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Research Article

A New Subclass of Analytic Functions Related to Mittag-Leffler Type Poisson Distribution Series

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The object of this work is to an innovation of a class $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ in Y with negative coefficients, further determining coefficient estimates, neighborhoods, partial sums, convexity, and compactness of this specified class.

1. Introduction

Let $Y=\{\omega:|\omega|<1\}$ be an open unit disc in $\mathscr{C}$. Consider the analytic class function A that indicates j specified on the unit disk along with normalization

$$j(0)=0, j'(0)=1 \tag{1}$$

and has the form

$$j(\omega)=\omega+\sum_{n=2}^{\infty} o_n\omega^n, \tag{2}$$

indicated by S, the subclass of A lying of functions that are univalent in Y. A function $j\in A$ is stated in $k-\mathrm{UST}(\iota)$, and $k-\mathrm{UCV}(\iota)$, "the class of k-uniformly starlike functions and convex functions of order ι, $0\le\iota<1$," if and only if

$$\begin{aligned}&\Re\left\{\frac{\nu j'(\nu)}{j(\nu)}\right\}>k\left|\frac{\nu j'(\nu)}{j(\nu)}-1\right|+\iota,(k\ge 0),\\&\Re\left\{1+\frac{\nu j''(\nu)}{j'(\nu)}\right\}>k\left|\frac{\nu j''(\nu)}{j'(\nu)}\right|+\iota,(k\ge 0).\end{aligned} \tag{3}$$

The classes UCV and UST were introduced by Goodman [1] and studied by Ronning [2]. Due to Sakaguchi [3], the class ST_s of starlike functions w.r.t. symmetric points are defined as follows.

The function $j\in A$ is stated to be starlike w.r.t. symmetric points in Y

$$\Leftrightarrow\Re\left\{\frac{2\omega j'(\omega)}{j(\omega)-j(-\omega)}\right\}>0,(\omega\in Y). \tag{4}$$

Owa et al. [4] defined the class $\mathrm{ST}_s(\alpha,\varsigma)$ as complies

$$\Re\left\{\frac{(1-\varsigma)\omega j'(\omega)}{j(\omega)-j(\varsigma\omega)}\right\}>\alpha,(\omega\in Y), \tag{5}$$

where $0\le\alpha<1$, $|\varsigma|\le 1$, $\varsigma\ne 1$. Here, $\mathrm{ST}_s(0,-1)=\mathrm{ST}_s$ and $\mathrm{ST}_s(\alpha,-1)=\mathrm{ST}_s(\alpha)$ is named Sakaguchi function of order α.

In recent years, binomial distribution series, Pascal distribution series, Poisson distribution series, etc., play important role in GFT. The sufficient ways were innovated for ST, UCV for some special functions in the GFT. By the motivation of the works [5–13], we develop this work.

In [14], Porwal, Poisson distribution series, gives a gracious application on analytic functions; it exposed a new way of research in GFT. Subsequently, the authors turned on the distribution series of confluent hypergeometric, hypergeometric, binomial, and Pascal and prevail necessary and sufficient stipulation for certain classes of univalent functions.

Lately, Porwal and Dixit [15] innovate Mittag-Leffler type Poisson distribution and prevailed moments, mgf, which is an abstraction of Poisson distribution using the definition of this distribution. Bajpai [16] innovated Mittag-Leffler type Poisson distribution series and discussed about necessary and sufficient conditions.

The probability mass function for this is

$$P(\hbar, \tau, \upsilon ; n)(\omega) = \frac{\hbar^n}{E_{\tau,\upsilon}(\hbar)\Gamma(\tau n + \upsilon)}, (n = 0, 1, 2, \cdots), \quad (6)$$

where

$$E_{\tau,\upsilon}(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\tau n + \upsilon)}, (\tau, \upsilon \in \mathscr{C}, \mathfrak{R}(\tau) > 0, \mathfrak{R}(\upsilon) > 0). \quad (7)$$

The series (7) converges for all finite values of ω if $R(\tau) > 0, \mathfrak{R}(\upsilon) > 0$. This suggest that the series $E_{\tau,\upsilon}(\hbar)$ is convergent for $\tau, \upsilon, \hbar > 0$. For further details of the study, see [17]. It is easy to see that the series (7) are reduced to exponential series for $\tau = \upsilon = 1$.

A variable x is said to have Poisson distribution if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-\hbar}, \hbar e^{-\hbar}/1!, \hbar^2 e^{-\hbar}/2!, \hbar^3 e^{-\hbar}/3!, \cdots$, respectively, where $\hbar$ is called the parameter. Thus,

$$P(x = n) = \frac{\hbar^n e^{-\hbar}}{n!}, n = 0, 1, 2, \cdots. \quad (8)$$

This motivates researchers (see [15, 17, 18], etc.) to introduce a new probability distribution if it assumes nonnegative values and its probability mass function is given by (6). It is easy to see that $P(\hbar, \tau, \upsilon ; n)(\omega)$ given by (6) is the probability mass function because

$$P(\hbar, \tau, \upsilon ; n)(\omega) \geq 0, \sum_{n=0}^{\infty} P(\hbar, \tau, \upsilon ; n)(\omega) = 1. \quad (9)$$

It is worthy to note that for $\alpha = \beta = 1$, it reduces to the Poisson distribution.

Also note that

$$E_{\tau,\upsilon}(\omega) = \omega\Gamma(\upsilon)E_{\tau,\tau+\upsilon}(\omega). \quad (10)$$

In [18], Chakrabortya and Ong introduced and discussed about the Mittag-Leffler function distribution—a new generalization of hyper-Poisson distribution. The Mittag-Leffler type Poisson distribution series was innovated by Porwal and Dixit [15] and given as

$$K(\hbar, \tau, \upsilon)(\omega) = \omega + \sum_{n=2}^{\infty} \frac{\hbar^{n-1}}{\Gamma(\tau(\mathrm{n} - 1) + \upsilon)E_{\tau,\upsilon}(\hbar)}\omega^n. \quad (11)$$

Equation (11) is a normalization function in S, since $K(\hbar, \tau, \upsilon)(0) = 0$ and $K'(\hbar, \tau, \upsilon)(0) = 1$. After that, in [19], Porwal et al. discussed about the geometric properties of (11).

For $j \in A$ given by (2) and $l(\omega)$ given by

$$l(\omega) = \omega + \sum_{n=2}^{\infty} b_n \omega^n, \quad (12)$$

their convolution, indicated by $(j * l)$, is given by

$$(j * l)(\omega) = \omega + \sum_{n=2}^{\infty} o_n b_n \omega^n = (l * j)(\omega), (\omega \in Y). \quad (13)$$

Note that $j * l \in A$.

Next, we innovate the convolution operator

$$\mathscr{I}(\hbar, \tau, \upsilon)j(\omega) = K(\hbar, \tau, \upsilon) * j(\omega) = \omega + \sum_{n=2}^{\infty} \varphi_{\hbar}^n(\tau, \upsilon)o_n\omega^n, \quad (14)$$

where $\varphi_{\hbar}^n(\tau, \upsilon) = \hbar^{n-1}/(\Gamma(\tau(n-1) + \upsilon)E_{\tau,\upsilon}(\hbar))$.

Then, using linear operator $\mathscr{I}(\hbar, \tau, \upsilon)$, we exemplifier a contemporary subclass of functions in A.

Definition 1. If $j \in A$ is named in the class $k - \mathrm{UST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ if for all $\omega \in Y$

$$\mathfrak{R}\left\{\frac{(1-\varsigma)\omega(\mathscr{I}(\hbar, \tau, \upsilon)j(\omega))'}{\mathscr{I}(\hbar, \tau, \upsilon)j(\omega) - \mathscr{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)}\right\} \geq k\left|\frac{(1-\varsigma)\omega(\mathscr{I}(\hbar, \tau, \upsilon)j(\omega))'}{\mathscr{I}(\hbar, \tau, \upsilon)j(\omega) - \mathscr{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)} - 1\right| + \iota, \quad (15)$$

for $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \iota < 1$.

Moreover, we named that $j \in k - \mathrm{UST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is in the subclass $k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ if $j(\omega)$ is of the compiling form

$$j(\omega) = \omega - \sum_{n=2}^{\infty} o_n\omega^n, \quad o_n \geq 0, n \in \mathbb{N}, \omega \in Y. \quad (16)$$

In this work, we analyze the bounds for coefficient, partial sums, and some neighborhood outcomes of the class $k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$.

To claim our outcomes, we adopt lemmas [20].

Lemma 2. *Let w be a complex number. Then, $\alpha \leq \mathfrak{R}(w) \Leftrightarrow |w - (1 + \alpha)| \leq |w + (1 - \alpha)|$.*

Lemma 3. *Suppose a complex number w with real numbers α, ι. Then,*

$$\Re(w) > \alpha|w-1| + \iota \Leftrightarrow \Re\left\{w\left(1+\alpha e^{i\rho}\right) - \alpha e^{i\rho}\right\} > \iota, (-\pi < \rho \le \pi). \tag{17}$$

2. Coefficient Bounds

Theorem 4. *A function j given by (16) is in $k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$*

$$\Leftrightarrow \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)|o_n \le 1-\iota, \tag{18}$$

here $k \ge 0, |\varsigma| \le 1, \varsigma \ne 1, 0 \le \iota < 1$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$. The result is sharp for $j(\omega)$ is

$$j(\omega) = \omega - \frac{1-\iota}{\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid} \omega^n. \tag{19}$$

Proof. By Definition 1, we have

$$\Re\left\{\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))}{\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)}\left(1+ke^{i\rho}\right) - ke^{i\rho}\right\} \ge \iota, -\pi < \rho \le \pi. \tag{20}$$

Let $H(\omega) = (1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))(1+ke^{i\rho}) - ke^{i\rho}[\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)]$ and $K(\omega) = \mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)$.

By Lemma 2, (20) becomes

$$|H(\omega) + (1-\iota)K(\omega)| \ge |H(\omega) - (1+\iota)K(\omega)|, \text{for } 0 \le \iota < 1. \tag{21}$$

But

$$\begin{aligned} |H(\omega) + (1-\iota)K(\omega)| &= |(1-\varsigma) \\ &\cdot \left\{(2-\iota)\omega - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n + j_n(1-\iota))o_n\omega^n - ke^{i\rho}\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n - j_n)o_n\omega^n\right\}| \ge |1-\varsigma| \\ &\cdot \left\{(2-\iota) \mid \omega| - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n + j_n(1-\iota) \mid o_n \mid \omega^n| - k\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n \mid o_n \mid \omega^n \mid \right\}. \end{aligned} \tag{22}$$

Also

$$\begin{aligned} |H(\omega) - (1+\iota)K(\omega)| &= |(1-\varsigma) \\ &\cdot \left\{-\iota\omega - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n - j_n(1+\iota))o_n\omega^n - ke^{i\rho}\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)(n - j_n)o_n\omega^n\right\}| \le |1-\varsigma| \\ &\cdot \left\{\iota \mid \omega| + \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n(1+\iota) \mid o_n \mid \omega^n| + k\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon) \mid n - j_n \mid o_n \mid \omega^n \mid \right\}. \end{aligned} \tag{23}$$

So

$$\begin{aligned} &|H(\omega) + (1-\iota)K(\omega)| - |H(\omega) - (1+\iota)K(\omega)| \ge |1-\varsigma| \\ &\quad \times \left\{2(1-\iota) \mid \omega| - \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)[|n + j_n(1-\iota)| + |n - j_n(1+\iota)| + 2k \mid n - j_n \mid]o_n \mid \omega^n \mid \right\} \\ &\ge 2(1-\iota)|\omega| - \sum_{n=2}^{\infty} 2\varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)|o_n|\omega^n| \ge 0. \end{aligned} \tag{24}$$

Conversely, suppose (18) holds. Then, we have

$$\Re\left\{\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar, \tau, \upsilon)j(\omega))\left(1+ke^{i\rho}\right) - ke^{i\rho}[\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)]}{\mathcal{I}(\hbar, \tau, \upsilon)j(\omega) - \mathcal{I}(\hbar, \tau, \upsilon)j(\varsigma\omega)}\right\} \ge \iota. \tag{25}$$

Opting ω values on the +ve real axis, where $0 \le |\omega| = r < 1$, then

$$\Re\left\{\frac{(1-\iota) - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)\left[n\left(1+ke^{i\rho}\right) - j_n\left(\iota + ke^{i\rho}\right)\right]o_n\omega^{n-1}}{1 - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)j_n o_n\omega^{n-1}}\right\} \ge 0. \tag{26}$$

Since $\Re(-e^{i\rho}) \ge -|e^{i\rho}| = -1$, then

$$\Re\left\{\frac{(1-\iota) - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)[n(1+k) - j_n(\iota+k)]o_n r^{n-1}}{1 - \sum_{n=2}^{\infty}\varphi_{\hbar}^{n}(\tau, \upsilon)j_n o_n r^{n-1}}\right\} \ge 0. \tag{27}$$

Taking limit r tends to 1^-, we obtain our needed result.

Corollary 5. *If $j(\omega) \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$, then*

$$o_n \le \frac{1}{(1-\iota)^{-1}\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid}, \tag{28}$$

where $k \ge 0, |\varsigma| \le 1, \varsigma \ne 1, 0 \le \iota < 1$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$.

3. Neighborhood Properties

The notion of β-neighbourhood was innovated and studied by Goodman [21] and Ruscheweyh [22].

Definition 6. We define the β-neighborhood of a mapping $j \in A$ and indicate by $N_\beta(j)$ lying of all mappings $g(\omega) = \omega - \sum_{n=2}^{\infty} b_n\omega^n \in S(b_n \ge 0, n \in \mathbb{N})$ satisfies the condition

$$\sum_{n=2}^{\infty} \frac{\varphi_{\hbar}^{n}(\tau, \upsilon) \mid n(k+1) - j_n(k+\iota) \mid}{1-\iota}|o_n - b_n| \le 1-\beta, \tag{29}$$

where $k \ge 0, |\varsigma| \le 1, \varsigma \ne 1, 0 \le \iota < 1, \beta \ge 0$ and $j_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$.

Theorem 7. *Let $j(\omega) \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ and every real ρ we get $\iota(e^{i\rho} - 1) - 2e^{i\rho} \ne 0$. For any $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \beta(\beta \ge 0)$, if j fulfills*

$$\frac{j(\omega) + \varepsilon\omega}{1+\varepsilon} \in k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma), \tag{30}$$

then, $N_\beta(j) \subset k-\tilde{U}ST_s(\hbar, \upsilon, \tau, \iota, \varsigma)$.

Proof. It is evident that $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$

$$\Leftrightarrow \left| \frac{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota\right)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))}{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))(1+ke^{i\rho}) + (1-ke^{i\rho}-\iota)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))} \right| < 1, \tag{31}$$

where $-\pi \le \rho \le \pi$ for some $s \in \mathbb{C}$ and $|s| = 1$, we obtain

$$\frac{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota\right)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))}{(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))(1+ke^{i\rho}) + (1-ke^{i\rho}-\iota)(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega))} \neq s. \tag{32}$$

In other words,

$$\begin{aligned} &(1-s)(1-\varsigma)\omega(\mathcal{I}(\hbar,\tau,\upsilon)j(\omega))\left(1+ke^{i\rho}\right) - \left(ke^{i\rho}+1+\iota+s\left(-1+ke^{i\rho}+\iota\right)\right) \\ &\quad \times (\mathcal{I}(\hbar,\tau,\upsilon)j(\omega) - \mathcal{I}(\hbar,\tau,\upsilon)j(\varsigma\omega)) \neq 0 \Rightarrow \omega \\ &\quad - \sum_{n=2}^{\infty} \frac{\varphi_\hbar^n(\tau,\upsilon)\left((n-j_n)\left(1+ke^{i\rho}-ske^{i\rho}\right) - s(n+j_n) - j_n\iota(1-s)\right)}{\iota(s-1)-2s}\omega^n \neq 0. \end{aligned} \tag{33}$$

However, $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma) \Leftrightarrow (j * h)/\omega \neq 0, \omega \in Y - \{0\}$, where $h(\omega) = \omega - \sum_{n=2}^{\infty} c_n \omega^n$ and

$$\begin{aligned} c_n &= \frac{\varphi_\hbar^n(\tau,\upsilon)\left((n-j_n)\left(1+ke^{i\rho}-ske^{i\rho}\right) - s(n+j_n) - j_n\iota(1-s)\right)}{\iota(s-1)-2s} \Rightarrow |c_n| \\ &\le \frac{\varphi_\hbar^n(\tau,\upsilon)|n(1+k) - j_n(k+\iota)|}{1-\iota}, \end{aligned} \tag{34}$$

since $((j(\omega)+\varepsilon\omega)/(1+\varepsilon)) \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$; therefore, $\omega^{-1}((j(\omega)+\varepsilon\omega)/(1+\varepsilon) * h(\omega)) \neq 0$, which implies

$$\frac{(j*h)(\omega)}{(1+\varepsilon)\omega} + \frac{\varepsilon}{1+\varepsilon} \neq 0. \tag{35}$$

Now, suppose $|(j*h)(\omega)/\omega| < \beta$. Then, by (35),

$$\left| \frac{(j*h)(\omega)}{(1+\varepsilon)\omega} + \frac{\varepsilon}{1+\varepsilon} \right| > \frac{|\varepsilon| - \beta}{|1+\varepsilon|} \ge 0, \tag{36}$$

which contradicts by $|\varepsilon| < \beta$, and thus, we arrive $|(j*h)(\omega)/\omega| \ge \beta$.

If $g(\omega) = \omega - \sum_{2 \le n \le \infty} b_n \omega^n \in N_\beta(j)$, then

$$\begin{aligned} \beta - \left| \frac{(g*h)(\omega)}{\omega} \right| &\le \left| \frac{((j-g)*h)(\omega)}{\omega} \right| \\ &< \sum_{n=2}^{\infty} \frac{\varphi_\hbar^n(\tau,\upsilon)\,|\,n(1+k) - j_n(k+\iota)\,|}{1-\iota} |o_n - b_n| \le \beta. \end{aligned} \tag{37}$$

4. Partial Sums

Theorem 8. *If the function j is of the form (2) fulfill (18) then*

$$\Re\left\{\frac{j(\omega)}{j_m(\omega)}\right\} \ge 1 - \frac{1}{\chi_{m+1}}, \tag{38}$$

$$\chi_n = \begin{cases} 1, & \text{if } 2 \le n \le m; \\ \chi_{m+1}, & \text{if } m+1 \le n \le \infty, \end{cases} \tag{39}$$

where

$$\chi_n = \frac{\varphi_\hbar^n(\tau,\upsilon)\,|\,n(1+k) - j_n(k+\iota)\,|}{1-\iota}. \tag{40}$$

The estimate (38) is sharp, for every m, with

$$j(\omega) = \omega + \frac{\omega^{m+1}}{\chi_{m+1}}. \tag{41}$$

Proof. Now, we define $\wp$; we can define

$$\begin{aligned} \frac{1+\wp(\omega)}{1-\wp(\omega)} &= \chi_{m+1}\left\{\frac{j(\omega)}{j_m(\omega)} - \left(1 - \frac{1}{\chi_{m+1}}\right)\right\} \\ &= \left[\frac{1 + \sum_{2 \le n \le m} o_n \omega^{n-1} + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n \omega^{n-1}}{1 + \sum_{n=2}^{m} o_n \omega^{n-1}}\right]. \end{aligned} \tag{42}$$

Then, from (42), we attain

$$\begin{aligned} \wp(\omega) &= \frac{\chi_{m+1}\sum_{n=m+1}^{\infty} o_n \omega^{n-1}}{2 + 2\sum_{n=2}^{m} o_n \omega^{n-1} + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n \omega^{n-1}}, \\ |\wp(\omega)| &\le \frac{\chi_{m+1}\sum_{n=m+1}^{\infty} o_n}{2 - 2\sum_{n=2}^{m} o_n - \chi_{m+1}\sum_{n=m+1}^{\infty} o_n}. \end{aligned} \tag{43}$$

Now, $|\wp(\omega)| \leq 1$ if

$$2\chi_{m+1}\sum_{n=m+1}^{\infty} o_n \leq 2-2\sum_{n=2}^{m} o_n \Rightarrow \sum_{n=2}^{m} o_n + \chi_{m+1}\sum_{m+1}^{\infty} o_n \leq 1. \tag{44}$$

It is enough to prove that the LHS of (44) is bounded above by $\sum_{n=2}^{\infty}\chi_n o_n$, which implies

$$\sum_{n=2}^{m}(\chi_n - 1)o_n + \sum_{n=m+1}^{\infty}(\chi_n - \chi_{m+1})o_n \geq 0. \tag{45}$$

To show that the mapping disposed by (41) gives the exact result, we notice that for $\omega = re^{i\pi/n}$,

$$\frac{j(\omega)}{j_m(\omega)} = 1 + \frac{\omega^m}{\chi_{m+1}}. \tag{46}$$

Taking limit ω tends to 1^-, we have

$$\frac{j(\omega)}{j_m(\omega)} = 1 - \frac{1}{\chi_{m+1}}. \tag{47}$$

Hence, the proof is completed.

Theorem 9. *If j of the form (2) which fulfill (18) then*

$$\Re\left\{\frac{j_m(\omega)}{j(\omega)}\right\} \geq \frac{\chi_{m+1}}{1+\chi_{m+1}}. \tag{48}$$

The result is sharp with (41).

Proof. Define

$$\begin{aligned}\frac{1+\wp(\omega)}{1-\wp(\omega)} &= (1+\chi_{m+1})\left\{\frac{j_m(\omega)}{j(\omega)} - \frac{\chi_{m+1}}{1+\chi_{m+1}}\right\}\\ &= \left[\frac{1+\sum_{n=2}^{m} o_n\omega^{n-1} - \chi_{m+1}\sum_{n=m+1}^{\infty} o_n\omega^{n-1}}{1+\sum_{n=2}^{\infty} o_n\omega^{n-1}}\right],\end{aligned} \tag{49}$$

where

$$\begin{aligned}\wp(\omega) &= \frac{(1+\chi_{m+1})\sum_{n=m+1}^{\infty} o_n\omega^{n-1}}{-(2+2\sum_{n=2}^{m} o_n\omega^{n-1} - (1-\chi_{m+1})\sum_{n=m+1}^{\infty} o_n\omega^{n-1})},\\ |\wp(\omega)| &\leq \frac{(1+\chi_{m+1})\sum_{n=m+1}^{\infty} o_n}{2-2\sum_{n=2}^{m} o_n + (1-\chi_{m+1})\sum_{n=m+1}^{\infty} o_n} \leq 1.\end{aligned} \tag{50}$$

This last inequality is

$$\sum_{n=2}^{m} o_n + \chi_{m+1}\sum_{n=m+1}^{\infty} o_n \leq 1. \tag{51}$$

It is enough to prove that the LHS of (51) is bounded above by $\sum_{n=2}^{\infty}\chi_n o_n$, which implies

$$\sum_{2\leq n\leq m}(\chi_n - 1)o_n + \sum_{n=m+1}^{\infty}(\chi_n - \chi_{m+1})o_n \geq 0. \tag{52}$$

This completes the proof.

Theorem 10. *If j of the form (2) fulfill (18) then*

$$\Re\left\{\frac{j'(\omega)}{j_m'(\omega)}\right\} \geq 1 - \frac{m+1}{\chi_{m+1}}, \tag{53}$$

$$\Re\left\{\frac{j_m'(\omega)}{j'(\omega)}\right\} \geq \frac{\chi_{m+1}}{1+m+\chi_{m+1}}, \tag{54}$$

where

$$\chi_n \geq \left\{\begin{array}{ll} 1, & \text{if } 1\leq n\leq m \\ n\dfrac{\chi_{m+1}}{m+1}, & \text{if } m+1\leq n\leq\infty \end{array}\right\}. \tag{55}$$

and χ_n is given by (40). The computation in (53) and (54) are sharp with (41).

Theorem 11. *$k-\tilde{U}ST_s(\hbar,\upsilon,\iota,\varsigma)$ is a convex and compact subset of T.*

Proof. Suppose $j_d \in k-\tilde{U}ST_s(\hbar,\upsilon,\tau,\iota,\varsigma)$,

$$j_d(\omega) = \omega - \sum_{n=2}^{\infty}|a_{d,n}|\omega^n. \tag{56}$$

Then, for $0\leq\psi<1$, let $j_1, j_2 \in k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ be given by (56). Then,

$$\begin{aligned}\xi(\omega) &= \psi j_1(\omega) + (1-\psi)j_2(\omega) = \psi\left(\omega - \sum_{n=2}^{\infty}|a_{1,n}|\omega^n\right)\\ &\quad + (1-\psi)\left(\omega - \sum_{n=2}^{\infty}|a_{2,n}|\omega^n\right)\\ &= \omega - \sum_{n=2}^{\infty}\left(\psi|a_{1,n}| + (1-\psi)|a_{2,n}|\right)\omega^n,\end{aligned}$$

$$\begin{aligned}&\sum_{n=2}^{\infty}\varphi_\hbar^n(\tau,\upsilon)(n(k+1) - j_n(k+\iota))(\psi\,|\,a_{1,n}|+(1-\psi)\,|\,a_{2,n}\,|)\\ &\quad\leq \psi(1-\iota) + (1-\psi)(1-\iota) = 1-\iota.\end{aligned} \tag{57}$$

Then, $\xi(\omega) = \psi j_1(\omega) + (1-\psi)j_2(\omega) \in k-\tilde{U}ST_s(\hbar,\upsilon,\tau,\iota,\varsigma)$. Therefore, $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ is convex. Now, we have to show $k-\tilde{\mathrm{U}}\mathrm{ST}_s(\hbar,\upsilon,\tau,\iota,\varsigma)$ is compact.

For $j_d \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma), \varsigma \in \mathbb{N}$ and $|\omega| < r (0 < r < 1)$, then we arrive

$$|j_d(\omega)| \le r + \sum_{n=2}^{\infty} |a_{d,n}| r^n \le r + \sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)||a_{d,n}|r^n \le r + (1+r)r^n. \tag{58}$$

Therefore, $k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is uniformly bounded.

Let $j_d(\omega) = \omega - \sum_{n=2}^{\infty} | a_{d,n} | \omega^n, \omega \in Y, d \in \mathbb{N}$.

Also, let $j(\omega) = \omega - \sum_{n=2}^{\infty} o_n \omega^n$. Then, by Theorem 4, we get

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)||o_n| \le 1 - \iota. \tag{59}$$

Assuming $j_d \longrightarrow j$, then we have $a_{d,n} \longrightarrow o_n$ as $n \longrightarrow \infty$, $(d \in \mathbb{N})$.

Let $\{\rho_n\}$ be the array of partial sums of the series

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)||o_n|. \tag{60}$$

Then, $\{\rho_n\}$ is a nondecreasing array and by (59), it is bounded above by $1 - \iota$.

Thus, it is convergent and

$$\sum_{n=2}^{\infty} \varphi_{\hbar}^{n}(\tau, \upsilon)|n(k+1) - j_n(k+\iota)||a_{d,n}| = \lim_{n \to \infty} \rho_n \le 1 - \iota. \tag{61}$$

Therefore, $j \in k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ and the class $k - \tilde{\mathrm{U}}\mathrm{ST}_s(\hbar, \upsilon, \tau, \iota, \varsigma)$ is closed.

Data Availability

Our manuscript does not contain any data.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed equally to this work. And all the authors have read and approved the final version of the manuscript.

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