

Research Article

Novel Refinements via n -Polynomial Harmonically s -Type Convex Functions and Application in Special Functions

Saad Ihsan Butt ¹, Saima Rashid ², Muhammad Tariq,¹ and Miao-Kun Wang ³

¹COMSATS University Islamabad, Lahore Campus, Pakistan

²Department of Mathematics, Government College University, Faisalabad 38000, Pakistan

³Department of Mathematics, Huzhou University, Huzhou 313000, China

Correspondence should be addressed to Saima Rashid; saimarashid@gcuf.edu.pk and Miao-Kun Wang; wmk000@126.com

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In this work, we introduce the idea of n -polynomial harmonically s -type convex function. We elaborate the new introduced idea by examples and some interesting algebraic properties. As a result, new Hermite–Hadamard, some refinements of Hermite–Hadamard and Ostrowski type integral inequalities are established, which are the generalized variants of the previously known results for harmonically convex functions. Finally, we illustrate the applicability of this new investigation in special functions (hypergeometric function and special mean of real numbers).

1. Introduction

The theory of convexity presents an amazing, fascinating, and captivating field of research and also played significant and important roles in many areas, such as statistics, economics, optimization, management science, finance, engineering, game theory, and mathematical inequalities. In probability theory, a convex function applied to the expected value of a random variable is always bounded above by the expected value of the convex function of the random variable. This result, known as Jensen's inequality, can be used to deduce inequalities such as the arithmetic-geometric mean inequality and Hölder's inequality. Due to widespread views, robustness, and a lot of applications, the theory of convexity has become a rich source of motivation and absorbing field for researchers. Using the theory of convexity, mathematicians provided amazing tools and numerical techniques to tackle and to solve a wide class of problems, which arise in pure and applied sciences. This theory has a rich and paramount history and has been the focus and fixed point of intense study for over a century in mathematics. Many generalizations, variants, and extensions for the convexity have attracted the attention of many researchers, see [1–3]. This theory also played a meaningful role in the development of

the theory of inequalities. In diverse and opponent research, inequalities have a lot of applications in statistical problems, probability, and numerical quadrature formulas. Integral inequalities have an interesting mathematical model due to its robustness and pivotal applications in fractional integral calculus and mathematical analysis. In approximation theory, integral inequalities explain and brief the growth rate of competing mathematical analysis. For the applications, interested readers refer to [4–6]. Recently, Toplu et al. [7] introduced a generalization form of convexity namely n -polynomial convex functions.

It is well known that the harmonic mean is the special case of power mean. It is often used for situations when the average rates are desired and have a lot of applications in different fields of sciences which are statistics, computer science, trigonometry, geometry, probability, finance, and electric circuit theory. Harmonic mean is the most appropriate measure for rates and ratios because it equalizes the weights of each data point. Harmonic mean is used to define the harmonic convex set. In 2003, the first harmonic convex set was introduced by Shi and Zhang [8]. Harmonic convex function was first introduced and discussed by Anderson et al. [9]. Awan et al. [10] keeping his work on generalizations introduced a new generalized class of convex function called n -

polynomial harmonic convex function. For the literature and attention for the readers about harmonically convex functions, see [11–14]. Motivated and inspired by the ongoing activities and research in the convex analysis field, we find out that there exists a special class of functions known as s -type convex function. Recently, Rashid et al. [15] introduced n -polynomial s -type convex function. Eventually, many mathematicians put an effort, hardworking, and has collaborated with different ideas and concepts in the field of convex analysis. The amazing techniques and remarkable ideas of this article may inspire and motivate for further research in this pivotal, captivating, and valuable field. Before we start, we need the following necessary known definitions and literature.

2. Preliminaries

Let $\psi : I \rightarrow \mathbb{R}$ be a real-valued function. A function ψ is said to be convex if

$$\psi(\kappa\vartheta_1 + (1 - \kappa)\vartheta_2) \leq \kappa\psi(\vartheta_1) + (1 - \kappa)\psi(\vartheta_2), \quad (1)$$

holds for all $\vartheta_1, \vartheta_2 \in I$ and $\kappa \in [0, 1]$.

Definition 1 (see [16]). A function $\psi : H \subseteq (0, +\infty) \rightarrow \mathbb{R}$ is said to be harmonically convex if

$$\psi\left(\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1 - \kappa)\vartheta_1}\right) \leq \kappa\psi(\vartheta_1) + (1 - \kappa)\psi(\vartheta_2), \quad (2)$$

holds for all $\vartheta_1, \vartheta_2 \in H$, and $\kappa \in [0, 1]$.

For the harmonically convex function, İscan [16] provided the Hermite–Hadamard type inequality.

Theorem 2 (see [16]). *Let $\psi : H \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a harmonically convex function. If $\psi \in L[\vartheta_1, \vartheta_2]$ for all $\vartheta_1, \vartheta_2 \in H$ with $\vartheta_1 < \vartheta_2$, then*

$$\psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) \leq \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \leq \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2}. \quad (3)$$

Definition 3 (see [7]). A nonnegative function $\psi : I \rightarrow \mathbb{R}$ is called n -polynomial convex if

$$\begin{aligned} \psi(\kappa\vartheta_1 + (1 - \kappa)\vartheta_2) &\leq \frac{1}{n} \sum_{i=1}^n [1 - (1 - \kappa)^i] \psi(\vartheta_1) \\ &\quad + \frac{1}{n} \sum_{i=1}^n [1 - \kappa^i] \psi(\vartheta_2), \end{aligned} \quad (4)$$

holds for every $\vartheta_1, \vartheta_2 \in I, n \in \mathbb{N}$, and $\kappa \in [0, 1]$.

Definition 4 (see [15]). A function $\psi : I \rightarrow \mathbb{R}$ is said to be s -type convex function if

$$\psi(\kappa\vartheta_1 + (1 - \kappa)\vartheta_2) \leq [1 - s(1 - \kappa)]\psi(\vartheta_1) + [1 - s\kappa]\psi(\vartheta_2), \quad (5)$$

holds $\forall \vartheta_1, \vartheta_2 \in I, s \in [0, 1]$, and $\kappa \in [0, 1]$.

Definition 5 (see [15]). A function $\psi : I \rightarrow \mathbb{R}$ is called n -polynomial s -type convex if

$$\begin{aligned} \psi(\kappa\vartheta_1 + (1 - \kappa)\vartheta_2) &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\vartheta_1) \\ &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2), \end{aligned} \quad (6)$$

holds for every $\vartheta_1, \vartheta_2 \in I, n \in \mathbb{N}$, $s \in [0, 1]$, and $\kappa \in [0, 1]$.

Motivated by the above results, literature and ongoing activities, and research in this amazing and captivating field, we will give in Section 3 the idea and its algebraic properties of n -polynomial harmonically s -type convex function. In Section 4, we will derive the new version of Hermite–Hadamard inequality by using the newly introduced definition. As a result in Sections 5 and 6, we will give some refinements of the Hermite–Hadamard and Ostrowski type inequalities, and in Section 7, we will give some applications for our proposed new definition. Finally, a brief conclusion will be provided as well.

3. New n -Polynomial Harmonically s -Type Convex Function and Its Properties

We are going to introduce a new definition called an n -polynomial harmonically s -type convex function and will study some of their algebraic properties.

Definition 6. A nonnegative real-valued function $\psi : H \subseteq (0, +\infty) \rightarrow [0, +\infty)$ is called an n -polynomial harmonically s -type convex if

$$\begin{aligned} \psi\left(\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1 - \kappa)\vartheta_1}\right) &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\vartheta_1) \\ &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2), \end{aligned} \quad (7)$$

holds for every $\vartheta_1, \vartheta_2 \in I, n \in \mathbb{N}$, $s \in [0, 1]$, and $\kappa \in [0, 1]$.

Remark 7.

- (i) Taking $n = 1$ in Definition 6, we obtain the following new definition about harmonically s -type convex

function

$$\psi\left(\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1}\right) \leq [1 - (s(1-\kappa))]\psi(\vartheta_1) + [1 - (s\kappa)]\psi(\vartheta_2) \tag{8}$$

- (ii) Taking $s = 1$ in Definition 6, then, we get a definition, namely, n -polynomial harmonically convex function, which is defined by Awan et al. [10]
- (iii) Taking $n = 1$ and $s = 1$ in Definition 6, then, we get a definition, namely, harmonically convex function, which is introduced by İşcan [16]
- (iv) Taking $n = 2$ and $s = 1$ in Definition 6, we obtain the following new definition about 2-polynomial harmonically convex function

$$\psi\left(\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1}\right) \leq \left(\frac{3\kappa - \kappa^2}{2}\right)\psi(\vartheta_1) + \left(\frac{2 - \kappa - \kappa^2}{2}\right)\psi(\vartheta_2) \tag{9}$$

That is the beauty of this newly introduce definition if we choosing different values of n and s , as a result, we obtain new amazing integral inequalities and also found some results which connect with previous results.

Lemma 8. *The following inequalities $1/n\sum_{i=1}^n [1 - (s(1-\kappa))^i] \geq \kappa$ and $1/n\sum_{i=1}^n [1 - (s\kappa)^i] \geq (1-\kappa)$ are hold. If for all $\kappa \in [0, 1]$ and $s \in [0, 1]$.*

Proof. The proof is evident.

Proposition 9. *Let $I \subset (0, +\infty)$ be a harmonically convex set. Every harmonically convex function on a harmonically convex set is an n -polynomial harmonically s -type convex function.*

Proof. Using the definition of harmonically convex function and from the Lemma 8, since $\kappa \leq 1/n\sum_{i=1}^n [1 - (s(1-\kappa))^i]$ and $(1-\kappa) \leq 1/n\sum_{i=1}^n [1 - (s\kappa)^i]$ for all $\kappa \in [0, 1]$ and $s \in [0, 1]$, we have

$$\begin{aligned} \psi\left[\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1}\right] &\leq \kappa\psi(\vartheta_1) + (1-\kappa)\psi(\vartheta_2) \\ &\leq \frac{1}{n}\sum_{i=1}^n [1 - (s(1-\kappa))^i]\psi(\vartheta_1) \\ &\quad + \frac{1}{n}\sum_{i=1}^n [1 - (s\kappa)^i]\psi(\vartheta_2). \end{aligned} \tag{10}$$

Proposition 10. *Every n -polynomial harmonically convex function is an n -polynomial harmonically s -type convex function.*

Proof. Using the definition of an n -polynomial harmonically convex function and from (Remark 1.6, see [15]), we have $1/n\sum_{i=1}^n [1 - (1-\kappa)^i] \leq 1/n\sum_{i=1}^n [1 - (s(1-\kappa))^i]$ and $1/n\sum_{i=1}^n [1 - \kappa^i] \leq 1/n\sum_{i=1}^n [1 - (s\kappa)^i]$ for all $\kappa \in [0, 1], n \in \mathbb{N}$, and $s \in [0, 1]$.

$$\begin{aligned} \psi\left[\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1}\right] &\leq \frac{1}{n}\sum_{i=1}^n [1 - (1-\kappa)^i]\psi(\vartheta_1) \\ &\quad + \frac{1}{n}\sum_{i=1}^n [1 - \kappa^i]\psi(\vartheta_2) \\ &\leq \frac{1}{n}\sum_{i=1}^n [1 - (s(1-\kappa))^i]\psi(\vartheta_1) \\ &\quad + \frac{1}{n}\sum_{i=1}^n [1 - (s\kappa)^i]\psi(\vartheta_2). \end{aligned} \tag{11}$$

Proposition 11. *Every n -polynomial harmonically s -type convex function is harmonically h -convex function with $h(\kappa) = 1/n\sum_{i=1}^n [1 - (s(1-\kappa))^i]$.*

Proof.

$$\begin{aligned} \psi\left[\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1}\right] &\leq \frac{1}{n}\sum_{i=1}^n [1 - (s(1-\kappa))^i]\psi(\vartheta_1) \\ &\quad + \frac{1}{n}\sum_{i=1}^n [1 - (s\kappa)^i]\psi(\vartheta_2) \\ &\leq h(\kappa)\psi(\vartheta_1) + h(1-\kappa)\psi(\vartheta_2). \end{aligned} \tag{12}$$

Now, we make some examples via newly introduce definition n -polynomial harmonically s -type convex function.

Example 12. $\psi(x) = e^x$ is an increasing convex function, so it is harmonically convex function (see [17]). By using Proposition 9, it is an n -polynomial harmonically s -type convex function.

Example 13. $\psi(x) = \ln x$ is an increasing convex function on $(0, \infty)$, so it is harmonically convex function $\forall x \in (0, \infty)$ (see [17]). By using Proposition 9, it is an n -polynomial harmonically s -type convex function.

These are the clear advantages of the proposed new definition with respect to other known functions on the topic mentioned above. Now, we will study some of their algebraic properties.

Theorem 14. *Let $\psi, \varphi : H \subset (0, +\infty) \rightarrow [0, +\infty)$. If ψ and φ are two n -polynomial harmonically s -type convex functions, then*

- (1) $\psi + \varphi$ is an n -polynomial harmonically s -type convex function
- (2) For nonnegative real number $c, c\psi$ is an n -polynomial harmonically s -type convex function

Proof.

- (1) Let ψ and φ be an n -polynomial harmonically s -type convex, then

$$\begin{aligned}
 & (\psi + \varphi) \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) \\
 &= \psi \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) + \varphi \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) \\
 &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \varphi(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \varphi(\vartheta_2) \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] [\psi(\vartheta_1) + \varphi(\vartheta_1)] \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] [\psi(\vartheta_2) + \varphi(\vartheta_2)] \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] (\psi + \varphi)(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] (\psi + \varphi)(\vartheta_2)
 \end{aligned} \tag{13}$$

- (2) Let ψ be an n -polynomial harmonically s -type convex function, then

$$\begin{aligned}
 & (c\psi) \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) \\
 &\leq c \left[\frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\vartheta_1) \right. \\
 &\quad \left. + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2) \right] \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] c\psi(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] c\psi(\vartheta_2) \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] (c\psi)(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] (c\psi)(\vartheta_2)
 \end{aligned} \tag{14}$$

which completes the proof.

Theorem 15. Let $\varphi : H \rightarrow [0, +\infty)$ be harmonically convex function and $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is nondecreasing and n -polynomial s -type convex function. Then, the function $\psi \circ \varphi : H \rightarrow [0, +\infty)$ is an n -polynomial harmonically s -type convex.

Proof. For all $\vartheta_1, \vartheta_2 \in H$, $s \in [0, 1]$, and $\kappa \in [0, 1]$, we have

$$\begin{aligned}
 (\psi \circ \varphi) \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) &= \psi \left(\varphi \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) \right) \\
 &\leq \psi(\kappa \varphi(\vartheta_1) + (1 - \kappa) \varphi(\vartheta_2)) \\
 &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\varphi(\vartheta_1)) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\varphi(\vartheta_2)) \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] (\psi \circ \varphi)(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] (\psi \circ \varphi)(\vartheta_2),
 \end{aligned} \tag{15}$$

which completes the proof.

Theorem 16. Let $0 < \vartheta_1 < \vartheta_2, \psi_j : [\vartheta_1, \vartheta_2] \rightarrow [0, +\infty)$ be a family of an n -polynomial harmonically s -type convex functions and $\psi(u) = \sup_j \psi_j(u)$. Then, ψ is an n -polynomial harmonically s -type convex function, and $U = \{u \in [\vartheta_1, \vartheta_2] : \psi(u) < +\infty\}$ is an interval.

Proof. Let $\vartheta_1, \vartheta_2 \in U$, $s \in [0, 1]$, and $\kappa \in [0, 1]$, then

$$\begin{aligned}
 \psi \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) &= \sup_j \psi_j \left(\frac{\vartheta_1 \vartheta_2}{\kappa \vartheta_2 + (1 - \kappa) \vartheta_1} \right) \\
 &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \sup_j \psi_j(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \sup_j \psi_j(\vartheta_2) \\
 &= \frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] \psi(\vartheta_1) \\
 &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2) < +\infty,
 \end{aligned} \tag{16}$$

which completes the proof.

Remark 17. Interested readers can find many other nice properties of this new class of functions. We omit here the details.

4. Hermite–Hadamard Type Inequality via n -Polynomial Harmonically s -Type Convex Functions

The purpose of this section is to derive a new version of Hermite–Hadamard type using n -polynomial harmonically s -type convexity.

Theorem 18. Let $\psi : [\vartheta_1, \vartheta_2] \rightarrow [0, +\infty)$ be an n -polynomial harmonically s -type convex function. If $\psi \in L[\vartheta_1, \vartheta_2]$, then

$$\begin{aligned} \frac{n}{2\sum_{i=1}^n [1 - (s/2)^i]} \psi \left[\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2} \right] &\leq \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \\ &\leq \left[\frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{n} \right] \cdot \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \tag{17}$$

Proof. Since ψ is an n -polynomial harmonically s -type convex function, we have

$$\begin{aligned} \psi \left[\frac{xy}{\kappa y + (1-\kappa)x} \right] &\leq \frac{1}{n} \sum_{i=1}^n [1 - (s(1-\kappa))^i] \psi(x) \\ &\quad + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(y), \end{aligned} \tag{18}$$

which lead to

$$\psi \left[\frac{2xy}{x+y} \right] \leq \frac{1}{n} \sum_{i=1}^n \left[1 - \left(\frac{s}{2}\right)^i \right] \psi(x) + \frac{1}{n} \sum_{i=1}^n \left[1 - \left(\frac{s}{2}\right)^i \right] \psi(y). \tag{19}$$

Using the change of variables, we get

$$\begin{aligned} \psi \left[\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2} \right] &\leq \frac{1}{n} \sum_{i=1}^n \left[1 - \left(\frac{s}{2}\right)^i \right] \times \left\{ \psi \left[\frac{\vartheta_1\vartheta_2}{(\kappa\vartheta_2 + (1-\kappa)\vartheta_1)} \right] \right. \\ &\quad \left. + \psi \left[\frac{\vartheta_1\vartheta_2}{(\kappa\vartheta_1 + (1-\kappa)\vartheta_2)} \right] \right\}, \end{aligned} \tag{20}$$

Integrate the above inequality with respect to κ over $[0,1]$, then

$$\frac{n}{2\sum_{i=1}^n [1 - (s/2)^i]} \psi \left[\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2} \right] \leq \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx, \tag{21}$$

which completes the left side inequality. For the right side inequality, changing the variable of integration as $x = [\vartheta_1\vartheta_2 / \kappa\vartheta_2 + (1-\kappa)\vartheta_1]$ and using Definition 6 for the function ψ , we have

$$\begin{aligned} \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx &= \int_0^1 \psi \left[\frac{\vartheta_1\vartheta_2}{\kappa\vartheta_2 + (1-\kappa)\vartheta_1} \right] d\kappa \\ &\leq \int_0^1 \left[\frac{1}{n} \sum_{i=1}^n [1 - (s(1-\kappa))^i] \psi(\vartheta_1) \right. \\ &\quad \left. + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] \psi(\vartheta_2) \right] d\kappa \\ &= \frac{\psi(\vartheta_1)}{n} \sum_{i=1}^n \int_0^1 [1 - (s(1-\kappa))^i] d\kappa \\ &\quad + \frac{\psi(\vartheta_2)}{n} \sum_{i=1}^n \int_0^1 [1 - (s\kappa)^i] d\kappa \\ &= \left[\frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{n} \right] \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right], \end{aligned} \tag{22}$$

which completes the proof.

Corollary 19. Choosing $n = 1$ in Theorem 18, then

$$\begin{aligned} \frac{1}{(2-s)} \psi \left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2} \right) &\leq \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \\ &\leq \frac{(2-s)}{2} [\psi(\vartheta_1) + \psi(\vartheta_2)]. \end{aligned} \tag{23}$$

Remark 20. Choosing $s = 1$ in Theorem 18, then, we get the Theorem (2.3) in [10].

Remark 21. Choosing $n = 1$ and $s = 1$ in Theorem 18, then we get the Theorem (3) in [18].

5. Refinements of (H-H) Type Inequality via n -Polynomial Harmonically s -Type Convex Functions

In order to obtain some refinements of (H-H) type inequality using n -polynomial harmonically s -type convex functions, we need the following lemma.

Lemma 22 (see [10]). Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function and $\omega, \omega \in [0, 1]$. If $\psi' \in L[\vartheta_1, \vartheta_2]$, then the following identity holds

$$\begin{aligned} & \frac{\omega\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \omega - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \\ &= \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \int_0^1 \left[\frac{4(1 - \omega - \kappa)}{((1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1)^2} \psi' \right. \\ &\quad \cdot \left. \left(\frac{2\vartheta_1\vartheta_2}{(1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1} \right) + \frac{4(\omega - \kappa)}{(\kappa\vartheta_1 + (2 - \kappa)\vartheta_2)^2} \psi' \right. \\ &\quad \cdot \left. \left(\frac{2\vartheta_1\vartheta_2}{\kappa\vartheta_1 + (2 - \kappa)\vartheta_2} \right) \right] d\kappa. \end{aligned} \tag{24}$$

For simplicity, we denote

$$A_{\vartheta_1, \vartheta_2} = (1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1 \text{ and } B_{\vartheta_1, \vartheta_2} = \kappa\vartheta_1 + (2 - \kappa)\vartheta_2. \tag{25}$$

The following special functions will be used in sequel:

$$\Gamma(\vartheta) = \int_0^{+\infty} e^{-\kappa} \kappa^{\vartheta-1} d\kappa, \quad \vartheta > 0,$$

$$\beta(\vartheta_1, \vartheta_2) = \int_0^1 \kappa^{\vartheta_1-1} (1 - \kappa)^{\vartheta_2-1} d\kappa, \quad \vartheta_1, \vartheta_2 > 0,$$

$$\beta(\vartheta_1, \vartheta_2) = \frac{\Gamma(\vartheta_1)\Gamma(\vartheta_2)}{\Gamma(\vartheta_1 + \vartheta_2)}, \quad \vartheta_1, \vartheta_2 > 0,$$

$$\begin{aligned} {}_2F_1(\vartheta_1, \vartheta_2; \vartheta_3; \vartheta) &= \frac{1}{\beta(\vartheta_2, \vartheta_3 - \vartheta_2)} \\ &\quad \cdot \int_0^1 \kappa^{\vartheta_2-1} (1 - \kappa)^{\vartheta_3 - \vartheta_2 - 1} (1 - \vartheta\kappa)^{-\vartheta_1} d\kappa, \end{aligned} \tag{26}$$

where $\vartheta_3 > \vartheta_2 > 0$ and $|\vartheta| < 1$.

Theorem 23. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function such that $\psi' \in L[\vartheta_1, \vartheta_2]$ and $\omega, \omega \in [0, 1]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, then for $s \in [0, 1]$ and $p, q > 1$ with $1/p + 1/q = 1$, we have

$$\begin{aligned} & \left| \frac{\omega\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \omega - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right. \\ & \quad \left. - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1) \times \left[\rho_1^{1/p} \left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\ & \quad \left. + \rho_2^{1/p} \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \end{aligned} \tag{27}$$

where

$$\rho_1 = \int_0^1 |1 - \omega - \kappa|^p d\kappa = \frac{(1 - \omega)^{p+1} + \omega^{p+1}}{p + 1},$$

$$\rho_2 = \int_0^1 |\omega - \kappa|^p d\kappa = \frac{(1 - \omega)^{p+1} + \omega^{p+1}}{p + 1},$$

$$\begin{aligned} C'_1 &= \frac{1}{2n} \sum_{i=1}^n \frac{\beta(1, 1)}{(2\vartheta_1)^{2q}} {}_2F_1\left(2q, 1; 2; 1 - \frac{\vartheta_2}{\vartheta_1}\right) \\ &\quad - \frac{1}{2n} \sum_{i=1}^n s^i \frac{\beta(1, i + 1)}{(2\vartheta_1)^{2q}} {}_2F_1\left(2q, 1; i + 2; 1 - \frac{\vartheta_2}{\vartheta_1}\right), \end{aligned}$$

$$\begin{aligned} C'_2 &= \frac{1}{2n} \sum_{i=1}^n \frac{\beta(1, 1)}{(\vartheta_1 + \vartheta_2)^{2q}} {}_2F_1\left(2q, 1; 2; \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 + \vartheta_2}\right) \\ &\quad - \frac{1}{2n} \sum_{i=1}^n s^i \frac{\beta(i + 1, 1)}{(2\vartheta_1)^{2q}} {}_2F_1 \\ &\quad \cdot \left(2q, i + 1; i + 2; \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 + \vartheta_2}\right), \end{aligned}$$

$$\begin{aligned} C'_3 &= \frac{1}{n} \sum_{i=1}^n \frac{(\beta(1, 1))}{(2\vartheta_2)^{2q}} {}_2F_1\left(2q, 1; 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right) \\ &\quad - \frac{1}{2n} \sum_{i=1}^n s^i \frac{\beta(i + 1, i + 2)}{(2\vartheta_2)^{2q}} {}_2F_1 \\ &\quad \cdot \left(2q, i + 1; i + 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right), \end{aligned}$$

$$\begin{aligned} C'_4 &= \frac{1}{2n} \sum_{i=1}^n \frac{\beta(1, 1)}{(2\vartheta_2)^{2q}} {}_2F_1\left(2q, 1; 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right) \\ &\quad - \frac{1}{2n} \sum_{i=1}^n s^i \frac{\beta(1, i + 1)}{(2\vartheta_1)^{2q}} {}_2F_1 \\ &\quad \cdot \left(2q, 1; i + 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right). \end{aligned} \tag{28}$$

and $A_{\vartheta_1, \vartheta_2}, B_{\vartheta_1, \vartheta_2}$ are defined from (25).

Proof. From Lemma 22, Hölder's inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$, and properties of modulus, we have

$$\begin{aligned}
 & \left| \frac{\omega\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \omega - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right. \\
 & \quad \left. - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\
 & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \left[\int_0^1 \left| \frac{4(1 - \omega - \kappa)}{((1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1)^2} \right| |\psi'| \right. \\
 & \quad \cdot \left(\frac{2\vartheta_1\vartheta_2}{(1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1} \right) |d\kappa + \int_0^1 \left| \frac{4(\omega - \kappa)}{(\kappa\vartheta_1 + (2 - \kappa)\vartheta_2)^2} \right| |\psi'| \\
 & \quad \cdot \left(\frac{2\vartheta_1\vartheta_2}{\kappa\vartheta_1 + (2 - \kappa)\vartheta_2} \right) |d\kappa] \\
 & \leq \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1) \left\{ \left(\int_0^1 |1 - \omega - \kappa|^p d\kappa \right)^{1/p} \right. \\
 & \quad \times \left[\int_0^1 \frac{1}{A_{\vartheta_1, \vartheta_2}^{2q}} \left(\frac{1}{2n} \sum_{i=1}^n [1 - (s(1 - (1 - \kappa)))]^i \right) |\psi'(\vartheta_1)|^q \right. \\
 & \quad \left. + \frac{1}{2n} \sum_{i=1}^n [1 + (s\kappa)]^i |\psi'(\vartheta_2)|^q \right] d\kappa \Bigg]^{1/q} \\
 & \quad + \left(\int_0^1 |\omega - \kappa|^p d\kappa \right)^{1/p} \times \left[\int_0^1 \frac{1}{B_{\vartheta_1, \vartheta_2}^{2q}} \right. \\
 & \quad \cdot \left(\frac{1}{2n} \sum_{i=1}^n [2 - (s\kappa)]^i |\psi'(\vartheta_1)|^q + \frac{1}{2n} \sum_{i=1}^n \right. \\
 & \quad \left. \left. \cdot [1 - (s(1 - \kappa))]^i |\psi'(\vartheta_2)|^q \right) d\kappa \right]^{1/q} \Bigg\} \\
 & = \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1) \times \left[\rho_1^{1/p} \left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\
 & \quad \left. + \rho_2^{1/p} \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \tag{29}
 \end{aligned}$$

which completes the proof.

Corollary 24. Taking $n = 1$ in Theorem 23, then

$$\begin{aligned}
 & \left| \frac{\omega\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \omega - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right. \\
 & \quad \left. - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\
 & \leq \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1) \times \left[\rho_1^{1/p} \left(D'_1 |\psi'(\vartheta_1)|^q + D'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\
 & \quad \left. + \rho_2^{1/p} \left(D'_3 |\psi'(\vartheta_1)|^q + D'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \tag{30}
 \end{aligned}$$

where

$$\begin{aligned}
 D'_1 &= \frac{1}{2} \frac{\beta(1, 1)}{(2\vartheta_1)^{2q}} F_1\left(2q, 1; 2; 1 - \frac{\vartheta_2}{\vartheta_1}\right) \\
 & \quad - \frac{1}{2} s \frac{\beta(1, 2)}{(2\vartheta_1)^{2q}} F_1\left(2q, 1; 3; 1 - \frac{\vartheta_2}{\vartheta_1}\right), \\
 D'_2 &= \frac{1}{2} \frac{\beta(1, 1)}{(\vartheta_1 + \vartheta_2)^{2q}} F_1\left(2q, 1; 2; \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 + \vartheta_2}\right) \\
 & \quad - \frac{1}{2} s \frac{\beta(2, 1)}{(2\vartheta_1)^{2q}} F_1\left(2q, 2; 3; \frac{\vartheta_2 - \vartheta_1}{\vartheta_1 + \vartheta_2}\right), \\
 D'_3 &= \frac{\beta(1, 1)}{(2\vartheta_2)^{2q}} F_1\left(2q, 1; 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right) \\
 & \quad - \frac{1}{2} s \frac{\beta(2, 3)}{(2\vartheta_2)^{2q}} F_1\left(2q, 2; 3; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right), \\
 D'_4 &= \frac{1}{2} \frac{\beta(1, 1)}{(2\vartheta_2)^{2q}} F_1\left(2q, 1; 2; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right) \\
 & \quad - \frac{1}{2} s \frac{\beta(1, 2)}{(2\vartheta_1)^{2q}} F_1\left(2q, 1; 3; \frac{1}{2} \left(1 - \frac{\vartheta_2}{\vartheta_1}\right)\right). \tag{31}
 \end{aligned}$$

Corollary 25. Taking $\omega = \omega$ in Theorem 23, then

$$\begin{aligned}
 & \left| \omega \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + (1 - \omega) \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right. \\
 & \quad \left. - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\
 & \leq \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1) \rho^{1/p} \times \left[\left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\
 & \quad \left. + \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \tag{32}
 \end{aligned}$$

where $\rho_1 = \rho_2 = \rho$.

Corollary 26. Choosing $\omega = \omega = 0$ in Theorem 23, then

$$\begin{aligned}
 & \left| \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{2\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\
 & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{\sqrt[p]{p+1}} \times \left[\left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\
 & \quad \left. + \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right]. \tag{33}
 \end{aligned}$$

Corollary 27. Choosing $\bar{\omega} = \omega = 1/2$ in Theorem 23, then

$$\begin{aligned} & \left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{2\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)^p \sqrt[p+1]{\frac{4}{p+1}} \times \left[\left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\ & \quad \left. + \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right]. \end{aligned} \quad (34)$$

Corollary 28. Taking $\bar{\omega} = \omega = 1/3$ in Theorem 23, then

$$\begin{aligned} & \left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + 2\psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{3\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq 3\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)^p \sqrt[p+1]{4 \left(\frac{(2/3)^{p+1} + (1/3)^{p+1}}{p+1} \right)} \\ & \quad \times \left[\left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\ & \quad \left. + \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right]. \end{aligned} \quad (35)$$

Corollary 29. Taking $\bar{\omega} = \omega = 1$ in Theorem 23, then

$$\begin{aligned} & \left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{\sqrt[p+1]{p+1}} \times \left[\left(C'_1 |\psi'(\vartheta_1)|^q + C'_2 |\psi'(\vartheta_2)|^q \right)^{1/q} \right. \\ & \quad \left. + \left(C'_3 |\psi'(\vartheta_1)|^q + C'_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right]. \end{aligned} \quad (36)$$

Theorem 30. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function such that $\psi' \in L[\vartheta_1, \vartheta_2]$ and $\bar{\omega}, \omega \in [0, 1]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, then for $s \in [0, 1]$ and $p, q > 1$ with $1/p + 1/q = 1$, we have

$$\begin{aligned} & \left| \frac{\bar{\omega}\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \bar{\omega} - \omega}{2} \psi\left(\frac{\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p} \right. \\ & \quad \cdot \left(C'_5 |\psi'(\vartheta_1)|^q + C'_6 |\psi'(\vartheta_2)|^q \right)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \\ & \quad \left. \cdot \left(C'_7 |\psi'(\vartheta_1)|^q + C'_8 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \end{aligned} \quad (37)$$

where

$$\begin{aligned} C'_5 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - \bar{\omega} - \omega|^q [1 - (s(1 - (1 - \kappa)))^i]^i d\kappa, C'_6 \\ &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - \bar{\omega} - \omega|^q [1 + (s\kappa)^i]^i d\kappa, \\ C'_7 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |\omega - \kappa|^q [2 - (s\kappa)^i] d\kappa, C'_8 \\ &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |\omega - \kappa|^q [1 - (s(1 - \kappa))^i]^i d\kappa. \end{aligned} \quad (38)$$

Proof. Applying Lemma 22, Hölder's inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$, and properties of modulus, we have

$$\begin{aligned} & \left| \frac{\bar{\omega}\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \bar{\omega} - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \left[\int_0^1 \left| \frac{4(1 - \bar{\omega} - \kappa)}{((1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1)^2} \right| \right. \\ & \quad \cdot \left| \psi'\left(\frac{2\vartheta_1\vartheta_2}{(1 - \kappa)\vartheta_2 + (1 + \kappa)\vartheta_1}\right) \right| d\kappa + \int_0^1 \left| \frac{4(\omega - \kappa)}{(t\vartheta_1 + (2 - \kappa)\vartheta_2)^2} \right| \\ & \quad \cdot \left| \psi'\left(\frac{2\vartheta_1\vartheta_2}{\kappa\vartheta_1 + (2 - \kappa)\vartheta_2}\right) \right| d\kappa \Big] \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \left\{ 4 \left(\int_0^1 \frac{1}{A_{\vartheta_1, \vartheta_2}^{2p}} d\kappa \right)^{1/p} \right. \\ & \quad \times \left[\int_0^1 |1 - \bar{\omega} - \omega|^q \left(\frac{1}{2n} \sum_{i=1}^n [1 - (s(1 - (1 - \kappa)))^i]^i |\psi'(\vartheta_1)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{2n} \sum_{i=1}^n [1 + (s\kappa)^i]^i |\psi'(\vartheta_2)|^q \right) d\kappa \right]^{1/q} + 4 \left(\int_0^1 \frac{1}{B_{\vartheta_1, \vartheta_2}^{2p}} d\kappa \right)^{1/p} \\ & \quad \times \left[\int_0^1 |\omega - \kappa|^q \left(\frac{1}{2n} \sum_{i=1}^n [2 - (s\kappa)^i]^i |\psi'(\vartheta_1)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{2n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i]^i |\psi'(\vartheta_2)|^q \right) d\kappa \right]^{1/q} \Big\} \\ & = \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p} \right. \\ & \quad \cdot \left(C'_5 |\psi'(\vartheta_1)|^q + C'_6 |\psi'(\vartheta_2)|^q \right)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \\ & \quad \left. \cdot \left(C'_7 |\psi'(\vartheta_1)|^q + C'_8 |\psi'(\vartheta_2)|^q \right)^{1/q} \right], \end{aligned} \quad (39)$$

which completes the proof.

Corollary 31. Taking $n = 1$ in Theorem 30, then

$$\begin{aligned} & \left| \frac{\omega\psi(\vartheta_1) + \omega\psi(\vartheta_2)}{2} + \frac{2 - \omega - \omega}{2} \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p} \right. \\ & \quad \cdot (D'_5|\psi'(\vartheta_1)|^q + D'_6|\psi'(\vartheta_2)|^q)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \\ & \quad \cdot (D'_7|\psi'(\vartheta_1)|^q + D'_8|\psi'(\vartheta_2)|^q)^{1/q} \Big], \end{aligned} \tag{40}$$

where

$$\begin{aligned} D'_5 &= \frac{1}{2} \int_0^1 |1 - \omega - \omega|^q [1 - (s(1 - (1 - \kappa)))] d\kappa, \\ D'_6 &= \frac{1}{2} \int_0^1 |1 - \omega - \omega|^q [1 + (s\kappa)] d\kappa, \\ D'_7 &= \frac{1}{2} \int_0^1 |\omega - \kappa|^q [2 - (s\kappa)] d\kappa, \quad D'_8 \\ &= \frac{1}{2} \int_0^1 |\omega - \kappa|^q [1 - (s(1 - \kappa))] d\kappa. \end{aligned} \tag{41}$$

Corollary 32. Taking $\bar{\omega} = \omega$ in Theorem 30, then

$$\begin{aligned} & \left| \omega \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + (1 - \omega) \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p} \right. \\ & \quad \cdot (E'_1|\psi'(\vartheta_1)|^q + E'_2|\psi'(\vartheta_2)|^q)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \\ & \quad \cdot (E'_3|\psi'(\vartheta_1)|^q + E'_4|\psi'(\vartheta_2)|^q)^{1/q} \Big], \end{aligned} \tag{42}$$

where

$$\begin{aligned} E'_1 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - 2\omega|^q [1 - (s(1 - (1 - \kappa)))^i] d\kappa, \\ E'_2 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - 2\omega|^q [1 + (s\kappa)^i] d\kappa, \\ E'_3 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |\omega - \kappa|^q [2 - (s\kappa)^i] d\kappa, \\ E'_4 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 |\omega - \kappa|^q [1 - (s(1 - \kappa))^i] d\kappa. \end{aligned} \tag{43}$$

Corollary 33. Taking $\bar{\omega} = \omega = 0$ in Theorem 30, then

$$\begin{aligned} & \left| \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p} \right. \\ & \quad \cdot (E'_5|\psi'(\vartheta_1)|^q + E'_6|\psi'(\vartheta_2)|^q)^{1/q} \\ & \quad + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \\ & \quad \cdot (E'_7|\psi'(\vartheta_1)|^q + E'_8|\psi'(\vartheta_2)|^q)^{1/q} \Big], \end{aligned} \tag{44}$$

where

$$\begin{aligned} E'_5 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 [1 - (s(1 - (1 - \kappa)))^i] d\kappa \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\frac{i + 1 - s^i}{i + 1} \right], \\ E'_6 &= E'_6 = \frac{1}{2n} \sum_{i=1}^n \int_0^1 [1 + (s\kappa)^i] d\kappa \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\frac{i + 1 + s^i}{i + 1} \right], \\ E'_7 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 \kappa^q [2 - (s\kappa)^i] d\kappa \\ &= \frac{1}{2n} \sum_{i=1}^n \left[\frac{2}{q + 1} - \frac{s^i}{q + i + 1} \right], \\ E'_8 &= \frac{1}{2n} \sum_{i=1}^n \int_0^1 \kappa^q [1 - (s(1 - \kappa))^i] d\kappa. \end{aligned} \tag{45}$$

Corollary 34. Choosing $\bar{\omega} = \omega = 1/2$ in Theorem 30, then

$$\begin{aligned} & \left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + \psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{2\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right| \\ & \leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{2} \times \left[\frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p} \right. \\ & \quad \cdot (E'_9|\psi'(\vartheta_1)|^q + E'_{10}|\psi'(\vartheta_2)|^q)^{1/q} \Big], \end{aligned} \tag{46}$$

where

$$E'_9 = \frac{1}{2^{q+1}n} \sum_{i=1}^n \int_0^1 |1 - 2\kappa|^q [2 - (s\kappa)^i] d\kappa,$$

$$E'_{10} = \frac{1}{2^{q+1}n} \sum_{i=1}^n \int_0^1 |1 - 2\kappa|^q [1 - (s(1 - \kappa))^i] d\kappa. \quad (47)$$

Corollary 35. *Choosing $\bar{\omega} = \omega = 1/3$ in Theorem 30, then*

$$\left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} + 2\psi\left(\frac{2\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}\right) - \frac{3\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right|$$

$$\leq \frac{3\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p}$$

$$\cdot (G'_1|\psi'(\vartheta_1)|^q + G'_2|\psi'(\vartheta_2)|^q)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p}$$

$$\cdot (G'_3|\psi'(\vartheta_1)|^q + G'_4|\psi'(\vartheta_2)|^q)^{1/q} \right], \quad (48)$$

where

$$G'_1 = \frac{1}{3^q 2n} \sum_{i=1}^n \int_0^1 [1 - (s(1 - (1 - \kappa)))^i] d\kappa$$

$$= \frac{1}{3^q 2n} \sum_{i=1}^n \left[\frac{i + 1 - s^i}{i + 1} \right],$$

$$G'_2 = \frac{1}{3^q 2n} \sum_{i=1}^n \int_0^1 [1 + (s\kappa)^i] d\kappa$$

$$= \frac{1}{3^q 2n} \sum_{i=1}^n \left[\frac{i + 1 + s^i}{i + 1} \right],$$

$$G'_3 = \frac{1}{3^q 2n} \sum_{i=1}^n \int_0^1 |1 - 3\kappa|^q [2 - (s\kappa)^i] d\kappa,$$

$$G'_4 = \frac{1}{3^q 2n} \sum_{i=1}^n \int_0^1 |1 - 3\kappa|^q [1 - (s(1 - \kappa))^i] d\kappa. \quad (49)$$

Corollary 36. *Taking $\bar{\omega} = \omega = 1$ in Theorem 30, then*

$$\left| \frac{\psi(\vartheta_1) + \psi(\vartheta_2)}{2} - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(x)}{x^2} dx \right|$$

$$\leq \frac{\vartheta_1\vartheta_2(\vartheta_2 - \vartheta_1)}{4} \times \left[\frac{4}{(\vartheta_1 + \vartheta_2)^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{\vartheta_1 + \vartheta_2}\right) \right)^{1/p}$$

$$\cdot (G'_5|\psi'(\vartheta_1)|^q + G'_6|\psi'(\vartheta_2)|^q)^{1/q} + \frac{1}{\vartheta_1^2} \left({}_2F_1\left(2p, 1; 2; \frac{\vartheta_1 - \vartheta_2}{2\vartheta_1}\right) \right)^{1/p}$$

$$\cdot (G'_7|\psi'(\vartheta_1)|^q + G'_8|\psi'(\vartheta_2)|^q)^{1/q} \right], \quad (50)$$

where

$$G'_5 = \frac{1}{2n} \sum_{i=1}^n \int_0^1 [1 - (s(1 - (1 - \kappa)))^i] d\kappa$$

$$= \frac{1}{2n} \sum_{i=1}^n \left[\frac{i + 1 - s^i}{i + 1} \right],$$

$$G'_6 = \frac{1}{2n} \sum_{i=1}^n \int_0^1 [1 + (s\kappa)^i] d\kappa = \frac{1}{2n} \sum_{i=1}^n \left[\frac{i + 1 + s^i}{i + 1} \right],$$

$$G'_7 = \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - \kappa|^q [2 - (s\kappa)^i] d\kappa,$$

$$G'_8 = \frac{1}{2n} \sum_{i=1}^n \int_0^1 |1 - \kappa|^q [1 - (s(1 - \kappa))^i] d\kappa. \quad (51)$$

6. Ostrowski Type Inequalities via n -Polynomial Harmonically s -Type Convex Functions

In this section, we are going to add some new Ostrowski type inequalities via this newly introduced definition namely n -polynomial harmonically s -type convex function. In order to obtain the results, we need the following lemma.

Lemma 37 (see [19]). *Suppose a mapping $\psi : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is differentiable on I° , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$. If $\psi' \in L[\vartheta_1, \vartheta_2]$, then the following inequality holds:*

$$\psi(x) - \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du$$

$$= \frac{\vartheta_1\vartheta_2}{\vartheta_2 - \vartheta_1} \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa\vartheta_1 + (1 - \kappa)x)^2} \psi' \right.$$

$$\cdot \left(\frac{\vartheta_1 x}{(\kappa\vartheta_1 + (1 - \kappa)x)} \right) d\kappa - (\vartheta_2 - x)^2$$

$$\cdot \left. \int_0^1 \frac{\kappa}{(\kappa\vartheta_2 + (1 - \kappa)x)^2} \psi' \left(\frac{\vartheta_2 x}{(\kappa\vartheta_2 + (1 - \kappa)x)} \right) d\kappa \right\}, \quad (52)$$

for each $x \in [\vartheta_1, \vartheta_2]$.

Theorem 38. *Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$ and $\psi' \in L[\vartheta_1, \vartheta_2]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, for $q \geq 1$ and $s \in [0, 1]$, then*

$$\begin{aligned} & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| \\ & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \times \left\{ (x - \vartheta_1)^2 \left(\Lambda_1 |\psi'(x)|^q + \Lambda_2 |\psi'(\vartheta_1)|^q \right)^{1/q} \right. \\ & \quad \left. + (\vartheta_2 - x)^2 \left(\Lambda_3 |\psi'(x)|^q + \Lambda_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right\}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} \Lambda_1 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(q+1, 1)}{x^{2q}} {}_2F_1 \left(2q, q+1; q+2; 1 - \frac{\vartheta_1}{x} \right) \\ & \quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(q+1, i+1)}{x^{2q}} {}_2F_1 \\ & \quad \cdot \left(2q, q+1; q+2; 1 - \frac{\vartheta_1}{x} \right), \end{aligned}$$

$$\begin{aligned} \Lambda_2 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(q+1, 1)}{x^{2q}} {}_2F_1 \left(2q, q+1; q+2; 1 - \frac{\vartheta_1}{x} \right) \\ & \quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(q+i+1, 1)}{x^{2q}} {}_2F_1 \\ & \quad \cdot \left(2q, q+i+1; q+i+2; 1 - \frac{\vartheta_1}{x} \right), \end{aligned}$$

$$\begin{aligned} \Lambda_3 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(q+1, 1)}{x^{2q}} {}_2F_1 \left(2q, q+1; q+2; 1 - \frac{\vartheta_2}{x} \right) \\ & \quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(q+1, i+1)}{x^{2q}} {}_2F_1 \\ & \quad \cdot \left(2q, q+1; q+i+2; 1 - \frac{\vartheta_2}{x} \right), \end{aligned}$$

$$\begin{aligned} \Lambda_4 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(q+1, 1)}{x^{2q}} {}_2F_1 \left(2q, q+1; q+2; 1 - \frac{\vartheta_2}{x} \right) \\ & \quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(q+i+1, 1)}{x^{2q}} {}_2F_1 \\ & \quad \cdot \left(2q, q+i+1; q+i+2; 1 - \frac{\vartheta_2}{x} \right). \end{aligned} \tag{54}$$

Proof. Using Lemma 37, power mean inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$ and properties of modulus, we have

$$\begin{aligned} & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| \\ & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) \right| d\kappa + (\vartheta_2 - x)^2 \right. \\ & \quad \cdot \int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right| d\kappa \left. \right\} \\ & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 1 d\kappa \right)^{1-1/q} \left(\int_0^1 \frac{\kappa^q}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} |\psi'| \right. \\ & \quad \cdot \left. \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right)^q d\kappa \right)^{1/q} + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 1 d\kappa \right)^{1-1/q} \\ & \quad \cdot \left(\int_0^1 \frac{\kappa^q}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right|^q d\kappa \right)^{1/q} \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \\ & \quad \cdot \left(\int_0^1 \frac{\kappa^q [1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_1)|^q]}{(t\vartheta_1 + (1 - t)x)^{2q}} d\kappa \right)^{1/q} \\ & \quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \\ & \quad \cdot \left(\int_0^1 \frac{\kappa^q [1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_2)|^q]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\ & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa^q [1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\ & \quad + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa^q [1 - (s\kappa)^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_1)|^q \left. \right)^{1/q} \\ & \quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa^q [1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\ & \quad + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa^q [1 - (s\kappa)^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_2)|^q \left. \right)^{1/q} \\ & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left\{ (x - \vartheta_1)^2 \left(\Lambda_1 |\psi'(x)|^q + \Lambda_2 |\psi'(\vartheta_1)|^q \right)^{1/q} \right. \\ & \quad \left. + (\vartheta_2 - x)^2 \left(\Lambda_3 |\psi'(x)|^q + \Lambda_4 |\psi'(\vartheta_2)|^q \right)^{1/q} \right\}, \end{aligned} \tag{55}$$

which completes the proof.

Theorem 39. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$ and $\psi' \in L[\vartheta_1, \vartheta_2]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, for $q \geq 1$ and $s \in [0, 1]$, then for all $x \in [\vartheta_1, \vartheta_2]$, one has

$$\begin{aligned} & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| \\ & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{2} \right)^{1-1/q} \\ & \quad \times \left\{ (x - \vartheta_1)^2 \left(\Lambda_5 |\psi'(x)|^q + \Lambda_6 |\psi'(\vartheta_1)|^q \right)^{1/q} \right. \\ & \quad \left. + (\vartheta_2 - x)^2 \left(\Lambda_7 |\psi'(x)|^q + \Lambda_8 |\psi'(\vartheta_2)|^q \right)^{1/q} \right\}, \end{aligned} \tag{56}$$

where

$$\begin{aligned} \Lambda_5 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(2, 1)}{x^{2q}} {}_2F_1 \left(2q, 2; 3; 1 - \frac{\vartheta_1}{x} \right) \\ & \quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(2, i+1)}{x^{2q}} {}_2F_1 \left(2q, 2; i+1; 1 - \frac{\vartheta_1}{x} \right), \end{aligned}$$

$$\begin{aligned} \Lambda_6 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(2, 1)}{x^{2q}} {}_2F_1\left(2q, 2; 3; 1 - \frac{\vartheta_1}{x}\right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(i+2, 1)}{x^{2q}} {}_2F_1\left(2q, i+2; i+3; 1 - \frac{\vartheta_1}{x}\right), \\ \Lambda_7 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(2, 1)}{x^{2q}} {}_2F_1\left(2q, 2; 3; 1 - \frac{\vartheta_2}{x}\right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(2, i+1)}{x^{2q}} {}_2F_1\left(2q, 2; i+1; 1 - \frac{\vartheta_2}{x}\right), \\ \Lambda_8 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(2, 1)}{x^{2q}} {}_2F_1\left(2q, 2; 3; 1 - \frac{\vartheta_2}{x}\right) \\ &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(i+2, 1)}{x^{2q}} {}_2F_1\left(2q, i+2; i+3; 1 - \frac{\vartheta_2}{x}\right). \end{aligned} \tag{57}$$

Proof. Using Lemma 37, power mean inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$ and properties of modulus, we have

$$\begin{aligned} \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \psi(u) du \right| &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \\ &\cdot \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) \right| d\kappa \right. \\ &\quad \left. + (\vartheta_2 - x)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right| d\kappa \right\} \\ &\leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \kappa d\kappa \right)^{1-1/q} \\ &\quad \cdot \left(\int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - t)x)} \right) \right|^q d\kappa \right)^{1/q} \\ &\quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \kappa d\kappa \right)^{1-1/q} \\ &\quad \cdot \left(\int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right|^q d\kappa \right)^{1/q} \\ &\leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{2} \right)^{1-1/q} \\ &\quad \cdot \left(\int_0^1 \frac{\kappa [1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/m) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_1)|^q]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\ &\quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{2} \right)^{1-1/q} \\ &\quad \cdot \left(\int_0^1 \frac{\kappa [1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/m) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_2)|^q]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \end{aligned}$$

$$\begin{aligned} &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{2} \right)^{1-1/q} \\ &\quad \times \left\{ (x - \vartheta_1)^2 \right. \\ &\quad \cdot \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa [1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\ &\quad \left. + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa [1 - (s\kappa)^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_1)|^q \right)^{1/q} \\ &\quad \left. + (\vartheta_2 - x)^2 \right. \\ &\quad \cdot \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa [1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\ &\quad \left. + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{\kappa [1 - (s\kappa)^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_2)|^q \right)^{1/q} \left. \right\} \\ &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{2} \right)^{1-1/q} \\ &\quad \times \left\{ (x - \vartheta_1)^2 \left(\Lambda_5 |\psi'(x)|^q + \Lambda_6 |\psi'(\vartheta_1)|^q \right)^{1/q} \right. \\ &\quad \left. + (\vartheta_2 - x)^2 \left(\Lambda_7 |\psi'(x)|^q + \Lambda_8 |\psi'(\vartheta_2)|^q \right)^{1/q} \right\}, \end{aligned} \tag{58}$$

which completes the proof.

Theorem 40. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$ and $\psi' \in L[\vartheta_1, \vartheta_2]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, for $q \geq 1$ and $s \in [0, 1]$, then for all $x \in [\vartheta_1, \vartheta_2]$, one has

$$\begin{aligned} \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left\{ \left((x - \vartheta_1) \left\{ \frac{1}{\vartheta_1} - \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} \right\} \right)^{1-1/q} \right. \\ &\quad \cdot \left(\Lambda_9 |\psi'(x)|^q + \Lambda_{10} |\psi'(\vartheta_1)|^q \right)^{1/q} \\ &\quad + \left((\vartheta_2 - x) \left\{ \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} - \frac{1}{\vartheta_2} \right\} \right)^{1-1/q} \\ &\quad \cdot \left(\Lambda_{11} |\psi'(x)|^q + \Lambda_{12} |\psi'(\vartheta_2)|^q \right)^{1/q} \left. \right\}, \end{aligned} \tag{59}$$

where

$$\begin{aligned}
 \Lambda_9 &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(1, 1)}{x^{2q}} {}_2F_1\left(2q, 1; 2; 1 - \frac{\vartheta_1}{x}\right) \\
 &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(1, i+2)}{x^{2q}} {}_2F_1\left(2q, 1; i+2; 1 - \frac{\vartheta_1}{x}\right), \\
 \Lambda_{10} &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(1, 1)}{x^{2q}} {}_2F_1\left(2q, 1; 2; 1 - \frac{\vartheta_1}{x}\right) \\
 &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(i+1, 1)}{x^{2q}} {}_2F_1\left(2q, i+1; i+2; 1 - \frac{\vartheta_1}{x}\right), \\
 \Lambda_{11} &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(1, 1)}{x^{2q}} {}_2F_1\left(2q, 2; 3; 1 - \frac{\vartheta_2}{x}\right) \\
 &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(1, i+2)}{x^{2q}} {}_2F_1\left(2q, 1; i+2; 1 - \frac{\vartheta_2}{x}\right), \\
 \Lambda_{12} &= \frac{1}{n} \sum_{i=1}^n \frac{\beta(1, 1)}{x^{2q}} {}_2F_1\left(2q, 2; 3; 1 - \frac{\vartheta_2}{x}\right) \\
 &\quad - \frac{1}{n} \sum_{i=1}^n s^i \frac{\beta(i+1, 1)}{x^{2q}} {}_2F_1\left(2q, i+1; i+2; 1 - \frac{\vartheta_2}{x}\right).
 \end{aligned} \tag{60}$$

Proof. Using Lemma 37, power mean inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$ and properties of modulus, we have

$$\begin{aligned}
 \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \psi(u) du \right| &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \\
 &\cdot \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) \right| d\kappa \right. \\
 &\quad \left. + (\vartheta_2 - x)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right| d\kappa \right\} \\
 &\leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} d\kappa \right)^{1-1/q} \\
 &\quad \cdot \left(\int_0^1 \frac{1}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) \right|^q d\kappa \right)^{1/q} \\
 &\quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} d\kappa \right)^{1-1/q} \\
 &\quad \cdot \left(\int_0^1 \frac{1}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right|^q d\kappa \right)^{1/q} \\
 &\leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{x - \vartheta_1} \left\{ \frac{1}{\vartheta_1} - \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} \right\} \right)^{1-1/q}
 \end{aligned}$$

$$\begin{aligned}
 &\times \left(\int_0^1 \frac{[1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_1)|^q]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\
 &\quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{\vartheta_2 - x} \left\{ \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} - \frac{1}{\vartheta_2} \right\} \right)^{1-1/q} \\
 &\quad \times \left(\int_0^1 \frac{[1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_2)|^q]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\
 &\leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{x - \vartheta_1} \left\{ \frac{1}{\vartheta_1} - \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} \right\} \right)^{1-1/q} \\
 &\quad \times \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{[1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\
 &\quad \left. + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{[1 - (s\kappa)^i]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_1)|^q \right)^{1/q} \\
 &\quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{\vartheta_2 - x} \left\{ \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} - \frac{1}{\vartheta_2} \right\} \right)^{1-1/q} \\
 &\quad \times \left(\frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{[1 - (s(1 - \kappa))^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(x)|^q \right. \\
 &\quad \left. + \frac{1}{n} \sum_{i=1}^n \int_0^1 \frac{[1 - (s\kappa)^i]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa |\psi'(\vartheta_2)|^q \right)^{1/q} \\
 &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \\
 &\quad \cdot \left\{ \left((x - \vartheta_1) \left\{ \frac{1}{\vartheta_1} - \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} \right\} \right)^{1-1/q} (\Lambda_9 |\psi'(x)|^q + \Lambda_{10} |\psi'(\vartheta_1)|^q)^{1/q} \right. \\
 &\quad \left. + \left((\vartheta_2 - x) \left\{ \frac{\ln x - \ln \vartheta_1}{x - \vartheta_1} - \frac{1}{\vartheta_2} \right\} \right)^{1-1/q} (\Lambda_{11} |\psi'(x)|^q + \Lambda_{12} |\psi'(\vartheta_2)|^q)^{1/q} \right\},
 \end{aligned} \tag{61}$$

which completes the proof.

Theorem 41. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$ and $\psi' \in L[\vartheta_1, \vartheta_2]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, for $q > 1$, $1/p + 1/q = 1$ and $s \in [0, 1]$, then

$$\begin{aligned}
 \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| &\leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{p+1} \right)^{1/p} \\
 &\quad \times \left\{ (x - \vartheta_1)^2 (\Lambda_9 |\psi'(x)|^q + \Lambda_{10} |\psi'(\vartheta_1)|^q)^{1/q} \right. \\
 &\quad \left. + (\vartheta_2 - x)^2 (\Lambda_{11} |\psi'(x)|^q + \Lambda_{12} |\psi'(\vartheta_2)|^q)^{1/q} \right\},
 \end{aligned} \tag{62}$$

where $\Lambda_9, \Lambda_{10}, \Lambda_{11}$, and Λ_{12} are defined in Theorem 40.

Proof. Using Lemma 37, Hölder’s inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$, and properties of modulus, we have

$$\begin{aligned}
 & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \psi(u) du \right| \\
 & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \times \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} |\psi'| \right. \\
 & \quad \cdot \left. \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) | d\kappa + (\vartheta_2 - x)^2 \right. \\
 & \quad \cdot \left. \int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} |\psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) | d\kappa \right\} \\
 & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \kappa d\kappa \right)^{1/p} \left(\int_0^1 \frac{1}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} |\psi'| \right. \\
 & \quad \cdot \left. \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right)^q | d\kappa \right)^{1/q} + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \kappa d\kappa \right)^{1/p} \\
 & \quad \cdot \left(\int_0^1 \frac{1}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} |\psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right)^q | d\kappa \right)^{1/q} \\
 & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{p+1} \right)^{1/p} \\
 & \quad \cdot \left(\int_0^1 \frac{[1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_1)|^q]}{(\kappa \vartheta_1 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\
 & \quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{p+1} \right)^{1/p} \\
 & \quad \cdot \left(\int_0^1 \frac{[1/n \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q + (1/n) \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_2)|^q]}{(\kappa \vartheta_2 + (1 - \kappa)x)^{2q}} d\kappa \right)^{1/q} \\
 & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{p+1} \right)^{1/p} \times \left\{ (x - \vartheta_1)^2 (\Lambda_9 |\psi'(x)|^q + \Lambda_{10} |\psi'(\vartheta_1)|^q)^{1/q} \right. \\
 & \quad \left. + (\vartheta_2 - x)^2 (\Lambda_{11} |\psi'(x)|^q + \Lambda_{12} |\psi'(\vartheta_2)|^q)^{1/q} \right\}, \tag{63}
 \end{aligned}$$

which completes the proof.

Theorem 42. Let $\psi : [\vartheta_1, \vartheta_2] \subseteq (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable function on I^o , where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$ and $\psi' \in L[\vartheta_1, \vartheta_2]$. If the function $|\psi'|^q$ is an n -polynomial harmonically s -type convex, for $q > 1$, $1/p + 1/q = 1$ and $s \in [0, 1]$, then

$$\begin{aligned}
 & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \frac{\psi(u)}{u^2} du \right| \\
 & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right] \right)^{1/q} \\
 & \quad \times \left\{ (x - \vartheta_1)^2 \left(\frac{\beta(p+1, 1)}{x^{2p}} {}_2F_1 \left(2p, p+1, p+2, 1 - \frac{\vartheta_1}{x} \right) \right) \right. \\
 & \quad \cdot \left(\left\{ |\psi'(x)|^q + |\psi'(\vartheta_1)|^q \right\} \right)^{1/q} + (\vartheta_2 - x)^2 \\
 & \quad \cdot \left(\frac{\beta(1, 1+p)}{\vartheta_2^{2p}} {}_2F_1 \left(2p, 1; p+2, 1 - \frac{x}{\vartheta_2} \right) \right) \\
 & \quad \cdot \left. \left(\left\{ |\psi'(x)|^q + |\psi'(\vartheta_2)|^q \right\} \right)^{1/q} \right\}. \tag{64}
 \end{aligned}$$

Proof. Using Lemma 37, Hölder’s inequality, n -polynomial harmonically s -type convexity of $|\psi'|^q$, and properties of modulus, we have

$$\begin{aligned}
 & \left| \psi(x) - \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \psi(u) du \right| \\
 & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left\{ (x - \vartheta_1)^2 \int_0^1 \frac{\kappa}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} |\psi'| \right. \\
 & \quad \cdot \left. \left(\frac{\vartheta_1 x}{(\kappa \vartheta_1 + (1 - \kappa)x)} \right) | d\kappa + (\vartheta_2 - x)^2 \right. \\
 & \quad \cdot \left. \int_0^1 \frac{\kappa}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} |\psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) | d\kappa \right\} \\
 & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\int_0^1 \frac{\kappa^p}{(\kappa \vartheta_1 + (1 - \kappa)x)^2} d\kappa \right)^{1/p} \\
 & \quad \cdot \left(\int_0^1 \left| \psi' \left(\frac{\vartheta_1 x}{(\kappa a + (1 - \kappa)x)} \right) \right|^q | d\kappa \right)^{1/q} + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \\
 & \quad \cdot \left(\int_0^1 \frac{\kappa^p}{(\kappa \vartheta_2 + (1 - \kappa)x)^2} d\kappa \right)^{1/p} \\
 & \quad \cdot \left(\int_0^1 \left| \psi' \left(\frac{\vartheta_2 x}{(\kappa \vartheta_2 + (1 - \kappa)x)} \right) \right|^q | d\kappa \right)^{1/q} \\
 & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{\beta(p+1, 1)}{x^{2p}} {}_2F_1 \left(2p, p+1, p+2, 1 - \frac{\vartheta_1}{x} \right) \right) \\
 & \quad \times \left(\int_0^1 \left[\frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q \right. \right. \\
 & \quad \left. \left. + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_1)|^q \right] d\kappa \right)^{1/q} \\
 & \quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{\beta(1, 1+p)}{\vartheta_2^{2p}} {}_2F_1 \left(2p, 1; p+2, 1 - \frac{x}{\vartheta_2} \right) \right) \\
 & \quad \times \left(\int_0^1 \left[\frac{1}{n} \sum_{i=1}^n [1 - (s(1 - \kappa))^i] |\psi'(x)|^q \right. \right. \\
 & \quad \left. \left. + \frac{1}{n} \sum_{i=1}^n [1 - (s\kappa)^i] |\psi'(\vartheta_2)|^q \right] d\kappa \right)^{1/q} \\
 & \leq \frac{\vartheta_1 \vartheta_2 (x - \vartheta_1)^2}{\vartheta_2 - \vartheta_1} \left(\frac{\beta(p+1, 1)}{x^{2p}} {}_2F_1 \left(2p, p+1, p+2, 1 - \frac{\vartheta_1}{x} \right) \right) \\
 & \quad \times \left(\sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right] \left\{ \frac{|\psi'(x)|^q + |\psi'(\vartheta_1)|^q}{n} \right\} \right)^{1/q} \\
 & \quad + \frac{\vartheta_1 \vartheta_2 (\vartheta_2 - x)^2}{\vartheta_2 - \vartheta_1} \left(\frac{\beta(1, 1+p)}{\vartheta_2^{2p}} {}_2F_1 \left(2p, 1; p+2, 1 - \frac{x}{\vartheta_2} \right) \right) \\
 & \quad \times \left(\sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right] \left\{ \frac{|\psi'(x)|^q + |\psi'(\vartheta_2)|^q}{n} \right\} \right)^{1/q} \\
 & \leq \frac{\vartheta_1 \vartheta_2}{\vartheta_2 - \vartheta_1} \left(\frac{1}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right] \right)^{1/q}
 \end{aligned}$$

$$\begin{aligned} & \times \left\{ (x - \vartheta_1)^2 \left(\frac{\beta(p+1, 1)}{x^{2p}} {}_2F_1 \left(2p, p+1, p+2, 1 - \frac{\vartheta_1}{x} \right) \right) \right. \\ & \cdot \left(\left\{ |\psi'(x)|^q + |\psi'(\vartheta_1)|^q \right\} \right)^{1/q} + (\vartheta_2 - x)^2 \\ & \cdot \left(\frac{\beta(1, 1+p)}{\vartheta_2^{2p}} {}_2F_1 \left(2p, 1; p+2, 1 - \frac{x}{\vartheta_2} \right) \right) \\ & \left. \cdot \left(\left\{ |\psi'(x)|^q + |\psi'(\vartheta_2)|^q \right\} \right)^{1/q} \right\}, \end{aligned} \tag{65}$$

which completes the proof.

7. Applications

In this section, we recall the following special means for two positive real numbers ϑ_1, ϑ_2 where $\vartheta_1 < \vartheta_2$:

- (1) The arithmetic mean

$$A = A(\vartheta_1, \vartheta_2) = \frac{\vartheta_1 + \vartheta_2}{2} \tag{66}$$

- (2) The geometric mean

$$G = G(\vartheta_1, \vartheta_2) = \sqrt{\vartheta_1 \vartheta_2} \tag{67}$$

- (3) The harmonic mean

$$H = H(\vartheta_1, \vartheta_2) = \frac{2\vartheta_1 \vartheta_2}{\vartheta_1 + \vartheta_2} \tag{68}$$

- (4) The logarithmic mean

$$L = L(\vartheta_1, \vartheta_2) = \frac{\vartheta_2 - \vartheta_1}{\ln \vartheta_2 - \ln \vartheta_1} \tag{69}$$

- (5) The p -logarithmic mean

$$L_p = L_p(\vartheta_1, \vartheta_2) = \left(\frac{\vartheta_2^{p+1} - \vartheta_1^{p+1}}{(p+1)(\vartheta_2 - \vartheta_1)} \right)^{1/p}, \quad p \in \mathbb{R} \setminus \{0\} \tag{70}$$

- (6) The identric mean

$$I = I(\vartheta_1, \vartheta_2) = \frac{1}{e} \left(\frac{\vartheta_2^{\vartheta_2}}{\vartheta_1^{\vartheta_1}} \right)^{1/\vartheta_2 - \vartheta_1} \tag{71}$$

These means have a lot of applications in areas and different type of numerical approximations. However, the following simple relationship are known in the literature.

$$H(\vartheta_1, \vartheta_2) \leq G(\vartheta_1, \vartheta_2) \leq L(\vartheta_1, \vartheta_2) \leq I(\vartheta_1, \vartheta_2) \leq A(\vartheta_1, \vartheta_2). \tag{72}$$

Proposition 43. Let $0 < \vartheta_1 < \vartheta_2$. Then, we get the following inequality

$$\begin{aligned} \frac{n}{2 \sum_{i=1}^n [1 - (s/2)^i]} H(\vartheta_1, \vartheta_2) & \leq \frac{G^2(\vartheta_1, \vartheta_2)}{L(\vartheta_1, \vartheta_2)} \\ & \leq A(\vartheta_1, \vartheta_2) \frac{2}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \tag{73}$$

Proof. Taking $\psi(x) = x$ for $x > 0$ in Theorem 18, then inequality (73) is easily captured.

Proposition 44. Let $0 < \vartheta_1 < \vartheta_2$. Then we get the following inequality

$$\begin{aligned} \frac{n}{2 \sum_{i=1}^n [1 - (s/2)^i]} H^2(\vartheta_1, \vartheta_2) & \leq G^2(\vartheta_1, \vartheta_2) \\ & \leq A(\vartheta_1^2, \vartheta_2^2) \frac{2}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \tag{74}$$

Proof. Taking $\psi(x) = x^2$ for $x > 0$ in Theorem 18, then inequality (74) is easily captured.

Proposition 45. Let $0 < \vartheta_1 < \vartheta_2$. Then, we get the following inequality

$$\begin{aligned} \frac{n}{2 \sum_{i=1}^n [1 - (s/2)^i]} H(\vartheta_1, \vartheta_2) & \leq I(\vartheta_1, \vartheta_2) \\ & \leq G(\vartheta_1, \vartheta_2) \frac{2}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \tag{75}$$

Proof. Taking $\psi(x) = \ln x$ for $x > 0$ in Theorem 18, then inequality (75) is easily captured.

Proposition 46. Let $0 < \vartheta_1 < \vartheta_2$. Then, we get the following inequality

$$\begin{aligned} & \frac{n}{2 \sum_{i=1}^n [1 - (s/2)^i]} H^2(\vartheta_1, \vartheta_2) \ln H(\vartheta_1, \vartheta_2) \\ & \leq G^2(\vartheta_1, \vartheta_2) \ln I(\vartheta_1, \vartheta_2) \\ & \leq A(\vartheta_1^2 \ln \vartheta_1, \vartheta_2^2 \ln \vartheta_2) \frac{2}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \quad (76)$$

Proof. Taking $\psi(x) = x^2 \ln x$ for $x > 0$ in Theorem 18, then inequality (76) is easily captured.

Proposition 47. Let $0 < \vartheta_1 < \vartheta_2$. Then, we get the following inequality

$$\begin{aligned} & \frac{n}{2 \sum_{i=1}^n [1 - (s/2)^i]} H^{p+2}(\vartheta_1, \vartheta_2) \leq G^2(\vartheta_1, \vartheta_2) L_p^p(\vartheta_1, \vartheta_2) \\ & \leq A(\vartheta_1^{p+2}, \vartheta_2^{p+2}) \frac{2}{n} \sum_{i=1}^n \left[\frac{i+1-s^i}{i+1} \right]. \end{aligned} \quad (77)$$

Proof. Taking $\psi(x) = x^{p+2}$ for $x > 0$ in Theorem 18, then inequality (77) is easily captured.

8. Conclusion

We have introduced and investigated some algebraic properties of a new class of functions namely n -polynomial harmonically s -type convex. We showed that this class of functions had some nice properties, which other convex functions had as well. We proved that our new class of n -polynomial harmonically s -type convex function is very larger with respect to the known class of functions, like n -polynomial convex and n -polynomial harmonically convex. A new version of Hermite–Hadamard type inequality and an integral identity for the differentiable functions are obtained. In recent years, many researchers put the effort into the theory of inequalities to bring a new dimension to mathematical analysis and applied mathematics with different features in the literature. Due to widespread views and applications, the theory of inequalities has become an attractive, interesting, and absorbing field for the researchers. It is high time to find the applications of these inequalities along with efficient numerical methods. We believe that our newly introduced idea and concept will have very deep research in this captivating field of analysis and inequalities. The amazing techniques and wonderful ideas of this article can be extended and generalized on the coordinates along with fractional integral calculus. Our aim in the future is that we will approach and continue our research work in this direction furthermore.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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