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Research Article

Sharp Estimation Type Inequalities for Lipschitzian Mappings in Euclidean Sense on a Disk

M. Rostamian Delavar,[1] **S. S. Dragomir**,[2] **and M. De La Sen**[3]

[1]*Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P. O. Box 1339, Bojnord 94531, Iran*
[2]*Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia*
[3]*Institute of Research and Development of Processes, University of Basque Country, Campus of Leioa (Bizkaia)-Aptdo. 644-Bilbao, Bilbao 48080, Spain*

Correspondence should be addressed to M. Rostamian Delavar; m.rostamian@ub.ac.ir

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Some sharp trapezoid and midpoint type inequalities for Lipschitzian bifunctions defined on a closed disk in Euclidean sense are obtained by the use of polar coordinates. Also, bifunctions whose partial derivative is Lipschitzian are considered. A new presentation of Hermite-Hadamard inequality for convex function defined on a closed disk and its reverse are given. Furthermore, two mappings $H(t)$ and $h(t)$ are considered to give some generalized Hermite-Hadamard type inequalities in the case that considered functions are Lipschitzian in Euclidean sense on a disk.

1. Introduction and Preliminaries

Consider that $D(C, R)$ is a closed disk in the plane centered at the point $C = (a, b)$ having the radius $R > 0$. In [1] (see also [2]), the Hermite-Hadamard inequality for a convex function defined on $D(C, R)$ has been obtained as follows:

Theorem 1. *If the mapping $\mathcal{F} : D(C, R) \to R$ is convex on $D(C, R)$, then one has the inequality*

$$\mathcal{F}(\mathcal{C}) \leq \frac{1}{\pi\mathcal{R}^2}\int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(x, y)dxdy \leq \frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{F}(\psi)dl(\psi), \quad (1)$$

where $\partial(C, R)$ is the circle centered at the point $C = (a, b)$ with radius $\mathcal{R}$. The above inequalities are sharp.

First of all, we give the following result which is including a new presentation of (1) and its reverse as well:

Theorem 2. *For a continuous function $\mathcal{F}$ defined on a convex subset $\mathcal{A} \subset \mathbb{R}^2$,*

(1) if $\mathcal{F}$ is convex on $\mathcal{A}$, then for any $\mathcal{D}(\mathcal{C}, \mathcal{R}) \subset \mathcal{A}$, we have

$$\int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(x, y)dA \leq \frac{1}{R}\int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{F}(x, y)(y-b)^2 d\partial, \quad (2)$$

where $\partial(C, R)$ is the boundary of $D(C, R)$

(2) if (2) holds for all $\mathcal{D}(\mathcal{C}, \mathcal{R}) \subset \mathcal{A}$, then $\mathcal{F}$ is convex

Proof.

(1) Consider the change of coordinates $\mathcal{M} : [a - \mathcal{R}, a + \mathcal{R}] \times [0, 1] \to \mathcal{D}(\mathcal{C}, \mathcal{R})$ defined as

$$\mathcal{M}(x, s) = \left(x - a, (2s-1)\sqrt{\mathcal{R}^2 - (x-a)^2} + b\right). \quad (3)$$

It follows that

$$\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(x,y)dA = 2\int_{a-\mathcal{R}}^{a+\mathcal{R}}\int_0^1 \mathcal{F}\left(s\left(x,\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)+(1-s)\left(x,-\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)\right) \times \sqrt{\mathcal{R}^2-(x-a)^2}dsdx \le 2\int_{a-\mathcal{R}}^{a+\mathcal{R}}\int_0^1 s\mathcal{F} \times \left(x,\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)\sqrt{\mathcal{R}^2-(x-a)^2}dsdx + 2\int_{a-\mathcal{R}}^{a+\mathcal{R}}\int_0^1 (1-s)\mathcal{F} \times \left(x,-\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)\sqrt{\mathcal{R}^2-(x-a)^2}dsdx = \int_{a-\mathcal{R}}^{a+\mathcal{R}}\mathcal{F} \times \left(x,\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)\sqrt{\mathcal{R}^2-(x-a)^2}dx + \int_{a-\mathcal{R}}^{a+\mathcal{R}}\mathcal{F} \times \left(x,-\sqrt{\mathcal{R}^2-(x-a)^2}+b\right)\sqrt{\mathcal{R}^2-(x-a)^2}dx. \tag{4}$$

Now consider $y=\pm\sqrt{\mathcal{R}^2-(x-a)^2}+b$ in above integrals with $\sqrt{1+(\partial y/\partial x)^2}=\mathcal{R}/(\sqrt{\mathcal{R}^2-(x-a)^2})=\mathcal{R}/(y-b)$ to obtain the desired result

(2) Suppose that there exist $X_1, X_2 \in \mathcal{A}$ and $s\in(0,1)$ such that

$$\mathcal{F}(sX_1+(1-s)x_2) > s\mathcal{F}(X_1)+(1-s)\mathcal{F}(X_2) \tag{5}$$

Since $\mathcal{F}$ is continuous on $\mathcal{A}$, then there exists $\mathcal{R}>0$ and a point $\mathcal{C}_0=(a_0,b_0)$ in convex combination of X_1 and X_2 such that (5) holds on whole of $\mathcal{D}(\mathcal{C}_0,\mathcal{R})\subset\mathcal{A}$. Now, if we follow the proof of part (1) for $\mathcal{F}$ by the use of (5) on $\mathcal{D}(\mathcal{C}_0,\mathcal{R})$ and $\partial(\mathcal{C}_0,\mathcal{R})$, then we have

$$\iint_{\mathcal{D}(\mathcal{C}_0,\mathcal{R})}\mathcal{F}(x,y)dA > \frac{1}{\mathcal{R}}\int_{\partial(\mathcal{C}_0,\mathcal{R})}\mathcal{F}(x,y)(y-b)^2 d\partial. \tag{6}$$

This contradiction proves the convexity of $\mathcal{F}$ on $\mathcal{A}$.

We remind that the classic form of Hermite-Hadamard inequality (see [3–5]) for a real valued convex function f defined on $[a,b]$ is the following:

$$f\left(\frac{a+b}{2}\right)(b-a) \le \int_a^b f(x)dx \le (b-a)\frac{f(a)+f(b)}{2}. \tag{7}$$

Generally, in the literature associated to any Hermite-Hadamard type inequality, there exist two inequalities which we call them trapezoid and mid-point type inequalities. The names "trapezoid" and "midpoint" comes from two classic inequalities (due to their geometric interpretation) related to the Hermite-Hadamard inequality obtained in [6, 7], respectively:

$$\left|\int_a^b f(x)dx-(b-a)\frac{f(a)+f(b)}{2}\right| \le \frac{1}{8}(b-a)^2\left(|f'(a)|+|f'(b)|\right), \tag{8}$$

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)^2\left(|f'(a)|+|f'(b)|\right), \tag{9}$$

where $f: I^\circ \subseteq \mathbb{R}\to\mathbb{R}$ is a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$ and $|f'|$ is convex on $[a,b]$. For more results about convex functions, related inequalities, and generalizations of (7)–(9), see [8–23] and references therein.

Recently, in [17], the authors obtained the trapezoid and midpoint type inequality related to (1) as follows, respectively:

Theorem 3. *Consider a set $I\subset\mathbb{R}^2$ with $\mathcal{D}(\mathcal{C},\mathcal{R})\subset I^\circ$. Suppose that the mapping $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ has continuous partial derivatives in the disk $\mathcal{D}(\mathcal{C},\mathcal{R})$ with respect to the variables ρ and φ in polar coordinates. If for any constant $\varphi\in[0,2\pi]$, the function $|\partial\mathcal{F}/\partial\rho|$ is convex with respect to the variable ρ on $[0,\mathcal{R}]$ then*

$$\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma)-\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy\right| \le \frac{1}{6\pi}\int_{\partial(\mathcal{C},\mathcal{R})}\left|\frac{\partial\mathcal{F}}{\partial r}\right|(\gamma)dl(\gamma), \tag{10}$$

$$\left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy-f(C)\right| \le \frac{2}{3\pi}\int_{\partial(\mathcal{C},\mathcal{R})}\left|\frac{\partial\mathcal{F}}{\partial r}\right|(\gamma)dl(\gamma). \tag{11}$$

Note that inequality (10) is sharp.

As we can see in (8) and (9), the classic trapezoid and midpoint type inequalities have been obtained for the functions whose the first derivative absolute values are convex. In [22, 24], the authors considered Lipschitzian mappings instead of those whose the first derivative absolute values are convex to obtain some midpoint and trapezoid type inequalities:

Theorem 4 [24]. *Let $f: I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be an M-Lipschitzian mapping on I and $a,b\in I$ with $a<b$. Then, we have the inequalities*

$$\left|\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)\right| \le \frac{M}{4}(b-a), \qquad \left|\int_a^b f(x)dx-(b-a)\frac{f(a)+f(b)}{2}\right| \le \frac{M}{3}(b-a). \tag{12}$$

Motivated by above works and results, we obtain some trapezoid and midpoint type inequalities related to (1) for Lipschitzian mappings (in Euclidean sense) defined on the disk $\mathcal{D}(\mathcal{C},\mathcal{R})$ in a plane. Also we investigate trapezoid and mid-point type inequalities in the case that in polar coordinates (ρ,φ), the derivative of considered function with respect to the variable ρ is Lipschitzian. Furthermore, two mappings $H(t)$ and $h(t)$ are considered to give some generalized Hermite-Hadamard type inequalities in the case that the functions are Lipschitzian on a disk $\mathscr{D}(\mathscr{C},\mathscr{R})$.

Here, we should mention that in [25], we can find some inequalities for the integral mean of Hölder continuous functions defined on disks in a plane which in a special case leads to trapezoid and midpoint type inequalities for a kind of Lipschitzian mappings as the following:

Theorem 5. *If $f:\mathscr{D}(\mathscr{C},\mathscr{R})\to\mathbb{R}$ satisfies the condition*

$$|f(a,b)-f(x,y)|\le M_1|x-a|+M_2|y-b|,\ (x,y)\in\mathscr{D}(\mathscr{C},\mathscr{R}), \tag{13}$$

where $M_1,M_2>0$, then we have the inequalities

$$\left|\frac{1}{2\pi R}\int_{\partial(\mathscr{C},\mathscr{R})}f(\gamma)dl(\gamma)-\frac{1}{\pi R^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})}f(x,y)dxdy\right|\le\frac{2R}{3\pi}(M_1+M_2), \tag{14}$$

$$\left|\frac{1}{\pi R^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})}f(x,y)dxdy-f(C)\right|\le\frac{4R}{3\pi}(M_1+M_2). \tag{15}$$

The main point is that the Euclidean Lipschitz condition used in this paper is a stronger condition than (13) in the case that $M_1=M_2$ and so our results obtained in (20) and (29) will provide more accurate estimation compared to (14) and (15). Furthermore, we obtain new trapezoid and midpoint type inequalities of our function is Lipschitzian.

2. Main Results

In this section, first, we obtain some trapezoid and midpoint type inequalities related to (1) for the case that our considered function is Lipschitzian (in Euclidean sense). Second, we obtain some trapezoid and mid-point type inequalities related to (1) for the case that the partial derivative of our function with respect to the variable ρ in polar coordinates (ρ,ϕ) is Lipschitzian (in Euclidean sense).

Definition 6 [26]. A function $\mathscr{F}:I\subset\mathbb{R}^2\to\mathbb{R}$ is said to satisfy a Lipschitz condition (briefly $\mathscr{K}$-Lipschitzian) on I with respect to a norm $\|\bullet\|$, if there exists a constant $\mathscr{K}>0$ such that

$$|\mathscr{F}(X_1)-\mathscr{F}(X_2)|\le\mathscr{K}\|X_1-X_2\|, \tag{16}$$

for any $X_1,X_2\in I$.

If $\mathscr{F}:\mathscr{D}(\mathscr{C},\mathscr{R})\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathscr{K}>0$ and the Euclidean norm $\|\bullet\|$, then for any $X_1=(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)$ and $X_2=(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)$, we have

$$\begin{aligned}|\mathscr{F}(X_1)-\mathscr{F}(X_2)|&=|\mathscr{F}(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)\\&\quad-\mathscr{F}(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)|\\&\le\mathscr{K}\|(\rho_1\cos\varphi_1-\rho_2\cos\varphi_2,\rho_1\sin\varphi_1-\rho_2\sin\varphi_2)\|\\&=\mathscr{K}\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\varphi_1-\varphi_2)},\end{aligned} \tag{17}$$

for any $\rho_1,\rho_2\in[0,\mathscr{R}]$ and $\varphi_1,\varphi_2\in[0,2\pi]$. Also, it is obvious that if $\mathscr{F}:I\subseteq\mathbb{R}^2\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathscr{K}>0$ on I, then, it is continuous and so integrable on I.

2.1. $\mathscr{F}$ Is Lipschitzian. The first result of this section is the trapezoid type inequality related to (1) for the case that our considered function is Lipschitzian. We start with a lemma.

Lemma 7. *Define a function $\mathscr{F}:\mathscr{D}(\mathscr{C},\mathscr{R})\to\mathbb{R}$ as*

$$\mathscr{F}(X)=\mathscr{F}(a+\rho\cos\varphi,b+\rho\sin\varphi)=\mathscr{K}(\mathscr{R}-\rho), \tag{18}$$

for fixed $\mathscr{K}>0$ and all $0\le\rho\le\mathscr{R}$, $0\le\varphi\le2\pi$. Then, the function $\mathscr{F}$ is $\mathscr{K}$-Lipschitzian.

Proof. Consider $X_1=(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)$ and $X_2=(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)$, for $\rho_1,\rho_2\in[0,\mathscr{R}]$ and $\varphi_1,\varphi_2\in[0,2\pi]$. So

$$\begin{aligned}|\mathscr{F}(X_1)-\mathscr{F}(X_2)|&=|\mathscr{F}(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)\\&\quad-\mathscr{F}(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)|\\&=\mathscr{K}|\rho_2-\rho_1|=\mathscr{K}\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2}\\&\le\mathscr{K}\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\varphi_1-\varphi_2)}\\&=\mathscr{K}\|(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)\\&\quad-(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)\|\\&=\mathscr{K}\|X_1-X_2\|,\end{aligned} \tag{19}$$

for all $X_1,X_2\in\mathscr{D}(\mathscr{C},\mathscr{R})$.

Theorem 8. *Suppose that the mapping $\mathscr{F}:\mathscr{D}(\mathscr{C},\mathscr{R})\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathscr{K}>0$ and the Euclidean norm $\|\bullet\|$. Then,*

$$\left|\frac{1}{2\pi\mathscr{R}}\int_{\partial(\mathscr{C},\mathscr{R})}\mathscr{F}(\psi)dl(\psi)-\frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})}\mathscr{F}(x,y)dxdy\right|\le\frac{\mathscr{K}\mathscr{R}}{3}, \tag{20}$$

where $\partial(\mathscr{C},\mathscr{R})$ is the boundary of $\mathscr{D}(\mathscr{C},\mathscr{R})$ and $\psi:[0,2\pi]\to\mathbb{R}^2$ is its corresponding curve. Also, inequality (20) is sharp.

Proof. Since $\mathscr{F}$ is Lipschitzian with respect to $\mathscr{K} > 0$ and the Euclidean norm on $\mathscr{D}(\mathscr{C}, \mathscr{R})$, then, we have

$$\begin{aligned}
&\left|\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi)\rho d\rho d\varphi \right.\\
&\quad \left. - \int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\mathscr{R}\cos\varphi, b+\mathscr{R}\sin\varphi)\rho d\rho d\varphi\right| \\
&\le \int_0^{2\pi}\int_0^{\mathscr{R}} |\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) \\
&\quad - \mathscr{F}(a+\mathscr{R}cos\varphi, b+\mathscr{R}sin\varphi)|\rho d\rho d\varphi \\
&\le \mathscr{K}\int_0^{2\pi}\int_0^{\mathscr{R}} \|((\rho-\mathscr{R})\cos\varphi, (\rho-\mathscr{R})\sin\varphi)\|\rho d\rho d\varphi \\
&= \mathscr{K}\int_0^{2\pi}\int_0^{\mathscr{R}} \rho\sqrt{\rho^2+\mathscr{R}^2-2\rho\mathscr{R}}d\rho d\varphi \\
&= \mathscr{K}\int_0^{2\pi}\int_0^{\mathscr{R}} \rho(\mathscr{R}-\rho)d\rho d\varphi = \frac{\mathscr{K}\pi\mathscr{R}^3}{3}.
\end{aligned} \tag{21}$$

Now, consider the constant $\mathscr{R}$ and the curve $\psi : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\psi(\varphi): \begin{cases} x(\varphi) = a + \mathscr{R}\cos\varphi, \\ y(\varphi) = b + \mathscr{R}\sin\varphi, \end{cases} \quad \varphi \in [0, 2\pi]. \tag{22}$$

It is clear that $\psi([0, 2\pi]) = \partial(\mathscr{C}, \mathscr{R})$, and then by integrating, we obtain that

$$\begin{aligned}
\int_{\partial(\mathscr{C},\mathscr{R})} \mathscr{F}(\psi)dl(\psi) &= \int_0^{2\pi} \mathscr{F}(x(\varphi), y(\varphi))\left(\left[\frac{\partial x(\varphi)}{\partial\varphi}\right]^2 + \left[\frac{\partial y(\varphi)}{\partial\varphi}\right]^2\right)^{1/2} d\varphi \\
&= \mathscr{R}\int_0^{2\pi} \mathscr{F}(a+\mathscr{R}cos\varphi, b+\mathscr{R}\sin\varphi)d\varphi,
\end{aligned} \tag{23}$$

where $[(\partial x(\varphi))/\partial\varphi]^2 + [(\partial y(\varphi))/\partial\varphi]^2 = (\mathscr{R}^2\sin^2\varphi + \mathscr{R}^2\cos^2\varphi)^{1/2} = \mathscr{R}$, and $(\partial x(\varphi))/\partial\varphi, (\partial y(\varphi))/\partial\varphi$ are derivatives of "$x(\varphi)$" and "$y(\varphi)$" with respect to φ, respectively. So the fact that

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\mathscr{R}\cos\varphi, b+\mathscr{R}\sin\varphi)\rho d\rho d\varphi \\
&\quad = \frac{\mathscr{R}^2}{2}\int_0^{2\pi} \mathscr{F}(a+\mathscr{R}\cos\varphi, b+\mathscr{R}\sin\varphi)d\varphi,
\end{aligned} \tag{24}$$

implies that

$$\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\mathscr{R}\cos\varphi, b+\mathscr{R}\sin\varphi)\rho d\rho d\varphi = \frac{\mathscr{R}}{2}\int_{\partial(\mathscr{C},\mathscr{R})} \mathscr{F}(\psi)dl(\psi). \tag{25}$$

Also, by the use of polar coordinates, we get to

$$\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy = \int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi)\rho d\rho d\varphi. \tag{26}$$

Finally, by replacing (25) and (26) in (21) and then dividing the result with "$\pi\mathscr{R}^2$," we deduce the desired result. To prove sharpness of (20), consider the function $\mathscr{F} : \mathscr{D}(\mathscr{C}, \mathscr{R}) \to \mathbb{R}$ defined by

$$\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) = \mathscr{K}(\mathscr{R}-\rho), \tag{27}$$

for fixed $\mathscr{K} > 0$ and all $0 \le \rho \le \mathscr{R}$, $0 \le \varphi \le 2\pi$. The function f is $\mathscr{K}$-Lipschitzian by Lemma 7. It is not hard to see that $\mathscr{F}(a+\rho\cos\varphi, b+\rho sin\varphi) \ge 0$ for all $0 \le \rho \le \mathscr{R}$ and also for the case that $\rho = \mathscr{R}$, we have $\mathscr{F}(a+\mathscr{R}\cos\varphi, b+\mathscr{R}\sin\varphi) = 0$. Now applying these results in (21) implies that

$$\begin{aligned}
&\left|\frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy - \frac{1}{2\pi\mathscr{R}}\int_{\partial(\mathscr{C},\mathscr{R})} \mathscr{F}(\psi)dl(\psi)\right| \\
&\quad = \frac{1}{\pi\mathscr{R}^2}\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi)\rho d\rho d\varphi \\
&\quad = \frac{1}{\pi\mathscr{R}^2}\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{K}(\mathscr{R}-\rho)\rho d\rho d\varphi = \frac{\mathscr{K}\mathscr{R}}{3}.
\end{aligned} \tag{28}$$

The following result is the midpoint type inequality related to (1) for Lipschitzian functions defined on a closed disk.

Theorem 9. *Suppose that the mapping $\mathscr{F} : \mathscr{D}(\mathscr{C}, \mathscr{R}) \to \mathbb{R}$ is Lipschitzian with respect to a constant $\mathscr{K} > 0$ and the Euclidean norm $\|\bullet\|$. Then,*

$$\left|\frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy - \mathscr{F}(\mathscr{C})\right| \le \frac{2\mathscr{K}\mathscr{R}}{3}. \tag{29}$$

Furthermore, inequality (29) is sharp.

Proof. Since the mapping $\mathscr{F}$ satisfies a Lipschitz condition with respect to a constant $\mathscr{K} > 0$ and the Euclidean norm on $\mathscr{D}(\mathscr{C}, \mathscr{R})$, we have

$$|\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) - \mathscr{F}(a,b)| \le \mathscr{K}\|(\rho\cos\varphi, \rho\sin\varphi)\| = \mathscr{K}\rho, \tag{30}$$

for all $\rho \in [0, \mathscr{R}]$ and $\varphi \in [0, 2\pi]$. It follows that

$$\begin{aligned}
&\left|\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi)\rho d\rho d\varphi - \int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{F}(a,b)\rho d\rho d\varphi\right| \\
&\quad \leq \int_0^{2\pi}\int_0^{\mathscr{R}} |\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) - \mathscr{F}(a,b)|\rho d\rho d\varphi \\
&\quad \leq \mathscr{K}\int_0^{2\pi}\int_0^{\mathscr{R}} \rho^2 d\rho d\varphi = \frac{2\mathscr{K}\pi\mathscr{R}^3}{3}.
\end{aligned} \tag{31}$$

By the use of identity (26) in inequality (31), we obtain that

$$\left|\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy - \pi\mathscr{R}^2 f(\mathscr{C})\right| \leq \frac{2\mathscr{K}\pi\mathscr{R}^3}{3}. \tag{32}$$

Finally, it is enough to divide (32) with "$\pi\mathscr{R}^2$" to get the result. For the sharpness of (29), consider the function $\mathscr{F} : \mathscr{D}(\mathscr{C},\mathscr{R}) \to \mathbb{R}$ defined by

$$\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) = \mathscr{K}\rho, \tag{33}$$

for $\mathscr{K} > 0$, $0 \leq \rho \leq \mathscr{R}$ and $0 \leq \varphi \leq 2\pi$. By a similar method used in the proof of Lemma 7, the function $\mathscr{F}$ is $\mathscr{K}$-Lipschitzian. Also, it is obvious that $\mathscr{F}(a+\rho\cos\varphi, b+\rho\sin\varphi) \geq 0$ and $\mathscr{F}(a,b) = 0$. So, we have

$$\begin{aligned}
&\left|\frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy - f(\mathscr{C})\right| \\
&\quad = \frac{1}{\pi\mathscr{R}^2}\int_0^{2\pi}\int_0^{\mathscr{R}} f(a+\rho\cos\varphi, b+\rho\sin\varphi)\rho d\rho d\varphi \\
&\quad = \frac{1}{\pi\mathscr{R}^2}\int_0^{2\pi}\int_0^{\mathscr{R}} \mathscr{K}\rho^2 d\rho d\varphi = \frac{2\mathscr{K}\mathscr{R}}{3},
\end{aligned} \tag{34}$$

showing that inequality (29) is sharp.

Corollary 10. *Suppose that $\mathscr{U} \subset \mathbb{R}^2$ is an open set with $\mathscr{D}(\mathscr{C},\mathscr{R}) \subset \mathscr{U}$. If $\mathscr{F}$ is a convex function defined on $\mathscr{U}$, then Theorem D of Section 41 in [26] implies that $\mathscr{F}$ satisfies a Lipschitz condition on $\mathscr{D}(\mathscr{C},\mathscr{R})$ with respect to a constant $\mathscr{K} > 0$ and so from inequalities (20) and (29) along with inequality (1), we have the following results:*

$$\begin{aligned}
0 \leq \frac{1}{2\pi\mathscr{R}}\int_{\partial(\mathscr{C},\mathscr{R})} \mathscr{F}(\psi)dl(\psi) - \frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy \leq \frac{\mathscr{K}\mathscr{R}}{3}, \\
0 \leq \frac{1}{\pi\mathscr{R}^2}\iint_{\mathscr{D}(\mathscr{C},\mathscr{R})} \mathscr{F}(x,y)dxdy - \mathscr{F}(\mathscr{C}) \leq \frac{2\mathscr{K}\mathscr{R}}{3}.
\end{aligned} \tag{35}$$

In the following example, for a given function, it is illustrated how we can obtain a Lipschitz constant $\mathscr{K}$ for a real valued bifunction defined on a disk.

Example 11. Consider $\mathscr{F}(x,y) = (x-a)^n + (y-b)^n$, $(x,y) \in \mathscr{D}(\mathscr{C},\mathscr{R})$, $n \in \mathbb{N}$. We find a Lipschitz constant for $\mathscr{F}$ as follows:

For $X_1, X_2 \in \mathscr{D}(\mathscr{C},\mathscr{R})$, consider the path $\eta : [0,1] \to \mathscr{D}(\mathscr{C},\mathscr{R})$ from X_2 to X_1 in $\mathscr{D}(\mathscr{C},\mathscr{R})$ as

$$\eta(s) = sX_1 + (1-s)X_2, \tag{36}$$

for $s \in [0,1]$. The fundamental theorem of calculus implies that

$$|\mathscr{F}(X_1) - \mathscr{F}(X_2)| = |\mathscr{F}(\eta(1)) - \mathscr{F}(\eta(0))| = \left|\int_0^1 \frac{d\mathscr{F}(\eta(s))}{ds} ds\right|. \tag{37}$$

Also, the chain rule for differentiation implies that

$$\frac{d\mathscr{F}(\eta(s))}{ds} = \nabla\mathscr{F}(\eta(s)).\eta'(s) = \nabla\mathscr{F}(\eta(s))(X_1 - X_2), \tag{38}$$

where ∇f is the gradient vector of $\mathscr{F}$. So,

$$\begin{aligned}
\left|\int_0^1 \frac{d\mathscr{F}(\eta(s))}{ds} ds\right| &= \left|\int_0^1 \nabla\mathscr{F}(\eta(s))(X_1 - X_2)ds\right| \\
&\leq \|X_1 - X_2\| \int_0^1 \|\nabla\mathscr{F}(\eta(s))\| ds \\
&\leq \|X_1 - X_2\| \sup_{w\in\mathscr{D}(\mathscr{C},\mathscr{R})} \|\nabla\mathscr{F}(w)\|,
\end{aligned} \tag{39}$$

which implies that

$$|\mathscr{F}(X_1) - \mathscr{F}(X_2)| \leq \|X_1 - X_2\| \sup_{w\in\mathscr{D}(\mathscr{C},\mathscr{R})} \|\nabla\mathscr{F}(w)\|. \tag{40}$$

Now, we conclude that $\mathscr{K} = \sup_{w\in\mathscr{D}(\mathscr{C},\mathscr{R})} \|\nabla\mathscr{F}(w)\|$ (if exists) is a Lipschitz constant for $\mathscr{F}$. Therefore, for any $(x,y) \in \mathscr{D}(\mathscr{C},\mathscr{R})$, we have

$$\nabla\mathscr{F}(x,y) = \left(n(x-a)^{n-1}, n(y-b)^{n-1}\right), \tag{41}$$

and then by the use of polar transformation, we get

$$\begin{aligned}
\|\nabla\mathscr{F}(w)\| &= \sqrt{\left(n(x-a)^{n-1}\right)^2 + \left(n(y-b)^{n-1}\right)^2} \\
&= n\sqrt{(\rho^2\cos^2\varphi)^{n-1} + (\rho^2\sin^2\varphi)^{n-1}} \\
&\leq n\sqrt{(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi)^{n-1}} = n\rho^{n-1} \leq n\mathscr{R}^{n-1}.
\end{aligned} \tag{42}$$

So, we can choose $\mathscr{K} = \sup_{w\in\mathscr{D}(\mathscr{C},\mathscr{R})} \|\nabla\mathscr{F}(w)\| = n\mathscr{R}^{n-1}$ as a Lipschitz constant for $\mathscr{F}$ on $\mathscr{D}(\mathscr{C},\mathscr{R})$.

Remark 12. According to the above example, if we have a function $\mathscr{F} : \mathscr{D}(\mathscr{C},\mathscr{R}) \to \mathbb{R}$ such that $\mathscr{K} = \sup_{w\in\mathscr{D}(\mathscr{C},\mathscr{R})} \|\nabla\mathscr{F}(w)\| < \infty$ with respect to the Euclidean norm $\|\bullet\|$, then we can consider $\mathcal{K}$ as a Lipschitz constant and then obtain inequalities (20) and (29).

2.2. $\partial\mathcal{F}/\partial\rho$ Is Lipschitzian. In this part, we investigate the trapezoid and midpoint type inequalities in the case that in polar coordinates (ρ,φ), the partial derivative of considered function with respect to the variable ρ is Lipschitzian in the Euclidean norm $\|\bullet\|$.

Theorem 13. *Consider a set $I\subset\mathbb{R}^2$ with $\mathcal{D}(\mathcal{C},\mathcal{R})\subset I^\circ$ and a mapping $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ such that $\partial\mathcal{F}/\partial\rho$ (partial derivative of $\mathcal{F}$ with respect to the variable ρ in polar coordinates) is Lipschitzian with respect to a constant $\mathcal{K}>0$ and the Euclidean norm $\|\bullet\|$. Then,*

$$\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy\right|\le\frac{\mathcal{K}\mathcal{R}^2}{4}. \tag{43}$$

Proof. For any fixed $\varphi\in[0,2\pi]$, if we set

$$\begin{cases}x(\rho)=a+\rho\cos\varphi,\\ y(\rho)=b+\rho\sin\varphi,\end{cases} \tag{44}$$

then we obtain that $([(\partial x(\rho))/\partial\rho]^2+[(\partial x(\rho))/\partial\rho]^2)^{1/2}=(\sin^2(\varphi)+\cos^2(\varphi))^{1/2}=1$, where $(\partial x(\rho))/\partial\rho$, $(\partial y(\rho))/\partial\rho$ are the derivatives of $x(\rho),y(\rho)$, respectively, with respect to the variable ρ in $[0,\mathcal{R}]$. By the above facts, using integration by parts and identities (25) and (26) obtained in Theorem 8, we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho^2d\rho d\varphi\\&=\mathcal{R}^2\int_0^{2\pi}\mathcal{F}(a+\mathcal{R}\cos\varphi,b+\mathcal{R}\sin\varphi)d\varphi\\&\quad-2\int_0^{2\pi}\int_0^{\mathcal{R}}\mathcal{F}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho d\rho d\varphi\\&=\mathcal{R}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-2\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy.\end{aligned} \tag{45}$$

On the other hand, we have

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho^2d\rho d\varphi\\&=\int_0^{\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho^2d\rho d\varphi\\&\quad+\int_\pi^{2\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho^2d\rho d\varphi\\&=\int_0^{\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\rho^2d\rho d\varphi\\&\quad-\int_0^{\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a-\rho\cos\varphi,b-\rho\sin\varphi)\rho^2d\rho d\varphi.\end{aligned} \tag{46}$$

So from (45) and (46), we obtain that

$$\begin{aligned}&\left|\mathcal{R}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-2\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy\right|\\&\le\int_0^{\pi}\int_0^{\mathcal{R}}\left|\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)\right.\\&\quad\left.-\frac{\partial\mathcal{F}}{\partial\rho}(a-\rho\cos\varphi,b-\rho\sin\varphi)\right|\rho^2d\rho d\varphi\\&\le\mathcal{K}\int_0^{\pi}\int_0^{\mathcal{R}}\|(2\rho\cos\varphi,2\rho\sin\varphi)\|\rho^2d\rho d\varphi=\frac{\pi\mathcal{K}\mathcal{R}^4}{2}.\end{aligned} \tag{47}$$

Finally, it is enough to divide (47) with "$2\pi R^2$" to get the desired result.

The following is a trapezoid type inequality for the case that the partial derivative of considered function with respect to the variable "ρ" is Lipschitzian with respect to the Euclidean norm $\|\bullet\|$.

Theorem 14. *Consider a set $I\subset\mathbb{R}^2$ with $\mathcal{D}(\mathcal{C},\mathcal{R})\subset I^\circ$ and a mapping $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ such that $\partial\mathcal{F}/\partial\rho$ (partial derivative of f with respect to the variable ρ in polar coordinates) is Lipschitzian with respect to a constant $\mathcal{K}>0$ and the Euclidean norm $\|\bullet\|$. Then,*

$$\left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy-\mathcal{F}(\mathcal{C})\right|\le\frac{3\mathcal{K}\mathcal{R}^2}{4}. \tag{48}$$

Proof. Using the description provided in the beginning of Theorem 13, it is not hard to see that

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)d\rho d\varphi\\&=\int_0^{2\pi}\mathcal{F}(a+\mathcal{R}\cos\varphi,b+\mathcal{R}\sin\varphi)d\varphi-2\pi\mathcal{F}(\mathcal{C}).\end{aligned} \tag{49}$$

Also by the use of (23) in Theorem 8, we have

$$\int_0^{2\pi}\mathcal{F}(a+\mathcal{R}\cos\varphi,b+\mathcal{R}\sin\varphi)d\varphi=\frac{1}{\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi). \tag{50}$$

On the other hand, we have

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)d\rho d\varphi\\&=\int_0^{\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi)d\rho d\varphi\\&\quad-\int_0^{\pi}\int_0^{\mathcal{R}}\frac{\partial\mathcal{F}}{\partial\rho}(a-\rho\cos\varphi,b-\rho\sin\varphi)d\rho d\varphi.\end{aligned} \tag{51}$$

Then,

$$\frac{1}{\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-2\pi\mathcal{F}(\mathcal{C}) = \int_0^{\pi}\int_0^{\mathcal{R}}\left[\frac{\partial\mathcal{F}}{\partial\rho}(a+\rho\cos\varphi,b+\rho\sin\varphi) - \frac{\partial\mathcal{F}}{\partial\rho}(a-\rho\cos\varphi,b-\rho\sin\varphi)\right]d\rho d\varphi. \quad (52)$$

Since $\partial\mathcal{F}/\partial\rho$ is $\mathcal{K}$-Lipschitzian, we have that

$$\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-\mathcal{F}(\mathcal{C})\right| \le \frac{\mathcal{K}\mathcal{R}^2}{2}. \quad (53)$$

Now triangle inequality and inequality (43) imply that

$$\begin{aligned}&\left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy-\mathcal{F}(\mathcal{C})\right| \\ &\le\left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy-\frac{1}{2\pi\mathcal{R}}\times\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)\right|+\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-\mathcal{F}(C)\right| \\ &\le\frac{\mathcal{K}\mathcal{R}^2}{4}+\frac{\mathcal{K}\mathcal{R}^2}{2}=\frac{3\mathcal{K}\mathcal{R}^2}{4}.\end{aligned} \quad (54)$$

Here, we provide two examples in connection with results obtained in this subsection.

Example 15. Define a function $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ *by*

$$\mathcal{F}(a+\rho\cos\varphi,b+\rho\sin\varphi)=\mathcal{M}\rho^2, \quad (55)$$

for $\mathcal{M}>0$, $0\le\rho\le\mathcal{R}$, and $0\le\varphi\le 2\pi$. Since $\partial\mathcal{F}/\partial\rho=2\mathcal{M}\rho$, then according to Remark 12, we can consider $\mathcal{K}=\sup_{w\in\mathcal{D}(\mathcal{C},\mathcal{R})}\|\nabla\mathcal{F}(w)\|=2\mathcal{M}<\infty$ as a Lipschitz constant with respect to the Euclidean norm $\|\bullet\|$. On the other hand,

$$\begin{aligned}&F(\mathcal{C})=F(a,b)=0,\\ &\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy=\frac{\mathcal{M}\pi\mathcal{R}^4}{2},\\ &\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)=2\mathcal{M}\pi\mathcal{R}^3.\end{aligned} \quad (56)$$

So,

$$\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy=\frac{\mathcal{M}\mathcal{R}^2}{2}, \quad (57)$$

which shows that (43) is sharp. Also,

$$\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\psi)dl(\psi)-\mathcal{F}(\mathcal{C})=\mathcal{M}\mathcal{R}^2, \quad (58)$$

which implies that (53) is sharp.

Example 16. Consider $a,b>0$, $0<\mathcal{R}\le\min\{a,b\}$ and $0\le\rho\le\mathcal{R}$. For $n\in\mathbb{N}$ and polar function $\mathcal{F}(\rho,\varphi)=(a-\rho)^n+(b-\rho)^n$ which is defined on $\mathcal{D}((a,b),\mathcal{R})$, by some calculations we can conclude that

$$\nabla\left(\frac{\partial\mathcal{F}}{\partial\rho}\right)(\rho,\varphi)=n(n-1)\left((a-\rho)^{n-2}+(b-\rho)^{n-2},0\right), \quad (59)$$

and then

$$\mathcal{K}=\sup_{0\le\rho\le\mathcal{R},0\le\varphi\le2\pi}\left\|\nabla\left(\frac{\partial\mathcal{F}}{\partial\rho}\right)(\rho,\varphi)\right\|=n(n-1)\left(a^{n-2}+b^{n-2}\right), \quad (60)$$

is a Lipschitz constant for $\mathcal{F}$. Then, from inequality (53), we have that

$$\left|A\left((a-\mathcal{R})^n,(b-\mathcal{R})^n\right)-A\left(a^n,b^n\right)\right|\le\frac{n(n-1)A\left(a^{n-2},b^{n-2}\right)\mathcal{R}^2}{2}, \quad (61)$$

where $A(a,b)=(a+b)/2$ is arithmetic mean of a and b. Also for the function $\mathcal{F}$, by the use of (43) and (48), we can obtain other arithmetic mean type inequalities.

3. Mappings H and h

In this section, by the use of two mappings $H(t)$: $[0,1]\to\mathbb{R}$ and $h(t)$: $[0,1]\to\mathbb{R}$ defined in [1], we give some generalized Hermite-Hadamard type inequalities in the case that considered functions are Lipschitzian with respect to Euclidean norm $\|\bullet\|$ on a disk $\mathcal{D}(\mathcal{C},\mathcal{R})$:

$$\begin{aligned}&H(t)=\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(t\mathcal{C}+(1-t)(x,y))dxdy,\\ &h(t)=\begin{cases}\dfrac{1}{2\pi t\mathcal{R}}\displaystyle\int_{\partial(\mathcal{C},t\mathcal{R})}, \quad \mathcal{F}(\gamma)dl(\gamma(t)), & t\in(0,1],\\ \mathcal{F}(\mathcal{C}), & t=0.\end{cases}\end{aligned} \quad (62)$$

By the use of some properties for the mappings h and H, we give some refinements for trapezoid and midpoint type inequalities obtained in previous sections for $\mathcal{K}$-Lipschitzian mappings $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$.

Theorem 17. *Suppose that the mapping $\mathcal{F}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{K}>0$ and the Euclidean norm $\|\bullet\|$. Then, the mapping H is Lipschitzian with respect to "$2\mathcal{K}\mathcal{R}/3$" and the mapping h is Lipschitzian with respect to "$\mathcal{K}\mathcal{R}$." The following inequalities also for all $t\in($*

0, 1) hold.

$$\left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(t\mathcal{C}+(1-t)(x,y))dxdy - t\mathcal{F}(\mathcal{C}) - \frac{1-t}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(x,y)dxdy\right| \leq \frac{4\mathcal{K}\mathcal{R}t(1-t)}{3}, \tag{63}$$

$$\left|\frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma(t)) - \frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{F}(t\mathcal{C}+(1-t)(x,y))dxdy\right| \leq \frac{2\mathcal{K}\mathcal{R}t}{3}, \tag{64}$$

$$\left|\frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma(t)) - \frac{t}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma) - (1-t)\mathcal{F}(\mathcal{C})\right| \leq 2\mathcal{K}\mathcal{R}t(1-t). \tag{65}$$

Proof. Consider the following relations for $t_1, t_2 \in [0,1]$, which prove the first part of this theorem:

$$\begin{aligned}|H(t_1)-H(t_2)| &\leq \frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}|\mathcal{F}(t_1\mathcal{C}+(1-t_1)(x,y)) - \mathcal{F}(t_2\mathcal{C}+(1-t_2)(x,y))|dxdy \\ &\leq \frac{\mathcal{K}\,|\,t_1-t_2\,|}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\|(a-x,b-y)\|dxdy \\ &= \frac{\mathcal{K}\,|\,t_1-t_2\,|}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\|(\rho cos\varphi, \rho sin\varphi)\|\rho d\rho d\varphi \\ &= \frac{\mathcal{K}\,|\,t_1-t_2\,|}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\rho^2 d\rho d\varphi = \frac{2\mathcal{K}\mathcal{R}}{3}|t_1-t_2|.\end{aligned} \tag{66}$$

Also,

$$\begin{aligned}|h(t_1)-h(t_2)| &= \left|\frac{1}{2\pi t_1\mathcal{R}}\int_{\partial(\mathcal{C},t_1\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma(t_1)) - \frac{1}{2\pi t_2\mathcal{R}}\cdot\int_{\partial(\mathcal{C},t_2\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma(t_2))\right| \\ &= \frac{1}{\pi\mathcal{R}^2}\left|\int_0^{2\pi}\int_0^{\mathcal{R}}\mathcal{F}(a+t_1\mathcal{R}\cos\varphi, b+t_1\mathcal{R}\sin\varphi)\rho d\rho d\varphi - \int_0^{2\pi}\int_0^{\mathcal{R}}\mathcal{F}(a+t_2\mathcal{R}\cos\varphi, b+t_2\mathcal{R}\sin\varphi)\rho d\rho d\varphi\right| \\ &\leq \frac{\mathcal{K}\,|\,t_1-t_2\,|}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\|(\mathcal{R}\cos\varphi, \mathcal{R}\sin\varphi)\|\rho d\rho d\varphi \\ &= \mathcal{K}\mathcal{R}|t_1-t_2|.\end{aligned} \tag{67}$$

For inequality (63), we use the definition of H and the fact that $\mathcal{F}$ is $\mathcal{K}$-Lipschitzian:

$$\begin{aligned}|H(t)-tH(1)-(1-t)H(0)| &\leq t|H(t)-H(1)| + (1-t)|H(t)-H(0)| \\ &\leq \frac{t}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}|\mathcal{F}(t\mathcal{C}+(1-t)(x,y)) - \mathcal{F}(\mathcal{C})|dxdy + \frac{1-t}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\cdot|\mathcal{F}(t\mathcal{C}+(1-t)(x,y)) - \mathcal{F}(x,y)|dxdy \\ &\leq \frac{2\mathcal{K}t(1-t)}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})}\|(x-a,y-b)\|dxdy \\ &= \frac{2\mathcal{K}t(1-t)}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\rho^2 d\rho d\varphi = \frac{4\mathcal{K}\mathcal{R}t(1-t)}{3}.\end{aligned} \tag{68}$$

To prove (64), if $t \in (0,1]$, we consider the following identity presented in [1],

$$H(t) = \frac{1}{\pi\mathcal{R}^2 t^2}\iint_{\mathcal{D}(\mathcal{C},t\mathcal{R})}\mathcal{F}(x,y)dxdy. \tag{69}$$

Now, consider transformation

$$\begin{cases} x(\rho) = a + t\rho\cos\varphi; & \rho \in [0,\mathcal{R}], \varphi \in [0,2\pi], t \in (0,1], \\ y(\rho) = b + t\rho\sin\varphi. \end{cases} \tag{70}$$

This implies that

$$H(t) = \frac{1}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\mathcal{F}(a+t\rho\cos\varphi, b+t\rho\sin\varphi)\rho d\rho d\varphi. \tag{71}$$

Also, we have

$$\begin{aligned}h(t) &= \frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})}\mathcal{F}(\gamma)dl(\gamma(t)) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(a+t\mathcal{R}\cos\varphi, b+t\mathcal{R}\sin\varphi)d\varphi \\ &= \frac{1}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\mathcal{F}(a+t\mathcal{R}\cos\varphi, b+t\mathcal{R}\sin\varphi)\rho d\rho d\varphi.\end{aligned} \tag{72}$$

So, we conclude that

$$\begin{aligned}|h(t)-H(t)| &\leq \frac{1}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}|\mathcal{F}(a+t\mathcal{R}\cos\varphi, b+t\mathcal{R}\sin\varphi) - \mathcal{F}(a+t\rho\cos\varphi, b+t\rho\sin\varphi)|\rho d\rho d\varphi \\ &\leq \frac{\mathcal{K}}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}\|(t(\mathcal{R}-\rho)\cos\varphi, t(\mathcal{R}-\rho)\sin\varphi)\|\rho d\rho d\varphi \\ &= \frac{\mathcal{K}t}{\pi\mathcal{R}^2}\int_0^{2\pi}\int_0^{\mathcal{R}}(\mathcal{R}-\rho)\rho d\rho d\varphi = \frac{\mathcal{K}t\mathcal{R}}{3}.\end{aligned} \tag{73}$$

Inequality (65) is a consequence of the fact that

$$|h(t) - th(1) - (1-t)h(0)| \leq t|h(t) - h(1)| + (1-t)|h(t) - h(0)|. \tag{74}$$

The details are omitted.

The following results also are of interest:

Theorem 18. *Suppose that the mapping $\mathcal{F} : \mathcal{D}(\mathcal{C}, \mathcal{R}) \to \mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{K} > 0$ and the Euclidean norm $\|\bullet\|$. The following inequalities hold:*

$$|H(t) - H(0)| = \left|\frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(t\mathcal{C} + (1-t)(x,y))dxdy - \frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(x,y)dxdy\right| \leq \frac{2\mathcal{K}t\mathcal{R}}{3}, \tag{75}$$

for all $t \in (0, 1]$.

$$|H(1) - H(t)| = \left|\mathcal{F}(\mathcal{C}) - \frac{1}{\pi\mathcal{R}^2}\iint_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{F}(t\mathcal{C} + (1-t)(x,y))dxdy\right| \leq \frac{2\mathcal{K}(1-t)\mathcal{R}}{3}, \tag{76}$$

for all $t \in [0, 1)$.

$$|h(t) - h(0)| = \left|\frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})} \mathcal{F}(\gamma)dl(\gamma(t)) - \mathcal{F}(\mathcal{C})\right| \leq \mathcal{K}t\mathcal{R}, \tag{77}$$

for all $t \in (0, 1]$, and

$$|h(1) - h(t)| = \left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{F}(\gamma)dl(\gamma) - \frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})} \mathcal{F}(\gamma)dl(\gamma(t))\right| \leq \mathcal{K}(1-t)\mathcal{R}, \tag{78}$$

for all $t \in (0, 1)$.

Proof. It is enough to consider special cases for t_1 and t_2 in two inequalities (66) and (67) obtained in the proof of previous theorem.

Remark 19.

(1) For a convex function $\mathcal{F} : \mathcal{D}(\mathcal{C}, \mathcal{R}) \to \mathbb{R}$ *with* $\mathcal{K} = \sup_{w\in\mathcal{D}(\mathcal{C},\mathcal{R})} \|\nabla\mathcal{F}(w)\| < \infty$, if we consider (1), some results obtained in [1], and the following inequality

$$\mathcal{F}(\mathcal{C}) \leq \frac{1}{\pi\mathcal{R}^2t^2}\iint_{\mathcal{D}(\mathcal{C},t\mathcal{R})} \mathcal{F}(x,y)dxdy \leq \frac{1}{2\pi t\mathcal{R}}\int_{\partial(\mathcal{C},t\mathcal{R})} \mathcal{F}(\gamma)dl(\gamma(t)) \leq h(1), t \in (0,1), \tag{79}$$

then, we deduce that (64) and (75)–(78) hold without using absolute value symbol.

(2) If we consider $t = 1$ in inequality (75) or consider $t = 0$ in inequality (76), then, we recapture inequality (29) in Theorem 9. Also from inequality (77), we obtain this new inequality

$$\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{F}(\gamma)dl(\gamma) - \mathcal{F}(\mathcal{C})\right| \leq \mathcal{K}\mathcal{R}, \tag{80}$$

where $\mathcal{F} : \mathcal{D}(\mathcal{C}, \mathcal{R}) \to \mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{K} > 0$ and the Euclidean norm $\|\bullet\|$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this article.

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