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Research Article

Uniformly Nonsquare in Orlicz Space Equipped with the Mazur-Orlicz F-Norm

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The definition of uniformly nonsquareness in Banach spaces is extended to F-normed spaces. Most of the results from this paper concern (uniformly) nonsquareness in the sense of James or in the sense of Schäffer in Orlicz spaces equipped with the Mazur-Orlicz F-norm. It is well known that uniform nonsquareness in the sense of Schäffer and in the sense of James are equivalent in Banach spaces. In this paper, we found that uniform nonsquareness in the sense of James and in the sense of Schäffer are not equivalent for F-normed spaces. Criteria for Orlicz spaces equipped with the Mazur-Orlicz F-norm to be nonsquare and uniformly nonsquare in the sense of James or in the sense of Schäffer are given.

1. Introduction and Preliminaries

As well known, Orlicz space is a generalization of classical Lebesgue space. The theory of Orlicz space has important applications in control theory, fixed point theory, ergodic theory, probability theory, and theory of vector analytic function and has been intensively developed during the last decades. In 2018, Cui et al. discussed the monotonicity of Orlicz space that generated by the monotone continuous function equipped with Mazur-Orlicz F-norm (see [1, 2]). In 2020, Bai et al. given criteria that Orlicz spaces that generated by the monotone function equipped with Mazur-Orlicz F-norm have strictly monotonicity and upper locally uniform monotonicity, and they get the conclusion that $\|\lambda x + (1-\lambda)y\|_F \le 1$ for each $x, y \in S(L_\Phi(\mu))$ and $\lambda \in (0,1)$ if and only if Φ is convex function on R. So, in order to studying geometric properties of Orlicz spaces equipped with the Mazur-Orlicz F-norm, we need to assume that Φ is convex see [3].

Inspired B-convex spaces, in 1964, the definition of uniformly nonsquare in normed linear space was introduced by James (see [4]). In 1976, the concept of uniformly nonsquare in normed linear space was introduced by Schäffer (see [5]). A lot of results concerning with uniformly nonsquare in Banach space are known. Among the great number of papers concerning this topic, we list here a little [4–10]. Particularly, whether or not uniformly nonsquare Banach space has fixed point of nonexpansive mapping has been discussed as an open problem. Until 2005, Garca-Falset et al. solved the open problem and obtained that uniformly nonsquare Banach space has fixed point property (see [11]).

The aim of this paper is to give criteria that Orlicz spaces equipped with the *Mazur Orlicz F*-norm are nonsquare and uniformly nonsquare in the sense of James or in the sense of Schäffer.

1.1. Introduction. Denoted by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively. Let $\mathbb{R}_+ := [0,+\infty)$. Given any real linear space X, the functional $\|\cdot\|: X \to \mathbb{R}_+$ is called an F-norm if the following conditions are satisfied:

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|-x\| = \|x\|$ for all $x \in X$

(iii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$

(iv) $\|\lambda_n x_n - \lambda x\| \to 0$ whenever $\|x_n - x\| \to 0$ and $\lambda_n \to \lambda$ for any $x \in X$, $(x_n)_{n=1}^{\infty}$ in X and $\lambda \in \mathbb{R}$ and $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{R}$

We say that $X = (X, \|\cdot\|)$ is an F-normed space if it is complete with respect to the F-norm topology.

Definition 1. Let (G, Σ, μ) be a finite measure space and L^0 be the space of all (equivalence) classes of Σ-measurable real-valued functions defined on G. A function Φ: $(-\infty, +\infty) \to [0, \infty]$ is called an Orlicz function if $\Phi(u) > 0$ for all $u \neq 0$, even, convex, and $\lim_{u\to\infty} \Phi(u) = \infty$. Any Orlicz function Φ determines a mapping $I_\Phi : L^0 \to [0,+\infty]$ defined by the formula $I_\Phi(f) = \int_G \Phi(f(t))d\mu$ called the modular. The order ideal $L^\Phi = \{f \in L^0: I_\Phi(rf) < \infty \text{ for some } r > 0\}$ in L^0 is called an Orlicz space.

The space L^Φ is an F-normed space with respect to the following lattice F-norm, called the Mazur-Orlicz F-norm [12]:

$$\|f\|_F = \inf \{\lambda > 0 : I_\Phi(f/\lambda) \leq \lambda\}. \tag{1}$$

Definition 2. We say that Φ satisfies Δ_2-condition ($\Phi \in \Delta_2$, for short) there are constants $K > 0$ and $u_0 > 0$ such that

$$\Phi(2u) \leq K\Phi(u), \tag{2}$$

whenever $|u| \geq u_0$.

Definition 3. Let Φ be an Orlicz function, p be the right derivative of Φ, and q be the right-inverse function of p. Then, we call

$$\Psi(v) = \int_0^{|v|} q(s)ds, \tag{3}$$

the complementary function of Φ.

Definition 4. We say Φ satisfies ∇_2-condition ($\Phi \in \nabla_2$, for short) there exist $u_0 > 0$ and $\delta_1 > 0$ such that

$$\Phi(2u) \geq (2 + \delta_1)\Phi(u), \tag{4}$$

whenever $|u| \geq u_0$. It is well known that $\Phi \in \nabla_2$ if and only if $\Psi \in \Delta_2$.

Lemma 5. *(see [13]).*
$\Phi \in \Delta_2$ if and only if there exist constants $\alpha > 0$ and $u_0 > 0$ such that

$$\frac{up(u)}{\Phi(u)} < \alpha, \tag{5}$$

whenever $|u| \geq u_0$.

The characteristic of convexity of X is defined by $\varepsilon_0(X) = \sup \{\varepsilon \in [0, 2]: \delta_X(\varepsilon) = 0\}$, where $\delta_X(\varepsilon) = \inf \{1 - 1/2\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$ is the Clarkson modulus of convexity of X. A Banach space X is said to be uniformly nonsquare whenever $\varepsilon_0(X) < 2$. Since X satisfies the definition of a finite tree, we can get that X is superreflexive (see [4]). Now, we extend the definition of nonsquare and uniformly nonsquare to F-normed space.

Definition 6. An F-normed space X is said to be nonsquare in the sense of James if for any $r > 0$ it is verified that

$$\min \left\{\|x + y\|_F, \|x - y\|_F\right\} < 2r, \tag{6}$$

whenever $\|x\|_F = r$, $\|y\|_F = r$.

Definition 7. An F-normed space X is said to be nonsquare in the sense of Schäffer if for any $r > 0$ it is verified that

$$\min \left\{\|x + y\|_F, \|x - y\|_F\right\} > r, \tag{7}$$

whenever $\|x\|_F = r$, $\|y\|_F = r$.

Definition 8. An F-normed space X is said to be uniformly nonsquare in the sense of James if for any $r > 0$ there exists $\delta > 0$ such that

$$\max \left\{\min \left\{\|x + y\|_F, \|x - y\|_F\right\}, \|x\|_F = r, \|y\|_F = r\right\} < 2r - \delta, \tag{8}$$

whenever $\|x\|_F = r$, $\|y\|_F = r$.

Definition 9. An F-normed space X is said to be uniformly nonsquare in the sense of Schäffer if for any $r > 0$ there exists $\delta > 0$ such that

$$\max \left\{\min \left\{\|x + y\|_F, \|x - y\|_F\right\}, \|x\|_F = r, \|y\|_F = r\right\} > r + \delta, \tag{9}$$

whenever $\|x\|_F = r$, $\|y\|_F = r$.

2. Main Results

Lemma 10. *Suppose that $\Phi \in \Delta_2$. If there exists $\varepsilon > 0$ such that $I_\Phi(x/r) \geq r + \varepsilon$ for anyr > 0, then there exists $\delta > 0$ which satisfy $\|x\|_F > r + \delta$.*

Proof. If there exists a sequence $\{x_n\} \subset L_\Phi$ such that $I_\Phi(x_n/r) \geq r + \varepsilon$ and $\|x_n\|_F \to r$, then

$$I_\Phi\left(\frac{x_n}{r}\right) \to r, \tag{10}$$

thanks to $\Phi \in \Delta_2$, a contradiction.

Theorem 11. *L_Φ is nonsquare in the sense of Schäffer if and only if $\Phi \in \Delta_2$.*

Proof. Necessity. Assume that $\Phi \notin \Delta_2$. We show that there exists a strictly increasing sequence numbers $\{u_n\}$ such that for each $n \in N$, $\lim_{n\to\infty} u_n = +\infty$ and

$$\Phi\left(\left(1+\frac{1}{n}\right)u_n\right) > 2^n\Phi(u_n). \quad (11)$$

We take $G_0, G_1 \subset G$ and $G_1 \cap G_0 = \varnothing$ which satisfy $\mu(G_0) = \mu(G_1) > 0$. There exists a sequence $\{G_n^0\}_{n=1}^{\infty}$ of pairwise disjoint sets in G_0, for some $n_1 \in N$, such that $\mu(G_1^0) \geq (1/\Phi(u_{n_1})) \cdot (r/2^2)$, $\mu(G_n^0) = (r/2) \cdot (1/2^n\Phi(u_{n+n_1}))$, $(n = 1, 2 \cdots)$. Let $x = r\sum_{n=1}^{\infty} u_{n+n_1}\chi_{G_n^0}$. Then, supp $(x) \subset G_0$, and we can get the following

$$\begin{aligned} I_\Phi\left(\frac{x}{r}\right) &= \int_{G_0}\Phi\left(\sum_{n=1}^{\infty}u_{n+n_1}(t)\chi_{G_n^0}\right)dt = \sum_{n=1}^{\infty}\int_{G_n^0}\Phi\left(u_{n+n_1}(t)\right)dt \\ &= \sum_{n=1}^{\infty}\Phi\left(u_{n+n_1}\right)\mu\left(G_n^0\right) = \sum_{n=1}^{\infty}\Phi\left(u_{n+n_1}\right)\frac{1}{2^n\Phi\left(u_{n+n_1}\right)}\cdot\frac{r}{2} \\ &= \frac{r}{2}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{r}{2}. \end{aligned} \quad (12)$$

On the other hand, for any $\lambda \in (0, 1)$, there exists $n_0 \in N$ such that

$$\frac{1}{\lambda} \geq 1 + \frac{1}{n+n_1}, \quad (13)$$

whenever $n \geq n_0$.

Combing (11) and (13), the following inequalities are found to be true:

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda r}\right) &\geq \sum_{n=n_0}^{\infty}\Phi\left(\frac{u_{n+n_1}}{\lambda}\right)\mu\left(G_n^0\right) \geq \sum_{n=n_0}^{\infty}\Phi\left(\left(1+\frac{1}{n+n_1}\right)u_{n+n_1}\right)\mu\left(G_n^0\right) \\ &\cdot > \sum_{n=n_0}^{\infty}2^{n+n_1}\Phi\left(u_{n+n_1}\right)\mu\left(G_n^0\right) = \sum_{n=n_0}^{\infty}2^{n+n_1}\frac{1}{2^n}\cdot\frac{r}{2} \\ &= \frac{r}{2}\cdot 2^{n_1}\sum_{n=n_0}^{\infty}1 = +\infty. \end{aligned} \quad (14)$$

Similar to the selection method of sequence $\{G_n^0\}_{n=1}^{\infty}$, the selection of sequence $\{G_n^1\}_{n=1}^{\infty}$ of pairwise disjoint sets in G_1 satisfies $\mu(G_n^1) = (r/2) \cdot (1/2^{n+n_1}\Phi(u_{n+n_1}))$, $(n = 1, 2 \cdots)$. Supposing $y = r\sum_{n=1}^{\infty} u_{n+n_1}\chi_{G_n^1}$, we verity that $I_\Phi(y/r) = r/2$ and $I_\Phi(y/\lambda r) = +\infty$. According to the definition of F-norm, we can obtain that $\|x\|_F = \|y\|_F = r$. Furthermore, by supp $(x) \cap$ supp $(y) = \varnothing$, we deduce that

$$I_\Phi\left(\frac{x+y}{r}\right) = I_\Phi\left(\frac{x-y}{r}\right) = I_\Phi\left(\frac{x}{r}\right) + I_\Phi\left(\frac{y}{r}\right) = r, \quad (15)$$

which implies that $\|x+y\|_F = \|x-y\|_F = r$. Namely

$$\max\left\{\|x+y\|_F, \|x-y\|_F\right\} = r, \quad (16)$$

contradiction with $\max\{\|x+y\|_F, \|x-y\|_F\} > r$, the proof of necessity is completed.

Sufficiency. Assume on the contrary that L_Φ is not nonsquare in the sense of Schäffer, then L_Φ has elements x, y satisfying $\|x\|_F = r$, $\|y\|_F = r$ and $\|x+y\|_F = r$, $\|x-y\|_F = r$ for some $r > 0$. Using $\Phi \in \Delta_2$, we conclude that $I_\Phi((x+y)/r) = I_\Phi((x-y)/r) = r$.

Due to the convexity of Φ, we have the following inequality

$$\begin{aligned} \alpha(t) &= \frac{1}{2}\left(\Phi\left(\frac{x(t)+y(t)}{r}\right) + \Phi\left(\frac{x(t)-y(t)}{r}\right)\right) \\ &\cdot \geq \max\left\{\Phi\left(\frac{x(t)}{r}\right), \Phi\left(\frac{y(t)}{r}\right)\right\}, \end{aligned} \quad (17)$$

and then

$$\begin{aligned} I_\Phi(\alpha(t)) &= \frac{1}{2}\left[\left(\int_G\Phi\left(\frac{x(t)+y(t)}{r}dt\right) + \int_G\Phi\left(\frac{x(t)-y(t)}{r}\right)dt\right]\right. \\ &= \frac{1}{2}\left[I_\Phi\left(\frac{x+y}{r}\right) + I_\Phi\left(\frac{x-y}{r}\right)\right] = \frac{1}{2}(r+r) = r. \end{aligned} \quad (18)$$

Observing that the Orlicz function Φ has the property $\Phi(|x|-|y|) \leq |\Phi(x) - \Phi(y)|$, the following inequality can be obtained

$$\begin{aligned} I_\Phi\left(\frac{|x|-|y|}{r}\right) &= \int_G\Phi\left(\frac{|x(t)|-|y(t)|}{r}\right)dt \leq \left|\int_G\Phi\left(\frac{x(t)}{r}\right)dt\right. \\ &\cdot \left.-\int_G\Phi\left(\frac{y(t)}{r}\right)dt\right| \leq \int_G\left[\alpha(t) - \Phi\left(\frac{x(t)}{r}\right)\right]dt \\ &\cdot + \int_G\left[\alpha(t) - \Phi\left(\frac{y(t)}{r}\right)\right]dt = 0. \end{aligned} \quad (19)$$

Hence $|x| = |y|$ holds.

Since $\Phi(u) = \int_0^u p(t)dt$, the following formula can be obtained

$$\Phi(2u) = \int_0^{2u}p(t)dt = \int_0^u p(t)dt + \int_u^{2u}p(t)dt > 2\int_0^u p(t)dt = 2\Phi(u), \quad (20)$$

whenever $u > 0$.

From the above analysis,

$$\int_G \Phi\left(\frac{x(t)+y(t)}{r}\right)dt + \int_G \Phi\left(\frac{x(t)-y(t)}{r}dt\right) \geq \int_G \Phi\left(\frac{|x(t)|+|y(t)|}{r}dt\right) \cdot = \int_G \Phi\left(\frac{2\,|\,x(t)\,|}{r}\right)dt > 2\int_G \Phi\left(\frac{|x(t)\,|}{r}dt\right) = 2r. \tag{21}$$

Therefore, we obtain that

$$\frac{I_\Phi((x+y)/r) + I_\Phi((x-y)/r)}{2} > r. \tag{22}$$

It means that $\max\{I_\Phi((x+y)/r), I_\Phi((x-y)/r)\} > r$, hence $\max\{\|x+y\|_F, \|x-y\|_F\} > r$, a contradiction. The proof is complete.

Theorem 12. *The following conditions are equivalent:*

(i) L_Φ is uniformly nonsquare in the sense of James

(ii) L_Φ is nonsquare in the sense of James

(iii) $\Phi \in \Delta_2$

Proof. It is obvious that (i) ⇒ (ii). We only need to prove (ii) ⇒ (iii) and (iii) ⇒ (i).

(ii) ⇒ (iii). Assume that Φ does not satisfy Δ_2-condition. Since the idea is similar to the proof of Theorem 11, we can find$G_0, G_1 \subset G$, $\mu(G_0) > 0$, $\mu(G_1) > 0$, and $G_1 \cap G_0 = \varnothing$. Construct $x, y \in L_\Phi$ such that supp $(x) \subset G_0$ and supp $(y) \subset G_1$. Then, $I_\Phi(x/r) = r/2$ and $I_\Phi(y/r) = r/2$ and $I_\Phi(x/\lambda r) = +\infty$ and $I_\Phi(y/\lambda r) = +\infty$ for any $\lambda \in (0,1)$. For convenience, denote $u = x + y$ and $v = x - y$ which shows that $\|u\|_F = \|v\|_F = r$.

Otherwise, we let $u + v = 2x$ and $u - v = 2y$; then, $I_\Phi((u + v)/2r) = r/2$, and $I_\Phi((u-v)/2r) = r/2$. By the arbitrariness of $\lambda \in (0,1)$, we obtain that $I_\Phi((u+v)/2\lambda r) = +\infty$ and $I_\Phi((u-v)/2\lambda r) = +\infty$.

Then, by the definition of Mazur-Orlicz F-norm, we conclude that

$$\|u+v\|_F = \|u-v\|_F = 2r. \tag{23}$$

It shows that L_Φ is not a nonsquare in the sense of James. Namely, we have $\Phi \in \Delta_2$.

We prove the implication (iii) ⇒ (i). Obviously, by the convexity of Φ, we have

$$I_\Phi\left(\frac{x+y}{2r}\right) \leq \frac{1}{2}\left(I_\Phi\left(\frac{x}{r}\right) + I_\Phi\left(\frac{y}{r}\right)\right) = r. \tag{24}$$

For the sake of simplicity, we set $z = x + y$. Next, we want to prove that the inequality

$$\sup\left\{\|z\|_F : I_\Phi\left(\frac{z}{2r}\right) \leq r\right\} < 2r, \tag{25}$$

holds.

Since $I_\Phi(z/2r) \leq r < 2r$, we have $\|z\|_F \leq 2r$,which implies that $\sup\{\|z\|_F : I_\Phi(z/2r) \leq r\} \leq 2r$ holds.

Without loss of generality, we may assume $\sup\{\|z\|_F : I_\Phi(z/2r) \leq r\} = 2r$. Then, there exists $\{z_n\} \subset L_\Phi$ which satisfies $I_\Phi(z_n/2r) \leq r$ and $\|z_n\|_F \to 2r$ as $n \to \infty$.

Since $\Phi \in \Delta_2$, for a fixed $\delta = 1/4$ find $0 < \varepsilon < 1$ such that

$$\Phi((1+\varepsilon)u) < (1+\delta)\Phi(u), \tag{26}$$

for every $u \geq u_0$, where $u_0 = (1/2)\Phi^{-1}(r/4\mu(G))$. Namely,

$$\Phi((1+\varepsilon)u) < \frac{5}{4}\Phi(u). \tag{27}$$

Since $\|z_n\|_F \to 2r$ as $n \to \infty$, we can obtain $\lim_{n\to\infty}(2r/\|z_n\|_F) = 1$. Then, there exists $n_0 \in N$ such that

$$\frac{2r}{\|z_n\|_F} \leq 1 + \varepsilon, \tag{28}$$

whenever $n \geq n_0$.

Combining conditions (27) and (28) and $I_\Phi(z_n/2r) \leq r$, we derive a contradiction:

$$\begin{aligned} 2r = I_\Phi\left(\frac{z_n}{\|z_n\|_F}\right) &= I_\Phi\left(\frac{2r}{\|z_n\|_F}\cdot\frac{z_n}{2r}\right) \leq \int_G \Phi\left((1+\varepsilon)\frac{z_n(t)}{2r}\right)dt \\ &= \int_{G_{u_0}} \Phi\left((1+\varepsilon)\frac{z_n(t)}{2r}\right)dt + \int_{G|G_{u_0}} \Phi\left((1+\varepsilon)\frac{z_n(t)}{2r}\right)dt \\ &\cdot \leq \Phi(2u_0)\mu(G) + \frac{5}{4}\int_G \Phi\left(\frac{z_n(t)}{2r}\right)dt = \frac{3r}{2} < 2r, \end{aligned} \tag{29}$$

where $G_{u_0} = \{t \in G : (z_n(t)/2r) \leq u_0\}$.

We get that $\sup\{\|x+y\|_F : I_\Phi((x+y)/2r) \leq r\} < 2r$. Hence, we can obtained that

$$\min\left\{\|x+y\|_F, \|x-y\|_F : \|x\|_F = \|y\|_F = r\right\} \leq \sup \left\{\|x+y\|_F : I_\Phi\left(\frac{x+y}{2r}\right) \leq r\right\} < 2r,, \tag{30}$$

which shows that L_Φ is uniformly nonsquare in the sense of James.

Theorem 13. *L_Φ is uniformly nonsquare in the sense of Schäffer if and only if $\Phi \in \Delta_2$ and $\Phi \in \nabla_2$.*

Proof. Since the necessity of $\Phi \in \Delta_2$ is similar to the one given in Theorem 11, we only state the results without the proof. We only need to prove the necessity of ∇_2-condition.

Suppose that ∇_2-condition is invalid. There exists a strictly increasing sequence $\{\xi_n\}_{n=1}^\infty$ which satisfy $\lim_{n\to\infty}\xi_n = +\infty$ and

$$\Phi(\xi_n) \geq \frac{1-(1/n)}{2}\Phi(2\xi_n). \tag{31}$$

Without loss of generality, we may take $G_n \subset G$ with

$$\Phi(\xi_n)\mu(G_n) = r \, (n = 1, 2 \cdots). \tag{32}$$

Let $x_n = r\xi_n\chi_{G_n}$, it means that $I_\Phi(x_n/r) = r$. Then, $\|x_n\|_F = r$.

Divided G_n into disjoint two subsets G_n' and G_n'' with $\mu(G_n') = \mu(G_n'') = (1/2)\mu(G_n)$.

Suppose $y_n = r(\xi_n\chi_{G_n'} - \xi_n\chi_{G_n''})$, we also have $I_\Phi(y_n/r) = r$ and $\|y_n\|_F = r$.

Consequently, by the convexity of Φ, we deduce that

$$\begin{aligned} I_\Phi\left(\frac{x_n + y_n}{r}\right) &= I_\Phi\left(2\xi_n\chi_{G_n'}\right) = \Phi(2\xi_n)\frac{1}{2}\mu(G_n) \ge 2\Phi(\xi_n)\frac{1}{2}\mu(G_n) \\ &= \Phi(\xi_n)\mu(G_n) = r, \end{aligned} \tag{33}$$

then the following inequality is found to be true

$$\|x_n + y_n\|_F \ge r. \tag{34}$$

To prove the reverse inequalities, notice that by (31) and (32), we can obtain the following inequality

$$\begin{aligned} I_\Phi\left(\frac{x_n + y_n}{r}\right) &= I_\Phi\left(2\xi_n\chi_{G_n'}\right) \le \frac{2}{1 - (1/n)}\Phi(\xi_n)\mu\left(G_n'\right) \\ &= \frac{1}{1 - (1/n)}\Phi(\xi_n)\mu(G_n) = r \cdot \frac{1}{1 - (1/n)}. \end{aligned} \tag{35}$$

That is $I_\Phi((1 - (1/n))(x_n + y_n)/(1 - (1/n))r) \le r/1 - (1/n)$, it follows that

$$\left\|\frac{n}{n-1}(x_n + y_n)\right\|_F \le \frac{r}{1 - (1/n)}. \tag{36}$$

Next, we will prove that $\lim_{n\to\infty} \|(1/n-1)(x_n + y_n)\|_F = 0$. Otherwise, we assume that $\overline{\lim_{n\to\infty}} \|(1/n-1)(x_n + y_n)\|_F > 0$. Without loss of generality, we may suppose there exists $\varepsilon_0 > 0$, which satisfies the following inequality

$$\lim_{n\to\infty} \left\|\frac{1}{n-1}(x_n + y_n)\right\|_F > \varepsilon_0. \tag{37}$$

Take $n_0 \in N$ large enough for which $(r/(n-1)\varepsilon_0) < 1$ whenever $n \ge n_0$. By the convexity of Φ and (37), this lead to a contradiction:

$$\begin{aligned} \varepsilon_0 &< \lim_{n\to\infty} I_\Phi\left(\frac{(1/(n-1))(x_n + y_n)}{\varepsilon_0}\right) = \lim_{n\to\infty} I_\Phi\left(\frac{r}{(n-1)\varepsilon_0}\left(\frac{x_n + y_n}{r}\right)\right) \\ &\cdot \le \lim_{n\to\infty} \frac{r}{(n-1)\varepsilon_0} I_\Phi\left(\frac{x_n + y_n}{r}\right) \\ &\cdot \le \lim_{n\to\infty} \frac{r}{(n-1)\varepsilon_0} \cdot \frac{r}{1 - (1/n)} = \lim_{n\to\infty} \frac{r^2 n}{(n-1)^2\varepsilon_0} \to 0. \end{aligned} \tag{38}$$

By inequality (36), we have the following results

$$\begin{aligned} \|x_n + y_n\|_F &= \left\|\frac{n}{n-1}(x_n + y_n) - \frac{1}{n-1}(x_n + y_n)\right\|_F \le \left\|\frac{n}{n-1}(x_n + y_n)\right\|_F \\ &\cdot - \left\|\frac{1}{n-1}(x_n + y_n)\right\|_F \le \frac{r}{1 - (1/n)} - \left\|\frac{x_n + y_n}{n-1}\right\|_F. \end{aligned} \tag{39}$$

It is easily to see that

$$\overline{\lim_{n\to\infty}} \|x_n + y_n\|_F \le r. \tag{40}$$

From (34) and (40), it follows that $\lim_{n\to\infty} \|x_n + y_n\|_F = r$.

Using the method of the proof of $\lim_{n\to\infty} \|x_n + y_n\|_F = r$, we can obtain that $\lim_{n\to\infty} \|x_n - y_n\|_F = r$. This implies that L_Φ is not uniformly nonsquare in the sense of Schäffer, that is $\Phi \in \nabla_2$.

Sufficiency. Assume for the contrary that L_Φ is not uniformly nonsquare in the sense of Schäffer. We can find subsequences $\{x_n\}, \{y_n\} \subset L_\Phi$ satisfying $\|x_n\|_F = r$ and $\|y_n\|_F = r$ and $\|x_n + y_n\|_F \to r$ and $\|x_n - y_n\|_F \to r$ for some $r > 0$.

Without loss of generality, we may take $u_1 > 0$ such that

$$\Phi(u_1)m(G) \le \frac{r}{2}. \tag{41}$$

Since $\Phi \in \nabla_2$, there exists $\delta_1 > 0$ such that

$$\Phi(2u) \ge (2 + \delta_1)\Phi(u), \tag{42}$$

whenever $u \ge u_1$.

Put

$$\alpha_n(t) = \frac{1}{2}\left(\Phi\left(\frac{x_n(t) + y_n(t)}{r}\right) + \Phi\left(\frac{x_n(t) - y_n(t)}{r}\right)\right) \, (n = 1, 2 \cdots). \tag{43}$$

The following equation can be obtained by integrating both sides of the above formula

$$\int_G \alpha_n(t)dt = \frac{1}{2}\left(\int_G \Phi\left(\frac{x_n(t) + y_n(t)}{r}\right)dt + \int_G \Phi\left(\frac{x_n(t) - y_n(t)}{r}\right)dt\right) \to r. \tag{44}$$

Since $\Phi \in \Delta_2$, there exists $K \ge 2$ and $u_0 > 0$ such that

$$\Phi(2u) \le K\Phi(u) + 1, \tag{45}$$

whenever $u \ge u_0$.

Hence, we have the following results

$$\begin{aligned} I_\Phi\left(\frac{|x_n|+|y_n|}{r}\right) &= I_\Phi\left(\frac{2|x_n|+2|y_n|}{2r}\right) \le \frac{1}{2}\left(I_\Phi\left(\frac{2x_n}{r}\right)+I_\Phi\left(\frac{2y_n}{r}\right)\right) \\ &\cdot \le \frac{1}{2}\left(KI_\Phi\left(\frac{x_n}{r}\right)+1+KI_\Phi\left(\frac{y_n}{r}\right)+1\right) \\ &= \frac{1}{2}(K\|x_n\|_F+1+K\|y_n\|_F+1) = Kr+1. \end{aligned} \tag{46}$$

Put

$$\begin{aligned} L &= Kr+1, \\ \varepsilon &= \frac{\delta_1 r}{6}. \end{aligned} \tag{47}$$

Using $\Phi \in \Delta_2$, there exists $\delta > 0$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon, \tag{48}$$

whenever $I_\Phi(x) \le L$ and $I_\Phi(y) \le \delta$.

Obviously,

$$\begin{aligned} I_\Phi\left(\frac{|x_n|-|y_n|}{r}\right) &\le \int_G \left|\Phi\left(\frac{x_n(t)}{r}\right)-\Phi\left(\frac{y_n(t)}{r}\right)\right| dt \\ &\le \int_G \left(\alpha_n(t)-\Phi\left(\frac{x_n(t)}{r}\right)\right)dt \\ &\cdot + \int_G \left(\alpha_n(t)-\Phi\left(\frac{y_n(t)}{r}\right)\right)dt \longrightarrow 0. \end{aligned} \tag{49}$$

We can choose $n_0 \in N$ such that

$$I_\Phi\left(\frac{|x_n|-|y_n|}{r}\right) \le \delta, \tag{50}$$

whenever $n \ge n_0$.

Therefore, we have

$$\left|I_\Phi\left(\frac{|x_n|+|y_n|}{r}+\frac{|x_n|-|y_n|}{r}\right)-I_\Phi\left(\frac{|x_n|+|y_n|}{r}\right)\right| < \varepsilon, \tag{51}$$

this shows that

$$\left|I_\Phi\left(\frac{2x_n}{r}\right)-I_\Phi\left(\frac{|x_n|+|y_n|}{r}\right)\right| < \varepsilon. \tag{52}$$

Define $G_{u_1} = \{t \in G : |x_n(t)/r| \ge u_1\}$, then we derive that

$$\begin{aligned} &\int_G \Phi\left(\frac{x_n(t)+y_n(t)}{r}\right)dt + \int_G \Phi\left(\frac{x_n(t)-y_n(t)}{r}\right)dt \ge \int_G \Phi\left(\frac{|x_n(t)|+|y_n(t)|}{r}\right)dt \\ &\cdot \ge \int_G \Phi\left(\frac{2x_n(t)}{r}\right)dt - \varepsilon = \int_{G_{u_1}} \Phi\left(\frac{2x_n(t)}{r}\right)dt \\ &+ \int_{G|G_{u_1}} \Phi\left(\frac{2x_n(t)}{r}\right)dt - \varepsilon \ge (2+\delta_1)\int_{G_{u_1}} \Phi\left(\frac{x_n(t)}{r}\right)dt \\ &+ 2\int_{G|G_{u_1}} \Phi\left(\frac{x_n(t)}{r}\right)dt - \varepsilon = 2\int_G \Phi\left(\frac{x_n(t)}{r}\right)dt \\ &+ \delta_1 \int_{G_{u_1}} \Phi\left(\frac{x_n(t)}{r}\right)dt - \varepsilon = 2\int_G \Phi\left(\frac{x_n(t)}{r}\right)dt \\ &+ \delta_1\left(\int_G \Phi\left(\frac{x_n(t)}{r}\right)dt - \int_{G|G_{u_1}} \Phi\left(\frac{x_n(t)}{r}\right)dt\right) - \varepsilon \ge 2r \\ &+ \frac{r\delta_1}{2} - \varepsilon \ge 2r + \frac{1}{3}r\delta_1. \end{aligned} \tag{53}$$

It follows that $\max\{I_\Phi((x_n+y_n)/r), I_\Phi(x_n-y_n)/r\} \ge r + \delta_1 r/6$ holds. Thanks to Lemma 10, there exists $\delta' > 0$ such thatmax $\{\|x_n+y_n\|_F, \|x_n-y_n\|_F\} \ge r+\delta'$, which contradicts with the assumption $\|x_n+y_n\|_F \to r$ and $\|x_n-y_n\|_F \to r$. So L_Φ is uniformly nonsquare in the sense of Schäffer. The proof is complete.

Example 14. Put

$$\Phi(u) = e^{|u|} - |u| - 1. \tag{54}$$

The derivative of the function $\Phi(u)$ is $p(t) = e^t - 1$. Obviously,

$$\lim_{u\to\infty} \frac{u \cdot p(u)}{\Phi(u)} = \lim_{u\to\infty} \frac{u(e^u-1)}{e^u-u-1} = +\infty. \tag{55}$$

So, from the Lemma 1.3, we immediately have $\Phi(u)$ does not satisfy the Δ_2-condition.

On the other hand, we can get that $q(s) = \ln(s+1)$. Thus

$$\Psi(v) = \int_0^v q(s)ds = \int_0^v \ln(s+1)ds = (1+v)\ln(1+v) - v \quad (v \ge 0). \tag{56}$$

Then

$$\lim_{v\to\infty} \frac{v \cdot q(v)}{\Psi(v)} = \lim_{v\to\infty} \frac{v \cdot \ln(v+1)}{(1+v)\ln(1+v)-v} = 1 + \lim_{v\to\infty} \frac{v}{(v+1)(\ln(1+v))} = 1. \tag{57}$$

Using Lemma 1.3 in [6], we have $\Psi(v)$ satisfy the Δ_2 -condition, i.e., $\Phi \in \nabla_2$. Then, L_Ψ is uniformly nonsquare in the sense of James and is not uniformly nonsquare in the sense of Schäffer. The proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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