Research Article

Exciting Fixed Point Results on a Novel Space with Supportive Applications

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The goal of this paper is to present a new space, a complex valued controlled rectangular b -metric space (for short, v_C -metric space). Some examples and topological properties of v_C -metric spaces are given. Also, some related common fixed point results are discussed. Our results generalize a lot of works in this direction. Moreover, we apply the theoretical results to find a unique solution of a complex valued Atangana-Baleanu fractional integral operator and a system of complex linear equations. Finally, a numerical example to find the current that passes through the RLC circuit is illustrated.

1. Introduction

In 2011, the concept of a complex valued metric space has been initiated by Azam et al. [1]. Later, Al-Mezel et al. [2] emphasized that this space is just a particular case of cone metric spaces [3, 4]. Several fixed point results have been investigated on complex valued metric spaces; see [5–9]. The notion of a b -metric space, as a generalization of metric spaces, was introduced by Bakhtin [10] and Czerwik [11]. Combining the above concepts, Rao et al. [12] initiated the notion of complex valued b -metric spaces. For related results on this setting, see [13, 14]. Going in the same direction, recently, Ullah et al. [15] presented complex valued extended b -metric spaces to extend the idea of extended b -metric spaces [16].

On the other hand, in the last few years, the fixed point theory became an essential tool to resolve several problems in nonlinear analysis, including fractional calculus. The contribution of our paper is to solve a complex valued Atangana-Baleanu fractional integral operator and a system of complex linear equations. The idea is to introduce the concept of a

new space, the so-called complex valued controlled rectangular b -metric space (for short, v_C -metric space). We also provide a numerical example illustrating the fact that the current passes through the RLC circuit.

2. Statements about v_C -Metric Spaces

Let $\mathbb{C}$ be the set of complex numbers and $\eta_1, \eta_2 \in \mathbb{C}$. Define the below partial order $\preceq$ on $\mathbb{C}$ as

$$\eta_1 \preceq \eta_2 \text{ iff } \Re(\eta_1) \leq \Re(\eta_2) \text{ and } \Im(\eta_1) \leq \Im(\eta_2). \quad (1)$$

Hence, $\eta_1 \preceq \eta_2$ if one of the below statements holds:

$$(\beth_1) \Re(\eta_1) = \Re(\eta_2) \text{ and } \Im(\eta_1) < \Im(\eta_2)$$

$$(\beth_2) \Re(\eta_1) < \Re(\eta_2) \text{ and } \Im(\eta_1) = \Im(\eta_2)$$

$$(\beth_3) \Re(\eta_1) < \Re(\eta_2) \text{ and } \Im(\eta_1) < \Im(\eta_2)$$

$$(\beth_4) \Re(\eta_1) = \Re(\eta_2) \text{ and } \Im(\eta_1) = \Im(\eta_2)$$

Also, we can write $\eta_1 \prec \eta_2$ if $\eta_1 \neq \eta_2$ and one of $(\beth_1)$, $(\beth_2)$, and $(\beth_3)$ holds. By following the same pattern, we write $\eta_1 < \eta_2$ if only $(\beth_3)$ is fulfilled. This implies that

- (h₁) $0 \leq \eta_1 \leq \eta_2 \Rightarrow |\eta_1| < |\eta_2|$
- (h₂) $\eta_1 \leq \eta_2$ and $\eta_2 < \eta_3 \Rightarrow \eta_1 < \eta_3$
- (h₃) $0 \leq \eta_1 \leq \eta_2 \Rightarrow |\eta_1| < |\eta_2|$
- (h₄) if $l, m \in \mathbb{R}$, $0 \leq l \leq m$ and $\eta_1 \leq \eta_2$, then $l\eta_1 \leq m\eta_2$

This important part is devoted to present new definitions and examples of a $v_{\mathbb{C}}$ -metric space.

Definition 1. A mapping $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$ is called a $v_{\mathbb{C}}$ -metric on a nonempty set ∇ , if it satisfies the following hypotheses, for all $l, m \in \nabla$,

- (v_{C1}) $v_{\mathbb{C}}(l, m) = 0 \Leftrightarrow l = m$
- (v_{C2}) $v_{\mathbb{C}}(l, m) = v_{\mathbb{C}}(m, l)$
- (v_{C3}) there is a function $\rho : \nabla \times \nabla \rightarrow [1, \infty)$ such that $v_{\mathbb{C}}(l, m) \leq \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)]$ (the so-called controlled rectangular inequality), for all distinct points $i, j \in \nabla \setminus \{l, m\}$

Then, $(\nabla, v_{\mathbb{C}})$ is called a $v_{\mathbb{C}}$ -metric space.

Example 1. Assume that (∇, ω) is a complex valued extended rectangular b -metric with coefficient $s \geq 1$ and $\tau : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing continuous function with $\mu \leq \tau(\mu)$, for all $\mu > 0$ and $\tau(0) = 0$. Suppose that $v_{\mathbb{C}}(m, l) = \tau(\omega(m, l))$ for all $l, m \in \nabla$. For any two distinct points $i, j \in \nabla \setminus \{l, m\}$, one can obtain that

$$\begin{aligned} v_{\mathbb{C}}(m, l) &= \tau(\omega(m, l)) \leq \tau(s\omega(l, i) + s\omega(i, j) + s\omega(j, m)) \\ &\leq \tau(\tau(\omega(l, i)) + \tau(\omega(i, j)) + \tau(\omega(j, m))) \\ &= \rho(sv_{\mathbb{C}}(l, i) + sv_{\mathbb{C}}(i, j) + sv_{\mathbb{C}}(j, m)). \end{aligned} \quad (2)$$

Thus, the pair $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space with $\rho(\mu) = (\tau s)\mu$.

Example 2. Let $\nabla = [0, \infty)$. Define $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$, $\rho : \nabla \rightarrow [0, \infty)$ by $v_{\mathbb{C}}(m, l) = |m - il|^2$, and $\rho(l, m) = l + m + 3^{\ell-1}$, $\ell > 1$, respectively. The conditions (v_{C1}) and (v_{C2}) are clear and the condition (v_{C3}) follows immediately by the Jensen inequality

$$\begin{aligned} v_{\mathbb{C}}(m, l) &= (|m - il|)^2 \leq 3^{\ell-1} [|m - ip|^2 + |p - iq|^2 + |q - il|^2] \\ &\leq (l + m + 3^{\ell-1}) [|m - ip|^2 + |p - iq|^2 + |q - il|^2] \\ &= \rho[v_{\mathbb{C}}(m, p) + v_{\mathbb{C}}(p, q) + v_{\mathbb{C}}(q, l)], \end{aligned} \quad (3)$$

for all $l, m, p, q \in \nabla$. This implies that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space with $\rho(l, m) = l + m + 3^{\ell-1}$. Note that in general $v_{\mathbb{C}}$ is not continuous.

Example 3. Let $\nabla = [0, 1]$. Define $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$, $\rho : \nabla \times \nabla \rightarrow [1, \infty)$ by $v_{\mathbb{C}}(m, l) = (4/3)(|m - l|^2 + i|m - l|^2)$, and $\rho(l, m) = l + m + 4$, respectively. The conditions (v_{C1}) and (v_{C2}) are trivial. We only show the condition (v_{C3}). For all $l, m, p, q, r, s \in \nabla$, we have

$$\begin{aligned} v_{\mathbb{C}}(m, l) &= \frac{4}{3}(|m - l|^2 + i|m - l|^2) \\ &= \frac{4}{3}(|m - p + p - q + q - l|^2 + i|m - p + p - q + q - l|^2) \\ &\leq \frac{4}{3}(|m - p|^2 + |p - q|^2 + |q - l|^2 + 2|m - p||p - q| \\ &\quad + 2|m - p||q - l| + 2|p - q||q - l| + i[|m - p|^2 \\ &\quad + |p - q|^2 + |q - l|^2 + 2|m - p||p - q| + 2|m - p||q - l| \\ &\quad + 2|p - q||q - l|]) \leq \frac{4}{3}(|m - p|^2 + |p - q|^2 + |q - l|^2 \\ &\quad + |m - p|^2 + |p - q|^2 + |m - p|^2 + |q - l|^2 + |p - q|^2 \\ &\quad + |q - l|^2 + i[|m - p|^2 + |p - q|^2 + |q - l|^2 + |m - p|^2 \\ &\quad + |p - q|^2 + |m - p|^2 + |q - l|^2 + |p - q|^2 + |q - l|^2]) \\ &= 4(|m - p|^2 + |p - q|^2 + |q - l|^2 + i[|m - p|^2 + |p - q|^2 \\ &\quad + |q - l|^2]) \leq (l + m + 4) \left(\frac{4}{3} [|m - p|^2 + i|m - p|^2] \right. \\ &\quad \left. + \frac{4}{3} [|p - q|^2 + i|p - q|^2] + \frac{4}{3} [|q - l|^2 + i|q - l|^2] \right) \\ &= \rho(l, m)[v_{\mathbb{C}}(m, p) + v_{\mathbb{C}}(p, q) + v_{\mathbb{C}}(q, l)]. \end{aligned} \quad (4)$$

Thus, $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space with $\rho(l, m) = l + m + 4$.

Example 4. Let $\nabla = Y_1 \cup Y_2$, where $Y_1 = \{1/n : n \in \mathbb{N}\}$ and $Y_2 = \{0, 2\}$, and choose a function $\rho(l, m) = l + m + 2$ for all $l, m \in \nabla$. Define a function $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$ by

$$v_{\mathbb{C}}(l, m) = \begin{cases} 0, & \text{if } m = l, \\ i, & \text{if } l \neq m, l, m \in Y_1 \text{ or } l, m \in Y_2, \\ im, & \text{if } l \in Y_2 \text{ and } m \in Y_1, \\ l, & \text{if } l \in Y_1 \text{ and } m \in Y_2. \end{cases} \quad (5)$$

It is clear that the pair $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space with $\rho(l, m) = l + m + 2$. If we take the sequence $\{1/n\}_{n \in \mathbb{N}}$, it is easy to prove that this sequence converges to both 0 and 2, so it is not a Cauchy sequence. Moreover, $v_{\mathbb{C}}$ is not continuous, since

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0, \lim_{n \rightarrow \infty} v\left(\frac{1}{n}, \frac{1}{3}\right) = i \neq \frac{i}{3} = v\left(0, \frac{1}{3}\right). \quad (6)$$

This means that there exists a sequence in $(\nabla, v_{\mathbb{C}})$ which is not convergent to a unique point in ∇ . Hence, $(\nabla, v_{\mathbb{C}})$ is not metrizable.

Example 5. Assume that $\nabla = \{1, 2, 3, 4\}$. Define $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$ by

$$\begin{aligned}
v_{\mathbb{C}}(1, 3) &= v_{\mathbb{C}}(3, 1) = 1 - i, & v_{\mathbb{C}}(1, 4) &= v_{\mathbb{C}}(4, 1) = 1 + i, \\
v_{\mathbb{C}}(1, 2) &= v_{\mathbb{C}}(2, 1) = 4 + 4i, & v_{\mathbb{C}}(2, 3) &= v_{\mathbb{C}}(3, 2) = i, \\
v_{\mathbb{C}}(2, 4) &= v_{\mathbb{C}}(4, 2) = 2 + 2i, & v_{\mathbb{C}}(3, 4) &= v_{\mathbb{C}}(4, 3) = 1 + 2i, \\
v_{\mathbb{C}}(1, 1) &= v_{\mathbb{C}}(2, 2) = v_{\mathbb{C}}(3, 3) = v_{\mathbb{C}}(4, 4) = 0.
\end{aligned} \tag{7}$$

Define $\rho : \nabla \times \nabla \rightarrow [1, \infty)$ by

$$\begin{aligned}
\rho(1, 3) &= \rho(3, 1) = 1, & \rho(1, 4) &= \rho(4, 1) = \frac{3}{2}, \\
\rho(1, 2) &= \rho(2, 1) = 2, & \rho(2, 3) &= \rho(3, 2) = \frac{4}{3}, \\
\rho(2, 4) &= \rho(4, 2) = \frac{5}{4}, & \rho(3, 4) &= \rho(4, 3) = \frac{8}{7}.
\end{aligned} \tag{8}$$

Now, we examine the condition $(v_{\mathbb{C}3})$ in the following cases:

- (i) When $l = m$, then the result follows immediately
- (ii) When $(l = 1, m = 2)$ and $(i = 3, j = 4)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(1, 2)| = |4 + 4i| \leq |8 + 6i| = 2|4 + 3i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{9}$$

- (iii) When $(l = 1, m = 3)$ and $(i = 2, j = 4)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(1, 3)| = |1 - i| \leq |7 + 8i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{10}$$

- (iv) When $(l = 1, m = 4)$ and $(i = 2, j = 3)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(1, 4)| = |1 + i| \leq \left| \frac{15}{2} + \frac{21}{2}i \right| = \frac{3}{2}|5 + 7i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{11}$$

- (v) When $(l = 2, m = 3)$ and $(i = 1, j = 4)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(2, 3)| = |i| \leq \left| 8 + \frac{28}{3}i \right| = \frac{4}{3}|6 + 7i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{12}$$

- (vi) When $(l = 2, m = 4)$ and $(i = 1, j = 3)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(2, 4)| = |2 + 2i| \leq \left| \frac{15}{2} + \frac{25}{4}i \right| = \frac{5}{4}|6 + 5i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{13}$$

- (vii) When $(l = 3, m = 4)$ and $(i = 1, j = 2)$, then

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(3, 4)| = |1 + 2i| \leq \left| 8 + \frac{40}{7}i \right| = \frac{8}{7}|7 + 5i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{14}$$

Note that

$$\begin{aligned}
v_{\mathbb{C}}(l, m) &= |v_{\mathbb{C}}(1, 2)| = |4 + 4i| > |4 + 3i| \\
&= \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)].
\end{aligned} \tag{15}$$

Thus, according to the above cases, $v_{\mathbb{C}}$ is not a complex valued rectangular b -metric for the function ρ , but $v_{\mathbb{C}}$ is a $v_{\mathbb{C}}$ -metric (conditions $(v_{\mathbb{C}1})$ and $(v_{\mathbb{C}2})$ are trivial).

Example 6. Let $\nabla = \mathbb{C}^n$ and define $v_{\mathbb{C}}(l, m) = (4/3) \max_{1 \leq t \leq n} |l_t - m_t|^2$, where $l, m \in \nabla$, then $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space with the function $\rho(l, m) = 4 + l + m$.

Verification: initially, we show that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space by discussing the following conditions. Assume that $l, m \in \nabla$ and $i, j \in \nabla \setminus \{l, m\}$, where $i \neq j$.

$(v_{\mathbb{C}1})$ Since $|l_t - m_t| \geq 0$, one writes $(4/3) \max_{1 \leq t \leq n} |l_t - m_t| \geq 0$. So that $v_{\mathbb{C}}(l, m) \geq 0$.

$(\Rightarrow)$ Let $v_{\mathbb{C}}(l, m) = 0$, then $(4/3) \max_{1 \leq t \leq n} |l_t - m_t| = 0 \Rightarrow l_t = m_t$ for $1 \leq t \leq n \Rightarrow l = m$.

$(\Leftarrow)$ If $l = m$, then for each $1 \leq t \leq n$, $|l_t - m_t| = 0 \Rightarrow (4/3) \max_{1 \leq t \leq n} |l_t - m_t| = 0 \Rightarrow v_{\mathbb{C}}(l, m) = 0$.

$(v_{\mathbb{C}2})$

$$v_{\mathbb{C}}(l, m) = (4/3) \max_{1 \leq t \leq n} |l_t - m_t| = (4/3) \max_{1 \leq t \leq n} |m_t - l_t| = v_{\mathbb{C}}(m, l).$$

$(v_{\mathbb{C}3})$ Since for all $1 \leq t \leq n$, one gets

$$\begin{aligned}
\frac{4}{3}|l_t - m_t|^2 &= \frac{4}{3}|l_t - i_t + i_t - j_t + j_t - m_t|^2 \\
&\leq 4(|l_t - i_t|^2 + |i_t - j_t|^2 + |j_t - m_t|^2) \\
&\leq (4 + m + l) \left[\frac{4}{3}|l_t - i_t|^2 + \frac{4}{3}|i_t - j_t|^2 + \frac{4}{3}|j_t - m_t|^2 \right].
\end{aligned} \tag{16}$$

Taking the maximum on two sides for all $1 \leq t \leq n$, we have

$$\begin{aligned} \frac{4}{3} \max_{1 \leq t \leq n} |l_t - m_t|^2 &\leq (4 + m + l) \left[\frac{4}{3} \max_{1 \leq t \leq n} |l_t - i_t|^2 + \frac{4}{3} \max_{1 \leq t \leq n} |i_t - j_t|^2 \right. \\ &\quad \left. + \frac{4}{3} \max_{1 \leq t \leq n} |j_t - m_t|^2 \right]. \end{aligned} \quad (17)$$

This leads to $v_{\mathbb{C}}(l, m) \leq \rho(l, m)[v_{\mathbb{C}}(l, i) + v_{\mathbb{C}}(i, j) + v_{\mathbb{C}}(j, m)]$. From the above condition, we conclude that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space.

Finally, we show the completeness property of $(\nabla, v_{\mathbb{C}})$. It means that for every Cauchy sequence in $(\nabla, v_{\mathbb{C}})$ is convergent. Let $\{m_n\}$ be a Cauchy sequence in $(\nabla, v_{\mathbb{C}})$, then for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that

$$v_{\mathbb{C}}(m_n, m_j) < \varepsilon, \forall n, j > n_0. \quad (18)$$

That is, for $1 \leq t \leq n$, it satisfies

$$v_{\mathbb{C}}(m_n, m_j) = \frac{4}{3} \max_{1 \leq t \leq n} |m_t^{(n)} - m_t^{(j)}|^2 < \varepsilon. \quad (19)$$

Hence, for every t , $(l_t^{(1)}, l_t^{(2)}, \dots)$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $l_t^{(n)} \rightarrow l_t$ for $n \rightarrow \infty$. Now, define $m = (m_1, m_2, \dots) \in \mathbb{C}^n$ so that $m_n \rightarrow m$. Passing $j \rightarrow \infty$ in (19), we have $(4/3) \max_{1 \leq t \leq n} |m_t^{(n)} - m_t|^2 < \varepsilon$, for $1 \leq t \leq n$, i.e., $v_{\mathbb{C}}(m_n, m) < \varepsilon$. This implies that $m_n \rightarrow m$, so $(\nabla, v_{\mathbb{C}})$ is complete.

Now, we introduce topological properties of a $v_{\mathbb{C}}$ -metric space.

Definition 2. Assume that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space, $\{m_n\}$ is a sequence in ∇ and $m \in \nabla$. For every $c \in \mathbb{C}$ with $0 < c$, there exists $N \in \mathbb{N}$ such that

- (i) for all $n \geq N$, $v_{\mathbb{C}}(m_n, m) < c$, then m_n is called convergent, $\{m_n\}$ converges to m , or we say that m is a limit point of $\{m_n\}$ and write $\lim_{n \rightarrow \infty} m_n = m$ or $\{m_n\} \xrightarrow{n \rightarrow \infty} m$
- (ii) for all $n \geq N$, $v_{\mathbb{C}}(m_n, m_{n+j}) < c$, $j \in \mathbb{N}$, then $\{m_n\}$ is called a Cauchy sequence
- (iii) if every Cauchy sequence in ∇ is convergent, then the pair $(\nabla, v_{\mathbb{C}})$ is called a complete $v_{\mathbb{C}}$ -metric space

Lemma 1. Assume that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space and $\{m_n\}$ is a sequence in ∇ , then $\{m_n\}$ is a Cauchy sequence iff $|v_{\mathbb{C}}(m_n, m_j)| \xrightarrow{n, j \rightarrow \infty} 0$, where $n, j \in \mathbb{N}$.

Proof. $(\Rightarrow)$ Let $\{m_n\}$ be a Cauchy sequence in ∇ , then for every $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ so that

$$v_{\mathbb{C}}(m_n, m_j) < \varepsilon, \forall n, j > n_0. \quad (20)$$

Since $\varepsilon \in \mathbb{C}$, there are $\alpha, \beta \in \mathbb{R}$ such that $\varepsilon = \alpha + i\beta$. Consider $\delta = |\varepsilon| = \sqrt{\alpha^2 + \beta^2}$, and take the absolute value in two

sides of (20), we get $|v_{\mathbb{C}}(m_n, m_j)| < \delta$. From definition of the limit, we get $|v_{\mathbb{C}}(m_n, m_j)| \rightarrow 0$ as $n, j \rightarrow \infty$.

$(\Leftarrow)$ Let $|v_{\mathbb{C}}(m_n, m_j)| \rightarrow 0$ as $n, j \rightarrow \infty$. It follows that $|v_{\mathbb{C}}(m_n, m_j)| < \delta$, for every $\delta > 0$. Since $v_{\mathbb{C}}(m_n, m_j) \in \mathbb{C}$, there exists $\varepsilon \in \mathbb{C}$ such that $\delta = |\varepsilon|$. It holds that $v_{\mathbb{C}}(m_n, m_j) < \varepsilon$, for every $\varepsilon > 0$. Thus, there is $n_0 \in \mathbb{N}$ so that for $n, j > n_0$, $\{m_n\}$ is a complex valued Cauchy sequence.

Lemma 2. Assume that $(\nabla, v_{\mathbb{C}})$ is a $v_{\mathbb{C}}$ -metric space and $\{m_n\}$ is a sequence in ∇ , then m is the limit point of $\{m_n\}$ iff $|v_{\mathbb{C}}(m_n, m)| \xrightarrow{n \rightarrow \infty} 0$.

Proof. Passing $j \rightarrow \infty$ in the proof of Lemma 1, we get the result.

Remark 1. If we take $v_{\mathbb{C}}(l, m) = s \geq 1$, for all $m, l \in \nabla$, then an $v_{\mathbb{C}}$ -metric space reduces to a complex valued rectangular b-metric space.

The below lemma is very important in the equal.

Lemma 3. Let $(\nabla, v_{\mathbb{C}})$ be a $v_{\mathbb{C}}$ -metric space and $\{m_n\}$ be a Cauchy sequence in ∇ such that $m_n \neq m_j$, whenever $n \neq j$. Then, $\{m_n\}$ converges at most to one point.

Proof. Suppose that the sequence $\{m_n\}$ has two limit points $m, m^* \in \nabla$, that is, $\lim_{n \rightarrow \infty} v_{\mathbb{C}}(m_n, m) = 0 = \lim_{n \rightarrow \infty} v_{\mathbb{C}}(m_n, m^*)$. Since $\{m_n\}$ is Cauchy, from $(v_{\mathbb{C}3})$, for $m_n \neq m_j$, whenever $n \neq j$, we can write

$$\begin{aligned} |v_{\mathbb{C}}(m, m^*)| &\leq \rho(m, m^*) [|v_{\mathbb{C}}(m, m_n)| + |v_{\mathbb{C}}(m_n, m_j)| \\ &\quad + |v_{\mathbb{C}}(m_j, m^*)|] \rightarrow 0 \text{ as } n, j \rightarrow \infty. \end{aligned} \quad (21)$$

It follows that $|v_{\mathbb{C}}(m, m^*)| = 0$, i.e., $m = m^*$. Thus, $\{m_n\}$ converges at most to one point.

The following diagram shows the latest generalizations of complex valued metric spaces. However, the inverse implications need not be true.

$$\begin{array}{ccccc} \text{c.v.m.s} & \rightarrow & \text{c.v.b.m.s} & \rightarrow & \text{extended c.v.b.m.s} \\ \downarrow & & \downarrow & & \downarrow \\ \text{c.v.rectangular m.s} & \rightarrow & \text{c.v.rectangular b.m.s} & \rightarrow & \text{c.v.controlled rectangular b.m.s} \end{array} \quad (22)$$

where c.v.m.s. refers to a complex valued metric space.

3. Exciting Fixed Point Results on $v_{\mathbb{C}}$ -Metric Spaces

This section is aimed at introducing two fixed point theorems in the context of $v_{\mathbb{C}}$ -metric spaces.

Theorem 1. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let the mappings $\Theta, \Xi : \nabla \rightarrow \nabla$ satisfy

$$\kappa^{\mu} v_{\mathbb{C}}(\Theta l, \Xi m) \leq \zeta v_{\mathbb{C}}(l, m) + \omega \frac{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Xi m)}{1 + v_{\mathbb{C}}(l, m)}, \forall l, m \in \nabla, \quad (23)$$

where $0 < v_{\mathbb{C}}(l, m)$, $\kappa, \mu \geq 1$ and ζ, ω are nonnegative real numbers with $\zeta + \omega < 1$.

Assume also

$$\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < \frac{1}{D}, \text{ where } D = \frac{\zeta}{\kappa^{\mu} - \omega} < 1, \quad (24)$$

and for all $m \in \nabla$, $\lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite. Then, Θ and Ξ have a common fixed point (cfp).

Proof. Let m_0 be an arbitrary point such that the sequence $\{m_n = \Xi^n m_0\}$ in ∇ satisfies the conditions of theorem and iterates below as follows:

$$m_{2n+1} = \Theta m_{2n} \text{ and } m_{2n+2} = \Xi m_{2n+1}, n \geq 0. \quad (25)$$

Assume that

$$\begin{aligned} \kappa^{\mu} v_{\mathbb{C}}(m_{2n+1}, m_{2n+2}) &= \kappa^{\mu} v_{\mathbb{C}}(\Theta m_{2n}, \Xi m_{2n+1}) \zeta v_{\mathbb{C}}(m_{2n}, m_{2n+1}) \\ &\leq \omega \frac{v_{\mathbb{C}}(m_{2n}, \Theta m_{2n}) v_{\mathbb{C}}(m_{2n+1}, \Xi m_{2n+1})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})} \\ &= \zeta v_{\mathbb{C}}(m_{2n}, m_{2n+1}) \\ &\quad + \omega \frac{v_{\mathbb{C}}(m_{2n}, m_{2n+1}) v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})} \zeta v_{\mathbb{C}} \\ &\leq \omega v_{\mathbb{C}}(m_{2n+1}, m_{2n+2}). \end{aligned} \quad (26)$$

This implies that

$$v_{\mathbb{C}}(m_{2n+1}, m_{2n+2}) \leq \frac{\zeta}{\kappa^{\mu} - \omega} v_{\mathbb{C}}(m_{2n}, m_{2n+1}). \quad (27)$$

Similarly,

$$v_{\mathbb{C}}(m_{2n+2}, m_{2n+3}) \leq \frac{\zeta}{\kappa^{\mu} - \omega} v_{\mathbb{C}}(m_{2n+1}, m_{2n+2}). \quad (28)$$

Now, with $\zeta/(\kappa^{\mu} - \omega) = D$, we can write

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+1})| &\leq D |v_{\mathbb{C}}(m_{n-1}, m_n)| \\ &\leq D^2 |v_{\mathbb{C}}(m_{n-2}, m_{n-1})| \\ &\vdots \\ &\leq D^n |v_{\mathbb{C}}(m_0, m_1)|. \end{aligned} \quad (29)$$

Now, we want to show that $\{m_n\}$ is a Cauchy sequence. So, we distinguish two cases.

Case 1. Let j be odd, that is, $j = 2k + 1$ for any $k \geq 1$. Now, by $(v_{\mathbb{C}3})$ for any $n \in \mathbb{N}$, we get

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+j})| &= |v_{\mathbb{C}}(m_n, m_{n+2k+1})| \leq \rho(m_n, m_{n+2k+1}) \\ &\quad \cdot [|v_{\mathbb{C}}(m_n, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, m_{n+2})| \\ &\quad + |v_{\mathbb{C}}(m_{n+2}, m_{n+2k+1})|] \leq \rho(m_n, m_{n+2k+1}) \\ &\quad \cdot [|v_{\mathbb{C}}(m_n, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, m_{n+2})|] \\ &\quad + \rho(m_n, m_{n+2k+1}) \\ &\quad \cdot |v_{\mathbb{C}}(m_{n+2}, m_{n+2k+1})| \leq \rho(m_n, m_{n+2k+1}) \\ &\quad \cdot [D^n |v_{\mathbb{C}}(m_0, m_1)| + D^{n+1} |v_{\mathbb{C}}(m_0, m_1)|] \\ &\quad + \rho(m_n, m_{n+2k+1}) |v_{\mathbb{C}}(m_{n+2}, m_{n+2k+1})| \\ &\quad + \rho(m_n, m_{n+2k+1}) (D^n + D^{n+1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &\quad + \rho(m_n, m_{n+2k+1}) |v_{\mathbb{C}}(m_{n+2}, m_{n+2k+1})| \\ &= \rho(m_n, m_{n+2k+1}) (D^n + D^{n+1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &\leq \rho(m_n, m_{n+2k+1}) \rho(m_{n+2}, m_{n+2k+1}) (D^{n+2} + D^{n+3}) \\ &\quad \cdot |v_{\mathbb{C}}(m_0, m_1)| + \dots \\ &\quad + \rho(m_n, m_{n+2k+1}) \dots \rho(m_{n+2k-2}, m_{n+2k+1}) \\ &\quad \cdot (D^{n+2k-2} + D^{n+2k-1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &\quad + \rho(m_n, m_{n+2k+1}) \dots \rho(m_{n+2k-2}, m_{n+2k+1}) D^{n+2k} \\ &\quad \cdot |v_{\mathbb{C}}(m_0, m_1)| = D^n (1 + D) |v_{\mathbb{C}}(m_0, m_1)| \\ &\quad \cdot \sum_{i=0}^{k-1} D^{2i} \prod_{t=0}^i \rho(m_{n+2t}, m_{n+2k+1}) \\ &\quad + D^{n+2k} \prod_{t=0}^{k-1} \rho(m_{n+2t}, m_{n+2k+1}) |v_{\mathbb{C}}(m_0, m_1)|. \end{aligned} \quad (30)$$

This implies that

$$\sum_{i=0}^{k-1} D^{2i} \prod_{t=0}^i \rho(m_{n+2t}, m_{n+2k+1}) \leq \sum_{i=0}^{k-1} D^{2i} \prod_{t=0}^i \rho(m_{2t}, m_{n+2k+1}). \quad (31)$$

As, in view of (24), we get $\lim_{n,k \rightarrow \infty} v_{\mathbb{C}}(m_n, m_k) D < 1$; therefore, by the ratio test, we conclude the convergence of the series $\sum_{i=0}^{\infty} D^{2i} \prod_{t=0}^i \rho(m_{2t}, m_{n+2k+1})$ for all $k \in \mathbb{N}$. Consider

$$\mathfrak{R} = \sum_{i=0}^{\infty} D^{2i} \prod_{t=0}^i \rho(m_{2t}, m_{n+2k+1}), \mathfrak{R}_n = \sum_{i=0}^n D^{2i} \prod_{t=0}^i \rho(m_{2t}, m_{n+2k+1}). \quad (32)$$

It follows by (30) that

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+2k+1})| &\leq D^n (1 + D) |v_{\mathbb{C}}(m_0, m_1)| [\mathfrak{R}_{k-1} - \mathfrak{R}_{n-1}] \\ &\quad + D^{n+2k} \prod_{t=0}^{k-1} \rho(m_{n+2t}, m_{n+2k+1}) |v_{\mathbb{C}}(m_0, m_1)|. \end{aligned} \quad (33)$$

Passing $n \rightarrow \infty$ in (33), we can write

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_n, m_{n+2k+1})| = 0. \quad (34)$$

Case 2. Let j be even, that is, $j = 2k$ for any $k \geq 1$. Then,

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+j})| &= |v_{\mathbb{C}}(m_n, m_{n+2k})| \leq \rho(m_n, m_{n+2k}) \\ &\cdot [|v_{\mathbb{C}}(m_n, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, m_{n+2})| \\ &+ |v_{\mathbb{C}}(m_{n+2}, m_{n+2k})|] \leq \rho(m_n, m_{n+2k}) \\ &\cdot [|v_{\mathbb{C}}(m_n, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, m_{n+2})|] \\ &+ \rho(m_n, m_{n+2k}) |v_{\mathbb{C}}(m_{n+2}, m_{n+2k})| \leq \rho(m_n, m_{n+2k}) \\ &\cdot [D^n |v_{\mathbb{C}}(m_0, m_1)| + D^{n+1} |v_{\mathbb{C}}(m_0, m_1)|] \\ &+ \rho(m_n, m_{n+2k}) |v_{\mathbb{C}}(m_{n+2}, m_{n+2k})| \\ &= \rho(m_n, m_{n+2k}) (D^n + D^{n+1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &+ \rho(m_n, m_{n+2k}) |v_{\mathbb{C}}(m_{n+2}, m_{n+2k+1})| \\ &\leq \rho(m_n, m_{n+2k}) (D^n + D^{n+1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &+ \rho(m_n, m_{n+2k}) \rho(m_{n+2}, m_{n+2k}) (D^{n+2} + D^{n+3}) \\ &\cdot |v_{\mathbb{C}}(m_0, m_1)| + \dots + \rho(m_n, m_{n+2k}) \dots \rho \\ &\cdot (m_{n+2k-2}, m_{n+2k}) (D^{n+2k-2} + D^{n+2k-1}) |v_{\mathbb{C}}(m_0, m_1)| \\ &+ \rho(m_n, m_{n+2k}) \dots \rho(m_{n+2k-2}, m_{n+2k}) D^{n+2k} \\ &\cdot |v_{\mathbb{C}}(m_0, m_1)| = D^n (1 + D) |v_{\mathbb{C}}(m_0, m_1)| \\ &\cdot \sum_{i=0}^{k-1} D^{2i} \prod_{t=0}^i \rho(m_{n+2t}, m_{n+2k}) + D^{n+2k-2} \prod_{t=0}^{k-1} \rho \\ &\cdot (m_{n+2t}, m_{n+2k}) |v_{\mathbb{C}}(m_0, m_2)|. \end{aligned} \quad (35)$$

Similar to Case 1, we conclude that

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+2k})| &\leq D^n (1 + D) |v_{\mathbb{C}}(m_0, m_1)| [\mathfrak{R}_{k-1} - \mathfrak{R}_{n-1}] D^{n+2k-2} \\ &\prod_{t=0}^{k-1} \rho(m_{n+2t}, m_{n+2k}) |v_{\mathbb{C}}(m_0, m_2)|. \end{aligned} \quad (36)$$

Applying the limit on (36) as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_n, m_{n+2k})| = 0. \quad (37)$$

It follows from (34) and (37) that $\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_n, m_{n+j})| = 0$, $j \in \mathbb{N}$, which illustrates that the sequence $\{m_n\}$ is Cauchy in ∇ . The completeness of $(\nabla, v_{\mathbb{C}})$ leads that there exists an element $\xi \in \nabla$ such that $\{m_n\}$ is complex valued convergent to ξ , i.e.,

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_n, \xi)| = 0. \quad (38)$$

By using controlled rectangular inequality ($v_{\mathbb{C}3}$), we can write

$$v_{\mathbb{C}}(\xi, m_{n+2}) \leq \rho(\xi, m_{n+2}) [v_{\mathbb{C}}(\xi, m_n) + v_{\mathbb{C}}(m_n, m_{n+1}) + v_{\mathbb{C}}(m_{n+1}, m_{n+2})]. \quad (39)$$

Passing the limit as $n \rightarrow \infty$ in (39) and using (38), we have

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_{n+2}, \xi)| = 0 = \lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_{n+1}, \xi)|. \quad (40)$$

Now, let us demonstrate that $\xi = \Theta\xi$, that is, ξ is a fixed point of Θ ; otherwise, $0 < v_{\mathbb{C}}(\Theta\xi, \xi)$, and for $n \in \mathbb{N}$, by (23), we have

$$\begin{aligned} \kappa^\mu |v_{\mathbb{C}}(\Theta\xi, \xi)| &\leq \kappa^\mu \rho(\Theta\xi, \xi) [|v_{\mathbb{C}}(\Theta\xi, m_{n+2})| \\ &+ |v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \xi)|] \\ &= \kappa^\mu \rho(\Theta\xi, \xi) [|v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \xi)|] \\ &+ \kappa^\mu \rho(\Theta\xi, \xi) |v_{\mathbb{C}}(\Theta\xi, \Xi m_{n+1})| \leq \kappa^\mu \rho(\Theta\xi, \xi) |v_{\mathbb{C}} \\ &\cdot (m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \xi)| + \kappa^\mu \rho(\Theta\xi, \xi) \\ &\cdot \left(\zeta |v_{\mathbb{C}}(\xi, m_{n+1})| + \omega \frac{|v_{\mathbb{C}}(\xi, \Theta\xi)| |v_{\mathbb{C}}(m_{n+1}, \Xi m_{n+1})|}{1 + |v_{\mathbb{C}}(\xi, m)|} \right) \\ &\leq \kappa^\mu \rho(\Theta\xi, \xi) [|v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \xi)|] \\ &+ \kappa^\mu \rho(\Theta\xi, \xi) \\ &\cdot \left(\zeta |v_{\mathbb{C}}(\xi, m_{n+1})| + \omega \frac{|v_{\mathbb{C}}(\xi, \Theta\xi)| |v_{\mathbb{C}}(m_{n+1}, m_{n+2})|}{1 + |v_{\mathbb{C}}(\xi, m_{n+1})|} \right). \end{aligned} \quad (41)$$

Taking the limit in (41) as $n \rightarrow \infty$ and applying (38) and (40), we conclude that $|v_{\mathbb{C}}(\Theta\xi, \xi)| = 0$, a contradiction. Hence, $\Theta\xi = \xi$. Similarly, we can prove $\xi = \Xi\xi$. Observe that, in view of Lemma 3, the sequence $\{m_n\}$ converges uniquely at point $\xi \in \nabla$.

To prove ξ is the unique cfp of Θ and Ξ , suppose that η is a distinct cfp of Θ and Ξ , then $0 < v_{\mathbb{C}}(\xi, \eta)$. So from (23), we get

$$\begin{aligned} \kappa^\mu |v_{\mathbb{C}}(\xi, \eta)| &= \kappa^\mu |v_{\mathbb{C}}(\Theta\xi, \Xi\eta)| \leq \zeta |v_{\mathbb{C}}(\xi, \eta)| \\ &+ \omega \frac{|v_{\mathbb{C}}(\xi, \Theta\xi)| |v_{\mathbb{C}}(\eta, \Xi\eta)|}{1 + |v_{\mathbb{C}}(\xi, \eta)|} \\ &= \zeta |v_{\mathbb{C}}(\xi, \eta)| + \omega \frac{|v_{\mathbb{C}}(\xi, \xi)| |v_{\mathbb{C}}(\eta, \eta)|}{1 + |v_{\mathbb{C}}(\xi, \eta)|} \\ &= \zeta |v_{\mathbb{C}}(\xi, \eta)| \Rightarrow (\kappa^\mu - \zeta) |v_{\mathbb{C}}(\xi, \eta)| \leq 0. \end{aligned} \quad (42)$$

This implies that $v_{\mathbb{C}}(\xi, \eta) = 0$, a contradiction. Thus, $\xi = \eta$. Hence, Θ and Ξ have a unique cfp. This finishes the proof.

The following theorem is more general than Theorem 1.

Theorem 2. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let the mappings $\Theta, \Xi : \nabla \rightarrow \nabla$ satisfy

$$\begin{aligned} \kappa^\mu v_{\mathbb{C}}(\Theta l, \Xi m) &\leq \alpha \max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Xi m)}{1 + v_{\mathbb{C}}(l, m)}, \right. \\ &\left. \frac{v_{\mathbb{C}}(l, \Xi m) \omega(m, \Theta l)}{1 + v_{\mathbb{C}}(l, m)} \right\}, \end{aligned} \quad (43)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta \in (0, 1)$, where $0 < v_{\mathbb{C}}(l, m)$. If $\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < 1/A$, where $A = \alpha/\kappa^\mu < 1$ and for all m

$\in \nabla, \lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite. Then, Θ and Ξ have a unique cfp.

Proof. Let $m_* \in \nabla$ be an arbitrary point. Define a sequence $\{m_n\}$ as follows:

$$m_{2n+1} = \Theta m_{2n} \text{ and } m_{2n+2} = \Xi m_{2n+1}, n \geq 0. \quad (44)$$

Consider

$$\begin{aligned} \kappa^\mu v_{\mathbb{C}}(m_{2n+1}, m_{2n+2}) &= \kappa^\mu v_{\mathbb{C}}(\Theta m_{2n}, \Xi m_{2n+1}) \\ &\leq \alpha \max \left\{ v_{\mathbb{C}}(m_{2n}, m_{2n+1}), \frac{v_{\mathbb{C}}(m_{2n}, \Theta m_{2n}) v_{\mathbb{C}}(m_{2n+1}, \Xi m_{2n+1})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})}, \right. \\ &\quad \left. \frac{v_{\mathbb{C}}(m_{2n}, \Xi m_{2n+1}) v_{\mathbb{C}}(m_{2n+1}, \Theta m_{2n})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})} \right\} \\ &\leq \alpha \max \left\{ v_{\mathbb{C}}(m_{2n}, m_{2n+1}), \frac{v_{\mathbb{C}}(m_{2n}, m_{2n+1}) v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})}, \right. \\ &\quad \left. \frac{v_{\mathbb{C}}(m_{2n}, m_{2n+2}) v_{\mathbb{C}}(m_{2n+1}, m_{2n+1})}{1 + v_{\mathbb{C}}(m_{2n}, m_{2n+1})} \right\} \\ &\leq \alpha \max \{v_{\mathbb{C}}(m_{2n}, m_{2n+1}), v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})\}. \end{aligned} \quad (45)$$

If $|v_{\mathbb{C}}(m_{2n}, m_{2n+1})| \neq |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})|$, we can write

$$\begin{aligned} \kappa^\mu |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})| &\leq \alpha |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})|, \\ &\Rightarrow (\kappa^\mu - \alpha) |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})| = 0. \end{aligned} \quad (46)$$

This is a contradiction. Hence,

$$\kappa^\mu |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})| \leq \alpha |v_{\mathbb{C}}(m_{2n}, m_{2n+1})|, \quad (47)$$

or

$$|v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})| \leq \frac{\alpha}{\kappa^\mu} |v_{\mathbb{C}}(m_{2n}, m_{2n+1})| = A |v_{\mathbb{C}}(m_{2n}, m_{2n+1})|. \quad (48)$$

Similarly, we can obtain

$$|v_{\mathbb{C}}(m_{2n+2}, m_{2n+3})| \leq A |v_{\mathbb{C}}(m_{2n+1}, m_{2n+2})|. \quad (49)$$

In general, we have

$$\begin{aligned} |v_{\mathbb{C}}(m_n, m_{n+1})| &\leq A |v_{\mathbb{C}}(m_{n-1}, m_n)| \\ &\leq A^2 |v_{\mathbb{C}}(m_{n-2}, m_{n-1})| \\ &\vdots \\ &\leq A^2 |v_{\mathbb{C}}(m_0, m_1)|. \end{aligned} \quad (50)$$

Using the same scenario as Cases 1 and 2 of Theorem 1, we conclude that the sequence $\{m_n\}$ is Cauchy in ∇ . Since $(\nabla, v_{\mathbb{C}})$ is complete, there is an element $\sigma \in \nabla$ such that

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_n, \sigma)| = 0. \quad (51)$$

By Lemma 3, a sequence $\{m_n\}$ converges uniquely at point $\sigma \in \nabla$. By the condition $(v_{\mathbb{C}3})$, we get

$$v_{\mathbb{C}}(\sigma, m_{n+2}) \leq \rho(\sigma, m_{n+2}) [v_{\mathbb{C}}(\sigma, m_n) + v_{\mathbb{C}}(m_n, m_{n+1}) + v_{\mathbb{C}}(m_{n+1}, m_{n+2})]. \quad (52)$$

Passing the limit as $n \rightarrow \infty$ in (52) and using (51), we have

$$\lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_{n+2}, \sigma)| = 0 = \lim_{n \rightarrow \infty} |v_{\mathbb{C}}(m_{n+1}, \sigma)|. \quad (53)$$

Now, we show that ξ is a fixed point of Θ ; otherwise, $0 < v_{\mathbb{C}}(\Theta \xi, \xi)$, and for $n \in \mathbb{N}$, by (43), we can get

$$\begin{aligned} \kappa^\mu |v_{\mathbb{C}}(\Theta \sigma, \sigma)| &\leq \kappa^\mu \rho(\Theta \sigma, \sigma) [|v_{\mathbb{C}}(\Theta \sigma, m_{n+2})| \\ &\quad + |v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \sigma)|] \\ &= \kappa^\mu \rho(\Theta \sigma, \sigma) [|v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \xi)|] \\ &\quad + \kappa^\mu \rho(\Theta \sigma, \sigma) |v_{\mathbb{C}}(\Theta \sigma, \Xi m_{n+1})| \leq \kappa^\mu \rho(\Theta \sigma, \sigma) \\ &\quad \cdot [|v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \sigma)|] \\ &\quad + \alpha \kappa^\mu \rho(\Theta \sigma, \sigma) \max \left\{ |v_{\mathbb{C}}(\sigma, m_{n+1})|, \frac{|v_{\mathbb{C}}(\sigma, \Theta \sigma)| |v_{\mathbb{C}}(m_{n+1}, \Xi m_{n+1})|}{1 + |v_{\mathbb{C}}(\sigma, m_{n+1})|}, \right. \\ &\quad \left. \frac{|v_{\mathbb{C}}(\sigma, \Xi m_{n+1})| |\omega(m_{n+1}, \Theta \sigma)|}{1 + |v_{\mathbb{C}}(\sigma, m_{n+1})|} \right\} \\ &\leq \kappa^\mu \rho(\Theta \sigma, \sigma) [|v_{\mathbb{C}}(m_{n+2}, m_{n+1})| + |v_{\mathbb{C}}(m_{n+1}, \sigma)|] \\ &\quad + \alpha \kappa^\mu \rho(\Theta \sigma, \sigma) \max \left\{ \zeta |v_{\mathbb{C}}(\xi, m_{n+1})| + \omega \frac{|v_{\mathbb{C}}(\sigma, \Theta \sigma)| |v_{\mathbb{C}}(m_{n+1}, m_{n+2})|}{1 + |v_{\mathbb{C}}(\xi, m_{n+1})|}, \right. \\ &\quad \left. \frac{|v_{\mathbb{C}}(\sigma, m_{n+2})| |\omega(m_{n+1}, \Theta \sigma)|}{1 + |v_{\mathbb{C}}(\sigma, m_{n+1})|} \right\}. \end{aligned} \quad (54)$$

After entering the limit as $n \rightarrow \infty$ in (54) and using (51) and (53), we conclude that $|v_{\mathbb{C}}(\Theta \sigma, \sigma)| = 0$, a contradiction. Hence, $\Theta \sigma = \sigma$. Similarly, we can prove $\sigma = \Xi \sigma$.

The uniqueness comes easy from the condition (51), and it is similar to Theorem 1. Therefore, the mappings Θ and Ξ have a unique cfp in ∇ . This ends the proof.

4. Examples and Some Sequences on the Main Theorems

This part is devoted to obtain some sequences of Theorems 1 and 2.

If we consider $\Theta = \Xi$ in Theorem 1, we have the below result.

Corollary 1. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let the mapping $\Theta : \nabla \rightarrow \nabla$ satisfy

$$\kappa^\mu v_{\mathbb{C}}(\Theta l, \Theta m) \leq \zeta v_{\mathbb{C}}(l, m) + \omega \frac{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Theta m)}{1 + v_{\mathbb{C}}(l, m)}, \forall l, m \in \nabla, \quad (55)$$

where $0 < v_{\mathbb{C}}(l, m), \kappa, \mu \geq 1$ and ζ, ω are nonnegative real numbers with $\zeta + \omega < 1$.

Assume also

$$\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < \frac{1}{D}, \quad (56)$$

where $D = (\zeta/\kappa^\mu - \omega) < 1$ and $\lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite for all $m \in \nabla$. Then, Θ has a unique fixed point.

Assume that $\kappa = \mu = 1$ and $\omega = 0$ in Corollary 1, we get the important below result.

Corollary 2. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let the mapping $\Theta : \nabla \rightarrow \nabla$ satisfy

$$v_{\mathbb{C}}(\Theta l, \Theta m) \leq \zeta v_{\mathbb{C}}(l, m), \forall l, m \in \nabla, \quad (57)$$

where ζ is a non-negative real number with $\zeta < 1$. Assume also

$$\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < \frac{1}{D}, \quad (58)$$

where $D = (\zeta/\kappa^\mu - \omega) < 1$ and $\lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite for all $m \in \nabla$. Then, Θ has a unique fixed point.

Remark 2. If we take $\Theta = \Xi$ in Theorem 2, we have the following immediate consequences.

Corollary 3. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let the mapping $\Theta : \nabla \rightarrow \nabla$ satisfy

$$\kappa^\mu v_{\mathbb{C}}(\Theta l, \Theta m) \leq \alpha \max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta l)v_{\mathbb{C}}(m, \Theta m)}{1 + v_{\mathbb{C}}(l, m)}, \frac{v_{\mathbb{C}}(l, \Theta m)\omega(m, \Theta l)}{1 + v_{\mathbb{C}}(l, m)} \right\}, \quad (59)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta \in (0, 1)$, where $0 < v_{\mathbb{C}}(l, m)$. Assume that $\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < 1/A$, where $A = \alpha/\kappa^\mu < 1$ and for all $m \in \nabla, \lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite. Then, Θ has a unique fixed point.

Corollary 4. Let $(\nabla, v_{\mathbb{C}})$ be a complete $v_{\mathbb{C}}$ -metric space and let $\Theta : \nabla \rightarrow \nabla$. Suppose that there exists $n \in \mathbb{N}$ such that

$$\kappa^\mu v_{\mathbb{C}}(\Theta^n l, \Theta^n m) \leq \alpha \max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta^n l)v_{\mathbb{C}}(m, \Theta^n m)}{1 + v_{\mathbb{C}}(l, m)}, \frac{v_{\mathbb{C}}(l, \Theta^n m)v_{\mathbb{C}}(m, \Theta^n l)}{1 + v_{\mathbb{C}}(l, m)} \right\}, \quad (60)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta \in (0, 1)$, where $0 < v_{\mathbb{C}}(l, m)$. Assume that $\lim_{n \rightarrow \infty} \rho(m_n, m_{n+1}) < 1/A$, where $A = \alpha/\kappa^\mu < 1$ and for all $m \in \nabla, \lim_{n \rightarrow \infty} \rho(m_n, m)$ exists and is finite. Then, Θ possesses a unique fixed point.

Proof. By Corollary 3, we have $\kappa \in \nabla$ such that

$$\Theta^n \kappa = \kappa. \quad (61)$$

So

$$\begin{aligned} \kappa^\mu v_{\mathbb{C}}(\Theta \kappa, \kappa) &= \kappa^\mu v_{\mathbb{C}}(\Theta \Theta^n \kappa, \Theta^n \kappa) = \kappa^\mu v_{\mathbb{C}}(\Theta^n \Theta \kappa, \Theta^n \kappa) \\ &\leq \alpha \max \left\{ v_{\mathbb{C}}(\Theta \kappa, \kappa), \frac{v_{\mathbb{C}}(\Theta \kappa, \Theta^n \Theta \kappa)v_{\mathbb{C}}(\kappa, \Theta^n \kappa)}{1 + v_{\mathbb{C}}(\Theta \kappa, \kappa)}, \frac{v_{\mathbb{C}}(\Theta \kappa, \Theta^n \kappa)v_{\mathbb{C}}(\kappa, \Theta^n \Theta \kappa)}{1 + v_{\mathbb{C}}(\Theta \kappa, \kappa)} \right\} \\ &\leq \alpha \max \left\{ v_{\mathbb{C}}(\Theta \kappa, \kappa), \frac{v_{\mathbb{C}}(\Theta \kappa, \Theta \Theta^n \kappa)v_{\mathbb{C}}(\kappa, \Theta^n \kappa)}{1 + v_{\mathbb{C}}(\Theta \kappa, \kappa)}, \frac{v_{\mathbb{C}}(\Theta \kappa, \Theta^n \kappa)v_{\mathbb{C}}(\kappa, \Theta \Theta^n \kappa)}{1 + v_{\mathbb{C}}(\Theta \kappa, \kappa)} \right\} = \alpha v_{\mathbb{C}}(\Theta \kappa, \kappa). \end{aligned} \quad (62)$$

Therefore, the result follows.

The below examples support Theorems 1 and 2.

Example 7. Assume that all requirements of Example 3 hold. It is clear that $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space with $\rho(l, m) = l + m + 4$. Define the mappings $\Theta, \Xi : \nabla \rightarrow \nabla$ by $\Theta l = (1/2)l$ and $m = (1/2)m, m, l \in \nabla$. We Only prove the contractive condition ((23)) of Theorem 1 with $\kappa = 2$ and $\mu = 1$. Indeed, for all $l, m \in \nabla$,

$$\begin{aligned} \kappa^\mu v_{\mathbb{C}}(\Theta l, \Xi m) &= 2v_{\mathbb{C}}\left(\frac{l}{2}, \frac{m}{2}\right) = \frac{8}{3} \left(\left| \frac{l}{2} - \frac{m}{2} \right|^2 + i \left| \frac{l}{2} - \frac{m}{2} \right|^2 \right) \\ &= \frac{2}{3} (|l - m|^2 + i |l - m|^2) \\ &\leq \frac{2}{3} v_{\mathbb{C}}(l, m) \left(\frac{3}{4} + \frac{1}{6} \right) v_{\mathbb{C}}(l, m) \leq \zeta v_{\mathbb{C}}(l, m) \\ &\quad + \omega \frac{v_{\mathbb{C}}(l, \Theta l)v_{\mathbb{C}}(m, \Xi m)}{1 + v_{\mathbb{C}}(l, m)}. \end{aligned} \quad (63)$$

Hence, condition (23) of Theorem 1 is satisfied with $\zeta = 3/4$ and $\omega = 1/6$ such that $(3/4) + (1/6) = 11/12 < 1$ and the point $0 \in \nabla$ is the unique cfp of Θ and Ξ .

Example 8. Suppose that the statements of Example 2 are fulfilled. Define $\Theta, \Xi : \nabla \rightarrow \nabla$ by $\Theta(l) = l/4$ and $\Xi(m) = m/4$ for all $l, m \in \nabla$. It is clear that $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space with $\rho(l, m) = l + m + 3^{l-1}, l > 1$. Now, we will go to fulfill condition ((43)) for $\kappa = 2$ and $\mu = 1$. We have

$$\begin{aligned} \kappa^\mu v_{\mathbb{C}}(\Theta l, \Xi m) &= 2v_{\mathbb{C}}\left(\frac{l}{4}, \frac{m}{4}\right) = 2 \left| \frac{l}{4} - i \frac{m}{4} \right|^2 \\ &\leq 2 \left(\frac{1}{4} \right)^2 |l - im|^2 = \frac{1}{8} v_{\mathbb{C}}(l, m) \\ &\leq \frac{1}{8} \max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta l)v_{\mathbb{C}}(m, \Xi m)}{1 + v_{\mathbb{C}}(l, m)}, \frac{v_{\mathbb{C}}(l, \Xi m)\omega(m, \Theta l)}{1 + v_{\mathbb{C}}(l, m)} \right\}. \end{aligned} \quad (64)$$

Hence, all required assumptions of Theorem 2 are fulfilled with $\alpha = 1/8 < 1$ and 0 is the unique cfp of Θ and Ξ .

Remark 3. (i) Theorems 1 and 2 extend some previous works in the different spaces. Some of them are as follows.

In Theorem 1

- (i) when $\bar{\omega} = 0, \kappa^\mu = s > 1$ and $\Theta = \Xi$, we have the work of Ege [17] in complex valued rectangular b -metric spaces, where $\zeta \in [0, 1/s]$
- (ii) when $\kappa = \mu = 1$ and $\Theta = \Xi$, we have the exciting results of Azam [1] in complex valued metric spaces
- (iii) if we reduce a complete $v_{\mathbb{C}}$ -metric space to a complete complex valued b -metric space with $s\zeta + \bar{\omega} < 1$, we get directly the work of Mukheimer [13]
- (iv) when $\kappa = \mu = 1$ and $\zeta, \bar{\omega} : \mathbb{C}_+ \rightarrow [0, 1)$ are two mappings in a complete complex valued metric space, we have the results of Sintunavarat et al. [5]

In Theorem 2

- (i) if we reduce a complete $v_{\mathbb{C}}$ -metric space to a complete complex valued metric space with $\alpha \in (0, 1)$ and $\kappa = \mu = 1$, we obtain the results of Hammad [6]
- (ii) the conclusion of Theorems 1 and 2 is still valid by using one of the following contractive conditions:

($\star_1$)

$$\kappa^\mu v_{\mathbb{C}}(\Theta l, \Theta m) \leq \zeta v_{\mathbb{C}}(m, \Theta m) + \bar{\omega} \frac{v_{\mathbb{C}}^2(l, m)}{1 + v_{\mathbb{C}}(l, m)}, \quad (65)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta, \bar{\omega}$ are nonnegative real numbers with $\zeta + \bar{\omega} < 1, \Theta = \Xi$ and $0 < v_{\mathbb{C}}(l, m)$.

($\star_2$)

$$\kappa^\mu v_{\mathbb{C}}(\Theta l, \Theta m) \leq \zeta v_{\mathbb{C}}(l, m) + \bar{\omega} \frac{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Theta m)}{v_{\mathbb{C}}(l, m) + v_{\mathbb{C}}(l, \Theta m) + v_{\mathbb{C}}(m, \Theta l)}, \quad (66)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta, \bar{\omega}$ are nonnegative real numbers with $\zeta + \bar{\omega} < 1, \Theta = \Xi$ and $v_{\mathbb{C}}(l, m) + v_{\mathbb{C}}(l, \Theta m) + v_{\mathbb{C}}(m, \Theta l) \neq 0$.

($\star_3$)

$$\kappa^\mu \int_0^{v_{\mathbb{C}}(\Theta l, \Xi m)} \gamma(t) dt \leq \zeta \int_0^{v_{\mathbb{C}}(l, m)} \gamma(t) dt + \bar{\omega} \int_0^{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Xi m) / (1 + v_{\mathbb{C}}(l, m))} \gamma(t) dt, \quad (67)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta, \bar{\omega}$ are nonnegative real numbers with $\zeta + \bar{\omega} < 1$, and $0 < v_{\mathbb{C}}(l, m)$, where $\gamma : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping on a compact subset of $[0, \infty)$ and for all $\epsilon > 0, \int_0^\epsilon \gamma(t) dt > 0$. This direction involves fixed point results of integral type.

($\star_4$)

$$\kappa^\mu e^{v_{\mathbb{C}}(\Theta l, \Xi m)} \leq \alpha e^{\max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta l) v_{\mathbb{C}}(m, \Xi m)}{1 + v_{\mathbb{C}}(l, m)}, \frac{v_{\mathbb{C}}(l, \Xi m) v_{\mathbb{C}}(m, \Theta l)}{1 + v_{\mathbb{C}}(l, m)} \right\}}, \quad (68)$$

for all $l, m \in \nabla, \kappa, \mu \geq 1$ and $\zeta \in (0, 1)$, where $0 < v_{\mathbb{C}}(l, m)$, where $e : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping on a compact subset of $[0, \infty)$ and for all $\epsilon > 0, e^\epsilon > 0$. This direction concerns fixed point results of exponential type.

5. Solving a Complex Valued Atangana-Baleanu Fractional Integral Operator

Let ∇ be the set of all complex analytic functions and $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}$ be defined as

$$v_{\mathbb{C}}(m(r), l(r)) = \frac{4}{3} (|m(r) - l(r)|^2 + i|m(r) - l(r)|^2), \quad (69)$$

for all $m(r), l(r) \in \nabla$. Also, define $\rho : \nabla \times \nabla \rightarrow [1, \infty)$ by $\rho(l(r), m(r)) = l(r) + m(r) + 4$. Then, $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space. We will use the form ${}_{\mathbb{C}}^{AB} Y_r^v m(r)$ to express the complex valued Atangana-Baleanu integral (AB-fractional integral), where c is a fixed complex number, m is a complex function which is analytic on an open star-domain $\mathbb{R}$ centered at $c, v \in \mathbb{C}$ and $r \in \mathbb{R} \setminus \{c\}$. It is defined by

$${}_{\mathbb{C}}^{AB} Y_r^v m(r) = \frac{1-v}{\beta(v)} m(r) + \frac{v}{\beta(v)} {}_{\mathbb{C}}^{RL} Y_r^v m(r), \quad (70)$$

where ${}_{\mathbb{C}}^{RL} Y_r^v m(r)$ refers to the Reimann-Liouville fractional integral equation for a complex analytical function m .

Another form of the complex valued Atangana-Baleanu integral is presented by Fernandez [18] as follows:

$${}_{\mathbb{C}}^{AB} Y_r^v m(r) = \frac{1}{2\pi i \beta(v)} \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) m(u) du, \quad (71)$$

where $u, r \in \mathbb{R} \setminus \{c\}; v \in \mathbb{C}/\mathbb{N}$ and H refers to the Hankel contour [19].

Define the operator $\Theta_{Ab} : \nabla \rightarrow \nabla$ by

$$\Theta_{Ab} m(r) = \frac{1}{2\pi i \beta(v)} \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) m(u) du. \quad (72)$$

Now, we will seek for the existence and uniqueness of a fixed point for the operator (72), which is considered as the unique solution for the operator (70) under the following condition:

$$\frac{1}{\pi^2 \beta^2(v)} \left(\frac{|v-1|(r-u) \ln(|r-u|)}{|r-u|} - \Gamma(1-v)(u-r)^v \right)^2 \leq 1. \quad (73)$$

Consider

$$\begin{aligned}
& v_{\mathbb{C}}(\Theta_{Ab}m(r), \Theta_{Ab}m(r)) \\
&= \frac{4}{3} (|\Theta_{Ab}m(r) - \Theta_{Ab}l(r)|^2 + i|\Theta_{Ab}m(r) - \Theta_{Ab}l(r)|^2) \\
&= \frac{4}{3} \left(\left| \frac{1}{2\pi i \beta(v)} \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) m(u) dr - \frac{1}{2\pi i \beta(v)} \right. \right. \\
&\quad \cdot \left. \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) l(u) dr \right|^2 + i \left| \frac{1}{2\pi i \beta(v)} \right. \\
&\quad \cdot \left. \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) m(u) dr - \frac{1}{2\pi i \beta(v)} \right. \\
&\quad \cdot \left. \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) l(u) dr \right|^2 \Bigg) \\
&= \frac{4}{3} \left(\left| \left[\frac{1}{2\pi i \beta(v)} \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) dr \right] (m(u) - l(u)) \right|^2 \right. \\
&\quad \left. + i \left| \left[\frac{1}{2\pi i \beta(v)} \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) dr \right] (m(u) - l(u)) \right|^2 \right) \\
&\lesssim \frac{1}{3} \left(\frac{1}{|\pi i \beta(v)|^2} \left| \int_H \left(\frac{1-v}{u-r} + \frac{v\Gamma(1-v)}{(u-r)^{1-v}} \right) dr \right|^2 \right. \\
&\quad \cdot (|m(u) - l(u)|^2 + i|m(u) - l(u)|^2) \Bigg) \\
&\lesssim \frac{1}{3} \left(\frac{1}{\pi^2 \beta^2(v)} \left(\frac{|v-1|(r-u) \ln(|r-u|)}{|r-u|} - \Gamma(1-v)(u-r)^v \right)^2 \right) \\
&\quad \times \frac{4}{3} (|m(u) - l(u)|^2 + i|m(u) - l(u)|^2) \\
&\lesssim \frac{1}{3} v_{\mathbb{C}}(m(r), l(r)) = \zeta v_{\mathbb{C}}(m(r), l(r)).
\end{aligned} \tag{74}$$

Thus, the conditions of Corollary 2 are satisfied. Hence, Θ_{AB} has a unique fixed point, which concludes that the fractional integral (70) has a unique solution.

6. Solving a System of Complex Linear Equations

In this part, we shall try to find a unique solution to a system of complex linear equations and apply it to solve the RLC circuit. Consider the below system:

$$\begin{cases} \vartheta_{11}m_1 + \vartheta_{12}m_2 + \dots + \vartheta_{1n}m_n = \ell_1 \\ \vartheta_{21}m_1 + \vartheta_{22}m_2 + \dots + \vartheta_{2n}m_n = \ell_2 \\ \vdots \\ \vartheta_{n1}m_1 + \vartheta_{n2}m_2 + \dots + \vartheta_{nn}m_n = \ell_n \end{cases}, \tag{75}$$

in n linear equations and n variables, where $m \in \nabla = \mathbb{C}^n$ and ϑ_{ts} is the complex coefficients matrix. Assume that $\nabla = \mathbb{C}^n$. Define $v_{\mathbb{C}} : \nabla \times \nabla \rightarrow \mathbb{C}, \rho : \nabla \rightarrow [1, \infty)$ by $v_{\mathbb{C}}(l, m) = (4/3) \max_{1 \leq t \leq n} |l_t - m_t|^2$ and $\rho(l, m) = l + m + 4$, respectively. According to Example 6, the pair $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space.

FIGURE 1: RLC circuit.

The following theorem is the main result of this section.

Theorem 3. Suppose that $(\nabla, v_{\mathbb{C}})$ is a complete $v_{\mathbb{C}}$ -metric space, then the system (75) has only one solution if $\sum_s^n |\vartheta_{ts}| < 0.46$ for $t = 1, 2, \dots, n$.

Proof. Define the linear mapping $\Theta : \nabla \rightarrow \nabla$ by $\Theta(m) = Qm + L$, where $m = (m_1, m_2, \dots, m_n), L = (\ell_1, \ell_2, \dots, \ell_n) \in \mathbb{C}^n$ and

$$Q = \begin{pmatrix} \vartheta_{11} & \vartheta_{12} & \dots & \vartheta_{1n} \\ \vartheta_{21} & \vartheta_{22} & \dots & \vartheta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta_{31} & \vartheta_{32} & \dots & \vartheta_{3n} \end{pmatrix}. \tag{76}$$

If we can prove that Θ is a contraction mapping, the proof is over. Consider $\kappa = 3$ and $\mu = 1$, then we have

$$\begin{aligned}
\kappa^\mu v_{\mathbb{C}}(\Theta l, \Theta m) &= 3 \times \frac{4}{3} \max_{1 \leq t \leq n} \left| \left(\sum_s^n \vartheta_{ts} l_s + L_j \right) - \left(\sum_s^n \vartheta_{ts} m_s + L_j \right) \right|^2 \\
&= 4 \max_{1 \leq t \leq n} \left(\left| \sum_s^n \vartheta_{ts} (l_s - m_s) \right|^2 \right) \lesssim 4 \max_{1 \leq t \leq n} \left(\sum_s^n |\vartheta_{ts}|^2 |l_s - m_s|^2 \right) \\
&\lesssim 4 \max_{1 \leq t \leq n} \sum_s^n |\vartheta_{ts}|^2 \times \max_{1 \leq s \leq n} |l_s - m_s|^2 \lesssim 0.8464 \max_{1 \leq s \leq n} |l_s - m_s|^2 \\
&\lesssim 0.8464 \times \left(\frac{4}{3} \max_{1 \leq s \leq n} |l_s - m_s|^2 \right) = \alpha v_{\mathbb{C}}(l, \Theta) \\
&\lesssim \alpha \max \left\{ v_{\mathbb{C}}(l, m), \frac{v_{\mathbb{C}}(l, \Theta) v_{\mathbb{C}}(m, \Theta m)}{1 + v_{\mathbb{C}}(l, m)}, \right. \\
&\quad \left. \frac{v_{\mathbb{C}}(l, \Theta m) \omega(m, \Theta l)}{1 + v_{\mathbb{C}}(l, m)} \right\},
\end{aligned} \tag{77}$$

where $\alpha = 0.8464 < 1$, then Θ is a contraction mapping. Thus, by Corollary 3, the system (75) has a unique solution.

To illustrate the usefulness of Theorem 3, we give the following example.

Example 9. Consider the RLC circuit containing three as follows.

The symbols R, C, V , and I refer to the resistance, capacity, voltage, and current, respectively. Also, let λ denote the imaginary number with current. Let us take the values $V_1 = 0.7 + 1.2\lambda$ and $V_2 = 1.65 + 0.3\lambda$ and $\omega = 1$. It is obvious that the RLC circuit forms a complex valued linear equation system that has one solution.

Verification: from Kirchhoff voltage law “the sum of all voltage in a closed loop is zero”; in this example, we observe it in three loops (Figure 1). From our simple physical information, we know that $V = IR = I\Phi_C = I\Phi_L$, where $\Phi_C = 1/\lambda\omega C$ and $\Phi_L = \lambda\omega L$.

- (i) From loop 1, we have $\Phi_{C_1} = 1/\lambda\omega C_1 = 1/(\lambda \times 1 \times 9) = (1/9\lambda) \times (\lambda/\lambda) = -0.111\lambda$ and

$$\begin{cases} -V_1 + R_1 I_1 + \Phi_{C_1} I_1 - \Phi_{C_1} I_2 = 0, \\ \Rightarrow -(0.7 + 1.2\lambda) + 0.04I_1 - 0.111\lambda I_1 + 0.111\lambda I_2 = 0, \\ \Rightarrow (0.04 - 0.111\lambda)I_1 + 0.111\lambda I_2 = 0.7 + 1.2\lambda. \end{cases} \quad (78)$$

- (ii) From loop 2, we have $\Phi_{C_2} = 1/\lambda\omega C_2 = 1/7\lambda = -0.142\lambda, \Phi_L = \lambda\omega L = 0.14\lambda$ and

$$\begin{cases} \Phi_{C_1} I_2 - \Phi_{C_1} I_1 + R_2 I_2 + \Phi_{C_2} I_2 + \Phi_L I_2 - \Phi_L I_3 = 0, \\ \Rightarrow -0.111\lambda I_2 + 0.111\lambda I_1 + 0.1I_2 - 0.142\lambda I_2 + 0.14\lambda I_2 - 0.14\lambda I_3 \approx 0, \\ \Rightarrow 0.111\lambda I_1 + (0.1 - 0.112\lambda)I_2 - 0.14\lambda I_3 = 0. \end{cases} \quad (79)$$

- (iii) From loop 3, we have $\Phi_{C_3} = 1/\lambda\omega C_3 = 1/24\lambda = -0.042\lambda$, and $\sum V = 0$; hence,

$$\begin{cases} \Phi_L I_3 - \Phi_L I_2 + \Phi_{C_3} I_3 + R_3 I_3 - V_2 = 0, \\ \Rightarrow 0.14\lambda I_3 - 0.14\lambda I_2 - 0.042\lambda I_3 + 0.015I_3 - (1.65 + 0.3\lambda) = 0, \\ \Rightarrow -0.14\lambda I_2 + (0.015 + 0.098\lambda)I_3 = 1.65 + 0.3\lambda. \end{cases} \quad (80)$$

So we have the below complex linear equation system:

$$\begin{cases} (0.04 - 0.111\lambda)I_1 + 0.111\lambda I_2 = 0.7 + 1.2\lambda, \\ 0.111\lambda I_1 + (0.1 - 0.112\lambda)I_2 - 0.14\lambda I_3 = 0, \\ -0.14\lambda I_2 + (0.015 + 0.098\lambda)I_3 = 1.65 + 0.3\lambda. \end{cases} \quad (81)$$

By testing the sum of the coefficients I_{pq} , where $p = 1, 2, 3$ and $q = 1, 2, 3$, we have

- (i) for $p = 1$,
 $\sum_{q=1}^3 |I_{1q}| = \sqrt{0.04^2 + 0.111^2} + 0.11 \approx 0.223 < 0.46$

- (ii) for $p = 2$,

$$\sum_{q=1}^3 |I_{2q}| = 0.111 + \sqrt{0.1^2 + 0.112^2} + 0.14 \approx 0.401 < 0.46$$

- (iii) for $p = 3$,
 $\sum_{q=1}^3 |I_{3q}| = 0.14 + \sqrt{0.015^2 + 0.098^2} \approx 0.239 < 0.46$

Hence, according to Theorem 3, the system (81) has a unique solution in the form of $I_1 = -0.9001 + 19.8529\lambda$, $I_2 = 2.7565 + 13.2222\lambda$, and $I_3 = 6.5320 + 3.0520\lambda$.

7. Conclusion

In this paper, we ensured the existence of a unique common fixed point for several contraction type self-mappings in the class of complex valued controlled rectangular b -metric spaces. Our presented results complement and generalize several significant known fixed point theorems in literature. As applications, we resolve a complex valued Atangana-Baleanu fractional integral operator and a system of complex linear equations, and we presented a related numerical example. We think our paper will be the source of motivation for researchers to improve and extend our results for new applications.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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