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Research Article

$(\in, \in \vee \breve{q})$-Bipolar Fuzzy b-Ideals of BCK/BCI-Algebras

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In this paper, the idea of $(\in, \in \vee \breve{q})$-bipolar fuzzy b-ideals and an $(\in, \in \vee \breve{q})$-bipolar fuzzy ideals of BCK/BCI-algebras is delivered, and their related properties are investigated with the aid of some examples. We also provide the connection between $(\in, \in \vee \breve{q})$-bipolar fuzzy ideals and bipolar fuzzy ideals and $(\in, \in \vee \breve{q})$-bipolar fuzzy b-ideals and bipolar fuzzy b-ideals by way of counterexamples.

1. Introduction

In the real world, there are several difficult problems in engineering, medical science, economics, environment, social science, and other various fields involving incalculable data. Problems of this nature which are frequently encountered in our daily lives cannot be solved by classical mathematical methods. In 1965, the belief of fuzzy sets turned into was first delivered with the aid of Zadeh [1], which is an effective hand set for modelling indecision and elusiveness in numerous issues springing up inside the field of technology. For the ultimate four decades, the fuzzy idea has ended up as a very lively area of study, and plenty of developments have been made within the concept of fuzzy sets to find the fuzzy analogues of the classical set theory. The study of BCK-algebras was initiated by Imai and Iséki [2, 3] in 1966 as showed in the concept of set-theoretic difference and propositional calculi. After that, many researchers investigated the fuzzification of ideals and subalgebras in BCK/BCI-algebras.

Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is [1, 1]. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree (0, 1] of an element indicates that the element somewhat satisfies the property, and the membership degree [1, 0) of an element indicates that the element somewhat satisfies the implicit counterproperty. The idea which lies behind such description is connected with the existence of "bipolar information" (e.g., positive information and negative information) about the given set. Positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. Actually, a wide variety of human decision-making is based on double-sided or bipolar judgmental thinking on a positive side and a negative side. For instance, cooperation and competition, friendship and hostility, common interests and conflict of interests, effect and side effect, likelihood and unlikelihood, and feedforward and feedback are often the two sides in decision and coordination. In traditional Chinese medicine, "yin" and "yang" are the two sides. Yin is the feminine or negative side of a system, and yang is the masculine or positive side of a system. The coexistence, equilibrium, and harmony of the two sides are considered a key for the mental and physical health of a person as well as for the stability and prosperity of a social system. Thus, bipolar fuzzy sets indeed have potential impacts on many fields, including artificial intelligence, computer science, information science, cognitive science, decision science, management science, economics, neural science, quantum computing, medical science, and social science. In 1998, the notion of bipolar fuzzy sets was proposed by Zhang [5, 6] as a generalization of fuzzy sets [1].

Bipolar-valued fuzzy units, which can be added by means of Lee [7], are an augmentation of fuzzy sets whose participation degree extend is broadened from the stretch $[0, 1]$ to $[-1, 1]$. Lee [8] involved the research with interval-valued fuzzy units, intuitionistic fuzzy units, and bipolar-valued fuzzy sets. Bipolar fuzzy sets have various applications in fuzzy algebras. For example, bipolar fuzzy ideals [4] in LA-semigroups, bipolar fuzzy subalgebras and ideals [7] of BCK/BCI-algebras, bipolar fuzzy a-ideals in BCK/BCI-algebras, and bipolar valued fuzzy BCK/BCI-algebras [9] are some of them. Bhakat and Das [10, 11] utilized the idea of $(\breve{\alpha}, \breve{\beta})$-fuzzy subgroups by the usage of $\in$ and $\breve{q}$ among a fuzzy factor and a fuzzy subset. It is visible that $(\in, \in\vee\breve{q})$-fuzzy subgroups are a critical generalization of Rosenfeld's [12] fuzzy subgroup. Similar types of $(\in, \in\vee q)$-fuzzy ideals of BCI-algebras are introduced by Zhan et al. [5]. Zhan et al. [13, 14] introduced three-way multiattribute decision-making based on outranking relations and multicriteria decision-making method based on a fuzzy rough set model with fuzzy α-neighborhoods. Senapati et al. [15] introduced cubic intuitionistic implicative ideals of BCK/BCI-algebras.

Jana et al. [16, 17] introduced generalizations of $(\in, \in\vee q)$-intuitionistic fuzzy subalgebras and ideals of BCK/BCI-algebras with thresholds. Jana et al. [18] introduced the concept of $(\in, \in\vee\breve{q})$-bipolar fuzzy BCK/BCI-subalgebras and $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals of BCK/BCI-algebras. These works are enough to motivate us, and, to the best of our knowledge, no other works are available on $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals and b-ideals in BCK/BCI-algebras and other fuzzy algebraic structures. For this reason, we have developed the theoretical study of $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals of BCK/BCI-algebras and $(\in, \in\vee\breve{q})$-bipolar fuzzy b-ideals of BCK/BCI-algebras. In this paper, the concepts of $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals are presented, and properties are established. Moreover, $(\in, \in\vee\breve{q})$-bipolar fuzzy b-ideals of BCI/BCK-algebras are proposed, and their properties are examined in detail.

2. Preliminaries

Definition 1. An algebra $(\breve{X}\,; \diamond, 0)$ of kind $(2, 0)$ could be a BCK-algebra if it fulfills for all $\breve{x}, \breve{y}, \breve{z} \in \breve{X}$,

$(K_1)((\breve{x}\diamond\breve{y})\diamond(\breve{x}\diamond\breve{z}))\diamond(\breve{z}\diamond\breve{y}) = 0$,

$(K_2)(\breve{x}\diamond(\breve{x}\diamond\breve{y}))\diamond\breve{y} = 0$,

$(K_3)\breve{x}\diamond\breve{x} = 0$,

$(K_4)0\diamond\breve{x} = 0$,

$(K_5)\breve{x}\diamond\breve{y} = 0$ and $\breve{y}\diamond\breve{x} = 0 \Rightarrow \breve{x} = \breve{y}$.

We remind the reader that $\breve{X}$ signifies a BCK/BCI-algebra unless otherwise specified.

A nonempty subset $\breve{A}$ of $\breve{X}$ is called an ideal of $\breve{X}$ if it satisfies

(I_1) $0 \in \breve{A}$,

(I_2) $\forall \breve{x}, \breve{y} \in \breve{X}, \breve{x}\diamond\breve{y} \in A, \breve{y} \in A \Rightarrow \breve{x} \in \breve{A}$.

A nonempty subset $\breve{A}$ of $\breve{X}$ is called a b-ideal of $\breve{X}$ if it satisfies (I_1) and

(I_3) $\forall \breve{x}, \breve{y}, \breve{z} \in \breve{X}, (\breve{x} * \breve{z})\diamond\breve{y} \in \breve{A}, \breve{y} \in \breve{A} \Rightarrow \breve{x} \in \breve{A}$.

A bipolar fuzzy set (BFS) in $\breve{X}$ is denoted by $\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$, where $\xi_{\breve{n}}, \xi_{\breve{p}}$ are the maps from $\breve{X}$ to $[-1, 0]$ and from $\breve{X}$ to $[0, 1]$, respectively.

3. $(\in, \in\vee\breve{q})$-Bipolar Fuzzy Ideals

In this section, we investigate an $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals of BCK/BCI-algebras.

Definition 2 (see [18]). A BFS$\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$ is a BFI of $\breve{X}$ if it satisfies the subsequent assertions:

(i) $(\forall \breve{x} \in \breve{X})(\xi_{\breve{n}}(0) \leq \xi_{\breve{n}}(\breve{x}),\ \xi_{\breve{p}}(0) \geq \xi_{\breve{p}}(\breve{x}))$

(ii) $(\forall \breve{x}, \breve{y} \in \breve{X})(\xi_{\breve{n}}(\breve{x}) \leq \xi_{\breve{n}}(\breve{x}\diamond\breve{y})\vee\xi_{\breve{n}}(\breve{y}))$

(iii) $(\forall \breve{x}, \breve{y} \in \breve{X})(\xi_{\breve{p}}(\breve{x}) \geq \xi_{\breve{p}}(\breve{x}\diamond\breve{y}) \wedge \xi_{\breve{p}}(\breve{y}))$

Definition 3. Let $\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$ be a BFS in $\breve{X}$ of the form

$$\begin{aligned} \xi_{\breve{n}}(\breve{y}) &= \begin{cases} \breve{m} \in [-1, 0), & \text{if } \breve{y} = \breve{x}, \\ 0, & \text{if } \breve{y} \neq \breve{x}, \end{cases} \\ \xi_{\breve{p}}(\breve{y}) &= \begin{cases} \breve{s} \in (0, 1], & \text{if } \breve{y} = \breve{x}, \\ 0, & \text{if } \breve{y} \neq \breve{x}. \end{cases} \end{aligned} \tag{1}$$

A bipolar fuzzy point with help $\breve{x}$ and values $\breve{m}$ and $\breve{s}$ is signified by $(\breve{x}, \breve{m}, \breve{s})$. In a bipolar fuzzy point $(\breve{x}, \breve{m}, \breve{s})$ and a BFS$\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$ in a set $\breve{X}$, we offer significance to the symbol $((\breve{x}, \breve{m})\Phi\xi_{\breve{n}}, (\breve{x}, \breve{s})\Phi\xi_{\breve{p}})$, where $\Phi \in \{\in, \breve{q}, \in\vee\breve{q}, \in\wedge\breve{q}\}$.

To say that $(\breve{x}, \breve{m}) \in \xi_{\breve{n}}$(respectively, $(\breve{x}, \breve{m})\breve{q}\xi_{\breve{n}}$) and $(\breve{x}, \breve{s}) \in \xi_{\breve{p}}$(respectively, $(\breve{x}, \breve{s})\breve{q}\xi_{\breve{p}}$) means that $\xi_{\breve{n}}(\breve{x}) \leq \breve{m}$ (respectively, $\xi_{\breve{n}}(\breve{x}) + \breve{m} < -1$) and $\xi_{\breve{p}}(\breve{x}) \geq \breve{s}$ (respectively, $\xi_{\breve{p}}(\breve{x}) + \breve{s} > 1$), and in this case, we say that $(\breve{x}, \breve{m})$ and $(\breve{x}, \breve{s})$ are said to belong to (respectively, be quasicoincident with) a BFS$\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$.

To say that $(\breve{x}, \breve{m}) \in \wedge\breve{q}$ (respectively, $(\breve{x}, \breve{m}\in\wedge\breve{q})$ and $(\breve{x}, \breve{s}) \in \wedge\breve{q}$ (respectively, $(\breve{x}, \breve{s}\in\wedge\breve{q})$ and imply $(\breve{x}, \breve{m}) \in \xi_{\breve{n}}$ or $(\breve{x}, \breve{m})\breve{q}\xi_{\breve{n}}$ (respectively, $(\breve{x}, \breve{m}) \in \xi_{\breve{n}}$ and $(\breve{x}, \breve{s}) \in \xi_{\breve{p}}$ or $(\breve{x}, \breve{s})\breve{q}\xi_{\breve{p}}$ (respectively, $(\breve{x}, \breve{s}) \in \xi_{\breve{n}}$).

To say that $((\breve{x}, \breve{m})\bar{\Phi}\xi_{\breve{n}}, (\breve{x}, \breve{s})\bar{\Phi}\xi_{\breve{p}})$ imply $(\breve{x}, \breve{m})\Phi\xi_{\breve{n}}$ does not hold and $(\breve{x}, \breve{s})\Phi\xi_{\breve{p}}$ does not hold, where $\Phi \in \{\in, \breve{q}, \in\vee\breve{q}, \in\wedge\breve{q}\}$.

Definition 4. A BFS$\xi = (\breve{X}\,; \xi_{\breve{n}}, \xi_{\breve{p}})$ is an $(\in, \in\vee\breve{q})$ − BFI of $\breve{X}$ if it satisfies the subsequent assertions:

(i) $(\breve{x}\diamond\breve{y}, \breve{m}) \in \xi_{\breve{n}}, (\breve{y}, \breve{n}) \in \xi_{\breve{n}} \Rightarrow (\breve{x}, \breve{m}\vee\breve{n}) \in \vee\breve{q}\xi_{\breve{n}}$, for all $\breve{x}, \breve{y} \in \breve{X}$ and $\breve{m}, \breve{n} \in [-1, 0)$

(ii) $(\breve{x}\diamond\breve{y}, \breve{t}) \in \xi_{\breve{n}}, (\breve{y}, \breve{s}) \in \xi_{\breve{n}} \Rightarrow (\breve{x}, \breve{t} \wedge \breve{s}) \in \vee\breve{q}\xi_{\breve{p}}$, for all $\breve{x}, \breve{y} \in \breve{X}$ and $\breve{t}, \breve{s} \in (0, 1]$

Example 5. Consider a BCI/BCK-algebra $\check{X} = \{0, \check{w}, \check{x}, \check{y}, \check{z}\}$ with the subsequent Cayley table:

o	0	$\check{w}$	$\check{x}$	$\check{y}$	$\check{z}$
0	0	0	0	0	0
$\check{w}$	$\check{w}$	0	$\check{w}$	0	$\check{w}$
$\check{x}$	$\check{x}$	$\check{x}$	0	0	0
$\check{y}$	$\check{y}$	w	$\check{y}$	0	$\check{y}$
$\check{z}$	$\check{z}$	$\check{z}$	$\check{z}$	$\check{z}$	0

(2)

Define a BFSξ of $\check{X}$ as follows:

o	0	$\check{w}$	$\check{x}$	$\check{y}$	$\check{z}$
$\xi_{\check{n}}$	-0.8	-0.5	-0.3	-0.6	-0.2
$\xi_{\check{p}}$	0.6	0.3	0.1	0.2	0.1

(3)

Hence, $\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in \vee \check{q}) - \text{BFI}$ of $\check{X}$.

Theorem 6. *A BFS$\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in \vee \check{q}) - BFI$ of $\check{X}$ if and only if it satisfiesif and only if it satisfies*

$$\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2},$$
$$\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \tag{4}$$

for all $\check{x}, \check{y} \in \check{X}$.

Proof. Let $\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ be an $(\in, \in \vee \check{q}) - \text{BFI}$ of $\check{X}$ and $\check{x}, \check{y} \in \check{X}$. If $\xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) > -1/2$ and $\xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) < 1/2$, then $\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y})$ and $\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y})$.

Assume that $\xi_{\check{n}}(\check{x}) > \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y})$ and $\xi_{\check{p}}(\check{x}) < \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y})$. Let us take $\check{m} \in \neg\xi$ and $t \in \xi$ such that $\xi_{\check{n}}(\check{x}) > \check{m} > \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y})$ and $\xi_{\check{p}}(\check{x}) < \check{t} < \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y})$. Then, $(\check{x} \diamond \check{y}, \check{m}) \in \xi_{\check{n}}, (\check{y}, \check{m}) \in \xi_{\check{n}}$ and $(\check{x} \diamond \check{y}, \check{t}) \in \xi_{\check{p}}, (\check{y}, \check{t}) \in \xi_{\check{p}}$ but $(\check{x}, \check{m} \vee \check{m}) = (\check{x}, \check{m}) \in, \overline{\in \vee \check{q}} \xi_{\check{n}}$ and $(\check{x}, \check{t} \wedge \check{t}) = (\check{x}, \check{t}) \in, \overline{\in \vee \check{q}} \xi_{\check{p}}$, a contradiction.

Hence, $\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y})$ whenever $\xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) > -1/2$ and $\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y})$ whenever $\xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) < 1/2$.

Suppose that $\xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \le -1/2$ and $\xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \ge 1/2$. Then, $(\check{x} \diamond \check{y}, -1/2) \in \xi_{\check{n}}, (\check{y}, -1/2) \in \xi_{\check{n}}$ and $(\check{x} \diamond \check{y}, 1/2) \in \xi_{\check{p}}, (\check{y}, 1/2) \in \xi_{\check{p}}$, which imply that

$$\left(\check{x}, -\frac{1}{2} \wedge -\frac{1}{2}\right) = \left(\check{x}, -\frac{1}{2}\right) \in \vee \check{q} \xi_{\check{n}},$$
$$\left(\check{x}, \frac{1}{2} \wedge \frac{1}{2}\right) = \left(\check{x}, \frac{1}{2}\right) \in \vee \check{q} \xi_{\check{p}}. \tag{5}$$

Thus, $\xi_{\check{n}}(\check{x}) \le -1/2$ and $\xi_{\check{p}}(\check{x}) \ge 1/2$. Otherwise, $\xi_{\check{n}}(\check{x}) - 1/2 > -1/2 - 1/2 = -1$ and $\xi_{\check{p}}(\check{x}) + 1/2 < 1/2 + 1/2 = 1$, a contradiction. And so,

$$\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \vee \frac{-1}{2},$$
$$\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \tag{6}$$

for all $\check{x}, \check{y} \in \check{X}$.

On the contrary, assume that $(\in, \in \vee \check{q}) - \text{BFI}$ of $\check{X}$ is valid. Let $\check{x}, \check{y} \in \check{X}$ and $\check{s}, \check{t} \in (0, 1]$ and $\check{m}, \check{n} \in \neg\xi$ such that $(\check{x} \diamond \check{y}, \check{m}) \in \xi_{\check{n}}, (\check{y}, \check{n}) \in \xi_{\check{n}}$ and $(\check{x} \diamond \check{y}, \check{s}) \in \xi_{\check{p}}, (\check{y}, \check{t}) \in \xi_{\check{p}}$. Then, $\xi_{\check{n}}(\check{x} \diamond \check{y}) \le \check{m}, \xi_{\check{n}}(\check{x} \diamond \check{y}) \le \check{n}$ and $\xi_{\check{p}}(\check{x} \diamond \check{y}) \ge \check{s}, \xi_{\check{p}}(\check{y}) \le \check{t}$.

If $\xi_{\check{n}}(\check{x}) > \check{m} \vee n$ and $\xi_{\check{p}}(\check{x}) < \check{s} \vee \check{t}$, then $\xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \le -1/2$ and $\xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \ge 1/2$.

Otherwise, we get

$$\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \le \xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \le \check{m} \vee \check{n},$$
$$\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \ge \xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \ge \check{s} \wedge \check{t}, \tag{7}$$

a contradiction. In that case,

$$\xi_{\check{n}}(\check{x}) + \check{m} \vee \check{n} < 2\xi_{\check{n}}(\check{x}) \le 2\left(\xi_{\check{n}}(\check{x} \diamond \check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2}\right) = -1,$$
$$\xi_{\check{n}}(\check{x}) + \check{s} \wedge \check{t} > 2\xi_{\check{p}}(\check{x}) \ge 2\left(\xi_{\check{p}}(\check{x} \diamond \check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2}\right) = 1. \tag{8}$$

Hence, $(\check{x}, \check{m} \vee \check{n}) \in \vee \check{q} \xi_{\check{n}}$ and $(\check{x}, \check{s} \wedge \check{t}) \in \vee \check{q} \xi_{\check{p}}$.

Therefore, ξ is an $(\in, \in \vee \check{q}) - \text{BFI}$ of $\check{X}$.

Lemma 7. *Every BFI is an $(\in, \in \vee \check{q}) - BFI$ of $\check{X}$.*

The opposite of Lemma 7 is not correct in general, justified in the subsequent Example 8.

Example 8. Consider a BCI-algebra $\check{X} = \{0, \check{w}, \check{x}, \check{y}\}$ with Cayley table:

o	0	$\check{w}$	$\check{x}$	$\check{y}$
0	0	$\check{w}$	$\check{x}$	$\check{y}$
$\check{w}$	$\check{w}$	0	$\check{y}$	$\check{x}$
$\check{x}$	$\check{x}$	$\check{y}$	0	$\check{w}$
$\check{y}$	$\check{y}$	$\check{x}$	$\check{w}$	0

(9)

Define a BFSξ of $\check{X}$ as follows:

o	0	$\check{w}$	$\check{x}$	$\check{y}$
$\xi_{\check{n}}$	-0.75	-0.25	-0.65	-0.25
$\xi_{\check{p}}$	0.75	0.65	0.65	0.55

(10)

Hence, $(\in, \in \vee \check{q}) - \text{BFI}$ of $\check{X}$ but is not *BFI* of $\check{X}$ because $\xi_{\check{p}}(\check{y}) = 0.55 \ngeq 0.65 = \xi_{\check{n}}(\check{y} \diamond \check{w}) \wedge \xi(\check{w})$.

4. $(\in, \in\vee\check{q})$-Bipolar Fuzzy b-Ideals

In this section, we investigate an $(\in, \in\vee\check{q})$-bipolar fuzzy b-ideals of BCK/BCI-algebras.

Definition 9. A BFS$\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is a BFBI of $\check{X}$ if it satisfies Definition 2 (i) and the subsequent assertions:

(i) $(\forall \check{x}, \check{y} \in \check{X}) \quad (\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}))$

(ii) $(\forall \check{x}, \check{y} \in \check{X}) \quad (\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}))$

Definition 10. A BFS$\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is called an $(\in, \in\vee\check{q})$-BFBI of $\check{X}$ if it satisfies the subsequent assertions:

(i) $((\check{x}\diamond\check{z})\diamond\check{y}, \check{m}) \in \xi_{\check{n}}, (\check{y}, \check{n}) \in \xi_{\check{n}} \Rightarrow (\check{x}, \check{m}\vee\check{n}) \in \vee\check{q}\xi_{\check{n}}$, for all $\check{x}, \check{y}, \check{z} \in \check{X}$ and $\check{m}, \check{n} \in [-1, 0)$

(ii) $((\check{x}\diamond\check{z})\diamond\check{y}, \check{s}) \in \xi_{\check{n}}, (\check{y}, \check{t}) \in \xi_{\check{n}} \Rightarrow (\check{x}, \check{s}\wedge\check{t}) \in \vee\check{q}\xi_{\check{p}}$, for all $\check{x}, \check{y}, \check{z} \in \check{X}$ and $\check{s}, \check{t} \in (0, 1]$

Example 11. Let $\check{X} = \{0, \check{w}, \check{x}, \check{y}, \check{z}\}$ be a BCK-algebra in Example 5 and a BFSξ of $\check{X}$ defined by

o	0	$\check{w}$	$\check{x}$	$\check{y}$	$\check{z}$
$\xi_{\check{n}}$	-0.75	-0.45	-0.25	-0.55	-0.15
$\xi_{\check{p}}$	0.55	0.25	0.05	0.15	0.05

(11)

Hence, $\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q})$-BFBI of $\check{X}$ as well as BFBI of $\check{X}$.

Theorem 12. *A BFS$\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is a BFBI of $\check{X}$ if and only if the following assertions are valid:*

(i) $(\check{x}, \check{m}) \in \xi_{\check{n}} \Rightarrow (0, \check{m}) \in \xi_{\check{n}}$ and $(\check{x}, \check{s}) \in \xi_{\check{p}} \Rightarrow (0, \check{s}) \in \xi_{\check{p}}$, for all $\check{x} \in \check{X}, \check{m} \in [-1, 0), \check{s} \in (0, 1]$

(ii) $((\check{x}\diamond\check{z})\diamond\check{y}, \check{m}) \in \xi_{\check{n}}, (\check{y}, \check{n}) \in \xi_{\check{n}} \Rightarrow (\check{x}, \check{m}\vee\check{n}) \in \xi_{\check{n}}$, for all $\check{x}, \check{y} \in \check{X}$ and $\check{m}, \check{n} \in [-1, 0)$

(iii) $((\check{x}\diamond\check{z})\diamond\check{y}, \check{s}) \in \xi_{\check{p}}, (\check{y}, \check{t}) \in \xi_{\check{p}} \Rightarrow (\check{x}, \check{s}\wedge\check{t}) \in \xi_{\check{p}}$, for all $\check{x}, \check{y} \in \check{X}$ and $\check{s}, \check{t} \in (0, 1]$

Proof. Assume that Definition 2(i) is valid and $\check{x} \in \check{X}$, $\check{s} \in (0, 1]$, $\check{m} \in [-1, 0)$ such that $(\check{x}, \check{m}) \in \xi_{\check{n}}$ and $(\check{x}, \check{s}) \in \xi_{\check{p}}$. Then, $\xi_{\check{n}}(0) \le \xi_{\check{n}}(\check{x}) \le \check{m}$ and $\xi_{\check{p}}(0) \ge \xi_{\check{p}}(\check{x}) \ge \check{s}$, and so $(0, \check{m}) \in \xi_{\check{n}}$ and $(0, \check{s}) \in \xi_{\check{p}}$.

Since $(\check{x}, \xi(\check{x})) \in \xi_{\check{n}}$ and $(\check{x}, \xi(\check{x})) \in \xi_{\check{p}}$ for all $\check{x} \in \check{X}$, it follows from (i) that $(0, \xi(\check{x})) \in \xi_{\check{n}}$ and $(0, \xi(\check{x})) \in \xi_{\check{p}}$ so that $\xi_{\check{n}}(0) \le \xi_{\check{n}}(\check{x})$ and $\xi_{\check{p}}(0) \ge \xi_{\check{p}}(\check{x})$ for all $\check{x} \in \check{X}$. Assume that Definition 9 holds.

Let $\check{x}, \check{y}, \check{z} \in \check{X}$, and $\check{m}, \check{n} \in [-1, 0)$, $\check{s}, \check{t} \in (0, 1]$ be such that $((\check{x}\diamond\check{z})\diamond\check{y}, \check{m}) \in \xi_{\check{n}}$, $(\check{y}, \check{n}) \in \xi_{\check{n}}$, and $((\check{x}\diamond\check{z})\diamond\check{y}, \check{s}) \in \xi_{\check{p}}$, $(\check{y}, \check{t}) \in \xi_{\check{p}}$. Then, $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \le \check{m}, \xi_{\check{n}}(\check{y}) \le \check{n}$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \ge \check{s}, \xi_{\check{p}}(\check{y}) \ge \check{t}$. It follows from Definition 9:

$$\begin{aligned}\xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \le \check{m}\vee\check{n},\\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \ge \check{s}\wedge\check{t}.\end{aligned} \tag{12}$$

So that $(\check{x}, \check{m}\vee\check{n}) \in \xi_{\check{n}}$ and $(\check{x}, \check{s}\wedge\check{t}) \in \xi_{\check{p}}$.

Again, suppose that (ii) and (iii) are valid. Also, for every $\check{x}, \check{y}, \check{z} \in \check{X}((\check{x}\diamond\check{z})\diamond\check{y}, \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})) \in \xi_{\check{n}}, (\check{y}, \xi_{\check{n}}(\check{y})) \in \xi_{\check{n}}$ and $((\check{x}\diamond\check{z})\diamond\check{y}, \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})) \in \xi_{\check{p}}, (\check{y}, \xi_{\check{p}}(\check{y}))$. Hence, $(\check{x}, \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y})) \in \xi_{\check{n}}$ and $(\check{x}, \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y})) \in \xi_{\check{p}}$ by (ii) and (iii), respectively, and thus,

$$\begin{aligned}\xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}),\\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}).\end{aligned} \tag{13}$$

Theorem 13. *A BFS$\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q})$-BFBI of $\check{X}$ if and only if it satisfies the subsequent assertions:*

(i) $\xi_{\check{n}}(0) \le \xi_{\check{n}}(\check{x}) \vee -1/2$ and $\xi_{\check{p}}(0) \ge \xi_{\check{p}}(\check{x}) \wedge 1/2$

(ii) $\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -1/2$

(iii) $\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge 1/2$

for all $\check{x}, \check{y}, \check{z} \in \check{X}$.

Proof. Suppose $\xi = (\check{X}\,; \xi_{\check{n}}, \xi_{\check{p}})$ be an $(\in, \in\vee\check{q})$-BFBI of $\check{X}$. Let $\check{x} \in \check{X}$ be such that $\xi_{\check{n}}(\check{x}) > -1/2$ and $\xi_{\check{p}}(\check{x}) < 1/2$. If $\xi_{\check{n}}(0) > \xi_{\check{n}}(\check{x})$ and $\xi_{\check{p}}(0) < \xi_{\check{p}}(\check{x})$, $\xi_{\check{n}}(0) > \check{m} > \xi_{\check{n}}(\check{x})$ and $\xi_{\check{p}}(0) < \check{s} < \xi_{\check{p}}(\check{x})$ for every $\check{m} \in (-1/2, 0)$ and $\check{s} \in (0, 1/2)$, so we get $(\check{x}, \check{m}) \in \xi_{\check{n}}$, $(0, \check{m})\bar{\in}\xi_{\check{n}}$ and $(\check{x}, \check{s}) \in \xi_{\check{p}}$, $(0, \check{s})\bar{\in}\xi_{\check{p}}$.

Since $\xi_{\check{n}}(0) + \check{m} > -1$ and $\xi_{\check{p}}(0) + \check{s} < 1$, so we have $(0, \check{m})\bar{\check{q}}\xi_{\check{n}}$ and $(0, \check{s})\bar{\check{q}}\xi_{\check{p}}$. It follows that $(0,\check{m})\in\bar{\vee}\check{q}\xi_{\check{n}}$ and $(0,\check{s})\in\bar{\vee}\check{q}\xi_{\check{p}}$, a contradiction. Hence, $\xi_{\check{n}}(0) \le \xi_{\check{n}}(\check{x})$ and $\xi_{\check{p}}(0) \ge \xi_{\check{p}}(\check{x})$. Now, if $\xi_{\check{n}}(\check{x}) \le -1/2$ and $\xi_{\check{p}}(\check{x}) \ge 1/2$, then $(\check{x},-1/2) \in \xi_{\check{n}}$ and $(\check{x}, 1/2) \in \xi_{\check{p}}$. Thus, $(0,-1/2) \in \vee\check{q}\xi_{\check{n}}$ and $(0, 1/2) \in \vee\check{q}\xi_{\check{p}}$. Thus, $\xi_{\check{n}}(0) \le -1/2$ and $\xi_{\check{p}}(0) \ge 1/2$.

Otherwise, $\xi_{\check{n}}(\check{x}) - 1/2 > -1/2 - 1/2 = -1$ and $\xi_{\check{p}}(\check{x}) + 1/2 < 1/2 + 1/2 = 1$, a contradiction. Consequently, $\xi_{\check{n}}(0) \le \{\xi_{\check{n}}(\check{x}),-1/2\}$ and $\xi_{\check{p}}(0) \ge \{\xi_{\check{p}}(\check{x}), 1/2\}$ for all $\check{x} \in \check{X}$.

Let $\check{x}, \check{y}, \check{z} \in \check{X}$. Suppose that $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) > -1/2$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) < 1/2$. Then, $\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y})$ and $\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y})$.

If not, then $\xi_{\check{n}}(\check{x}) > \check{m} > \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y})$ and $\xi_{\check{p}}(\check{x}) < \check{s} < \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y})$, for some $\check{m} \in (-1/2, 0), \check{s} \in (0, 1/2)$.

It follows that $((\check{x}\diamond\check{z})\diamond\check{y}, \check{m}) \in \xi_{\check{n}}$ and $(\check{y}, \check{m}) \in \xi_{\check{n}}$ but $(\check{x}, \check{m}\vee\check{m}) = (\check{x}, \check{m}) \in \vee\check{q}\xi_{\check{n}}$ and $((\check{x}\diamond\check{z})\diamond\check{y}, \check{s}) \in \xi_{\check{p}}$ and $(\check{y}, \check{s}) \in \xi_{\check{p}}$ but $(\check{x}, \check{s}\vee\check{s}) = (\check{x}, \check{s}) \in \vee\check{q}\xi_{\check{p}}$ which is a contradiction.

Hence, $\xi_{\check{n}}(\check{x}) \le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y})$ whenever $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) > -1/2$ and $\xi_{\check{p}}(\check{x}) \ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y})$ whenever $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) < 1/2$.

If $\xi_{\check{n}}(\check{x}\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\leq -1/2$, then $((\check{x}\diamond\check{z})\diamond\check{y},-1/2)\in\xi_{\check{n}}$ and $(\check{y},-1/2)\in\xi_{\check{n}}$, which imply that $(\check{x},-1/2)=(\check{x},-1/2\vee -1/2)\in\vee\check{q}\xi_{\check{n}}$ and if $\xi_{\check{p}}(\check{x}\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\geq 1/2$, then $((\check{x}\diamond\check{z})\diamond\check{y},1/2)\in\xi_{\check{p}}$ and $(\check{y},1/2)\in\xi_{\check{p}}$, which imply that $(\check{x},1/2)=(\check{x},1/2\wedge 1/2)\in\vee\check{q}\xi_{\check{p}}$.

Therefore, $\xi_{\check{n}}(\check{x})\leq -1/2$ and $\xi_{\check{p}}(\check{x})\geq 1/2$, because if $\xi_{\check{n}}(\check{x})>-1/2$ and $\xi_{\check{p}}(\check{x})<1/2$, then $\xi_{\check{n}}(\check{x})-1/2>-1/2-1/2=-1$ and $\xi_{\check{p}}(\check{x})+1/2<1/2+1/2=1$, which is a contradiction. Hence,

$$\begin{aligned}&\xi_{\check{n}}(\check{x})\leq\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -1/2,\\&\xi_{\check{p}}(\check{x})\geq\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge 1/2\end{aligned}\tag{14}$$

for all $\check{x},\check{y},\check{z}\in\check{X}$.

Conversely, assume that ξ satisfies the conditions of (i), (ii), and (iii). Let $\check{x}\in\check{X}$ and $\check{s}\in(0,1]$ and $\check{m}\in[-1,0)$ be such that $(\check{x},\check{m})\in\xi_{\check{n}}$ and $(\check{x},\check{s})\in\xi_{\check{p}}$. Then, $\xi_{\check{n}}(\check{x})\leq\check{m}$ and $\xi_{\check{p}}(\check{x})\geq\check{s}$.

Suppose that $\xi_{\check{n}}(0)\geq\check{m}$ and $\xi_{\check{p}}(0)\leq\check{s}$. If $\xi_{\check{n}}(\check{x})>-1/2$ and $\xi_{\check{p}}(\check{x})<1/2$, then $\xi_{\check{n}}(0)\leq\xi_{\check{n}}(\check{x})\vee -1/2=\xi_{\check{n}}(\check{x})\leq\check{m}$ and $\xi_{\check{p}}(0)\geq\xi_{\check{p}}(\check{x})\wedge 1/2=\xi_{\check{p}}(\check{x})\geq\check{s}$, a contradiction. Hence, we know that $\xi_{\check{n}}(\check{x})\leq -1/2$ and $\xi_{\check{p}}(\check{x})\geq 1/2$, and so, we get

$$\begin{aligned}&\xi_{\check{n}}(0)+\check{m}<2\xi_{\check{n}}(0)\leq\xi_{\check{n}}(\check{x})\vee -\frac{1}{2}=-1,\\&\xi_{\check{p}}(0)+\check{t}>2\xi_{\check{n}}(0)\geq\xi_{\check{p}}(\check{x})\wedge\frac{1}{2}=1.\end{aligned}\tag{15}$$

Thus, $(0,\check{m})\in\vee\xi_{\check{n}}$ and $(0,\check{s})\in\vee\xi_{\check{p}}$.

Let $\check{x},\check{y}\in\check{X}$, $\check{s},\check{t}\in(0,1]$ and $\check{m},\check{n}\in[1,0)$ be such that $((\check{x}\diamond\check{z})\diamond\check{y},\check{m})\in\xi_{\check{n}}$, $(\check{y},\check{n})\in\xi_{\check{n}}$ and $((\check{x}\diamond\check{z})\diamond\check{y},\check{s})\in\xi_{\check{p}}$, $(\check{y},\check{t})\in\xi_{\check{p}}$. Then, $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\leq\check{m}$, $\xi_{\check{n}}(\check{y})\leq n$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\geq\check{s}$, $\xi_{\check{p}}(\check{y})\geq\check{t}$.

Suppose that $\xi_{\check{n}}(\check{x})>\check{m}\vee n$ and $\xi_{\check{p}}(\check{x})<\check{s}\wedge\check{t}$. If $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\geq -1/2$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi(\check{y})\leq 1/2$. Then,

$$\begin{aligned}&\xi_{\check{n}}(\check{x})\leq\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -\frac{1}{2}\leq\xi_{\check{n}}(\check{x}\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\leq\check{m}\vee n,\\&\xi_{\check{p}}(\check{x})\geq\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge\frac{1}{2}\geq\xi_{\check{p}}(\check{x}\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\geq\check{s}\wedge\check{t},\end{aligned}\tag{16}$$

a contradiction. Thus, $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\leq -1/2$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\geq 1/2$. In that case,

$$\begin{aligned}&\xi_{\check{n}}(\check{x})+\check{m}\vee\check{n}<2\xi_{\check{n}}(\check{x})\leq 2\left(\xi_{\check{n}}(\check{x}\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -\frac{1}{2}\right)=-1,\\&\xi_{\check{n}}(\check{x})+\check{s}\wedge\check{t}>2\xi_{\check{p}}(\check{x})\geq 2\left(\xi_{\check{p}}(\check{x}\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge\frac{1}{2}\right)=1.\end{aligned}\tag{17}$$

Hence, $(\check{x},\check{m}\vee\check{n})\in\vee\check{q}\xi_{\check{n}}$ and $(\check{x},\check{s}\wedge\check{t})\in\vee\check{q}\xi_{\check{p}}$. So, ξ is an $(\in,\in\vee\check{q})$-BFBI of $\check{X}$.

Definition 14. Let $\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$ be a BFS of $\check{X}$ and $(\check{m},\check{s})\in[-1,0]\times[0,1]$, we define $U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})=\{x\in\check{X}\mid\xi_{\check{n}}(x)\leq\check{m}$ and $\xi_{\check{p}}(x)\geq\check{s}\}$ is called a $\check{m}$-level cut of $\xi_{\check{n}}$ and $\check{s}$-level cut of $\xi_{\check{p}}$ of the BFS$\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$.

Theorem 15. *A BFS$\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$ is an $(\in,\in\vee\check{q})-BFI$ of $\check{X}$ if and only if the level subset*

$$U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})=\{\check{x}\in\check{X}\mid\xi_{\check{n}}(\check{x})\leq\check{m}\ and\ \xi_{\check{p}}(\check{x})\geq\check{s}\}\tag{18}$$

is a BFI of $\check{X}$ for all $\check{m}\in[-1/2,0)$ and for all $\check{s}\in(0,1/2]$.

Proof. Assume that a $BFS\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$ is an $(\in,\in\vee\check{q})-BFI$ of $\check{X}$. Let $(\check{x}\diamond\check{z})\diamond\check{y},\check{y}\in U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})$ with $\check{m}\in[-1/2,0)$ and $\check{s}\in(0,1/2]$. Then $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\leq\check{m},\xi_{\check{n}}(\check{y})\leq\check{m}$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\geq\check{s},\xi_{\check{p}}(\check{y})\geq\check{s}$. Therefore, from Theorem 6 that

$$\begin{aligned}&\xi_{\check{n}}(\check{x})\leq\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{x})\vee -\frac{1}{2}\leq\check{m}\vee -\frac{1}{2}=\check{m},\\&\xi_{\check{p}}(\check{x})\geq\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{x})\wedge\frac{1}{2}\geq\check{s}\wedge\frac{1}{2}=\check{s},\end{aligned}\tag{19}$$

so that $\check{x}\in U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})$. Therefore, $U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})$ is a b-ideal of $\check{X}$.

Conversely, let ξ be a BFS of $\check{X}$ such that the set $U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})=\{\check{x}\in\check{X}\mid\xi_{\check{n}}\leq\check{m}$ and $\xi_{\check{p}}\geq\check{s}\}$ is a b-ideal of $\check{X}$ for all $\check{m}\in[-1/2,0)$ and $\check{s}\in(0,1/2]$. If there exist $(\check{x}\diamond\check{z})\diamond\check{y},\check{y}\in\check{X}$ such that $\xi_{\check{n}}>\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -1/2$ and $\xi_{\check{p}}(\check{x})<\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge 1/2$, then we take $\check{m}\in(-1,0)$ and $\check{s}\in(0,1)$ such that $\xi_{\check{n}}(x)>\check{m}>\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -1/2$ and $\xi_{\check{p}}(\check{x})<\check{s}<\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge 1/2$.

Thus, $(\check{x}\diamond\check{z})\diamond\check{y},\check{y}\in U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})$ with $\check{s}<1/2$ and $\check{m}>1/2$, and so $\check{x}\in U(\xi_{\check{n}},\xi_{\check{p}};\check{m},\check{s})$, i.e., $\xi_{\check{n}}(\check{x})\leq\check{m}$ and $\xi_{\check{p}}(\check{x})\geq\check{s}$ which is a contradiction. Therefore,

$$\begin{aligned}&\xi_{\check{n}}(\check{x})\leq\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y})\vee\xi_{\check{n}}(\check{y})\vee -\frac{1}{2},\\&\xi_{\check{p}}(\check{x})\geq\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y})\wedge\xi_{\check{p}}(\check{y})\wedge\frac{1}{2},\end{aligned}\tag{20}$$

for all $\check{x},\check{y}\in\check{X}$. Using Theorem 6, we conclude that ξ is an $(\in,\in\vee\check{q})$-BFBI of $\check{X}$.

Theorem 16. *Let $\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$ be an $(\in,\in\vee q)-BFBI$ of $\check{X}$, where $\xi_{\check{n}}(\check{x})>-1/2$ and $\xi_{\check{p}}(\check{x})<1/2$ for all $\check{x}\in\check{X}$. Then, $\xi=(\check{X};\xi_{\check{n}},\xi_{\check{p}})$ is an $(\in,\in)$-BFBI of $\check{X}$.*

Proof. The proof is straightforward using Theorem 6.

Theorem 17. *Let $\wedge$ be an index set and $\{(\xi_{i_{\check{n}}},\xi_{i_{\check{p}}})\mid i\in\wedge\}$ be a family of $(\in,\in\vee\check{q})-BFBI$ of $\check{X}$. Then, $\xi=\bigcap_{i\in\wedge}(\xi_{i_{\check{n}}},\xi_{i_{\check{p}}})$ is an $(\in,\in\vee\check{q})-BFBI$ of $\check{X}$.*

Proof. Let us take $(\check{x}\diamond\check{z})\diamond\check{y}, \check{y} \in \check{X}$ and $\check{m}_1, \check{m}_2 \in [-1, 0)$, and $\check{s}_1, \check{s}_2 \in (0, 1]$ be such that $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \leq \check{m}_1$ and $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \leq \check{m}_2$, $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \geq \check{s}_1$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \geq \check{s}_2$.

Assume that $\check{x}_{\check{m}_1\vee\check{m}_2} \overline{\in\vee q} \xi_{\check{n}}$ and $\check{x}_{\check{s}_1\wedge\check{s}_2} \overline{\in\vee q} \xi_{\check{p}}$. Then, $\xi_{\check{n}}(\check{x}) > \check{m}_1\vee\check{m}_2$ and $\xi_{\check{n}}(\check{x}) + \check{m}_1\vee\check{m}_2 \geq -1$, and $\xi_{\check{p}}(\check{x}) < \check{s}_1 \wedge \check{s}_2$ and $\xi_{\check{p}}(\check{x}) + \check{s}_1 \wedge \check{s}_2 \leq 1$, which implies

$$\begin{aligned} &\xi_{\check{n}}(\check{x}) > -\frac{1}{2}, \\ &\xi_{\check{p}}(\check{x}) < \frac{1}{2}. \end{aligned} \tag{21}$$

Now, we define

$$\begin{aligned} \Delta_1 &= \left\{ i\in\wedge | \check{x}_{\check{m}_1\vee\check{m}_2} \in \xi_{i_{\check{n}}} \text{ and } \check{x}_{\check{s}_1\wedge\check{s}_2} \in \xi_{i_{\check{p}}} \right\}, \\ \Delta_2 &= \left\{ \left[\left\{ i\in\wedge | \check{x}_{\check{m}_1\vee\check{m}_2} q \xi_{i_{\check{n}}} \right\} \cap \left\{ j\in\wedge | \check{x}_{\check{m}_1\vee\check{m}_2} \in \bar{\xi}_{i_{\check{n}}} \right\} \right] \text{ and } \right. \\ &\quad \left. \left[\left\{ i\in\wedge | \check{x}_{\check{s}_1\wedge\check{s}_2} q \xi_{i_{\check{p}}} \right\} \cap \left\{ j\in\wedge | \check{x}_{\check{s}_1\wedge\check{s}_2} \in \bar{\xi}_{i_{\check{p}}} \right\} \right] \right\}. \end{aligned} \tag{22}$$

Then, $\wedge = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \varnothing$.

If $\Delta_2 = \varnothing$, then $\check{x}_{\check{m}_1\vee\check{m}_2} \in \xi_{i_{\check{n}}}$ and $\check{x}_{\check{s}_1\wedge\check{s}_2} \in \xi_{i_{\check{p}}}$ for all $i \in \wedge$, i.e., $\xi_{i_{\check{n}}}(\check{x}) \leq \check{m}_1\vee\check{m}_2$ and $\xi_{i_{\check{p}}}(\check{x}) \geq \check{s}_1 \wedge \check{s}_2$ for all $i \in \wedge$, which indicate

$$\xi_{i_{\check{n}}}(\check{x}) \leq \check{m}_1\vee\check{m}_2 \text{ and } \xi_{i_{\check{p}}}(\check{x}) \geq \check{s}_1 \wedge \check{s}_2. \tag{23}$$

This is a contradiction. Hence, $\Delta_2 \neq \varnothing$, and so for every $i \in \Delta_2$, we have $\xi_{i_{\check{n}}}(\check{x}) > \check{m}_1\vee\check{m}_2$ and $\xi_{i_{\check{n}}}(\check{x}) + \check{m}_1\vee\check{m}_2 < -1$, and $\xi_{i_{\check{p}}}(\check{x}) < \check{s}_1 \wedge \check{s}_2$ and $\xi_{i_{\check{p}}}(\check{x}) + \check{s}_1 \wedge \check{s}_2 > 1$.

It follows that $\check{m}_1\vee\check{m}_2 < -1/2$ and $\check{s}_1 \wedge \check{s}_2 > 1/2$.

Now, $\check{x}_{\check{m}_1} \in \xi_{\check{n}}$ and $\check{x}_{\check{s}_1} \in \xi_{\check{p}}$ implies that $\xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \leq \check{m}_1$ and $\xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{z}) \geq \check{s}_1$, and thus,

$$\begin{aligned} &\xi_{i_{\check{n}}}((\check{x}\diamond\check{z})\diamond\check{y}) \leq \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \leq \check{m}_1 \leq \check{m}_1\vee\check{m}_2 < -\frac{1}{2}, \\ &\xi_{i_{\check{p}}}((\check{x}\diamond\check{z})\diamond\check{y}) \geq \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \geq \check{s}_1 \geq \check{s}_1 \wedge \check{s}_2 > \frac{1}{2} \end{aligned} \tag{24}$$

for all $i \in \wedge$.

Similarly, we get $\xi_{i_{\check{n}}} < -1/2$ and $\xi_{i_{\check{p}}}(\check{y}) > 1/2$ for all $i \in \wedge$.

We suppose that $\check{m} = \xi_{i_{\check{n}}} > -1/2$ and $\check{s} = \xi_{i_{\check{p}}} < 1/2$.

Taking that $\check{m} > n > -1/2$ and $\check{s} < t < 1/2$, we get

$$((\check{x}\diamond\check{y})\diamond\check{z})_n \in \xi_{i_{\check{n}}} \text{ and } \check{y}_n \in \xi_{i_{\check{n}}}, \text{ but } \check{x}_{n\vee n} = \check{x}_n \bar{\in} \vee q \xi_{i_{\check{n}}}, \tag{25}$$

$$((\check{x}\diamond\check{y})\diamond\check{z})_t \in \xi_{i_{\check{p}}} \text{ and } \check{y}_t \in \xi_{i_{\check{p}}}, \text{ but } \check{x}_{t\wedge t} = \check{x}_t \bar{\in} \vee q \xi_{i_{\check{p}}}. \tag{26}$$

This contradicts that $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q}) - BFBI$ of $\check{X}$.

Hence, $\xi_{i_{\check{n}}} \leq -1/2$ and $\xi_{i_{\check{p}}} \geq 1/2$ for all $i \in \wedge$, so $\xi_{\check{n}} \leq -1/2$ and $\xi_{\check{p}} \geq 1/2$ which contradicts (21).

Therefore, $\check{x}_{\check{m}_1\vee\check{m}_2} \in \vee q \xi_{\check{n}}$ and $\check{x}_{\check{s}_1\wedge\check{s}_2} \in \vee q \xi_{\check{p}}$, and consequently, $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in,\in\vee q) - BFBI$ of $\check{X}$.

For any $BFS\xi$ in $\check{X}$, where $m \in [1, 0)$ and $s \in (0, 1]$, we denote

$$\begin{aligned} \xi_{\check{m}_{\check{n}}} &= \left\{ \check{x} \in \check{X} \mid \check{x}_{\check{m}} \in \check{q}\xi_{\check{n}} \right\}, \\ \xi_{\check{s}_{\check{p}}} &= \left\{ \check{x} \in \check{X} \mid \check{x}_{\check{s}} \in \check{q}\xi_{\check{p}} \right\}, \\ [\xi]_{(\check{m},\check{s})} &= \left\{ \check{x} \in \check{X} \mid \check{x}_{\check{m}} \in \check{q}\xi_{\check{n}} \text{ and } \check{x}_{\check{s}} \in \check{q}\xi_{\check{p}} \right\}. \end{aligned} \tag{27}$$

Then, it is obvious that $[\xi]_{(\check{m},\check{s})} = U(\xi_{\check{n}}, \xi_{\check{p}}\,; \check{m}, \check{s}) \cup \xi_{\check{m}_{\check{n}}} \cup \xi_{\check{s}_{\check{p}}}$. Here, $[\xi]_{(\check{m},\check{s})}$ is an $(\in\vee q)$-level b-ideal of ξ.

Theorem 18. *Let $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ be a BFS in ξ. Then, $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q}) - BFBI$ of $\check{X}$ if and only if $[\xi]_{(\check{m},\check{s})}$ is a b-ideal of ξ for all $\check{m} \in [-1, 0)$ and $\check{s} \in (0, 1]$.*

Proof. Suppose that $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q}) - BFBI$ of $\check{X}$ and let $\check{x}, \check{y} \in [\xi]_{(\check{m},\check{s})}$ for $\check{m} \in [-1, 0)$ and $\check{s} \in (0, 1]$. Then $\check{x}_{\check{m}} \in \check{q}\xi_{\check{n}}, \check{y}_{\check{m}} \in \check{q}\xi_{\check{n}}$ and $\check{x}_{\check{s}} \in \check{q}\xi_{\check{p}}, \check{y}_{\check{s}} \in \check{q}\xi_{\check{p}}$. That is, $\xi_{\check{n}}(\check{x}) \leq \check{m}$ or $\xi_{\check{n}}(\check{x}) + \check{m} < -1$, $\xi_{\check{n}}(\check{y}) \leq \check{m}$ or $\xi_{\check{n}}(\check{y}) + \check{m} < -1$ and $\xi_{\check{p}}(\check{x}) \geq \check{s}$ or $\xi_{\check{p}}(\check{x}) + \check{s} > 1$, $\xi_{\check{p}}(\check{y}) \geq \check{s}$ or $\xi_{\check{p}}(\check{y}) + \check{s} > 1$. Using Theorem 6, we get

$$\begin{aligned} &\xi_{\check{n}}(\check{x}) \leq \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2}, \\ &\xi_{\check{p}}(\check{x}) \geq \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2}. \end{aligned} \tag{28}$$

Case 19. $\xi_{\check{n}}(\check{x}) \leq \check{m}$, $\xi_{\check{n}}(\check{y}) \leq \check{m}$ and $\xi_{\check{p}}(\check{x}) \geq \check{s}$, $\xi_{\check{p}}(\check{y}) \geq \check{s}$. If $\check{m} < -1/2$ and $t > 1/2$, then

$$\begin{aligned} &\xi_{\check{n}}(\check{x}) \leq \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} = -\frac{1}{2}, \\ &\xi_{\check{p}}(\check{x}) \geq \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} = \frac{1}{2}. \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} &\xi_{\check{n}}(\check{x}) + \check{m} < -\frac{1}{2} - \frac{1}{2} = -1, \\ &\xi_{\check{p}}(\check{x}) + \check{s} > \frac{1}{2} + \frac{1}{2} = 1, \end{aligned} \tag{30}$$

and so, $\check{x} \in \check{q}\xi_{\check{n}}$ and $\check{x} \in \check{q}\xi_{\check{p}}$. If $\check{m} \geq -1/2$ and $\check{s} \leq 1/2$, then

$$\begin{aligned} &\xi_{\check{n}}(\check{x}) \leq \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \leq \check{m}, \\ &\xi_{\check{p}}(\check{x}) \geq \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \geq \check{s}. \end{aligned} \tag{31}$$

Thus, $\check{x}_{\check{m}} \in \vee\check{q}\xi_{\check{n}}$ and $\check{x}_{\check{s}} \in \vee\check{q}\xi_{\check{p}}$. Therefore, $[\xi]_{(\check{m},\check{s})}$.

Case 20. $\xi_{\check{n}}(\check{x}) \le \check{m}$, $\xi_{\check{n}}(\check{y}) + \check{m} < -1$ and $\xi_{\check{p}}(\check{x}) \ge \check{s}$, $\xi_{\check{p}}(\check{y}) + \check{s} > 1$. If $\check{m} < -1/2$ and $t > 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} = \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \\ &= (-1-\check{m}) \vee -\frac{1}{2} = -1-\check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} = \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \\ &= (1-t) \wedge \frac{1}{2} = 1-t. \end{aligned} \tag{32}$$

Hence,

$$\begin{aligned} \xi_{\check{n}}(\check{x}) + \check{m} &< -\frac{1}{2} - \frac{1}{2} = -1, \\ \xi_{\check{p}}(\check{x}) + \check{s} &> \frac{1}{2} + \frac{1}{2} = 1, \end{aligned} \tag{33}$$

and so, $\check{x} \in \check{q}\xi_{\check{n}}$ and $\check{x} \in \check{q}\xi_{\check{p}}$. If $\check{m} \ge -1/2$ and $\check{s} \le 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \le \check{m} \vee (-1-\check{m}) \vee -\frac{1}{2} = \check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \ge \check{s} \wedge (1-\check{s}) \wedge \frac{1}{2} = \check{s}. \end{aligned} \tag{34}$$

Thus, $\check{x}_{\check{m}} \in \vee\check{q}\xi_{\check{n}}$ and $\check{x}_{\check{s}} \in \vee\check{q}\xi_{\check{p}}$. Therefore, $[\xi]_{(\check{m},\check{s})}$.

Case 21. $\xi_{\check{n}}(\check{x}) + \check{m} < -1$, $\xi_{\check{n}}(\check{y}) \le \check{m}$ and $\xi_{\check{p}}(\check{x}) + \check{s} > 1$, $\xi_{\check{p}}(\check{y}) \ge \check{s}$. If $\check{m} < -1/2$ and $t > 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} = \xi_{\check{n}}(\check{x}) \vee -\frac{1}{2} \\ &= (-1-\check{m}) \vee -\frac{1}{2} = -1-\check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} = \xi_{\check{p}}(\check{x}) \wedge \frac{1}{2} \\ &= (1-t) \wedge \frac{1}{2} = 1-t. \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} \xi_{\check{n}}(\check{x}) + \check{m} &< -\frac{1}{2} - \frac{1}{2} = -1, \\ \xi_{\check{p}}(\check{x}) + \check{s} &> \frac{1}{2} + \frac{1}{2} = 1, \end{aligned} \tag{36}$$

and so, $\check{x} \in \check{q}\xi_{\check{n}}$ and $\check{x} \in \check{q}\xi_{\check{p}}$. If $\check{m} \ge -1/2$ and $\check{s} \le 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \le (-1-\check{m}) \vee \check{m} \vee -\frac{1}{2} = \check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \ge (1-\check{s}) \wedge \check{s} \wedge \frac{1}{2} = \check{s}. \end{aligned} \tag{37}$$

Thus, $\check{x}_{\check{m}} \in \vee\check{q}\xi_{\check{n}}$ and $\check{x}_{\check{s}} \in \vee\check{q}\xi_{\check{p}}$. Therefore, $[\xi]_{(\check{m},\check{s})}$.

Case 22. $\xi_{\check{n}}(\check{x}) + \check{m} < -1$, $\xi_{\check{n}}(\check{y}) + \check{m} < -1$ and $\xi_{\check{p}}(\check{x}) + \check{s} > 1$, $\xi_{\check{p}}(\check{y}) + \check{s} > 1$. If $\check{m} < -1/2$ and $t > 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} = \xi_{\check{n}}(\check{x}) \vee -\frac{1}{2} \\ &= (-1-\check{m}) \vee -\frac{1}{2} = -1-\check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} = \xi_{\check{p}}(\check{x}) \wedge \frac{1}{2} \\ &= (1-t) \wedge \frac{1}{2} = 1-t. \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \xi_{\check{n}}(\check{x}) + \check{m} &< -\frac{1}{2} - \frac{1}{2} = -1, \\ \xi_{\check{p}}(\check{x}) + \check{s} &> \frac{1}{2} + \frac{1}{2} = 1, \end{aligned} \tag{39}$$

and so, $\check{x} \in \check{q}\xi_{\check{n}}$ and $\check{x} \in \check{q}\xi_{\check{p}}$. If $\check{m} \ge -1/2$ and $\check{s} \le 1/2$, then

$$\begin{aligned} \xi_{\check{n}}(\check{x}) &\le \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2} \le (-1-\check{m}) \vee -\frac{1}{2} = -\frac{1}{2} \le \check{m}, \\ \xi_{\check{p}}(\check{x}) &\ge \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2} \ge (1-\check{s}) \wedge \frac{1}{2} = \frac{1}{2} \ge \check{s}. \end{aligned} \tag{40}$$

Therefore, $\check{x}_{\check{m}} \in \vee\check{q}\xi_{\check{n}}$ and $\check{x}_{\check{s}} \in \vee\check{q}\xi_{\check{p}}$. Hence, $[\xi]_{(\check{m},\check{s})}$. Therefore, $[\xi]_{(\check{m},\check{s})}$ is a b-ideal of ξ.

Conversely, let $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ be a *BFS* in ξ and $\check{m} \in [-1, 0), \check{s} \in (0, 1]$. Then, $\xi = (\xi_{\check{n}}, \xi_{\check{p}})$ is an $(\in, \in\vee\check{q}) - BFBI$ of $\check{X}$ such that $[\xi]_{(\check{m},\check{s})}$ is a b-ideal of ξ. If possible, let

$$\begin{aligned} \xi_{\check{n}}(\check{x}) > \check{m} &\ge \xi_{\check{n}}((\check{x}\diamond\check{z})\diamond\check{y}) \vee \xi_{\check{n}}(\check{y}) \vee -\frac{1}{2}, \\ \xi_{\check{p}}(\check{x}) < \check{s} &\le \xi_{\check{p}}((\check{x}\diamond\check{z})\diamond\check{y}) \wedge \xi_{\check{p}}(\check{y}) \wedge \frac{1}{2}, \end{aligned} \tag{41}$$

for some $\check{m} \in (-1, 0), \check{s} \in (0, \check{s})$. Then, $(\check{x}\diamond\check{z})\diamond\check{y}, \check{y} \in U(\xi_{\check{n}}, \xi_{\check{p}}; \check{m}, \check{s}) \subseteq [\xi]_{(\check{m},\check{s})}$, which indicate $\check{x} \in [\xi]_{(\check{m},\check{s})}$. Thus, $\xi_{\check{n}}(x) \le \check{m}$ or $\xi_{\check{n}}(\check{x}) + \check{m} < -1$, $\xi_{\check{n}}(\check{y}) \le \check{m}$ or $\xi_{\check{n}}(\check{y}) + \check{m} < -1$ and $\xi_{\check{p}}(\check{x}) \ge \check{s}$ or $\xi_{\check{p}}(\check{x}) + \check{s} > 1$, $\xi_{\check{p}}(\check{y}) \ge \check{s}$ or $\xi_{\check{p}}(\check{y}) + \check{s} > 1$, and these are a contradiction. Hence,

$$\begin{aligned}\xi_{\breve{n}}(\breve{x}) &\leq \xi_{\breve{n}}((\breve{x}\diamond\breve{z})\diamond\breve{y})\vee\xi_{\breve{n}}(\breve{y})\vee-\frac{1}{2},\\ \xi_{\breve{p}}(\breve{x}) &\geq \xi_{\breve{p}}((\breve{x}\diamond\breve{z})\diamond\breve{y})\wedge\xi_{\breve{p}}(\breve{y})\wedge\frac{1}{2}\end{aligned} \quad (42)$$

for all $\breve{x}, \breve{y} \in \breve{X}$. Now, by using Theorem 6, we conclude that $\xi = (\xi_{\breve{n}}, \xi_{\breve{p}})$ is an $(\in, \in\vee\breve{q}) - BFBI$ of $\breve{X}$.

5. Conclusion

In this paper, the thought of $(\in, \in\vee\breve{q})$-bipolar fuzzy ideals, and $(\in, \in\vee\breve{q})$-bipolar fuzzy b-ideals are presented and described their valuable properties. We investigated the connection between $(\in, \in\vee\breve{q})$-bipolar fuzzy b-ideals and bipolar fuzzy b-ideals and also the relation of their corresponding ideals. In our future study of a bipolar fuzzy structure of BCK/BCI-algebra, we may consider the following topics: (i) bipolar fuzzy soft subalgebra of BCK/BCI-algebra, (ii) bipolar $(\in, \in\vee\breve{q})$-fuzzy soft subalgebra of BCK/BCI-algebra, and (iii) $(\in, \in\vee\breve{q})$-bipolar fuzzy soft (p and a)-ideals of BCK/BCI-algebra and their relations.

Data Availability

No data is used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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