Research Article

Estimates for Commutators of Bilinear Fractional p-Adic Hardy Operator on Herz-Type Spaces

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In the current article, we investigate the boundedness of commutators of the bilinear fractional p-adic Hardy operator on p-adic Herz spaces and p-adic Morrey-Herz spaces by considering the symbol function from central bounded mean oscillations and Lipschitz spaces.

1. Introduction

For every $x \neq 0$, there is a unique $\gamma = \gamma(x) \in \mathbb{Z}$ such that $x = p^\gamma m/n$, where $p \geq 2$ is a fixed prime number which is coprime to $m, n \in \mathbb{Z}$. The mapping $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_+$ defines a norm on $\mathbb{Q}$ with a range

$$\{0\} \cup \{p^\gamma : \gamma \in \mathbb{Z}\}. \tag{1}$$

It follows from Ostrowski's theorem (see [1]) that each nontrivial absolute value on $\mathbb{Q}$ is either the p-adic absolute value $|\cdot|_p$ or usual absolute value $|\cdot|$. The p-adic norm $|\cdot|_p$ is an ultrametric on $\mathbb{Q}$, that is

$$|x + y|_p \leq \max \left\{ |x|_p, |y|_p \right\}. \tag{2}$$

The field of p-adic numbers is represented by $\mathbb{Q}_p$ and is the completion of rational numbers with respect to the p-adic norm $|\cdot|_p$. Any p-adic number is written in series form (see [2]) as

$$\cdots + d_4 p^4 + d_3 p^3 + d_2 p^2 + d_1 p + d_0 + \frac{d_{-1}}{p} + \cdots + \frac{d_{-j+1}}{p^{j-1}} + \frac{d_{-j}}{p^j}, \tag{3}$$

where $d_k \in \mathbb{Z}/p\mathbb{Z}$. Hence, each member of $\mathbb{Q}_p$ is written in the form

$$\cdots d_k d_{k-1} \cdots d_1 d_0 . d_{-1} d_{-2} \cdots d_{-j}. \tag{4}$$

The higher dimensional vector space $\mathbb{Q}_p^n$ consists of tuples $\mathbf{x} = (x_1, \cdots, x_n)$, where $x_k \in \mathbb{Q}_p, k = 1, \cdots, n$, with the following norm

$$|\mathbf{x}|_p = \max_{1 \leq k \leq n} |x_k|_p. \tag{5}$$

For $\gamma \in \mathbb{Z}$ and $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{Q}_p^n$, we represent by

$$B_\gamma(\mathbf{a}) = \left\{ \mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^\gamma \right\}, \tag{6}$$

the closed ball with the center $\mathbf{a}$ and radius p^{γ} and by

$$S_{\gamma}(\mathbf{a}) = \left\{ \mathbf{x} \in \mathbb{Q}_p^n : ||\mathbf{x} - \mathbf{a}|_p = p^{\gamma} \right\}, \qquad (7)$$

the corresponding sphere. For $\mathbf{a} = \mathbf{0}$, we write $B_{\gamma}(\mathbf{0}) = B_{\gamma}$ and $S_{\gamma}(\mathbf{0}) = S_{\gamma}$. It is easy to see that the equalities

$$\mathbf{a}_0 + B_{\gamma} = B_{\gamma}(\mathbf{a}_0), \mathbf{a}_0 + S_{\gamma} = S_{\gamma}(\mathbf{a}_0), S_{\gamma}(\mathbf{a}_0) = B_{\gamma}(\mathbf{a}_0) \setminus B_{\gamma-1}(\mathbf{a}_0), \qquad (8)$$

hold for all $\mathbf{a}_0 \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$.

Since the space $\mathbb{Q}_p^n$ is locally compact commutative group under addition, so it leads to a translation-invariant Haar measure $d\mathbf{x}$ which is normalized as follows

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1, \qquad (9)$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. In addition, it is not hard to see that $|B_{\gamma}(\mathbf{a})| = p^{n\gamma}$, $|S_{\gamma}(\mathbf{a})| = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Recently, p-adic analysis has taken considerable attention in harmonic analysis defined on the p-adic field [3–9] and mathematical physics [10, 11]. Furthermore, applications of p-adic analysis have been found in quantum gravity [12, 13], string theory [14], spring glass theory [15], and quantum mechanics [11].

The Hardy operator was taken into consideration in [16] and is given as below:

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt, x > 0, \qquad (10)$$

satisfying the following inequality:

$$\|Hf\|_{L^p(\mathbb{R}^+)} \le \frac{p}{p-1} \|f\|_{L^p(\mathbb{R}^+)}, 1 < p < \infty. \qquad (11)$$

The generalization of (10) to n-dimensional Euclidian space was made in [17], which is given by:

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n} \int_{|\mathbf{t}| \le |\mathbf{x}|} f(\mathbf{t}) d\mathbf{t}, \qquad (12)$$

where $f \in L_{loc}^1(\mathbb{R}^n)$ and $\mathbf{x} = (x_1, \cdots, x_n)$. The boundedness of Hardy operator on $L^p(\mathbb{R}^n)$ was investigated in [18]. Without going into the detailed history regarding the boundedness of Hardy-type operators and their commutators on function spaces, we refer the readers to see [19–25] and the references therein.

Fractional calculus is one of the major fields in the modern ages due to its numerous applications in science and engineering, see for instance [26–29]. Also, fractional integral operators are an integral part of the mathematical analysis. In this sense, Wu [30] defined the p-adic fractional Hardy operator as:

$$H_{\beta}^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\beta}} \int_{|\mathbf{t}|_p \le |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t}, \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}, \qquad (13)$$

where $f \in L_{loc}(\mathbb{Q}_p^n)$ and $0 \le \beta < n$. Also, he gave the following definition of its commutators:

$$H_{\beta,b}^p = bH_{\beta}^p f - H_{\beta}^p(bf). \qquad (14)$$

If $\beta = 0$, the fractional p-adic Hardy type operator is the p-adic Hardy operator [31, 32]. The commutator estimates of fractional Hardy-type operators on Herz spaces were obtained in [30, 32]. The articles [33, 34] are also important with regard to the study of p-adic Hardy operators on function spaces.

Multilinear operators are studied in the analysis because of their natural appearance in numerous physical phenomenons and their purpose is not merely to generalize the theory of linear operators. We refer articles [35–37] for better comprehension of multilinear operators. The m-linear Hardy operator was defined by Fu et al. [19] and is given by:

$$H^m(f_1, \cdots, f_m)(\mathbf{x}) = \frac{1}{|\mathbf{x}|^{mn}} \int_{|(\mathbf{t}_1, \cdots, \mathbf{t}_m)| < |\mathbf{x}|} f_1(\mathbf{t}_1) \cdots f_m(\mathbf{t}_m) d\mathbf{t}_1 \cdots d\mathbf{t}_m, \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \qquad (15)$$

for $f_1, \cdots, f_m$ in $L_{loc}^1(\mathbb{R}^n)$. In the same paper, they worked out the precise norm of the very operator on Lebesgue spaces with power weights.

Now, we introduce the definition of m-linear fractional p-adic Hardy operator as

$$H_{\beta}^{p,m}(f_1, \cdots, f_m)(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{mn-\beta}} \int_{|(\mathbf{t}_1, \cdots, \mathbf{t}_m)|_p \le |\mathbf{x}|_p} f_1(\mathbf{t}_1) \cdots f_m(\mathbf{t}_m) d\mathbf{t}_1 \cdots d\mathbf{t}_m, \qquad (16)$$

$\mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}$, for $f_1, \cdots, f_m$ in $L_{loc}^1(\mathbb{Q}_p^n)$. The 2-linear fractional p-adic Hardy operator will be referred to as a bilinear fractional p-adic Hardy operator. If $\beta = 0$, we get the m-linear p-adic Hardy operator, see [38], where the authors obtained the sharp bounds of the m-linear p-adic Hardy operator and Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights. In [33], sharp bounds for the m-linear p-adic Hardy operator on the product of p-adic Lebesgue spaces have been obtained in an efficient way. Next, we define the commutator generated by the m-linear fractional p-adic Hardy operator as follows. Let $b_i \in L_{loc}^1(\mathbb{Q}_p^n)$ for $i = 1, \cdots, m$, then

$$H_{\beta,\vec{b}}^{p,m}(f_1, \cdots, f_m)(\mathbf{x}) = \sum_{i=1}^{m} H_{\beta,b_i}^{p,m}(f_1, \cdots, f_m)(\mathbf{x}), \qquad (17)$$

where

$$H_{\beta,b_i}^{p,m}(f_1,\cdots,f_m)(\mathbf{x}) = b_i(\mathbf{x})H_\beta^{p,m}(f_1,\cdots,f_m)(\mathbf{x})$$
$$- H_\beta^{p,m}(f_1,\cdots,f_{i-1},f_ib_i,f_{i+1},\cdots,f_m)(\mathbf{x}). \tag{18}$$

If $\beta = 0$, we get the commutator operator defined in [39] with $\mathbb{R}^n$ as underlying space.

The aim of this article is to establish the CMO (central bounded mean oscillation) and Lipschitz estimates for commutators of a bilinear fractional p-adic Hardy operator on p-adic function spaces such as p-adic Herz spaces and Morrey-Herz spaces. Before moving to our main results, let us specify that χ_k is the characteristic function of a sphere S_k, and C is a constant free from essential variables and its value may change at its multiple occurrences. It is imperative to recall the definition of homogeneous p-adic Herz spaces and homogeneous p-adic Morrey-Herz spaces, p-adic CMO spaces, and p-adic Lipschitz spaces.

Definition 1 [31]. Suppose $0 < q, r < \infty$ and $\alpha \in \mathbb{R}$. The homogeneous p-adic Herz space $\dot{K}_q^{\alpha,r}(\mathbb{Q}_p^n)$ is defined by

$$K_q^{\alpha,r}\left(\mathbb{Q}_p^n\right) = \left\{f \in L^q\left(\mathbb{Q}_p^n\right): \|f\|_{K_q^{\alpha,r}(\mathbb{Q}_p^n)} < \infty\right\}, \tag{19}$$

where

$$\|f\|_{K_q^{\alpha,r}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} p^{k\alpha r}\|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^r\right)^{1/q}. \tag{20}$$

Obviously, $\dot{K}_q^{0,q}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$ and $\dot{K}_q^{\alpha/q,q}(\mathbb{Q}_p^n) = L^q(|\mathbf{x}|_p^\alpha)$.

Definition 2 [40]. Suppose $0 < q, r < \infty, \lambda \geq 0$, and $\alpha \in \mathbb{R}$. The homogeneous p-adic Morrey-Herz space is defined as:

$$M\dot{K}_{r,q}^{\alpha,\lambda}\left(\mathbb{Q}_p^n\right) = \left\{f \in L_{\mathrm{loc}}^q\left(\mathbb{Q}_p^n \setminus \{0\}\right): \|f\|_{M\dot{K}_{r,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} < \infty\right\}, \tag{21}$$

where

$$\|f\|_{M\dot{K}_{r,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha r}\|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^r\right)^{1/r}. \tag{22}$$

It is eminent that $M\dot{K}_{r,q}^{\alpha,0}(\mathbb{Q}_p^n) = \dot{K}_q^{\alpha,r}(\mathbb{Q}_p^n)$ and $M\dot{K}_{q,q}^{\alpha/q,0}(\mathbb{Q}_p^n) = L^q(|\mathbf{x}|_p^\alpha)$.

Definition 3 [31]. Let $1 \leq q < \infty$. The space $\mathrm{CMO}^q(\mathbb{Q}_p^n)$ is defined to be the space of all measurable function $f \in L_{\mathrm{loc}}^q(\mathbb{Q}_p^n)$ if

$$\|f\|_{\mathrm{CMO}^q(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|f(\mathbf{x}) - f_{B_\gamma}\right|^q d\mathbf{x}\right)^{1/q} < \infty, \tag{23}$$

where $f_{B_\gamma} = (1/|B_\gamma|_H)\int_{B_\gamma}f(\mathbf{x})d\mathbf{x}$.

Definition 4 (see [40]). Suppose δ is a positive real number. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ is defined to be the space of all measurable function f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h}\in\mathbb{Q}_p^n,\mathbf{h}\neq 0} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{|\mathbf{h}|_p^\delta} < \infty. \tag{24}$$

2. CMO Estimates for $H_{\beta,\vec{b}}^{p,2}$

In the following section, we acquire the boundedness of commutators of bilinear fractional p-adic Hardy operator on homogeneous p-adic Herz spaces and Morrey-Herz spaces by considering the symbol function from CMO spaces. We start the section with few lemmas that are helpful to prove the main results.

Lemma 5 (see [30]). *Let b be a CMO function and $1 \leq q < r < \infty$, then*

$$\mathrm{CMO}^r\left(\mathbb{Q}_p^n\right) \subset \mathrm{CMO}^q\left(\mathbb{Q}_p^n\right) \text{ and } \|b\|_{\mathrm{CMO}^q(\mathbb{Q}_p^n)} \leq \|b\|_{\mathrm{CMO}^r(\mathbb{Q}_p^n)}. \tag{25}$$

Lemma 6 (see [30]). *Let b be a CMO function, $i, k \in \mathbb{Z}$, then*

$$\left|b(\mathbf{t}) - b_{B_k}\right| \leq \left|b(\mathbf{t}) - b_{b_i}\right| + p^n|i-k|\|b\|_{\mathrm{CMO}^1(\mathbb{Q}_p^n)}. \tag{26}$$

Now, we proceed to state our key results for this section.

Theorem 7. *Let $\alpha, \alpha_1, \alpha_2$ be arbitrary real numbers, $1 < p, p_1, p_2, q, q_1, q_2 < \infty, 0 \leq \beta < n, \alpha_1 + \alpha_2 = \alpha, (1/p_1) + (1/p_2) = (1/p),$ and $\beta/n = 1/q_1 + 1/q_2 - 1/q$. If for $i = 1, 2$ with $n/q_i' > \alpha_i$, then $H_{\beta,\vec{b}}^{p,2}$ is bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n),$ where $\vec{b} = (b_1, b_2), b_1, b_2 \in CM\,O^{\max\{q,q_1\}}.$*

Theorem 8. *Let $\alpha, \alpha_1, \alpha_2$ be arbitrary real numbers, $1 < p, p_1, p_2, q, q_1, q_2 < \infty, 0 \leq \beta < n, \alpha_1 + \alpha_2 = \alpha, \lambda_1 + \lambda_2 = \lambda, 0 \leq \lambda_1, \lambda_2 < \infty, (1/p_1) + (1/p_2) = (1/p),$ and $\beta/n = 1/q_1 + 1/q_2 - 1/q$. If for $i = 1, 2$ with $n/q_i' + \lambda_i > \alpha_i$, then $H_{\beta,\vec{b}}^{p,2}$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)$ to $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n),$ where $\vec{b} = (b_1, b_2), b_1, b_2 \in \mathrm{CMO}^{\max\{q,q_1\}}.$*

Note that Theorem 7 is a special case of Theorem 8. So, we only prove Theorem 8.

Proof. Let $(b_i)_{B_k}$ denotes the average of b_i on the ball B_k for $i = 1, 2$ and $k \in \mathbb{Z}$. By definition, we have

$$
\begin{aligned}
&\left\| H_{\beta,b_1}^{p,2} (f_1, f_2) \chi_k \right\|_{L^q(\mathbb{Q}_p^n)}^q \\
&= \int_{S_k} |\mathbf{x}|_p^{-q(2n-\beta)} \left| \int_{|(\mathbf{t}_1, \mathbf{t}_2)|_p \leq |\mathbf{x}|_p} (f_1(\mathbf{t}_1) f_2(\mathbf{t}_2))(b_1(\mathbf{x}) - b_1(\mathbf{t}_1)) d\mathbf{t}_1 d\mathbf{t}_2 \right|^q d\mathbf{x} \\
&\leq \int_{S_k} p^{-kq(2n-\beta)} \left(\int_{|(\mathbf{t}_1, \mathbf{t}_2)|_p \leq p^k} |(f_1(\mathbf{t}_1) f_2(\mathbf{t}_2))(b_1(\mathbf{x}) - b_1(\mathbf{t}_1))| d\mathbf{t}_1 d\mathbf{t}_2 \right)^q d\mathbf{x} \\
&\leq C p^{-kq(2n-\beta)} \int_{S_k} \left(\int_{|(\mathbf{t}_1, \mathbf{t}_2)|_p \leq p^k} \left| (f_1(\mathbf{t}_1) f_2(\mathbf{t}_2))(b_1(\mathbf{x}) - (b_1)_{B_k} \right| d\mathbf{t}_1 d\mathbf{t}_2 \right)^q d\mathbf{x} \\
&\quad + C p^{-kq(2n-\beta)} \int_{S_k} \left(\int_{|(\mathbf{t}_1, \mathbf{t}_2)|_p \leq p^k} \left| (f_1(\mathbf{t}_1) f_2(\mathbf{t}_2))(b_1(\mathbf{t}_1) - (b_1)_{B_k} \right| d\mathbf{t}_1 d\mathbf{t}_2 \right)^q d\mathbf{x} \\
&= I + II.
\end{aligned}
\tag{27}
$$

To evaluate I, we use $1/q_1 + 1/q_1{}' = 1, 1/q_2 + 1/q_2{}' = 1$, and $\beta/n = 1/q_1 + 1/q_2 - 1/q$. Applying Hölder's inequality to get

$$
\begin{aligned}
I &= C p^{-kq(2n-\beta)} \left(\int_{S_k} \left| b_1(\mathbf{x}) - (b_1)_{B_k} \right|^q d\mathbf{x} \right) \left(\int_{B_k} \int_{B_k} |(f_1(\mathbf{t}_1) f_2(\mathbf{t}_2))| d\mathbf{t}_1 d\mathbf{t}_2 \right)^q \\
&\leq C p^{-kq(2n-\beta)} |B_k|_H \left(\frac{1}{|B_k|_H} \int_{B_k} \left| b_1(\mathbf{x}) - (b_1)_{B_k} \right|^q d\mathbf{x} \right) \\
&\quad \cdot \left(\sum_{j=-\infty}^{k} \int_{S_j} |f_1(\mathbf{t}_1)| d\mathbf{t}_1 \times \sum_{i=-\infty}^{k} \int_{S_i} |f_2(\mathbf{t}_2)| d\mathbf{t}_2 \right)^q \\
&\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \|b_1\|_{\mathrm{CMO}^q(\mathbb{Q}_p^n)}^q \left\{ \sum_{j=-\infty}^{k} \left(\int_{S_j} |f_1(\mathbf{t}_1)|^{q_1} d\mathbf{t}_1 \right)^{1/q_1} \left(\int_{S_j} d\mathbf{t}_1 \right)^{1/q_1{}'} \right. \\
&\quad \left. \times \sum_{i=-\infty}^{k} \left(\int_{S_i} |f_2(\mathbf{t}_2)|^{q_2} d\mathbf{t}_2 \right)^{1/q_2} \left(\int_{S_i} d\mathbf{t}_2 \right)^{1/q_2{}'} \right\}^q \\
&\leq C \|b_1\|_{\mathrm{CMO}^q(\mathbb{Q}_p^n)}^q \left\{ \sum_{j=-\infty}^{k} p^{(j-k)n/q_1{}'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \cdot \sum_{i=-\infty}^{k} p^{(i-k)n/q_2{}'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right\}^q.
\end{aligned}
\tag{28}
$$

Now, we turn our attention towards estimating II :

$$
\begin{aligned}
II &= C p^{-kq(2n-\beta)} \int_{S_k} \left(\int_{B_k} \int_{B_k} \left| (f_1(\mathbf{t}_1) f_2(\mathbf{t}_2)) \left(b_1(\mathbf{t}_1) - (b_1)_{B_k} \right) \right| d\mathbf{t}_1 d\mathbf{t}_2 \right)^q d\mathbf{x} \\
&\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \left(\sum_{j=-\infty}^{k} \int_{S_j} \left| (f_1(\mathbf{t}_1)) \left(b_1(\mathbf{t}_1) - (b_1)_{B_j} \right) \right| d\mathbf{t}_1 \sum_{i=-\infty}^{k} \int_{S_i} |f_2(\mathbf{t}_2)| d\mathbf{t}_2 \right)^q \\
&\quad + C p^{knq(-1/q_2{}'-1/q_1{}')} \left(\sum_{j=-\infty}^{k} \int_{S_j} \left| (f_1(\mathbf{t}_1)) \left((b_1)_{B_k} - (b_1)_{B_j} \right) \right| d\mathbf{t}_1 \sum_{i=-\infty}^{k} \int_{S_i} |f_2(\mathbf{t}_2)| d\mathbf{t}_2 \right)^q \\
&= II_1 + II_2.
\end{aligned}
\tag{29}
$$

An easy application of Hölder's inequality simplifies the expression of II_1, that is:

$$
\begin{aligned}
II_1 &\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \left\{ \sum_{j=-\infty}^{k} \left(\int_{S_j} |f_1(\mathbf{t}_1)|^{q_1} d\mathbf{t}_1 \right)^{1/q_1} \left(\int_{S_j} \left| b_1(\mathbf{t}_1) - (b_1)_{B_j} \right|^{q_1{}'} d\mathbf{t}_1 \right)^{1/q_1{}'} \right. \\
&\quad \left. \times \sum_{i=-\infty}^{k} \left(\int_{S_i} |f_2(\mathbf{t}_2)|^{q_2} d\mathbf{t}_2 \right)^{1/q_2} \left(\int_{S_i} d\mathbf{t}_2 \right)^{1/q_2{}'} \right\}^q \\
&\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \left\{ \sum_{j=-\infty}^{k} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} p^{jn/q_1{}'} \left(\frac{1}{|B_j|_H} \int_{B_j} \left| b_1(\mathbf{t}_1) - (b_1)_{B_j} \right|^{q_1{}'} d\mathbf{t}_1 \right)^{1/q_1{}'} \right. \\
&\quad \left. \times \sum_{i=-\infty}^{k} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} p^{in/q_2{}'} \right\}^q \\
&\leq C \|b_1\|_{\mathrm{CMO}^{q_1}(\mathbb{Q}_p^n)}^q \left\{ \sum_{j=-\infty}^{k} p^{(j-k)n/q_1{}'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \cdot \sum_{i=-\infty}^{k} p^{(i-k)n/q_2{}'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right\}^q.
\end{aligned}
\tag{30}
$$

To estimate II_2, we use Lemma 6 along with the Hölder's inequality to have

$$
\begin{aligned}
II_2 &\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \|b_1\|_{\mathrm{CMO}^1(\mathbb{Q}_p^n)}^{q_2} \left(\sum_{j=-\infty}^{k} (k-j) \int_{S_j} |f_1(\mathbf{t}_1)| d\mathbf{t}_1 \times \sum_{i=-\infty}^{k} \int_{S_i} |f_2(\mathbf{t}_2)| d\mathbf{t}_2 \right)^q \\
&\leq C p^{knq(-1/q_2{}'-1/q_1{}')} \|b_1\|_{\mathrm{CMO}^1(\mathbb{Q}_p^n)}^q \left\{ \sum_{j=-\infty}^{k} (k-j) \left(\int_{S_j} |f_1(\mathbf{t}_1)|^{q_1} d\mathbf{t}_1 \right)^{1/q_1} \times \left(\int_{S_j} d\mathbf{t}_1 \right)^{1/q_1{}'} \right. \\
&\quad \left. \cdot \sum_{i=-\infty}^{k} \left(\int_{S_i} |f_2(\mathbf{t}_2)|^{q_2} d\mathbf{t}_2 \right)^{1/q_2} \left(\int_{S_i} d\mathbf{t}_2 \right)^{1/q_2{}'} \right\}^q \\
&\leq C \|b_1\|_{\mathrm{CMO}^1(\mathbb{Q}_p^n)}^q \times \left\{ \sum_{j=-\infty}^{k} (k-j) p^{(j-k)n/q_1{}'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \cdot \sum_{i=-\infty}^{k} p^{(i-k)n/q_2{}'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right\}^q.
\end{aligned}
\tag{31}
$$

Since $\alpha = \alpha_1 + \alpha_2, \lambda = \lambda_1 + \lambda_2$, and $1/p = 1/p_1 + 1/p_2$, by the definition of p-adic Morrey-Herz space along with Lemma 5 and Hölder's inequality, we are down to

$$
\begin{aligned}
&\left\| H_{\beta,b_1}^{p,2} (f_1, f_2) \right\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha q} \left\| H_{\beta,b}^{p,2} (f_1, f_2) \chi_k \right\|_{L^q(\mathbb{Q}_p^n)}^p \right)^{1/p} \\
&\leq C \|b_1\|_{\mathrm{CMO}^{\max\{q, q_1{}'\}}(\mathbb{Q}_p^n)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha p} \times \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n/q_1{}'} \right. \right. \\
&\quad \left. \left. \cdot \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \sum_{i=-\infty}^{k} p^{(i-k)n/q_2{}'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right)^p \right\}^{1/p} \\
&\leq C \|b_1\|_{\mathrm{CMO}^{\max\{q, q_1{}'\}}(\mathbb{Q}_p^n)} \times \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_1} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha_1 p_1} \right. \\
&\quad \cdot \left. \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n/q_1{}'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{p_1} \right\}^{1/p_1} \\
&\quad \times \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_2} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha_2 p_2} \left(\sum_{i=-\infty}^{k} p^{(i-k)n/q_2{}'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right)^{p_2} \right\}^{1/p_2} \\
&= C \|b_1\|_{\mathrm{CMO}^{\max\{q, q_1{}'\}}(\mathbb{Q}_p^n)} E_1 \times E_2,
\end{aligned}
\tag{32}
$$

where

$$
E_1 = \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_1} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha_1 p_1} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n/q_1{}'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \right)^{p_1} \right\}^{1/p_1},
$$

$$E_2 = \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_2} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha_2 p_2} \left(\sum_{i=-\infty}^{k} p^{(i-k)n/q_2'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right)^{p_2} \right\}^{1/p_2}.$$

(33)

To proceed further, we consider:

$$
\begin{aligned}
\left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} &= p^{j\alpha_1} p^{-j\alpha_1} \left(\left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1} \right)^{1/p_1} \\
&\leq p^{-j\alpha_1} \left(\sum_{l=-\infty}^{j} p^{l\alpha_1 p_1} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1} \right)^{1/p_1} \\
&\leq p^{j(\lambda_1 - \alpha_1)} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

(34)

Since $n/q_1' + \lambda_1 > \alpha_1$ and $\lambda_1 > 0$, ultimately, we have the following representation of E_1.

$$
\begin{aligned}
E_1 &\leq \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_1} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha_1 p_1} \left(\sum_{j=-\infty}^{k} (k-j) p^{(j-k)n/q_1'} p^{j(\lambda_1 - \alpha_1)} \right)^{p_1} \right)^{1/p_1} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)} \\
&\leq \sup_{k_0 \in \mathbb{Z}} p^{-k_0 \lambda_1} \left(\sum_{k=-\infty}^{k_0} p^{k\lambda_1 p_1} \left(\sum_{j=-\infty}^{k} (k-j) p^{(k-j)(\alpha_1 - n/q_1' - \lambda_1)} \right)^{p_1} \right)^{1/p_1} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)} \\
&= C \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

(35)

In a similar fashion, E_2 can be estimated as

$$E_2 \leq C \|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)}.$$

(36)

Making use of above inequalities in (32), we get

$$
\begin{aligned}
&\left\| H_{\beta,b_1}^{p,2}(f_1,f_2) \right\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} \\
&\quad \leq C \|b_1\|_{\mathrm{CMO}^{\max\{q,q_1'\}}(\mathbb{Q}_p^n)} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)} \|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

(37)

Similarly, we have

$$
\begin{aligned}
&\left\| H_{\beta,b_2}^{p,2}(f_1,f_2) \right\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} \\
&\quad \leq C \|b_2\|_{\mathrm{CMO}^{\max\{q,q_1'\}}(\mathbb{Q}_p^n)} \|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)} \|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

(38)

Therefore, we complete the proof of Theorem 8.

3. Lipschitz Estimates for $H_{\beta,\vec{b}}^{p,2}$ on p-Adic Herz Type Spaces

In the present section, we establish Lipschitz estimates for commutators of the bilinear fractional Hardy operator on homogeneous p-adic Herz spaces and Morrey-Herz spaces. The main results of the section are as follows.

Theorem 9. *Let $\alpha, \alpha_1, \alpha_2$ be arbitrary real numbers, $1 < p, p_1, p_2, q, q_1, q_2 < \infty, 0 \leq \beta < n, \alpha_1 + \alpha_2 = \alpha, (1/p_1) + (1/p_2) = (1/p),$*

and $(\beta + \gamma)/n = 1/q_1 + 1/q_2 - 1/q$. If for $i = 1, 2$ with $n/q_i' > \alpha_i$, then $H_{\beta,\vec{b}}^{p,2}$ is bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in \Lambda_\gamma(\mathbb{Q}_p^n)$.

Theorem 10. *Let $\alpha, \alpha_1, \alpha_2$ be any arbitrary real numbers, $1 < p, p_1, p_2, q, q_1, q_2 < \infty, 0 \leq \beta < n, \alpha_1 + \alpha_2 = \alpha, \lambda_1 + \lambda_2 = \lambda, 0 \leq \lambda_1, \lambda_2 < \infty, (1/p_1) + (1/p_2) = (1/p),$ and $(\beta + \gamma)/n = 1/q_1 + 1/q_2 - 1/q$. If for $i = 1, 2$ with $n/q_i' + \lambda_i > \alpha_i$, then $H_{\beta,\vec{b}}^{p,2}$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)$ to $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$, where $\vec{b} = (b_1, b_2), b_1, b_2 \in \Lambda_\gamma(\mathbb{Q}_p^n)$.*

Since Theorem 9 can easily be deduced from Theorem 10, so we opt for proof of later theorem.

Proof. Since $b_1 \in \Lambda_\gamma(\mathbb{Q}_p^n)$, therefore, we have

$$|b_1(\mathbf{x}) - b_1(\mathbf{t_1})| \leq |\mathbf{x} - \mathbf{t_1}|_p^\gamma \|b_1\|_{(\mathbb{Q}_p^n)}.$$

(39)

Next, consider

$$
\begin{aligned}
&\left\| H_{\beta,b_1}^{p,2}(f_1,f_2) \chi_k \right\|_{L^{q_2}(\mathbb{Q}_p^n)}^q \\
&= \int_{S_k} |\mathbf{x}|_p^{-q(2n-\beta)} \left| \int_{|(\mathbf{t_1},\mathbf{t_2})|_p \leq |\mathbf{x}|_p} (f_1(\mathbf{t_1}) f_2(\mathbf{t_2}))(b_1(\mathbf{x}) - b_1(\mathbf{t_1})) d\mathbf{t_1} d\mathbf{t_2} \right|^q d\mathbf{x} \\
&\leq \int_{S_k} p^{-kq(2n-\beta)} \left(\int_{|(\mathbf{t_1},\mathbf{t_2})|_p \leq p^k} |(f_1(\mathbf{t_1}) f_2(\mathbf{t_2}))(b_1(\mathbf{x}) - b_1(\mathbf{t_1}))| d\mathbf{t_1} d\mathbf{t_2} \right)^q d\mathbf{x} \\
&\leq C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{-kq(2n-\beta)} \int_{S_k} \left(\int_{|(\mathbf{t_1},\mathbf{t_2})|_p \leq p^k} |\mathbf{x} - \mathbf{t_1}|_p^\gamma |f_1(\mathbf{t_1}) f_2(\mathbf{t_2})| d\mathbf{t_1} d\mathbf{t_2} \right)^q d\mathbf{x} \\
&\leq C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{-kq(2n-\beta)} \int_{S_k} \left(\int_{|(\mathbf{t_1},\mathbf{t_2})|_p \leq p^k} |\mathbf{x}|_p^\gamma |f_1(\mathbf{t_1}) f_2(\mathbf{t_2})| d\mathbf{t_1} d\mathbf{t_2} \right)^q d\mathbf{x} \\
&= C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{-kq(2n-\beta-\gamma)} \int_{S_k} \left(\int_{|(\mathbf{t_1},\mathbf{t_2})|_p \leq p^k} |f_1(\mathbf{t_1}) f_2(\mathbf{t_2})| d\mathbf{t_1} d\mathbf{t_2} \right)^q d\mathbf{x} \\
&= I
\end{aligned}
$$

(40)

To evaluate I, we use $1/q_1 + 1/q_1' = 1, 1/q_2 + 1/q_2' = 1$, and $(\beta + \gamma)/n = 1/q_1 + 1/q_2 - 1/q$. Applying Hölder's inequality to get

$$
\begin{aligned}
I &= C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{-kq(2n-\beta-\gamma)+kn} \left(\int_{B_k} \int_{B_k} |(f_1(\mathbf{t_1}) f_2(\mathbf{t_2}))| d\mathbf{t_1} d\mathbf{t_2} \right)^q \\
&= C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{-kq(2n-\beta-\gamma)+kn} \left(\sum_{j=-\infty}^{k} \int_{S_j} |f_1(\mathbf{t_1})| d\mathbf{t_1} \cdot \sum_{i=-\infty}^{k} \int_{S_i} |f_2(\mathbf{t_2})| d\mathbf{t_2} \right)^q \\
&\leq C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} p^{knq(-1/q_2' - 1/q_1')} \left\{ \sum_{j=-\infty}^{k} \left(\int_{S_j} |f_1(\mathbf{t_1})|^{q_1} d\mathbf{t_1} \right)^{1/q_1} \right. \\
&\quad \left. \cdot \left(\int_{S_j} d\mathbf{t_1} \right)^{1/q_1'} \times \sum_{i=-\infty}^{k} \left(\int_{S_i} |f_2(\mathbf{t_2})|^{q_2} d\mathbf{t_2} \right)^{1/q_2} \left(\int_{S_i} d\mathbf{t_2} \right)^{1/q_2'} \right\}^q \\
&\leq C \|b_1\|_{\Lambda_\gamma(\mathbb{Q}_p^n)} \left\{ \sum_{j=-\infty}^{k} p^{(j-k)n/q_1'} \left\| f_1 \chi_j \right\|_{L^{q_1}(\mathbb{Q}_p^n)} \cdot \sum_{i=-\infty}^{k} p^{(i-k)n/q_2'} \|f_2 \chi_i\|_{L^{q_2}(\mathbb{Q}_p^n)} \right\}^q.
\end{aligned}
$$

(41)

By the definition of p-adic Morrey-Herz space, we have

$$
\begin{aligned}
&\left\| H_{\beta,b_1}^{p,2}(f_1,f_2) \right\|_{M\dot{K}_{p,q}^{\alpha,\lambda}\left(\mathbb{Q}_p^n\right)} \\
&= \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} p^{k\alpha p} \left\| H_{\beta,b}^{p,2}(f_1,f_2)\chi_k \right\|_{L^q\left(\mathbb{Q}_p^n\right)}^p \right)^{1/p} \\
&\leq C \|b_1\|_{\Lambda_\gamma\left(\mathbb{Q}_p^n\right)} \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} p^{k\alpha p} \left(\sum_{j=-\infty}^{k} p^{((j-k)n)/\left(q_1'\right)} \right. \right. \\
&\quad \left\| f_1 \chi_j \right\|_{L^{q_1}\left(\mathbb{Q}_p^n\right)} \times \sum_{i=-\infty}^{k} p^{((i-k)n)/q_2'} \|f_2 \chi_i\|_{L^{q_2}\left(\mathbb{Q}_p^n\right)} \bigg)^p \Bigg\}^{1/p}.
\end{aligned}
\tag{42}
$$

The rest of the proof follows from Theorem 8. So, we conclude the theorem.

4. Conclusion

Here, we obtained the CMO and Lipschitz estimates for the commutators of the bilinear fractional p-adic Hardy operator on p-adic Herz-type spaces.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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