x) = x$, and $p = \omega = 0$ and $(\alpha, m) = (1, 1)$ in (12), then ([5], Theorem 7]) is obtained

(xii) If we consider $\alpha' = \beta$ in the result of (xi), ([5], Corollary 8]) can be obtained

(xiii) If we consider $\alpha' = \beta = 1$ and $x = a$ or $x = b$ in the result of (xii), ([5], Corollary 12]) can be obtained

(xiv) If we consider $\alpha' = \beta = 1$ and $x = (a + b)/2$ in the result of (xii), ([5], Corollary 15]) can be obtained

To prove the next result, we need the following lemma.

Lemma 10 (see [24]). *Let $f : [a, b] \to \mathbb{R}$ be strongly (α, m)-convex function, $0 < a < mb$. If $f((a + mb - x)/m) = f(x)$ and $(\alpha, m) \in [0, 1]^2$ with $m \in (0, 1]$, then the following inequality holds:*

$$f\left(\frac{a+mb}{2}\right) \leq f(x)\left(\frac{1}{2^\alpha}+m\left(1-\frac{1}{2^\alpha}\right)\right)-\frac{\lambda}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right) \times (a+mb-x-mx)^2. \tag{21}$$

The following result provides upper and lower bounds of sum of operators (7) and (8) in the form of a Hadamard inequality.

Theorem 11. *With the assumptions of Theorem 7 in addition if $f(x) = f((a + mb - x)/m)$ with $m \in (0, 1]$, then we have*

$$\begin{aligned}&\frac{2^\alpha}{(1+m(2^\alpha-1))}\left(f\left(\frac{a+mb}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}1\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}1\right)(b,\omega;p)\right)+\frac{\lambda}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)\right.\\&\cdot\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}(a+mb-x-mx)^2\right)(a,\omega;p)\right)\\&\left.+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}(a+mb-x-mx)^2\right)(b,\omega;p)\right)\\&\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(a,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(b,\omega;p)\\&\leq\left(K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)+K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\right)\\&\times\left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\right.\\&\cdot\left(f(b)-mf\left(\frac{a}{m}\right)\right)^\alpha I_{b^-}g(a)-\frac{\lambda(mb-a)^2}{m(b-a)^\alpha}\\&\left.\cdot\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(a)}{(b-a)^\alpha}-\Gamma(\alpha+1)^\alpha I_{b^-}g(a)\right)\right).\end{aligned} \tag{22}$$

Proof. According to **P**, the following inequalities hold:

$$K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)g'(x) \leq K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)g'(x), \tag{23}$$

$$K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)g'(x) \leq K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)g'(x). \tag{24}$$

Using the definition of strongly (α, m)-convex function f, the following inequality holds:

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^\alpha f(b)+m\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)f\left(\frac{a}{m}\right). \tag{25}$$

Multiplying (23) and (25) and integrating the resulting inequality over $[a, b]$, one can obtain

$$\begin{aligned}&\int_a^b K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)f(x)d(g(x))\\&\leq mf\left(\frac{a}{m}\right)K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\int_a^b\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)d(g(x))\\&+f(b)K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\int_a^b\left(\frac{x-a}{b-a}\right)^\alpha d(g(x))\\&-\frac{\lambda}{m}(x-ma)^2K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\int_a^b\left(\frac{x-a}{b-a}\right)^\alpha\\&\cdot\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)d(g(x)).\end{aligned} \tag{26}$$

From the above inequality, one can obtain

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(b,\omega;p) \le K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right) \cdot \left((f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(f(b)-mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-}g(a) - \frac{\lambda(mb-a)^2}{m(b-a)^{\alpha}}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(a)}{(b-a)^{\alpha}} - \Gamma(\alpha+1)^{\alpha}I_{b^-}g(a))\right). \tag{27}$$

Adopting the same pattern of simplification as we did for (23) and (25), the following inequality can be obtained from (25) and (24)

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(a;p) \le K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)((f(b)g(b) - mf\left(\frac{a}{m}\right)g(a)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot \left(f(b)-mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-}g(a) - \frac{\lambda(mb-a)^2}{m(b-a)^{\alpha}}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(a)}{(b-a)^{\alpha}} - \Gamma(\alpha+1)^{\alpha}I_{b^-}g(a)\right)\Bigg). \tag{28}$$

From (27) and (28), the following inequality can be obtained:

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(b,\omega;p) + \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(a,\omega;p) \le \left(\left(f(b)g(b)-mf\left(\frac{a}{m}\right)g(a)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot \left(f(b)-mf\left(\frac{a}{m}\right)\right)^{\alpha} I_{b^-}g(a) - \frac{\lambda(mb-a)^2}{m(b-a)^{\alpha}} \cdot \left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(a)}{(b-a)^{\alpha}} - \Gamma(\alpha+1)^{\alpha}I_{b^-}g(a))\right) \cdot \left(K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right) + K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\right). \tag{29}$$

Multiplying both sides of (21) by $K_x^a(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi)d(g(x))$, and integrating over $[a,b]$, we have

$$f\left(\frac{a+mb}{2}\right)\int_a^b K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)d(g(x)) \le \left(\frac{1}{2^{\alpha}} + m\left(1-\frac{1}{2^{\alpha}}\right)\right) \times \int_a^b K_x^a \cdot \left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)f(x)d(g(x)) - \frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right) \times \int_a^b K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)(a+mb-x-mx)^2 d(g(x)). \tag{30}$$

From the above inequality, one can obtain

$$\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left(f\left(\frac{a+mb}{2}\right) \cdot \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}1\right)(a,\omega;p) + \frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right) \times \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}(a+mb-x-mx)^2\right)(a,\omega;p)\right) \le \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f\right)(a,\omega;p). \tag{31}$$

Similarly multiplying both sides of (21) by $K_b^x(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi)d(g(x))$, and integrating over $[a,b]$, we have

$$\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))}\left(f\left(\frac{a+mb}{2}\right) \cdot \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,b^-}1\right)(b,\omega;p) + \frac{\lambda}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right) \times \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}(a+mb-x-mx)^2\right)(b,\omega;p)\right) \le \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f\right)(b,\omega;p). \tag{32}$$

From inequalities (29), (31), and (32), inequality (22) can be obtained.

Corollary 12. *If we consider α = 1 and λ = 0 in (22), then the following inequality holds for m-convex function:*

$$\frac{2}{(1+m)}\left(f\left(\frac{a+mb}{2}\right)\left(\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}1\right) \cdot (a,\omega;p) + \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}1\right)(b,\omega;p)\right)\right) \le \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,b^-}f\right)(a,\omega;p) + \left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,a^+}f\right)(b,\omega;p) \le \left(K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right) + K_b^a\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\right) \cdot \left((g(b)-g(a))\left(f(b)+mf\left(\frac{a}{m}\right)\right)\right). \tag{33}$$

Remark 13. (i) If we consider $(\alpha,m)=(1,1)$ in (22), ([29], Theorem 5]) is obtained

(ii) If we consider $\phi(t)=t^{\alpha'}$ and $g(x)=x$, in (22), then ([24], Theorem 6]) is obtained

(iii) If we consider $\lambda=0$, $(\alpha,m)=(1,1)$ in (22), ([11], Theorem 22]) is obtained

(iv) If we consider and $\phi(t)=\Gamma(\alpha')t^{\alpha'+1}$ in (22), $p=\omega=0$, and $(\alpha,m)=(1,1)$ in (22), ([6], Theorem 14]) is obtained

(v) If we consider $\alpha'=\beta$ in the result of (iv), ([6], Corollary 15]) is obtained

(vi) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha')t^{(\alpha'/k)+1}$, (α,m) =(1,1), $g(x)=x$ and $p=\omega=0$ in (22), then ([4], Theorem 14]) can be obtained

(vii) If we consider $\alpha'=\beta$ in the result of (vi), then ([4], Corollary 6]) can be obtained

(viii) If we consider $\lambda=0$, $\phi(t)=\Gamma(\alpha')t^{\alpha'+1}p=\omega=0$, $(\alpha, m)=1$, and $g(t)=t$ in (22), ([5], Theorem 14]) can be obtained

(ix) If we consider $\alpha' = \beta$ in the result of (viii), ([5], Corollary 6]) can be obtained

Theorem 14. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $|f'|$ is strongly (α, m)-convex with $m \in (0, 1]$, $0 < a < mb$ and $g : [a, b] \to \mathbb{R}$ be differentiable and strictly increasing function, also let ϕ/x be an increasing function on $[a, b]$. Then, for unified integral operators, the following inequality holds:*

$$\begin{aligned}
&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+} f * g\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-} f * g\right)(x,\omega;p)\right| \\
&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right)\right. \\
&\quad - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^\alpha I_{a^+}g(x) \\
&\quad \left. - \frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\left(\Gamma(\alpha+1)^\alpha I_{a^+}g(x) - \frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right) \\
&\quad + K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l}, g;\phi\right)\left(\left(f(b)g(b) - mf\left(\frac{x}{m}\right)g(x)\right)\right. \\
&\quad - \frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(f(b) - mf\left(\frac{x}{m}\right)\right)^\alpha I_{b^-}g(x) \\
&\quad - \frac{\lambda(mb-x)^2}{m(b-x)^\alpha}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(x)}{(b-x)^\alpha} - \Gamma(\alpha+1)^\alpha I_{b^-}g(x))\right),
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+} f * g\right)(x,\omega;p) &:= \int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right) f'(t)\,d(g(t)), \\
\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-} f * g\right)(x,\omega;p) &:= \int_x^b K^x_t\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right) f'(t)\,d(g(t)).
\end{aligned} \tag{35}$$

Proof. Using the definition of strongly (α, m)-convexity for $|f'|$, the following inequality is valid:

$$\begin{aligned}
|f'(t)| \le \left(\frac{x-t}{x-a}\right)^\alpha |f'(a)| + m\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right| \\
- \lambda m\left(\frac{x-t}{x-a}\right)^\alpha\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left(\frac{x-ma}{m}\right)^2.
\end{aligned} \tag{36}$$

The inequality (36) can be written as follows:

$$\begin{aligned}
&-\left(\left(\frac{x-t}{x-a}\right)^\alpha |f'(a)| + m\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right|\right. \\
&\quad \left. - \lambda m\left(\frac{x-t}{x-a}\right)^\alpha\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left(\frac{x-ma}{m}\right)^2\right) \\
&\le f'(t) \le \left(\frac{x-t}{x-a}\right)^\alpha |f'(a)| + m\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right| \\
&\quad - \lambda m\left(\frac{x-t}{x-a}\right)^\alpha\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left(\frac{x-ma}{m}\right)^2.
\end{aligned} \tag{37}$$

We consider the second inequality of (37)

$$\begin{aligned}
f'(t) \le \left(\frac{x-t}{x-a}\right)^\alpha |f'(a)| + m\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right| \\
- \lambda m\left(\frac{x-t}{x-a}\right)^\alpha\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left(\frac{x-ma}{m}\right)^2.
\end{aligned} \tag{38}$$

Multiplying (13) and (38) and integrating over $[a, x]$, we can obtain

$$\begin{aligned}
&\int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right) f'(t)\,d(g(t)) \\
&\le |f'(a)| K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x \\
&\quad \cdot \left(\frac{x-t}{x-a}\right)^\alpha d(g(t)) + m\left|f'\left(\frac{x}{m}\right)\right| K^a_x \\
&\quad \cdot \left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x \left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right) d(g(t)) \\
&\quad - \frac{\lambda}{m}(x-ma)^2 K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\int_a^x \left(\frac{x-t}{x-a}\right)^\alpha \\
&\quad \cdot \left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right) d(g(t)).
\end{aligned} \tag{39}$$

From the above inequality, one can obtain

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+} f * g\right)(x,\omega;p) \\
&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right)\right. \\
&\quad - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^\alpha I_{a^+}g(x) \\
&\quad \left. - \frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\left(\Gamma(\alpha+1)^\alpha I_{a^+}g(x) - \frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right).
\end{aligned} \tag{40}$$

If we consider the left hand side from inequality (37) and adopt the same pattern as did for the right hand side inequality, then

$$\begin{aligned}
&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+} f * g\right)(x,\omega;p) \\
&\ge -K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l}, g;\phi\right)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x) - |f'(a)|g(a)\right)\right. \\
&\quad - \frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|f'\left(\frac{x}{m}\right)\right| - |f'(a)|\right)^\alpha I_{a^+}g(x) \\
&\quad \left. - \frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\left(\Gamma(\alpha+1)^\alpha I_{a^+}g(x) - \frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right).
\end{aligned} \tag{41}$$

From (40) and (41), the following inequality is obtained:

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f * g\right)(x,\omega;p)\right|\\&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\left(\left(m\left|f'\left(\frac{x}{m}\right)\right|g(x)-|f'(a)|g(a)\right)\right.\\&\quad-\frac{\Gamma(\alpha+1)}{(x-a)^\alpha}\left(m\left|f'\left(\frac{x}{m}\right)\right|-|f'(a)|\right)^\alpha I_{a^+}g(x)\\&\quad\left.-\frac{\lambda(x-ma)^2}{m(x-a)^\alpha}\left(\Gamma(\alpha+1)^\alpha I_{a^+}g(x)-\frac{\Gamma(2\alpha+1)^{2\alpha}I_{a^+}g(x)}{(x-a)^\alpha}\right)\right).\end{aligned} \tag{42}$$

Now, using again the definition of strongly (α,m)-convexity for f, the following inequality is valid:

$$\begin{aligned}|f'(t)|&\le\left(\frac{t-x}{b-x}\right)^\alpha|f'(b)|+m\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\left|f'\left(\frac{x}{m}\right)\right|\\&\quad-\lambda m\left(\frac{t-x}{b-x}\right)^\alpha\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\left(\frac{mb-x}{m}\right)^2.\end{aligned} \tag{43}$$

On the same procedure as we did for (13) and (36), one can obtain following inequality from (14) and (43):

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f * g\right)(x,\omega;p)\right|\\&\le K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\left(\left(|f'(b)|g(b)-m\left|f'\left(\frac{x}{m}\right)\right|g(x)\right)\right.\\&\quad-\frac{\Gamma(\alpha+1)}{(b-x)^\alpha}\left(|f'(b)|-m\left|f'\left(\frac{x}{m}\right)\right|\right)^\alpha I_{b^-}g(x)\\&\quad-\frac{\lambda(mb-x)^2}{m(b-x)^\alpha}\left(\frac{\Gamma(2\alpha+1)^{2\alpha}I_{b^-}g(x)}{(b-x)^\alpha}\right.\\&\quad\left.\left.-\Gamma(\alpha+1)^\alpha I_{b^-}g(x)\right)\right).\end{aligned} \tag{44}$$

From (42) and (44), inequality (34) can be obtained.

Corollary 15. *If we consider $\alpha = 1$ and $\lambda = 0$ in (34), then the following inequality holds for the m-convex function:*

$$\begin{aligned}&\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha',l,a^+}f * g\right)(x,\omega;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\beta,l,b^-}f * g\right)(x,\omega;p)\right|\\&\le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha',l},g;\phi\right)\left((g(x)-g(a))\left(\left|f'\left(\frac{x}{m}\right)\right|+|f'(a)|\right)\right)\\&\quad+K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\beta,l},g;\phi\right)\left((g(b)-g(x))\left(|f'(b)|+\left|f'\left(\frac{x}{m}\right)\right|\right)\right).\end{aligned} \tag{45}$$

Remark 16. (i) If we consider $\lambda = 0$ in (34), ([28], Theorem 14]) is obtained

(ii) If we consider $(\alpha, m) = (1, 1)$ in (34), ([29], Theorem 6]) is obtained

(iii) If we consider $\phi(t) = t^{\alpha'}$ and $g(x) = x$, in (34), then ([24], Theorem 5]) is obtained

(iv) If we consider $\lambda = 0$, $(\alpha, m) = (1, 1)$ in (34), then ([11], Theorem 25]) is obtained

(v) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha')t^{(\alpha'/k)+1}$ and $p = \omega = 0$ in (34), then ([10], Theorem 11]) can be obtained

(vi) If we consider $\alpha' = \beta$ in the result of (v), then ([10], Corollary 12]) can be obtained

(vii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha')t^{\alpha'+1}$ and $(\alpha, m) = (1, 1)$ in (34), then ([6], Theorem 11]) is obtained

(viii) If we consider $\alpha' = \beta$ in the result of (vii), then ([6], Corollary 12]) is obtained

(ix) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha')t^{(\alpha'/k)+1}$, $(\alpha, m) =(1,1)$, $g(x) = x$, and $p = \omega = 0$ in (34), then ([4], Theorem 11]) can be obtained

(x) If we consider $\alpha' = \beta$ in the result of (ix), then ([4], Corollary 4]) can be obtained

(xi) If we consider $\alpha' = \beta = k = 1$ and $x = a + b/2$, in the result of (ix), then ([4], Corollary 5]) can be obtained

(xii) If we consider $\lambda = 0$, $\phi(t) = \Gamma(\alpha')t^{\alpha'+1}$, $g(x) = x$, and $p = \omega = 0$ and $(\alpha, m) = (1, 1)$ in (34), then ([5], Theorem 11]) is obtained

(xiii) If we consider $\alpha' = \beta$ in the result of (xii), then ([5], Corollary 5]) can be obtained

3. Concluding Remarks

In this paper, integral inequalities for strongly (α, m)-convex functions are given by utilizing unified integral operators. Results for m-convex functions are presented in particular in the form of corollaries. In the form of remarks, many published results are highlighted as consequences of presented results. The reader can get integral and fractional integral inequalities for convex functions, strongly convex functions, (α, m)-convex functions, m-convex functions, and strongly m-convex functions.

Data Availability

There is no any data required for this paper

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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