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Research Article

Duality of Large Fock Spaces in Several Complex Variables and Compact Localization Operators

Youqi Liu and Xiaofeng Wang

School of Mathematics and Information Science and Key Laboratory of Mathematics and Interdisciplinary Sciences of the Guangdong Higher Education Institute, Guangzhou University, Guangzhou 510006, China

Correspondence should be addressed to Xiaofeng Wang; wxf@gzhu.edu.cn

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In this paper, dual spaces of large Fock spaces $\mathscr{F}^p_\phi$ with $0 < p < \infty$ are characterized. Also, algebraic properties and equivalent conditions for compactness of weakly localized operators are obtained on $\mathscr{F}^p_\phi(0 < p<\infty)$.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space. Let dv denote the Lebesgue volume measure on $\mathbb{C}^n$. For any two points $z = (z_1,\cdots,z_n)$ and $w = (w_1,\cdots,w_n)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and $|z| = \sqrt{|z_1|^2+\cdots+|z_n|^2}$.

For each $z \in \mathbb{C}^n$ and $r >0$,

$$B(z, r) = \{w \in \mathbb{C}^n : |w - z| < r\} \tag{1}$$

denotes the Euclidean ball centered at z with radius r.

Let Δ denote the Laplacian operator. Suppose $\phi : \mathbb{C}^n \longrightarrow \mathbb{R}$ is a C^2 plurisubharmonic function (see [1]). We say that ϕ belongs to the weight class W if ϕ satisfies the following statements:

(A) There exists $c > 0$ such that for $z \in \mathbb{C}^n$

$$\inf_{z\in\mathbb{C}^n} \sup_{w\in B(z,c)} \Delta\phi(w) > 0\,; \tag{2}$$

(B) For any $z \in \mathbb{C}^n$ and $r > 0$, $\Delta\phi$ satisfies the reverse-Hölder inequality

$$\|\Delta\phi\|_{L^\infty(B(z,r))} \le Cr^{-2n}\int_{B(z,r)} \Delta\phi dv \tag{3}$$

for some $0 < C < \infty$;

(C) The eigenvalues of H_ϕ are comparable, i.e., for every $z, u \in \mathbb{C}^n$, there exists $\delta_0 > 0$ such that

$$\left(H_\phi(z)u, u\right) \ge \delta_0\Delta\phi(z)|u|^2, \tag{4}$$

where

$$H_\phi = \left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z_k}}\right)_{j,k}. \tag{5}$$

Suppose $0 < p < \infty$, $\phi \in W$. The space L^p_ϕ consists of all Lebesgue measurable functions f on $\mathbb{C}^n$ for which

$$\|f\|_{p,\phi} = \left(\int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(z)} dv(z)\right)^{\frac{1}{p}} < \infty. \tag{6}$$

L^∞_ϕ is the set of all Lebesgue measurable functions f on $\mathbb{C}^n$ with

$$\|f\|_{\infty,\phi} = \sup_{z\in\mathbb{C}^n} |f(z)|e^{-\phi(z)} < \infty. \tag{7}$$

Let $\mathcal{H}(\mathbb{C}^n)$ be the family of all entire functions on $\mathbb{C}^n$. The large Fock space is defined as

$$\mathcal{F}^p_\phi = L^p_\phi \cap \mathcal{H}(\mathbb{C}^n). \tag{8}$$

$\mathcal{F}^p_\phi$ is a Banach space under $\|\cdot\|_{p,\phi}$ if $p \geq 1$, and $\mathcal{F}^p_\phi$ is a quasi-Banach space with distance $d(f,g) = \|f-g\|^p_{p,\phi}$ if $0 < p < 1$. Assume that $\phi(z) = |z|^2/2$, then $\mathcal{F}^p_\phi$ is the classical Fock space which has been studied in [2–4] for example. Also, the weight function ϕ on $\mathbb{C}^n$ with the restriction that $dd^c\phi \simeq d\, d^c|z|^2$ belongs to W, where $d = \bar\partial + \partial$ and $d^c = (\sqrt{-1}/4)(\bar\partial - \partial)$. See [5, 6] for more details.

Particularly, $\mathcal{F}^2_\phi$ is a reproducing kernel Hilbert space. That is, for any $f \in \mathcal{F}^2_\phi$, there exists a unique function $K_z \in \mathcal{F}^2_\phi$ so that $f(z) = \langle f, K_z\rangle_{\mathcal{F}^2_\phi}$, where

$$\langle f, g\rangle_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}^n} f(z)\bar{g}(z)e^{-2\phi(z)}dv(z), f, g \in \mathcal{F}^2_\phi. \tag{9}$$

We say that the function $K_z(\cdot)$ is the reproducing kernel of $\mathcal{F}^2_\phi$. It is well known that the orthogonal projection $P : L^2_\phi \longrightarrow \mathcal{F}^2_\phi$ is given by

$$Pf(z) = \int_{\mathbb{C}^n} f(w)K(z,w)e^{-2\phi(w)}dv(w), f \in L^2_\phi. \tag{10}$$

As we know if $1 \leq p < \infty$ and q is the conjugate exponents of p, then the dual space of L^p_ϕ can be identified with L^q_ϕ by the integral pairing $\langle,\rangle_{\mathcal{F}^2_\phi}$ defined by (9). In general, for $1 \leq p < \infty$, no less important than the Hahn-Banach theorem is the Bergman projection to explore the dual spaces of $\mathcal{F}^p_\phi$. However, there are some differences for these quasi-Banach spaces $\mathcal{F}^p_\phi (0 < p < 1)$. To do this, we will mainly apply Hörmander's solution of the $\bar\partial$ equation and the Lebesgue dominated convergence theorem to consider the duality of $\mathcal{F}^p_\phi\ (0 < p < 1)$.

The "weakly localized" operators were introduced for the first time in [7], and the authors studied the compactness of these operators on the Bergman space A^p and weighted the Bargmann-Fock space F^p_φ with $1 < p < \infty$. In fact, this kind of operators is interesting since these weakly localized operators contain Toeplitz operators which are induced by bounded symbols. Indeed, Toeplitz operators are a kind of significant operators, and these Toeplitz operators induced by diverse functions enjoy abundant properties, see more in [8, 9]. As a further research, Hu, Lv, and Wick characterized the compactness of these weakly localized operators on generalized Fock spaces F^p_φ with $0 < p \leq 1$, see [5]. Besides, in generalized Bergman space setting [10], there are two questions: whether Toeplitz operators induced by bounded symbols are weakly localized operators? Would these weakly localized operators form an algebra?

This paper is devoted to consider the compactness of these weakly localized operators on large Fock spaces $\mathcal{F}^p_\phi$ with $0 < p < \infty$. To ensure the validity of these fascinating operators, we show these localization operators contain Toeplitz operators induced by bounded symbols on $\mathcal{F}^p_\phi$, see Theorem 16. Meanwhile, we also give affirmative answer about the second question on our Fock spaces, see Theorem 15.

Notice that although in the one-dimensional case, the diverse weight function gives another Bergman metric, and the resulting Bergman disk will be changed. Furthermore, there is no inclusion relation between $\mathcal{F}^p_\phi$ and $\mathcal{F}^q_\phi$ if $p \neq q$. The above properties are much different from [5], so we have to apply more techniques to discuss the compactness of weakly localized operators in case $0 < p \leq 1$. For case $1 < p < \infty$, the ideas to study compact weakly localized operators in [7] are not entirely applicable to the situation we are discussing. Hence, we finally combine the skills in [5, 7] to consider the compactness of these operators on $\mathcal{F}^p_\phi (1 < p < \infty)$. Eventually, when $p > 1$, we bring new consequences even if $\mathcal{F}^p_\phi$ is the generalized Fock space in [5].

This paper is organized as follows. In Section 2, we give some lemmas which will play key roles in our proofs. In Section 3, we show some properties of projection and dual spaces of large Fock spaces $\mathcal{F}^p_\phi$ when $0 < p < \infty$. In Section 4, we conclude the algebraic properties and boundedness of localization operators. Finally, in Section 5, we consider the compactness of weakly localized operators on our Fock spaces.

Throughout this paper, we write $A \lesssim B$ for two quantities A and B if there is a constant $C > 0$ such that $A \leq CB$. Furthermore, $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ are satisfied.

2. Preliminaries and Basic Estimates

In this section, we will give some useful estimates for our proofs. For $z \in \mathbb{C}^n$, set

$$\rho_\phi(z) = \sup\left\{ r > 0 : \sup_{w\in B(z,r)} \Delta\phi(w) \leq r^{-2} \right\}. \tag{11}$$

In the following, we write $\rho(z)$ instead of $\rho_\phi(z)$ for short.

By [11] (see also [12]), we have the following consequences.

Lemma 1. *Let ϕ be as defined in (2). Then, the function ρ satisfies the following properties:*

(A) There exists $M > 0$ such that

$$\sup_{z\in\mathbb{C}^n} \rho(z) \leq M. \tag{12}$$

(B) The function ρ is Lipschitz, that is

$$|\rho(z)-\rho(w)| \le |z-w|,\ z, w \in \mathbb{C}^n. \tag{13}$$

(C) For $r \in (0,1)$ and $w \in B(z, r\rho(z))$, there holds

$$(1-r)\rho(z) \le \rho(w) \le (1+r)\rho(z). \tag{14}$$

(D) There exist $a, b > 0$ such that

$$|z|^{-a} \lesssim \rho(z) \lesssim |z|^b, \text{for } |z| > 1. \tag{15}$$

Let $r > 0$, we write $B^r(z) = B(z, r\rho(z))$ and $B(z) = B^1(z)$. In fact, it is easily obtained from estimate (14) that there is some constant c_r such that $c_r^{-1}\rho(z) \le \rho(w) \le c_r\rho(z)$, where $c_r = (1-r)^{-1}$ for any $r \in (0,1)$. That is for every $r \in (0,1)$, we have $\rho(w) \simeq \rho(z)$ whenever $w \in B^r(z)$. Besides, (14) and the triangle inequality give m_1 and m_2 so that

$$B(z) \subset B^{m_1}(w) \text{ and } B(w) \subset B^{m_2}(z), \text{whenever } w \in B(z). \tag{16}$$

Given $r > 0$, there is a sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathbb{C}^n$ such that $\{B^r(a_k)\}_k$ covers $\mathbb{C}^n$, and the balls $\{B^{r/5}(a_k)\}_k$ are pairwise disjoint. We say the sequence $\{a_k\}_k$ is an r-lattice. For the r-lattice $\{a_k\}_k$ and $m > 0$, there exists some integer N such that any z in $\mathbb{C}^n$ belongs to at most N balls of $\{B^{mr}(a_k)\}_k$. That is, for every $z \in \mathbb{C}^n$,

$$\sum_{k=1}^{\infty} \chi_{B^{mr}(a_k)}(z) \le N. \tag{17}$$

Now, we are going to state the properties of the reproducing kernel K_z. Let $\phi \in W$, and it follows from [11–13] that

(A) For $z, w \in \mathbb{C}^n$, there are constants $\epsilon, \alpha > 0$ such that

$$|K_z(w)| \lesssim \frac{e^{\phi(z)+\phi(w)}}{\rho(z)^n\rho(w)^n} e^{-\epsilon\left(\frac{|w-z|}{\rho(z)}\right)^{\alpha}}. \tag{18}$$

(B) For $z \in \mathbb{C}^n$, there exists $\beta \in (0,1)$ such that

$$|K_z(w)| \simeq \frac{e^{\phi(z)+\phi(w)}}{\rho(z)^{2n}}, \quad w \in B^{\beta}(z). \tag{19}$$

(C) For $0 < p \le \infty$, there holds

$$\|K_z\|_{p,\phi} \simeq e^{\phi(z)}\rho(z)^{2n(1/p-1)}, \quad z \in \mathbb{C}^n. \tag{20}$$

With the help of Lemmas 1 and 2 in [12], we get the following lemma.

Lemma 2. *Given $p, \alpha > 0$ and $k \in \mathbb{R}$, there exists $C > 0$ such that for $z \in \mathbb{C}^n$*

$$\int_{\mathbb{C}^n} \rho(w)^k e^{-p\left(\frac{|w-z|}{\rho(z)}\right)^{\alpha}} dv(w) \le C\rho(z)^{2n+k}. \tag{21}$$

For $r > 0$ and $z \in \mathbb{C}^n$, we write $(B^r(z))^c = \mathbb{C}^n \setminus B^r(z)$. Let $d\sigma = \rho^{-2n}dv$. It is directly from ([14], Lemma 2.7) that we have the next estimate.

Lemma 3. *For any $\alpha > 0$, $p > 0$, $k \ge 0$, $r \ge 1$, and $z \in \mathbb{C}^n$, there is a constant $C_{\alpha,p,k}(r) > 0$ such that*

$$\int_{(B^r(z))^c} |w-z|^k e^{-p\left(\frac{|w-z|}{\rho(z)}\right)^{\alpha}} d\sigma(w) \le C_{\alpha,p,k}(r)\rho(z)^k, \tag{22}$$

and $C_{\alpha,p,k}(r) \longrightarrow 0$ whenever $r \longrightarrow \infty$.

We will write $k_{p,z}(w) = K_z(w)/\|K_z\|_{p,\phi}$ for the normalized reproducing kernel at $z \in \mathbb{C}^n$, where $0 < p < \infty$ and $w \in \mathbb{C}^n$.

Lemma 4. *Let $0 < p \le 2$. Then, for every $z \in \mathbb{C}^n$, we have*

$$\int_{(B^r(z))^c} \left|k_{p,z}(w)e^{-\phi(w)}\right|^p dv(w) \to 0 \tag{23}$$

as $r \longrightarrow \infty$.

Proof. By joining (18) and (20), we have

$$\begin{aligned}\int_{(B^r(z))^c} \left|k_{p,z}(w)e^{-\phi(w)}\right|^p dv(w) &\simeq \int_{(B^r(z))^c} \left|\frac{K_z(w)}{e^{\phi(z)+\phi(w)}}\right|^p \rho(z)^{2np-2n} dv(w) \\ &\lesssim \rho(z)^{np-2n} \int_{(B^r(z))^c} \rho(w)^{2n-pn} e^{-p\left(\frac{|w-z|}{\rho(z)}\right)^{\alpha}} d\sigma(w) \\ &\lesssim \rho(z)^{np-2n} \int_{(B^r(z))^c} e^{-p\left(\frac{|w-z|}{\rho(z)}\right)^{\alpha}} d\sigma(w).\end{aligned} \tag{24}$$

Here, the last step is from the estimate (12). Thus, the assertion follows from Lemma 3 with $k = 0$ for any fixed $z \in \mathbb{C}^n$.

The next lemma is immediately from ([12], Lemma 4) (see also ([11], Lemma 2) for any $r > 0$.

Lemma 5. *For $0<p<\infty$, there is a constant $C>0$ such that for each $r > 0, f \in \mathcal{H}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, we have*

$$|f(z)|e^{-\phi(z)} \le \frac{c}{(r\rho(z))^{2n/p}} \left(\int_{B^r(z)} \left|f(w)e^{-\phi(w)}\right|^p dv(w)\right)^{1/p}. \tag{25}$$

For $r>0$ and some domain $\Omega \subset \mathbb{C}^n$, write $\Omega_r^+ = \bigcup_{z\in\Omega} B^r(z)$. Let $d(\cdot,\cdot)$ be the Euclidean distance, and we have the following lemma.

Lemma 6. *For $0 < p \le 1$, $0 < r < 1$, and $\alpha \in \mathbb{R}$, there is some constant C (depending on p, n, and r) such that for any domain $\Omega \subset \mathbb{C}^n$ and $f \in \mathscr{H}(\mathbb{C}^n)$,*

$$\left(\int_\Omega | f(w)e^{-\phi(w)}\rho(w)^\alpha | dv(w)\right)^p \le C\int_{\Omega_r^+} \left|f(w)e^{-\phi(w)}\rho(w)^\alpha\right|^p \rho(w)^{(2p-2)n} dv(w). \tag{26}$$

Proof. Consider the r-lattice $\mathscr{L} = \{z_1, z_2, \cdots, z_j, \cdots\}$ in $\mathbb{C}^n$. For $0 < r < 1$ and $z \in \mathbb{C}^n$, we get $\rho(w) \simeq \rho(z)$ whenever $w \in B^r(z)$. By letting $w \in B^r(z_j)$, it follows from (16) that

$$B^r(z_j) \subset B^{mr}(w) \subset B^{m^2 r}(z_j), \tag{27}$$

where $m = m(r) > 1$. Also notice that $(a+b)^p \le a^p + b^p$ for positive a, b and $0 < p \le 1$. Let r be sufficiently small so that $m^2 r < 1$. Thus, the above inequality, (17) and (25) show

$$\begin{aligned}
&\left(\int_\Omega \left|f(w)e^{-\phi(w)}\rho(w)^\alpha\right| dv(w)\right)^p \\
&\le \left(\sum_{z_j\in\mathscr{L}} \int_{\Omega\cap B^r(z_j)} | f(w)e^{-\phi(w)}\rho(w)^\alpha | dv(w)\right)^p \\
&\lesssim C \sum_{z_j\in\mathscr{L}, d(z_j,\Omega)<r\rho(z_j)} \sup_{|w-z_j|\le r\rho(z_j)} \left|f(w)e^{-\phi(w)}\right|^p \rho(w)^{(2n+\alpha)p} \\
&\lesssim C \sum_{z_j\in\mathscr{L}, d(z_j,\Omega)<r\rho(z_j)} \int_{B^{m^2 r}(z_j)} \left|f(z)e^{-\phi(z)}\right|^p \rho(z_j)^{(2n+\alpha)p-2n} dv(z) \\
&\simeq C \sum_{z_j\in\mathscr{L}, d(z_j,\Omega)<r\rho(z_j)} \int_{B^{m^2 r}(z_j)} \left|f(z)e^{-\phi(z)}\right|^p \rho(z)^{(2n+\alpha)p-2n} dv(z) \\
&\le C\int_{\Omega_r^+} \sum_{z_j\in\mathscr{L}, d(z_j,\Omega)<r\rho(z_j)} \chi_{B^{m^2 r}(z_j)}(w) \left|f(w)e^{-\phi(w)}\rho(w)^\alpha\right|^p \rho(w)^{(2p-2)n} dv(w) \\
&\le CN\int_{\Omega_r^+} \left|f(w)e^{-\phi(w)}\rho(w)^\alpha\right|^p \rho(w)^{(2p-2)n} dv(w),
\end{aligned} \tag{28}$$

which completes the proof.

3. Bergman Projection and Duality

The paper [12] points out that the Bergman projection P is bounded on $\mathscr{F}_\phi^p$ for $0 < p < \infty$. And there is no answer to whether $Pf = f$ on $\mathscr{F}_\phi^p$. In what follows, we use the classical H*örmander* theorem to prove that the projection P is an identity operator on $\mathscr{F}_\phi^p (0 < p < \infty)$.

Theorem 7 ([15], Theorem 4.2.6). *Let X be a pseudo-convex open set in $\mathbb{C}^n$, φ a plurisubharmonic function in X, and $a > 0$. If ψ is in $L^2_{(0,1)}$ locally in X and $\bar\partial\psi = 0$, then the equation $\bar\partial u = \psi$ has a solution $u \in L^2_{loc}(X)$ such that*

$$a\int_X |u(z)|^2 e^{-\varphi(z)}\left(1+|z|^2\right)^{-a} dv(z) \le \int_X |\psi(z)|^2 e^{-\varphi(z)}\left(1+|z|^2\right)^{2-a} dv(z). \tag{29}$$

For $1 < p < \infty$, we let q be the conjugate exponent of p such that $1/p + 1/q = 1$.

Theorem 8. *If $0 < p < \infty$ and $f \in \mathscr{F}_\phi^p$, then $Pf = f$.*

Proof. Suppose that $h_0(z) \in C^\infty(\mathbb{C}^n)$ satisfying $h_0(z) = 1$ if $|z| \le R (R > 1)$, $0 < h_0(z) < 1$ if $R < |z| < R+1$, $h_0(z) = 0$ if $|z| \ge R+1$ and

$$\left|\bar\partial h_0(z)\right|^2 \lesssim h_0(z). \tag{30}$$

Set $\Omega_j = \{z : |z| \le jR\}$ where $j = 1, 2, \cdots$. It follows that for any $z \in \Omega_j$,

$$h_{0,j}(z) := h_0\left(\frac{z}{j}\right) = 1. \tag{31}$$

Because of (15), there are $a, b > 0$ so that $|z|^{-a} \lesssim \rho(z) \lesssim |z|^b$ whenever $|z| > 1$. Indeed, by choosing $r > 0$ sufficiently small, we obtain $(\Omega_j^c)_r^+ \subset \{w : |w| > j(R-1)\}$ when j is large enough.

If $0 < p \le 1$, then by Lemma 6, we have

$$\begin{aligned}
\left|Pf(z) - P(fh_{0,j})(z)\right|^p &\le \left(\int_{|w|>jR} |f(w)K(z,w)| e^{-2\phi(w)} dv(w)\right)^p \\
&\lesssim \int_{(\Omega_j^c)_r^+} |f(w)K(z,w)|^p e^{-2p\phi(w)} \rho(w)^{(2p-2)n} dv(w).
\end{aligned} \tag{32}$$

This together with (18), Lemma 2 and Fubini's theorem give

$$\begin{aligned}
\left\|Pf - P(fh_{0,j})\right\|_{p,\phi}^p &\lesssim \int_{\mathbb{C}^n}\int_{(\Omega_j^c)_r^+} |f(w)K(z,w)|^p e^{-2p\phi(w)} \rho(w)^{(2p-2)n} e^{-p\phi(z)} dv(w) dv(z) \\
&= \int_{(\Omega_j^c)_r^+} |f(w)|^p e^{-2p\phi(w)} \rho(w)^{(2p-2)n} \int_{\mathbb{C}^n} |K(z,w)|^p e^{-p\phi(z)} dv(z) dv(w) \\
&\lesssim \int_{(\Omega_j^c)_r^+} |f(w)|^p e^{-p\phi(w)} dv(w) \le \int_{|w|>j(R-1)} |f(w)|^p e^{-p\phi(w)} dv(w).
\end{aligned} \tag{33}$$

We now let $1 < p < \infty$. Notice that estimate (18) and Lemma 2 indicate

$$\int_{\mathbb{C}^n} |K(z,w)| e^{-\phi(z)-\phi(w)} dv(w) \lesssim \int_{\mathbb{C}^n} \rho(z)^{-n} \rho(w)^{-n} e^{-\epsilon\left(\frac{|w-z|}{\rho(z)}\right)^\alpha} dv(w) < \infty. \tag{34}$$

So, Hölder's inequality and Fubini's theorem show

$$\begin{aligned}\|Pf-P(fh_{0,j})\|_{p,\phi}^p &\lesssim \int_{\mathbb{C}^n}\int_{\Omega_j^c}|f(w)|^p e^{-p\phi(w)}|K(z,w)|e^{-\phi(z)-\phi(w)}dv(w)\\ &\quad\times\left(\int_{\Omega_j^c}|K(z,w)|e^{-\phi(z)-\phi(w)}dv(w)\right)^{\frac{p}{q}}dv(z)\\ &\lesssim \int_{\Omega_j^c}|f(w)|^p e^{-p\phi(w)}\int_{\mathbb{C}^n}|K(z,w)|e^{-\phi(z)-\phi(w)}dv(z)dv(w)\\ &\lesssim \int_{\Omega_j^c}|f(w)|^p e^{-p\phi(w)}dv(w)\le \int_{|w|>j(R-1)}|f(w)|^p e^{-p\phi(w)}dv(w).\end{aligned} \tag{35}$$

And then, for $0<p<\infty$, we get

$$\lim_{j\longrightarrow\infty}\|Pf-P(fh_{0,j})\|_{p,\phi}^p \le C\lim_{j\longrightarrow\infty}\int_{|w|>j(R-1)}|f(w)|^p e^{-p\phi(w)}dv(w)=0. \tag{36}$$

This combined with (25) means that

$$|Pf(z)-P(fh_{0,j})(z)|\le Ce^{\phi(z)}\rho(z)^{-\frac{2n}{p}}\|Pf-P(fh_{0,j})\|_{p,\phi}\longrightarrow 0, \tag{37}$$

as $j\longrightarrow\infty$.

On the other hand, applying Theorem 7 with $a=2$ to the solution of $\bar\partial u=\psi$ in L_ϕ^2, we have

$$\int_X |u(w)|^2 e^{-2\phi(w)}\left(1+|w|^2\right)^{-2}dv(w)\le\int_X|\psi(w)|^2e^{-2\phi(w)}dv(w). \tag{38}$$

Hence, for $z\in\Omega_j$ and let j be sufficiently large, it follows immediately from esitmates (25), (30), and Lemma 2 that

$$\begin{aligned}&|f(z)-P(fh_{0,j})(z)|^2e^{-2\phi(z)}\rho(z)^{\frac{4n}{p}}\\ &=|fh_{0,j}(z)-P(fh_{0,j})(z)|^2e^{-2\phi(z)}\rho(z)^{\frac{4n}{p}}\\ &\lesssim\rho(z)^{2n\left(\frac{2}{p}-1\right)}\int_{B^r(z)}|fh_{0,j}(w)-P(fh_{0,j})(w)|^2\left(1+|w|^2\right)^{-2}e^{-2\phi(w)}e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\alpha}dv(w)\\ &\lesssim\rho(z)^{2n\left(\frac{2}{p}-1\right)}\int_{\mathbb{C}^n}|fh_{0,j}(w)-P(fh_{0,j})(w)|^2\left(1+|w|^2\right)^{-2}e^{-2\phi(w)}e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\alpha}dv(w)\\ &\lesssim\rho(z)^{2n\left(\frac{2}{p}-1\right)}\int_{\mathbb{C}^n}\left|\frac{1}{j}f(w)\bar\partial h_0\left(\frac{w}{j}\right)\right|^2e^{-2\phi(w)}e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\alpha}dv(w)\\ &\lesssim\frac{1}{j^2}\rho(z)^{2n\left(\frac{2}{p}-1\right)}\|f\|_{p,\phi}^2\int_{jR<|w|<j(R+1)}\rho(w)^{\frac{-4n}{p}}e^{-\left(\frac{|z-w|}{\rho(z)}\right)^\alpha}dv(w)\lesssim\frac{1}{j^2}\|f\|_{p,\phi}^2.\end{aligned} \tag{39}$$

By combining the above estimate and (37), we finally obtain

$$|Pf(z)-f(z)|\le|Pf(z)-P(fh_{0,j})(z)|+|P(fh_{0,j})(z)-f(z)|\longrightarrow 0, \tag{40}$$

as $j\longrightarrow\infty$. This ends the proof.

We now proceed to identify the dual space of $\mathcal{F}_\phi^p$ when $0<p<\infty$. Arguing as in [16], we let

$$\mathcal{F}_{s,\phi}^\infty=\left\{f\in\mathcal{H}(\mathbb{C}^n):\ \|f\|_{\infty,s,\phi}=\sup_{z\in\mathbb{C}^n}\left(|f(z)|\rho(z)^se^{-\phi(z)}\right)<\infty\right\}. \tag{41}$$

Theorem 9. *Suppose $0<p\le 1$. Then, $(\mathcal{F}_\phi^p)^*=\mathcal{F}_{2n-2n/p,\phi}^\infty$.*

Proof. For any $f\in\mathcal{F}_\phi^p$, consider $L_g(\cdot)=\langle\cdot,g\rangle$ where $g\in\mathcal{F}_{2n-2n/p,\phi}^\infty$. Then, (25) says

$$\begin{aligned}|L_g(f)|&\le\int_{\mathbb{C}^n}|f(z)g(z)|e^{-2\phi(z)}dv(z)\\ &\le\left(\sup_{z\in\mathbb{C}^n}|f(z)|e^{-\phi(z)}\rho(z)^{\frac{2n}{p}}\right)^{1-p}\int_{\mathbb{C}^n}|g(z)|\rho(z)^{2n-\frac{2n}{p}}e^{-\phi(z)}\left|f(z)e^{-\phi(z)}\right|^pdv(z)\\ &\le C\|g\|_{\infty,2n-\frac{2n}{p},\phi}\|f\|_{p,\phi}.\end{aligned} \tag{42}$$

The above inqeuality shows that L_g is a bounded linear functional on $\mathcal{F}_\phi^p$ and $\|L_g\|\le C\|g\|_{\infty,2n-2n/p,\phi}$.

For $w\in\mathbb{C}^n$, define $g(w)=L(K(\bar{\cdot},w))$ where L is a bounded linear functional on $\mathcal{F}_\phi^p$. Pick an $r>0$ such that $w+\Delta w\in B^r(w)$. For some $m>0$ and every $z\in\mathbb{C}^n$, using Cauchy's estimates, we have

$$\begin{aligned}\left|\frac{K(w+\Delta w,z)-K(w,z)}{\Delta w}\right|^p&\le\sup_{0\le t\le 1}\left|\frac{\partial K}{\partial w}(w+t\Delta w,z)\right|^p\\ &\le\frac{1}{(mr\rho(w))^p}\left(\sup_{|\xi-w|=mr\rho(w)}|K(\xi,z)|\right)^p\\ &\lesssim\frac{1}{r^{2n+p}\rho(w)^{2n+p}}\int_{B^{m^2r}(w)}|K(u,z)|^pdv(u).\end{aligned} \tag{43}$$

We note that for any fixed $w\in\mathbb{C}^n$, the function $\left(\int_{B^{m^2r}(w)}|K(u,\cdot)|^pdv(u)\right)^{1/p}$ is in L_ϕ^p. Fix w and z, and we get

$$\lim_{\Delta w\longrightarrow 0}\frac{K(w+\Delta w,z)-K(w,z)}{\Delta w}=\frac{\partial K}{\partial w}(w,z). \tag{44}$$

Thus Lebesgue dominated convergence theorem indicates

$$\lim_{\Delta w\longrightarrow 0}\left\|\frac{K(w+\Delta w,\cdot)-K(w,\cdot)}{\Delta w}-\frac{\partial K}{\partial w}(w,\cdot)\right\|_{p,\phi}=0. \tag{45}$$

Hence, for any $L\in(\mathcal{F}_\phi^p)^*$, we obtain $g(w)\in\mathcal{H}(\mathbb{C}^n)$ since

$$g'(w)=\lim_{\Delta w\longrightarrow 0}\frac{L(K(\cdot,\overline{w+\Delta w}))-L(K(\bar\cdot,w))}{\Delta w}=L\left(\overline{\frac{\partial K}{\partial \bar w}}(w,\cdot)\right), \tag{46}$$

and $|g(w)| \leq \|L\| \|K(\cdot, w)\|_{p,\phi} \leq C\|L\| e^{\phi(w)} \rho(w)^{2n(1/p-1)}$. The result is

$$g \in \mathcal{F}^{\infty}_{2n-\frac{2n}{p},\phi} \text{ and } \|g\|_{\infty,2n-\frac{2n}{p},\phi} \leq C\|L\|. \tag{47}$$

To complete the proof, it only remains to show that

$$L(f) = \int_{\mathbb{C}^n} f(w)\bar{g}(w) e^{-2\phi(w)} dv(w). \tag{48}$$

Let $\{a_n\}_n$ be an r-lattice. For $0 < R < \infty$, we consider

$$S_{r,R}(f)(z) = \sum_n K(z, a_n) \int_{\left(B^r(a_n)\setminus \bigcup_{j<n} B^r(a_j)\right) \bigcap B(0,R)} f(w) e^{-2\phi(w)} dv(w). \tag{49}$$

Since $\bar{B(0, R)}$ is compact, there exists $k > 0$ so that $\bigcup_{n=1}^{k} B^r(a_n) \supset B(0, R)$. Moreover, we see that $S_{r,R}(f) \in \mathcal{H}(\mathbb{C}^n)$ because it is actually a finite sum of analytic functions. And there is $R' > R$ such that $\bigcup_{n=1}^{k} B^r(a_n) \subset B(0, R')$. It follows from (43) that

$$\begin{aligned}
&\left|S_{r,R}(f)(z) - P\left(f\chi_{B(0,R)}\right)(z)\right| \\
&= \left|\sum_n \int_{\left(B^r(a_n)\setminus \bigcup_{j<n} B^r(a_j)\right)\bigcap B(0,R)} (K(z, a_n) - K(z, w)) f(w) e^{-2\phi(w)} dv(w)\right| \\
&\leq \left(\sup_{w\in B(0,R)} |f(w)e^{-2\phi(w)}|\right) \sum_n \int_{\left(B^r(a_n)\setminus \bigcup_{j<n} B^r(a_j)\right)\bigcap B(0,R)} |K(z, a_n) - K(z, w)| dv(w) \\
&\lesssim \left(\sup_{w\in B(0,R)} \left|f(w)e^{-2\phi(w)}\right|\right) \sum_{B^r(a_n)\bigcap B(0,R)} \left(\sup_{|\xi|<2R'} |K(\xi, z)|\right) [r\rho(a_n)]^{2n} \\
&\leq C(R)\left(r^{2n} \sup_{|\xi|<2R'} |K(\xi, z)|\right)
\end{aligned} \tag{50}$$

goes to 0 by letting $r \longrightarrow 0$, where $\chi_{B(0,R)}$ denotes the characteristic function for the ball $B(0, R)$. This means that

$$\lim_{r\longrightarrow 0} S_{r,R}(f) = P\left(f\chi_{B(0,R)}\right). \tag{51}$$

It is clear that, for each fixed $z \in \mathbb{C}^n$, $\sup_{|a_n|\leq 2R'} |K(a_n, z)|$ is in L^p_ϕ. Hence, by the following estimate,

$$\begin{aligned}
|S_{r,R}(f)(z)| &\leq \left(\sup_{w\in B(0,R)} \left|f(w)e^{-2\phi(w)}\right|\right) \sum_{\left(B^r(a_n)\setminus \bigcup_{j<n} B^r(a_j)\right)\bigcap B(0,R)} \int_{B^r(a_n)} |K(a_n, z)| dv(w) \\
&\leq C(R) \sup_{|a_n|\leq 2R'} |K(a_n, z)|,
\end{aligned} \tag{52}$$

and the Lebesgue dominated convergence theorem we deduce

$$\lim_{r\longrightarrow 0} \|S_{r,R}(f) - P\left(f\chi_{B(0,R)}\right)\|_{p,\phi} = 0. \tag{53}$$

Furthermore, we claim that

$$\begin{aligned}
&\left|L(S_{r,R}(f))(\cdot) - \int_{|w|\leq R} f(w) L(K(\cdot, w)) e^{-2\phi(w)} dv(w)\right| \\
&\leq \left(\sup_{w\in B^r(a_n)\bigcap B(0,R)} |L(K(\cdot, a_n) - K(\cdot, w))|\right) \int_{|w|\leq R} |f(w)| e^{-2\phi(w)} dv(w) \\
&\leq C(R)\left(\|L\| \sup_{w\in B^r(a_n)\bigcap B(0,R)} \|K(\cdot, a_n) - K(\cdot, w)\|_{p,\phi}\right) \longrightarrow 0,
\end{aligned} \tag{54}$$

as $r \longrightarrow 0$. Here, the last assertion follows from fact that $\|K(\cdot, a_n) - K(\cdot, w)\|_{p,\phi} \longrightarrow 0$ whenever $r \longrightarrow 0$. To see this, by letting $r \longrightarrow 0$ and $w \in B^r(a_n)$, we then get $K_w \longrightarrow K_{a_n}$. Indeed, (18) gives us a dominating function, and it is from Lemma 2 that the function is in L^p_ϕ since $\|(e^{\phi(w)+\phi(z)}/\rho(w)^n \rho(z)^n) e^{-\epsilon(|z-w|/\rho(w))^\alpha}\|_{p,\phi} \lesssim e^{\phi(w)} \rho(w)^{2n/p-2n}$ for any fixed w. Then, the desired assertion holds by Lebesgue dominated convergence theorem again. So, we have

$$\lim_{r\longrightarrow 0} L(S_{r,R}(f)) = \int_{|w|\leq R} f(w) L(K(\cdot, w)) e^{-2\phi(w)} dv(w). \tag{55}$$

Therefore, we have by Theorem 8, (51) and (55) that

$$\begin{aligned}
L(f) &= \lim_{R\longrightarrow\infty} L\left(P\left(f\chi_{B(0,R)}\right)\right) = \lim_{R\longrightarrow\infty} \lim_{r\longrightarrow 0} L(S_{r,R}(f)) \\
&= \lim_{R\longrightarrow\infty} \int_{|w|\leq R} f(w) L(K(\cdot, w)) e^{-2\phi(w)} dv(w) \\
&= \int_{\mathbb{C}^n} f(w) L(K(\cdot, w)) e^{-2\phi(w)} dv(w).
\end{aligned} \tag{56}$$

This finishes the proof.

Theorem 10. *Suppose $1 < p < \infty$. Then, $(\mathcal{F}^p_\phi)^* = \mathcal{F}^q_\phi$ under the pairing*

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\bar{g}(z) e^{-2\phi(z)} dv(z). \tag{57}$$

Proof. If $g \in \mathcal{F}^q_\phi$, define

$$L_g(\cdot) = \langle \cdot, g\rangle. \tag{58}$$

For any $f \in \mathcal{F}^p_\phi$, Hölder's inequality gives

$$|L_g(f)| \leq \int_{\mathbb{C}^n} |f(z)\bar{g}(z)| e^{-2\phi(z)} dv(z) \leq \|g\|_{q,\phi} \|f\|_{p,\phi}. \tag{59}$$

This means that L_g is a bounded linear functional on $\mathcal{F}^p_\phi$ and $\|L_g\| \leq \|g\|_{q,\phi}$.

On the other hand, let $L : \mathscr{F}^p_\phi \longrightarrow \mathbb{C}$ be a bounded linear functional. The Hahn-Banach extension theorem implies that L can be extended to a bounded linear functional $\tilde{L}$ on L^p_ϕ. It follows from the duality theory of L^p_ϕ that there is a function $G \in L^q_\phi$ such that $\|G\|_{q,\phi} \le \|\tilde{L}\| = \|L_g\|$ and

$$\tilde{L}(f) = \langle f, G\rangle, f \in L^p_\phi. \quad (60)$$

Set $g = PG$, then $\|g\|_{q,\phi} \le \|P\|\|G\|_{q,\phi}$ since P is bounded. Also, note that Theorem 8 shows $Pf = f$ for $f \in \mathscr{F}^p_\phi$. So, (60) indicates

$$\tilde{L}(f) = \langle f, G\rangle = \langle Pf, G\rangle = \langle f, PG\rangle = \langle f, g\rangle, f \in \mathscr{F}^p_\phi. \quad (61)$$

This completes the proof.

Corollary 11. *Suppose $0 < p < \infty$. Then, the linear span E of all reproducing kernel functions $K_z(\cdot)$ is dense in $\mathscr{F}^p_\phi$.*

Proof. Let $0 < p \le 1$. It is immediately from Theorem 8 and the proof of Theorem 9, for any $f \in \mathscr{F}^p_\phi$, that

$$\begin{aligned}&\lim_{R\longrightarrow\infty}\lim_{r\longrightarrow 0}\|S_{r,R}(f) - f\|_{p,\phi}\\ &\le \lim_{R\longrightarrow\infty}\lim_{r\longrightarrow 0}\left(\left\|S_{r,R}(f) - P\left(f\chi_{B(0,R)}\right)\right\|_{p,\phi} + \left\|P\left(f\chi_{B(0,R)}\right) - f\right\|_{p,\phi}\right) = 0.\end{aligned} \quad (62)$$

Next, we assume that $p > 1$. By Theorem 10 and the Hahn-Banach theorem, it suffices to show that for any $g \in E$, we have $f = 0$ if $f \in \mathscr{F}^q_\phi$ satisfies $\langle f, g\rangle = 0$. This follows from the fact that $f(z) = Pf(z) = \langle f, K_z\rangle = 0$ for every $z \in \mathbb{C}^n$.

4. Localization Operators

In this section, we will explore some properties of weakly localized operators on our Fock spaces. In particular, we will show the algebraic properties of these localization operators.

Before stating weakly localized operators, we consider firstly the following proposition.

Proposition 12. *Suppose $0 < p \le 1$. Then,*

$$\|K(\cdot, w)\|_{2n-\frac{2n}{p},\phi} \simeq e^{\phi(w)}\rho(w)^{-\frac{2n}{p}}. \quad (63)$$

Proof. It is from (11) that there exists some $r_0 > 0$ such that $\rho(u) > r_0$ for each $u \in \mathbb{C}^n$. Fix $w \in \mathbb{C}^n$, and we have

$$\rho(w)^{\frac{2n}{p}-n}\rho(u)^{n-\frac{2n}{p}}e^{-\epsilon\left(\frac{|u-w|}{\rho(w)}\right)^\alpha} < (r_0)^{n-\frac{2n}{p}}\rho(w)^{\frac{2n}{p}-n}e^{-\epsilon\left(\frac{|u-w|}{\rho(w)}\right)^\alpha}. \quad (64)$$

For every $w \in \mathbb{C}^n$, let $r > 0$ be sufficiently large and let $|u - w| \ge r$, and it follows that estimate (18) together with (64) gives

$$|K(u, w)|e^{-\phi(u)}\rho(u)^{2n-\frac{2n}{p}} < e^{\phi(w)}\rho(w)^{-\frac{2n}{p}}\left((r_0)^{n-\frac{2n}{p}}\rho(w)^{\frac{2n}{p}-n}e^{-\epsilon\left(\frac{r}{\rho(w)}\right)^\alpha}\right) \quad (65)$$

is a dimensionless. For fixed $w \in \mathbb{C}^n$, we get $r \simeq \beta\rho(w)$, where $\beta \in (0, 1)$. Hence, Theorem 9 together with (19) shows that

$$\begin{aligned}\|K(u, w)\|2n - \frac{2n}{p}, \phi &= \sup_{z\in\mathbb{C}^n}\left(|K(u, w)\mid e^{-\phi(u)}\rho(u)^{2n-\frac{2n}{p}}\right)\\ &\simeq \sup_{u\in B(w,r)}\left(|K(u, w)\mid e^{-\phi(u)}\rho(u)^{2n-\frac{2n}{p}}\right)\\ &\simeq e^{\phi(w)}\rho(w)^{-\frac{2n}{p}},\end{aligned} \quad (66)$$

which is the desired estimate.

Now, with the above preparations, we are ready for the definition of weakly localized operators.

Definition 13. Let $0 < p \le 1$. A linear operator T on $\mathscr{F}^p_\phi$ is called weakly localized for $\mathscr{F}^p_\phi$ if

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\left\langle Tk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)<\infty,\\ &\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\left\langle T^*k_{2n-\frac{2n}{p},z}, k_{p,w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)<\infty,\end{aligned} \quad (67)$$

$$\begin{aligned}&\lim_{r\longrightarrow\infty}\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}\left|\left\langle Tk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w) = 0,\\ &\lim_{r\longrightarrow\infty}\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}\left|\left\langle T^*k_{2n-\frac{2n}{p},z}, k_{p,w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w) = 0,\end{aligned} \quad (68)$$

where

$$k_{2n-\frac{2n}{p},\cdot}(u) \simeq \frac{K(u, \cdot)}{e^{\phi(\cdot)}\rho(\cdot)^{-(2n/p)}}, u \in \mathbb{C}^n. \quad (69)$$

Recall that, for $1 < p < \infty$, q is the conjugate exponent of p so that $1/p + 1/q = 1$.

Definition 14. Suppose $1 < p < \infty$. A linear operator T on $\mathscr{F}^p_\phi$ is called weakly localized for $\mathscr{F}^p_\phi$ if

$$\begin{aligned}&\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)<\infty,\\ &\sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n}|\langle T^*k_{q,z}, k_{p,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)<\infty,\end{aligned} \quad (70)$$

$$\lim_{r\longrightarrow\infty}\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}|\langle Tk_{p,z},k_{q,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)=0,$$
$$\lim_{r\longrightarrow\infty}\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}|\langle T^*k_{q,z},k_{p,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)=0. \tag{71}$$

Next, we are going to answer the questions raised at the beginning of the paper in our Fock spaces. In fact, each set of these weakly localized operators on $\mathscr{F}^p_\phi$ is an algebra.

Theorem 15. *Suppose $0<p<\infty$. Then, weakly localized operators on $\mathscr{F}^p_\phi$ form an algebra.*

Proof. Suppose operators T and S are weakly localized. So, it remains to show that TS is a weakly localized operator because the linear combination of two weakly localized operators is also a weakly localized operator.

We let $0<p\le 1$. It follows from (68) that there is some $r>0$ such that

$$\int_{\left(B^{\frac{r}{a}}(z)\right)^c}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)<\varepsilon, \tag{72}$$

where $a=(1+r)^{1/2}+1$ and any $\varepsilon>0$ (when $1<p<\infty$, estimate (71) gives an analogous representation). For $z,x\in\mathbb{C}^n$, if $x\in B^{r/a}(z)$ then $\rho(x)\le(1+(r/a))\rho(z)$ and by the triangle inequality, we have $B^{r/a}(x)\subset B^r(z)$. That is, $(B^r(z))^c\subset(B^{r/a}(x))^c$ whenever $x\in B^{r/a}(z)$.

By joining Lemma 6 and Fubini's theorem, we get

$$\begin{aligned}
&\int_{(B^r(z))^c}\left|\left\langle TSk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\\
&=\int_{(B^r(z))^c}\left|\left\langle Sk_{p,z},T^*k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\\
&\simeq\int_{(B^r(z))^c}\left|\langle SK_z,T^*K_w\rangle_{\mathscr{F}^2_\phi}\right|^p e^{-p\phi(z)-p\phi(w)}\rho(z)^{2np-2n}dv(w)\\
&=\int_{(B^r(z))^c}\left|\int_{\mathbb{C}^n}\langle SK_z,K_x\rangle_{\mathscr{F}^2_\phi}\langle K_x,T^*K_w\rangle_{\mathscr{F}^2_\phi}e^{-2\phi(x)}dv(x)\right|^p e^{-p\phi(z)-p\phi(w)}\rho(z)^{2np-2n}dv(w)\\
&\simeq\int_{(B^r(z))^c}\left|\int_{\mathbb{C}^n}\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\left\langle k_{p,x},T^*k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\rho(x)^{-2n}dv(x)\right|^p\rho(w)^{-2n}dv(w)\\
&\lesssim\int_{(B^r(z))^c}\int_{\mathbb{C}^n}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\left|\left\langle k_{p,x},T^*k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}dv(x)\rho(w)^{-2n}dv(w)\\
&=\int_{\mathbb{C}^n}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}\int_{(B^r(z))^c}\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)dv(x).
\end{aligned} \tag{73}$$

Since T and S are weakly localized, hence

$$\begin{aligned}
I_1&:=\int_{B^{\frac{r}{a}}(z)}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}\int_{(B^r(z))^c}\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)dv(x)\\
&\le\int_{B^{\frac{r}{a}}(z)}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}\int_{\left(B^{\frac{r}{a}}(x)\right)^c}\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)dv(x)\\
&<\varepsilon\int_{B^{\frac{r}{a}}(z)}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}dv(x),
\end{aligned}$$

$$\begin{aligned}
I_2&:=\int_{\left(B^{\frac{r}{a}}(z)\right)^c}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}\int_{(B^r(z))^c}\\
&\quad\cdot\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)dv(x)\\
&\le\int_{\left(B^{\frac{r}{a}}(z)\right)^c}\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}\int_{\mathbb{C}^n}\\
&\quad\cdot\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)dv(x)\\
&\le\left(\sup_{x\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\right)\int_{\left(B^{\frac{r}{a}}(z)\right)^c}\\
&\quad\cdot\left|\left\langle Sk_{p,z},k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(x)^{-2n}dv(x)\\
&<\varepsilon\left(\sup_{x\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\left\langle Tk_{p,x},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\right).
\end{aligned} \tag{74}$$

Therefore, by combining I_1 and I_2, we get

$$\int_{(B^r(z))^c}\left|\left\langle TSk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)<C\varepsilon, \tag{75}$$

where the constant C does not depend on ε. This means

$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}\left|\left\langle TSk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\longrightarrow 0, \tag{76}$$

when $r\longrightarrow\infty$. Meanwhile, we also get

$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}\left|\left\langle (TS)^*k_{2n-\frac{2n}{p},z},k_{p,w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\longrightarrow 0, \tag{77}$$

whenever $r\longrightarrow\infty$.

On the other hand, let now $1<p<\infty$, by Fubini's theorem, and we have

$$\begin{aligned}
&\int_{(B^r(z))^c}|\langle TSk_{p,z},k_{q,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\
&\simeq\int_{(B^r(z))^c}\left|\int_{\mathbb{C}^n}\langle Sk_{p,z},k_{q,x}\rangle_{\mathscr{F}^2_\phi}\langle k_{p,x},T^*k_{q,w}\rangle_{\mathscr{F}^2_\phi}\rho(x)^{-2n}dv(x)\right|\rho(w)^{-2n}dv(w)\\
&\le\int_{\mathbb{C}^n}|\langle Sk_{p,z},k_{q,x}\rangle_{\mathscr{F}^2_\phi}|\rho(x)^{-2n}\int_{(B^r(z))^c}|\langle Tk_{p,x},k_{q,w}\rangle_{\mathscr{F}^2_\phi}|\rho(w)^{-2n}dv(w)dv(x).
\end{aligned} \tag{78}$$

We split again the above integral on $\mathbb{C}^n$ into the corresponding integrals on $B^{r/a}(z)$ and $(B^{r/a}(z))^c$, then

$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}|\langle TSk_{p,z},k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w),$$
$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}|\langle (TS)^*k_{q,z},k_{p,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w) \tag{79}$$

all go to 0 as $r\longrightarrow\infty$. This ends the proof since others are obvious.

Let WL^ϕ_p denote the algebra generated by weakly localized operators for $\mathcal{F}^p_\phi$. Let T_f be a Toeplitz operator (see [8]) on $\mathcal{F}^p_\phi$, where f is called a symbol function. Then, each WL^ϕ_p contains some special Toeplitz operators.

Theorem 16. *Suppose $0<p<\infty$ and $f\in L^\infty$. Then, Toeplitz operator $T_f\in WL^\phi_p$.*

Proof. We first suppose $0<p\le 1$. Clearly, it suffices to prove that

$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}\left|\left\langle T_f k_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w) \tag{80}$$

converges to 0 as $r\longrightarrow\infty$.

Since $T_f k_{p,z}=P(fk_{p,z})$ for any fixed z, hence Lemma 4 gives that

$$\begin{aligned}&\int_{(B^r(z))^c}\left|\left\langle T_f k_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\\&\lesssim\|f\|^p_{L^\infty}\int_{(B^r(z))^c}\left|\left\langle k_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}^2_\phi}\right|^p\rho(w)^{-2n}dv(w)\\&\simeq\|f\|^p_{L^\infty}\int_{(B^r(z))^c}|k_{p,z}(w)|^pe^{-p\phi(w)}dv(w)\end{aligned} \tag{81}$$

goes to 0 whenever $r\longrightarrow\infty$.

Now, assume that $1<p<\infty$. It is easily obtained from (18), (20), and Lemma 2 that

$$\begin{aligned}&\int_{\mathbb{C}^n}|\langle T_f k_{p,z},k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\&\simeq\int_{\mathbb{C}^n}|f(w)||K_z(w)|e^{-\phi(z)-\phi(w)}\rho(z)^{2n-\frac{2n}{p}}\rho(w)^{-\frac{2n}{q}}dv(w)\\&\lesssim\|f\|_{L^\infty}\int_{\mathbb{C}^n}\rho(z)^{n-\frac{2n}{p}}\rho(w)^{-n-\frac{2n}{q}}e^{-\epsilon\left(\frac{|w-z|}{\rho(z)}\right)^\alpha}dv(w)\lesssim\|f\|_{L^\infty}.\end{aligned} \tag{82}$$

Thus, we only need to show

$$\sup_{z\in\mathbb{C}^n}\int_{(B^r(z))^c}|\langle T_f k_{p,z},k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\longrightarrow 0 \tag{83}$$

as $r\longrightarrow\infty$. In fact, $|w-z|\ge r\rho(z)$ if $w\in(B^r(z))^c$. This together with (82) indicates

$$\begin{aligned}&\int_{(B^r(z))^c}|\langle T_f k_{p,z},k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\&\lesssim\|f\|_{L^\infty}\int_{(B^r(z))^c}\rho(z)^{n-\frac{2n}{p}}\rho(w)^{-n-\frac{2n}{q}}e^{-\frac{\epsilon}{2}\left(\frac{|w-z|}{\rho(z)}\right)^\alpha}e^{-\frac{\epsilon}{2}r^\alpha}dv(w)\\&\lesssim e^{-\frac{\epsilon}{2}r^\alpha}\|f\|_{L^\infty}.\end{aligned} \tag{84}$$

Therefore, the desired conclusion follows when $r\longrightarrow\infty$. This ends the proof.

Remark. Moreover, Theorem 16 indicates that the identity operator is also in WL^ϕ_p. Namely, each algebra WL^ϕ_p possesses an unit.

We next consider the boundedness of operator $T\in WL^\phi_p$ for $0<p<\infty$.

Theorem 17. *If $0<p<\infty$ and $T\in WL^\phi_p$, then T is bounded on $\mathcal{F}^p_\phi$.*

Proof. First, we see that

$$\begin{aligned}Tf(z)&=\langle Tf,K_z\rangle_{\mathcal{F}^2_\phi}=\langle f,T^*K_z\rangle_{\mathcal{F}^2_\phi}\\&=\int_{\mathbb{C}^n}f(w)\langle K_w,T^*K_z\rangle_{\mathcal{F}^2_\phi}e^{-2\phi(w)}dv(w).\end{aligned} \tag{85}$$

Let $0<p\le 1$ and let

$$M_1=\sup_{w\in\mathbb{C}^n}\int_{\mathbb{C}^n}\left|\left\langle Tk_{p,w},k_{2n-\frac{2n}{p},z}\right\rangle_{\mathcal{F}^2_\phi}\right|^p\rho(z)^{-2n}dv(z). \tag{86}$$

Estimate (20) combined with Lemma 6 yields

$$\begin{aligned}\left|Tf(z)e^{-\phi(z)}\right|^p&\le\left(\int_{\mathbb{C}^n}|f(w)\langle K_w,T^*K_z\rangle_{\mathcal{F}^2_\phi}|e^{-2\phi(w)}e^{-\phi(z)}dv(w)\right)^p\\&\simeq\left(\int_{\mathbb{C}^n}|f(w)\left\langle K_w,T^*k_{2n-\frac{2n}{p},z}\right\rangle_{\mathcal{F}^2_\phi}|e^{-2\phi(w)}\rho(z)^{-\frac{2n}{p}}dv(w)\right)^p\\&\simeq\rho(z)^{-2n}\left(\int_{\mathbb{C}^n}\left|f(w)\left\langle k_{p,w},T^*k_{2n-\frac{2n}{p},z}\right\rangle_{\mathcal{F}^2_\phi}\right|e^{-\phi(w)}\rho(w)^{\frac{2n}{p}-2n}dv(w)\right)^p\\&\lesssim\rho(z)^{-2n}\int_{\mathbb{C}^n}\left|f(w)\left\langle k_{p,w},T^*k_{2n-\frac{2n}{p},z}\right\rangle_{\mathcal{F}^2_\phi}\right|^pe^{-p\phi(w)}dv(w).\end{aligned} \tag{87}$$

So, we conclude by Fubini's theorem that

$$\begin{aligned}\|Tf\|^p_{p,\phi}&\lesssim\int_{\mathbb{C}^n}|f(w)|^pe^{-p\phi(w)}\int_{\mathbb{C}^n}\left|\left\langle Tk_{p,w},k_{2n-\frac{2n}{p},z}\right\rangle_{\mathcal{F}^2_\phi}\right|^p\rho(z)^{-2n}dv(z)dv(w)\\&\le M_1\int_{\mathbb{C}^n}|f(w)|^pe^{-p\phi(w)}dv(w)=M_1\|f\|^p_{p,\phi}.\end{aligned} \tag{88}$$

We now assume that $p > 1$. Set

$$M_2 = \max\left\{\sup_{w\in\mathbb{C}^n}\int_{\mathbb{C}^n} |\langle Tk_{p,w}, k_{q,z}\rangle_{\mathscr{F}^2_\phi}| \rho(z)^{-2n} dv(z), \sup_{z\in\mathbb{C}^n}\int_{\mathbb{C}^n} \cdot \left|\langle T^* k_{q,z}, k_{p,w}\rangle_{\mathscr{F}^2_\phi}\right| \rho(w)^{-2n} dv(w)\right\}. \tag{89}$$

By Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned}\|Tf\|^p_{p,\phi} &\le \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} |f(w)\langle TK_w, K_z\rangle_{\mathscr{F}^2_\phi}| e^{-2\phi(w)} dv(w)\right)^p e^{-p\phi(z)} dv(z)\\ &\simeq \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} |f(w)| e^{-\phi(w)} |\langle Tk_{p,w}, k_{q,z}\rangle_{\mathscr{F}^2_\phi}| \rho(w)^{\frac{2n}{p}-2n}\rho(z)^{\frac{2n}{q}-2n} dv(w)\right)^p dv(z)\\ &\le \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)} |\langle Tk_{p,w}, k_{q,z}\rangle_{\mathscr{F}^2_\phi}| \rho(z)^{-2n} dv(w)\right)\\ &\quad\cdot\left(\int_{\mathbb{C}^n} |\langle T^* k_{q,z}, k_{p,w}\rangle_{\mathscr{F}^2_\phi}| \rho(w)^{-2n} dv(w)\right)^{\frac{p}{q}} dv(z)\\ &\lesssim M_2^{\frac{p}{q}}\int_{\mathbb{C}^n} |f(w)|^p e^{-p\phi(w)}\int_{\mathbb{C}^n} |\langle Tk_{p,w}, k_{q,z}\rangle_{\mathscr{F}^2_\phi}|\rho(z)^{-2n} dv(z) dv(w) \lesssim \|f\|^p_{p,\phi},\end{aligned} \tag{90}$$

which completes the proof.

Now, it follows from Theorem 17 that each $WL^\phi_p (0 < p \le 1)$ is analogous to a Banach algebra.

Theorem 18. *Suppose $0 < p \le 1$ and $T, S \in WL^\phi_p$. Then, $\|TS\|_{p,\phi} \lesssim \|T\|_{p,\phi}\|S\|_{p,\phi}$.*

Proof. Suppose $0 < p \le 1$. For every $z \in \mathbb{C}^n$, by the proof of Theorem 15, we see that

$$\begin{aligned}\|TSk_{p,z}\|^p_{p,\phi} &= \int_{\mathbb{C}^n} |TSk_{p,z}(w)|^p e^{-p\phi(w)} dv(w)\\ &\simeq \int_{\mathbb{C}^n}\left|\left\langle TSk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p \rho(w)^{-2n} dv(w)\\ &\lesssim \int_{\mathbb{C}^n}\left|\left\langle Sk_{p,z}, k_{2n-\frac{2n}{p},x}\right\rangle_{\mathscr{F}^2_\phi}\right|^p \rho(x)^{-2n}\int_{\mathbb{C}^n}\\ &\quad\cdot\left|\left\langle Tk_{p,x}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p \rho(w)^{-2n} dv(w) dv(x)\\ &\simeq \int_{\mathbb{C}^n} |Sk_{p,z}(x)|^p e^{-p\phi(x)}\int_{\mathbb{C}^n} |Tk_{p,x}(w)|^p e^{-p\phi(w)} dv(w) dv(x)\\ &= \int_{\mathbb{C}^n} \|Tk_{p,x}\|^p_{p,\phi} |Sk_{p,z}(x)|^p e^{-p\phi(x)} dv(x).\end{aligned} \tag{91}$$

Since $T, S \in WL^\phi_p$, then Theorem 17 says T and S are bounded on $\mathscr{F}^p_\phi$. Thus, the above estimate implies $\|TSk_{p,z}\|_{p,\phi} \lesssim \|T\|_{p,\phi}\|S\|_{p,\phi}$. This completes the proof since the supremum of $\|TSk_{p,z}\|_{p,\phi}$ is no more than C times $\|T\|_{p,\phi}\|S\|_{p,\phi}$.

Theorem 19. *If $0 < p \le 1$, then WL^ϕ_p is closed under the operator norm on $\mathscr{F}^p_\phi$.*

Proof. See Lemma 2.6 of [5]. We omit the details.

5. Equivalent Conditions for Compactness

For this section, we use the ideas in [5, 7] to characterize compactness of weakly localized operators on large Fock spaces. Indeed, it is more complex than [5] because Bergman metric works in a different way than in Euclidean metric.

We begin with the following preparations. Recall that, for fixed $r > 0$, there is an r-lattice $\{z_j\}_j$ such that $\{B^r(z_j)\}_j$ covers $\mathbb{C}^n$. Let $F_j = B^r(z_j) \setminus \bigcup_{i<j} B^r(z_i)$. It follows that $\{F_j\}_j$ is also a covering of $\mathbb{C}^n$ and $F_j \cap F_k = \varnothing (j \ne k)$. We write $(F_j)^+_r = \bigcup_{x\in F_j} B^r(x)$, and it is from estimate (16) that we consider

$$G_j = \left\{y : d(y, F_j) \le m^4 r\rho(z_j), m = m(r) > 1\right\}. \tag{92}$$

In what follows, we always define F_j and G_j as above. Also, there is some constant N such that

$$\sum_{j=1}^{\infty} \chi_{F_j}(w) \le \sum_{j=1}^{\infty} \chi_{(F_j)^+_r}(w) \le N \text{ for } any\, w \in \mathbb{C}^n. \tag{93}$$

Lemma 20. *If $0 < p \le 1$ and $T \in WL^\phi_p$, then for every $\varepsilon > 0$, there exists sufficiently large $r > 0$ such that for the covering $\{F_j\}_j$ (associated to r), we obtain*

$$\left\|T - P\left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}\right)\right\|_{\mathscr{F}^p_\phi \longrightarrow \mathscr{F}^p_\phi} < \varepsilon. \tag{94}$$

Proof. Since $T \in WL^\phi_p$, then there is some $r > 0$ sufficiently large such that

$$\sup_{u\in\mathbb{C}^n}\int_{(B^r(u))^c}\left|\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|^p \rho(w)^{-2n} dv(w) < \varepsilon. \tag{95}$$

Define $S = TP - \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. Then, Lemma 6 indicates that

$$\begin{aligned}|PSf(z)|^p &\le \left(\int_{\mathbb{C}^n}\left|Sf(w)K(z,w)e^{-2\phi(w)}\right| dv(w)\right)^p\\ &= \left(\int_{\mathbb{C}^n}\left|\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G^c_j}} f(w)\right| |K(z,w)| e^{-2\phi(w)} dv(w)\right)^p\\ &\le \left(\sum_{j=1}^{\infty}\int_{\mathbb{C}^n}\left|M_{\chi_{F_j}} TPM_{\chi_{G^c_j}} f(w)\right| |K(z,w)| e^{-2\phi(w)} dv(w)\right)^p\\ &\le \sum_{j=1}^{\infty}\left(\int_{F_j}\left|TPM_{\chi_{G^c_j}} f(w)\right| |K(z,w)| e^{-2\phi(w)} dv(w)\right)^p\\ &\lesssim \sum_{j=1}^{\infty}\int_{(F_j)^+_t}\left|TPM_{\chi_{G^c_j}} f(w)\right|^p |K(z,w)|^p e^{-2p\phi(w)} \rho(w)^{2np-2n} dv(w).\end{aligned} \tag{96}$$

Notice that

$$\begin{aligned}|TPM_{\chi_{G_j^c}}f(w)| &= |\left\langle TPM_{\chi_{G_j^c}}f, K_w\right\rangle_{\mathcal{F}_\phi^2}| = |\left\langle M_{\chi_{G_j^c}}f, T^*K_w\right\rangle_{\mathcal{F}_\phi^2}| \\ &= |\int_{\mathbb{C}^n}\left(M_{\chi_{G_j^c}}f\right)\overline{T^*K_w}e^{-2\phi(u)}dv(u)| \\ &\le \int_{G_j^c}|f(u)\langle K_u, T^*K_w\rangle_{\mathcal{F}_\phi^2}|e^{-2\phi(u)}dv(u).\end{aligned} \tag{97}$$

This together with, for some $t, s > 0$, estimate (20), Lemma 6, and Proposition 12 shows

$$\begin{aligned}|PSf(z)|^p &\lesssim \sum_{j=1}^{\infty}\int_{(F_j)_t^+}\left(\int_{G_j^c}\left|f(u)\langle K_u, T^*K_w\rangle_{\mathcal{F}_\phi^2}\right|e^{-2\phi(u)}dv(u)\right)^p \\ &\quad\cdot |K(z,w)|^p e^{-2p\phi(w)}\rho(w)^{2np-2n}dv(w) \\ &\simeq \sum_{j=1}^{\infty}\int_{(F_j)_t^+}|K(z,w)|^p e^{-p\phi(w)}\rho(w)^{2np-2n} \\ &\quad\times\left(\int_{G_j^c}\left|f(u)e^{-\phi(u)}\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|\rho(w)^{-\frac{2n}{p}}\rho(u)^{\frac{2n}{p}-2n}dv(u)\right)^p dv(w) \\ &\lesssim \sum_{j=1}^{\infty}\int_{(F_j)_t^+}|K(z,w)|^p e^{-p\phi(w)}\rho(w)^{2np-4n}\int_{(G_j^c)_s^+} \\ &\quad\cdot\left|f(u)e^{-\phi(u)}\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p dv(u)dv(w).\end{aligned} \tag{98}$$

We claim first that $(F_j)_t^+ \subset (B^r(u))^c$ if $w \in (F_j)_t^+$ and $u \in (G_j^c)_s^+$. Furthermore, applying (18), (93), Lemma 6, and Fubini's theorem, we obtain

$$\begin{aligned}\|PSf(z)\|_{p,\phi}^p &\lesssim \sum_{j=1}^{\infty}\int_{(G_j^c)_s^+}\left|f(u)e^{-\phi(u)}\right|^p\int_{(F_j)_t^+}\chi_{(B^r(u))^c}(w) \\ &\quad\cdot\left|\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p e^{-p\phi(w)}\rho(w)^{2np-4n} \\ &\quad\times\int_{\mathbb{C}^n}|K(z,w)|^p e^{-p\phi(z)}dv(z)dv(w)dv(u) \\ &\le N\int_{\mathbb{C}^n}\left|f(u)e^{-\phi(u)}\right|^p\int_{(B^r(u))^c} \\ &\quad\cdot\left|\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\rho(w)^{-2n}dv(w)dv(u) \\ &\lesssim \int_{\mathbb{C}^n}\left|f(u)e^{-\phi(u)}\right|^p \\ &\quad\cdot\left(\sup_{u\in\mathbb{C}^n}\int_{(B^r(u))^c}\left|\left\langle Tk_{p,u}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\rho(w)^{-2n}dv(w)\right)dv(u) \\ &< \varepsilon\|f\|_{p,\phi}^p.\end{aligned} \tag{99}$$

Since $PTP = T$ on $\mathcal{F}_\phi^p$, thus $PS = T - P(\sum_{j=1}^{\infty}M_{\chi_{F_j}}TPM_{\chi_{G_j}})$ is well defined.

Thus it only remains to prove that $(F_j)_t^+ \subset (B^r(u))^c$.

In fact, (16) gives $B^r(z_j) \subset B^{m^2r}(z_j) \subset B^{m^4r}(z_j)$ where $m = m(r) > 1$. We then choose a t satisfying $w \in (F_j)_t^+ \subset B^{m^2r}(z_j)$.

Note that for any $u \in \mathbb{C}^n$, there exists $M > 0$ so that $\rho(u) \le M$ by estimate (12). For fixed $r > 0$, it follows that $|u| - r\rho(u) > 0$ whenever $|u|$ is sufficiently large. So, for any $j > 0$, there is some $R > 0$ such that

$$|u| - r\rho(u) > m^2r\rho(z_j) + |z_j|, \tag{100}$$

whenever $|u| > R$. Thus, the triangle inequality implies $|u - z_j| \ge |u| - |z_j| > r\rho(u) + m^2r\rho(z_j)$. Further, the above inequality concludes that $(F_j)_t^+ \cap B^r(u) = \varnothing$.

Next, we assume that $|u| \le R$. For any fixed j, there is a r_0 satisfying $u \in B^{r_0}(z_j)$. Since ρ is a Lipschitz function, then $\rho(z_j) \ge (1/c_{r_0})\rho(u)$ where $c_{r_0} > 1$. Notice that $m = m(r) > 1$, thus it allows us to let $m > c_{r_0}$. It follows that $mr\rho(z_j) > r\rho(u)$. For $u \in (G_j^c)_s^+$ and $|u| \le R$, we can pick an appropriate s so that $s\rho(u) < mr\rho(z_j) - r\rho(u)$. Hence,

$$\begin{aligned}|u - z_j| &> (m^4+1)r\rho(z_j) - s\rho(u) > (m^4 - m + 1)r\rho(z_j) + r\rho(u) \\ &> m^2r\rho(z_j) + r\rho(u)\end{aligned} \tag{101}$$

shows $(F_j)_t^+ \cap B^r(u) = \varnothing$.

Therefore, the desired assertion holds, and the proof is finished.

Lemma 21. *If $1 < p < \infty$ and $T \in WL_p^\phi$, then for every $\varepsilon > 0$, there exists sufficiently large $r > 0$ such that for the covering $\{F_j\}_j$ (associated to r), we obtain*

$$\left\|TP - \left(\sum_{j=1}^{\infty}M_{\chi_{F_j}}TPM_{\chi_{G_j}}\right)\right\|_{\mathcal{F}_\phi^p\longrightarrow\mathcal{F}_\phi^p} < \varepsilon. \tag{102}$$

Proof. By (71), for any $\varepsilon > 0$ and $w \in \mathbb{C}^n$, there is some $r > 0$ such that

$$\int_{(B^r(w))^c}|\langle T^*k_{q,w}, k_{p,u}\rangle_{\mathcal{F}_\phi^2}|\rho(u)^{-2n}dv(u) < \varepsilon, \tag{103}$$

and (70) shows

$$\int_{\mathbb{C}^n}|\langle Tk_{p,u}, k_{q,w}\rangle_{\mathcal{F}_\phi^2}|\rho(w)^{-2n}dv(w) < M < \infty. \tag{104}$$

We also consider $S = TP - \sum_{j=1}^{\infty}M_{\chi_{F_j}}TPM_{\chi_{G_j}}$. For any fixed $w \in \mathbb{C}^n$, we assume that $w \in F_{j_0} \subset B^r(z_{j_0})$. Note first that if $u \in G_{j_0}^c$, then $|u - z_{j_0}| > (m^4+1)r\rho(z_{j_0})$ where $m = m(r) > 1$. For $w \in F_{j_0}$, we have $|w - z_{j_0}| < r\rho(z_{j_0})$ and $\rho(z_{j_0}) \ge c_r^{-1}\rho(w)$ since ρ is a Lipschitz function. Suppose $m^4 > c_r$, so $|u - w| > m^4r\rho(z_{j_0}) > r\rho(w)$, that is $G_{j_0}^c \subset (B^r(w))^c$. It follows from

Lemma 20 that

$$\begin{aligned}
|Sf(w)| &= |\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j^c}} \mathrm{f}(w)| \\
&\le \sum_{j=1}^{\infty} M_{\chi_{F_j}}(w) \int_{G_j^c} |f(u)\langle K_u, T^* K_w\rangle_{\mathscr{F}_\phi^2}| e^{-2\phi(u)} dv(u) \\
&\le \int_{(B^r(w))^c} |f(u)\langle K_u, T^* K_w\rangle_{\mathscr{F}_\phi^2}| e^{-2\phi(u)} dv(u).
\end{aligned} \tag{105}$$

This together with Hölder's inequality and Fubini's theorem implies

$$\begin{aligned}
\|Sf\|_{p,\phi}^p &\le \int_{\mathbb{C}^n} \left(\int_{(B^r(w))^c} | f(u) \langle TK_u, K_w\rangle_{\mathscr{F}_\phi^2} | e^{-2\phi(u)} dv(u) \right)^p e^{-p\phi(w)} dv(w) \\
&\simeq \int_{\mathbb{C}^n} \left(\int_{(B^r(w))^c} | f(u) | e^{-\phi(u)} | \langle Tk_{p,u}, k_{q,w}\rangle_{\mathscr{F}_\phi^2} | \rho(u)^{\frac{2n}{p}-2n} \rho(w)^{\frac{2n}{q}-2n} dv(u) \right)^p dv(w) \\
&\lesssim \int_{\mathbb{C}^n} \int_{(B^r(w))^c} |f(u)|^p e^{-p\phi(u)} |\langle Tk_{p,u}, k_{q,w}\rangle_{\mathscr{F}_\phi^2}| \rho(w)^{-2n} dv(u) \\
&\quad \times \left(\int_{(B^r(w))^c} | \langle T^* k_{q,w}, k_{p,u}\rangle_{\mathscr{F}_\phi^2} | \rho(u)^{-2n} dv(u) \right)^{\frac{p}{q}} dv(w) \\
&< \varepsilon^{\frac{p}{q}} \int_{\mathbb{C}^n} |f(u)|^p e^{-p\phi(u)} \int_{\mathbb{C}^n} |\langle Tk_{p,u}, k_{q,w}\rangle_{\mathscr{F}_\phi^2}| \rho(w)^{-2n} dv(u) dv(w) < M\varepsilon^{\frac{p}{q}} \|f\|_{p,\phi}^p,
\end{aligned} \tag{106}$$

which proves the desired result.

Lemma 22. *Suppose $0 < p < \infty$. For any bounded linear operator T on $\mathscr{F}_\phi^p$, there is some constant C such that*

$$\limsup_{m\longrightarrow\infty} \|PT_m\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p} \le C \limsup_{m\longrightarrow\infty} \sup_{w\in\bigcup_{j>m}(G_j)_r^+} \|Tk_{p,w}\|_{p,\phi}, \tag{107}$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$.

Proof. First, let $0 < p \le 1$. Suppose $f \in \mathscr{F}_\phi^p$ and $f \ne 0$, define

$$g_j = P\left(\frac{\chi_{G_j} f}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}} \right). \tag{108}$$

Since $Pf = f$ on $\mathscr{F}_\phi^p$, then by (85) we get

$$\begin{aligned}
|Tg_j(z)| &= |\int_{\mathbb{C}^n} g_j(w) TK_{\mathrm{w}}(z) e^{-2\phi(w)} dv(w)| \\
&= |\int_{\mathbb{C}^n} \frac{\chi_{G_j} f(w)}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}} TK_w(z) e^{-2\phi(w)} dv(w)| \\
&\le \int_{G_j} \frac{|f(w)||TK_w(z)| e^{-2\phi(w)}}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}} dv(w).
\end{aligned} \tag{109}$$

By combining the above estimate and Lemma 6, we obtain

$$\begin{aligned}
&\left| P\left(M_{\chi_{F_j}} Tg_j \right)(u) \right|^p \\
&= \left| \int_{F_j} \left(Tg_j \right)(z) K(u,z) e^{-2\phi(z)} dv(z) \right|^p \\
&\lesssim \int_{(F_j)_r^+} \left| \left(Tg_j \right)(z) \right|^p |K(u,z)|^p e^{-2p\phi(z)} \rho(z)^{2np-2n} dv(z) \\
&\lesssim \int_{(F_j)_r^+} |K(u,z)|^p e^{-2p\phi(z)} \\
&\quad \cdot \left(\int_{G_j} \frac{|f(w)||TK_w(z)| e^{-2\phi(w)}}{\left\| \chi_{(G_j)_r^+} f \right\|_{p,\phi}} dv(w) \right)^p \rho(z)^{2np-2n} dv(z) \\
&\lesssim \int_{(F_j)_r^+} |K(u,z)|^p e^{-2p\phi(z)} \rho(z)^{2np-2n} \\
&\quad \times \left(\int_{(G_j)_r^+} \frac{|f(w)|^p |TK_w(z)|^p e^{-2p\phi(w)}}{\left\| \chi_{(G_j)_r^+} f \right\|_{p,\phi}^p} \rho(w)^{2np-2n} dv(w) \right) dv(z).
\end{aligned} \tag{110}$$

Notice that there is a $r' \in (0,1)$ such that $(F_j)_r^+ \subset B^{r'}(z_j)$ (we can let r be small enough), then for $z \in B^{r'}(z_j)$, we get $\rho(z) \simeq \rho(z_j)$ by (14). Now, by (18), (25), Lemma 2, and Fubini's theorem, we have

$$\begin{aligned}
\|P\left(M_{\chi_{F_j}} Tg_j \right)\|_{p,\phi}^p &\lesssim \int_{(G_j)_r^+} \frac{|f(w)|^p}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}^p} e^{-2p\phi(w)} \rho(w)^{2np-2n} \\
&\quad \times \int_{(F_j)_r^+} |TK_w(z)|^p e^{-2p\phi(z)} \rho(z)^{2np-2n} \int_{\mathbb{C}^n} \\
&\quad \cdot |K(u,z)|^p e^{-p\phi(u)} dv(u) dv(w) dv(z) \\
&\simeq \int_{(G_j)_r^+} \frac{|f(w)|^p}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}^p} e^{-p\phi(w)} \int_{(F_j)_r^+} \\
&\quad \cdot \left| Tk_{p,w}(z) \right|^p e^{-p\phi(z)} dv(z) dv(w) \\
&\lesssim \left(\sup_{w\in(G_j)_r^+} \|Tk_{p,w}\|_{p,\phi}^p \right) \\
&\quad \cdot \int_{B^{r'}(z_j)} \rho(z)^{-2n} dv(z) \int_{(G_j)_r^+} \\
&\quad \cdot \frac{|f(w)|^p}{\|\chi_{(G_j)_r^+} f\|_{p,\phi}^p} e^{-p\phi(w)} dv(w) \\
&\lesssim \sup_{w\in(G_j)_r^+} \|Tk_{p,w}\|_{p,\phi}^p.
\end{aligned} \tag{111}$$

Therefore, we deduce

$$\begin{aligned}\|PT_mf\|_{p,\phi}^p &\le \sum_{j>m}\|PM_{\chi_{F_j}}TPM_{\chi_{G_j}}f\|_{p,\phi}^p = \sum_{j>m}\left\|P\left(M_{\chi_{F_j}}Tg_j\right)\right\|_{p,\phi}^p\|\chi_{(G_j)_r^+}f\|_{p,\phi}^p \\ &\lesssim \sum_{j>m}\sup_{w\in(G_j)_r^+}\|Tk_{p,w}\|_{p,\phi}^p\|\chi_{(G_j)_r^+}f\|_{p,\phi}^p \\ &\le N\left(\sup_{w\in\bigcup_{j>m}(G_j)_r^+}\|Tk_{p,w}\|_{p,\phi}^p\right)\|f\|_{p,\phi}^p.\end{aligned} \tag{112}$$

Now, $1<p<\infty$. It comes from the above proof that

$$\begin{aligned}|PT_mf(u)|^p &= \left|P\left(\sum_{j>m}M_{\chi_{F_j}}TPM_{\chi_{G_j}}f\right)(u)\right|^p \\ &\le \left(\sum_{j>m}\int_{F_j}|K(u,z)|e^{-2\phi(z)}\int_{G_j}|f(w)|\right. \\ &\quad\left.\cdot\left|\langle TK_w,K_z\rangle_{\mathcal{F}_\phi^2}\right|e^{-2\phi(w)}dv(w)dv(z)\right)^p.\end{aligned} \tag{113}$$

For fixed j and $r>0$, let $x\in F_j\setminus\{z_j\}$. Note that $F_j\subset B^r(z_j)$, we then get $F_j\subset B^{br}(x)$ and $G_j\subset B^{b^5r}(x)$ by (16), where $b=b(r)>1$. Furthermore, there exists some $r'>0$ such that $B^{b^5r}(x)\subset(G_j)_{r'}^+$. It follows from estimate (14) we have $\rho(z)\simeq\rho(x)$ if $z\in B^{br}(x)$ and $\rho(w)\simeq\rho(x)$ if $w\in B^{b^5r}(x)$ (here we also assume r is small enough so that $b^5r<1$). Hence, by estimates (18), (20), (25), (93), Minkowski's inequality, and Lemma 6, we obtain

$$\begin{aligned}\|PT_mf\|_{p,\phi} &\le N\left[\int_{\mathbb{C}^n}\left(\int_{B^{b^5r}(x)}|f(w)|e^{-2\phi(w)}\right.\right. \\ &\quad\left.\left.\times\int_{B^{br}(x)}|TK_w(z)|e^{-\phi(z)}|K(u,z)|e^{-\phi(z)-\phi(u)}dv(z)dv(w)\right)^p dv(u)\right]^{\frac{1}{p}} \\ &\lesssim \int_{B^{b^5r}(x)}|f(w)|e^{-2\phi(w)}\int_{B^{br}(x)}|TK_w(z)|e^{-\phi(z)} \\ &\quad\times\left(\int_{\mathbb{C}^n}|K(u,z)|^pe^{-p\phi(z)-p\phi(u)}dv(u)\right)^{\frac{1}{p}}dv(z)dv(w) \\ &\simeq \int_{B^{b^5r}(x)}|f(w)|e^{-2\phi(w)}\int_{B^{br}(x)}|TK_w(z)|e^{-\phi(z)}\rho(z)^{\frac{2n}{p}-2n}dv(z)dv(w) \\ &\simeq \int_{B^{b^5r}(x)}|f(w)|e^{-\phi(w)}\rho(w)^{\frac{2n}{p}-2n}\int_{B^{br}(x)}|Tk_{p,w}(z)|e^{-\phi(z)}\rho(z)^{\frac{2n}{p}-2n}dv(z)dv(w) \\ &\simeq \int_{B^{b^5r}(x)}|f(w)|e^{-\phi(w)}\rho(w)^{\frac{2n}{p}-2n}\|Tk_{p,w}\|_{p,\phi}\int_{B^{br}(x)}\rho(z)^{-2n}dv(z)dv(w) \\ &\lesssim \left(\sup_{w\in B^{b^5r}(x)}\|Tk_{p,w}\|_{p,\phi}\right)\|f\|_{p,\phi}\left(\rho(x)^{-2n}v\left(B^{br}(x)\right)\right)\int_{B^{b^5r}(x)}\rho(x)^{-2n}dv(w) \\ &\le C(b,r)\left(\sup_{w\in\bigcup_{j>m}(G_j)_{r'}^+}\|Tk_{p,w}\|_{p,\phi}\right)\|f\|_{p,\phi}.\end{aligned} \tag{114}$$

This completes the proof.

Theorem 23. *Suppose $0<p<\infty$ and $T\in WL_p^\phi$. Then, the following conditions are equivalent ($q:=2n-2n/p$ if $0<p\le 1$):*

(A) $\lim_{z\longrightarrow\infty}\|Tk_{p,z}\|_{p,\phi}=0$;

(B) $\lim_{z\longrightarrow\infty}\sup_{w\in B^r(z)}|\langle Tk_{p,z},k_{q,w}\rangle_{\mathcal{F}_\phi^2}|=0$ for any $r>0$;

(C) $\lim_{z\longrightarrow\infty}\sup_{w\in\mathbb{C}^n}|\langle Tk_{p,z},k_{q,w}\rangle_{\mathcal{F}_\phi^2}|=0$.

Proof. Suppose condition (A) holds. If $0<p\le1$, then (25) gives

$$\begin{aligned}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p &\simeq \left|Tk_{p,z}(w)e^{-\phi(w)}\rho(w)^{\frac{2n}{p}}\right|^p \\ &\le C\int_{B^r(w)}\left|Tk_{p,z}(u)\right|^pe^{-p\phi(u)}dv(u) \\ &\lesssim \|Tk_{p,z}\|_{p,\phi}^p.\end{aligned} \tag{115}$$

Similarly, when $1<p<\infty$, we obtain

$$\begin{aligned}\left|\langle Tk_{p,z},k_{q,w}\rangle\right|^p &\simeq \left|Tk_{p,z}(w)e^{-\phi(w)}\rho(w)^{2n-\frac{2n}{q}}\right|^p \\ &\lesssim \int_{B^r(w)}\left|Tk_{p,z}(u)\right|^pe^{-p\phi(u)}dv(u)\lesssim\|Tk_{p,z}\|_{p,\phi}^p.\end{aligned} \tag{116}$$

Then (A) implies (C).

Because $(C)\Rightarrow(B)$ is clear, so it remains to prove that the implication $(B)\Rightarrow(A)$. If $T\in WL_p^\phi$ and $0<p\le1$, then, by (68), there is some $r>0$ such that

$$\int_{(B^r(z))^c}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\rho(w)^{-2n}dv(w)<\varepsilon. \tag{117}$$

Note that by the definition of function ρ, we get $\rho\ge\alpha>0$. Then, (B) shows

$$\begin{aligned}\|Tk_{p,z}\|_{p,\phi}^p &= \int_{\mathbb{C}^n}\left|\left\langle Tk_{p,z},K_w\right\rangle_{\mathcal{F}_\phi^2}\right|^pe^{-p\phi(w)}dv(w) \\ &\simeq \int_{(B^r(z))^c\bigcup(B^r(z))}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\rho(w)^{-2n}dv(w) \\ &<\varepsilon+\int_{B^r(z)}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\rho(w)^{-2n}dv(w) \\ &\le\varepsilon+\sup_{w\in B^r(z)}\left|\left\langle Tk_{p,z},k_{2n-\frac{2n}{p},w}\right\rangle_{\mathcal{F}_\phi^2}\right|^p\int_{B^r(z)}\rho(w)^{-2n}dv(w)\lesssim\varepsilon,\end{aligned} \tag{118}$$

whenever $|z|$ is large enough. By hypothesis, (A) holds when $0<p\le1$.

Suppose $1 < p < \infty$. It follows from (20), (25), and Theorem 17 that

$$|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}| \simeq |\langle Tk_{p,z}, K_w\rangle_{\mathcal{F}^2_\phi}|e^{-\phi(w)}\rho(w)^{2n-\frac{2n}{q}} \lesssim \|Tk_{p,z}\|_{p,\phi} \lesssim 1. \tag{119}$$

Therefore, joining condition (B) and (71), we deduce

$$\begin{aligned}\|Tk_{p,z}\|^p_{p,\phi} &\simeq \int_{\mathbb{C}^n}\left|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}\right|^p \rho(w)^{-2n}dv(w)\\ &\lesssim \int_{(B^r(z))^c}|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\ &\quad+\int_{B^r(z)}|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\ &< \varepsilon + \int_{B^r(z)}|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\rho(w)^{-2n}dv(w)\\ &\lesssim \varepsilon + \left(\sup_{w\in B^r(z)}|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}|\right)\int_{B^r(z)}\rho(w)^{-2n}dv(w) \lesssim \varepsilon\end{aligned} \tag{120}$$

for sufficiently large z. This completes the proof.

It is similar to ([17], Lemma 3.2) that we have the following assertion about relatively compact. That is, for every $\varepsilon > 0$, there is some $R > 0$ such that

$$\sup_{f\in E}\int_{|z|\geq R}\left|f(z)e^{-\phi(z)}\right|^p dv(z) < \varepsilon \tag{121}$$

if and only if a bounded subset $E \subset \mathcal{F}^p_\phi$ is relatively compact. In what follows, we call $\mathcal{K}(\mathcal{F}^p_\phi)$ the set of compact operators on $\mathcal{F}^p_\phi$.

Theorem 24. *If $0 < p < \infty$ and $T \in \mathcal{K}(\mathcal{F}^p_\phi)$, then*

$$\lim_{R\longrightarrow\infty}\|PM_{\chi_{B(0,R)}}T - T\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi} = 0. \tag{122}$$

Proof. We omit the details for $0 < p \leq 1$, see ([5], Lemma 2.11).

If $1 < p < \infty$, then Hölder's inequality, Fubini's theorem, (18), and Lemma 2 imply

$$\begin{aligned}&\left\|\left(PM_{\chi_{B(0,R)}}T - T\right)(f)\right\|^p_{p,\phi}\\ &\leq \int_{\mathbb{C}^n}\left(\int_{|w|\geq R}|Tf(w)|e^{-\phi(w)}|K(z,w)|e^{-\phi(w)-\phi(z)}dv(w)\right)^p dv(z)\\ &\leq \int_{\mathbb{C}^n}\left(\int_{|w|\geq R}|Tf(w)|^p e^{-p\phi(w)}|K(z,w)|e^{-\phi(w)-\phi(z)}dv(w)\right)\\ &\quad\times\left(\int_{|w|\geq R}|K(z,w)|e^{-\phi(w)-\phi(z)}dv(w)\right)^{\frac{p}{q}}dv(z)\\ &\lesssim \int_{|w|\geq R}|Tf(w)|^p e^{-p\phi(w)}\int_{\mathbb{C}^n}|K(z,w)|e^{-\phi(w)-\phi(z)}dv(z)dv(w)\\ &\lesssim \int_{|w|\geq R}|Tf(w)|^p e^{-p\phi(w)}dv(w)\end{aligned} \tag{123}$$

converges to 0 as $R \longrightarrow \infty$. This finishes the proof.

Due to $PM_{\chi_{G_j}}$ that can be viewed as a Toeplitz operator induced by χ_{G_j}, then $PM_{\chi_{G_j}}$ is compact (the reason is similar to ([18], Lemma 3.1)).

Theorem 25. *Suppose $1 < p < \infty$. Then, there exists r such that*

$$\|T\|_e \leq C\lim\sup_{z\longrightarrow\infty}\sup_{w\in B^r(z)}\left|\langle Tk_{p,z}, k_{q,w}\rangle_{\mathcal{F}^2_\phi}\right| \tag{124}$$

where $\|T\|_e$ means the essential norm of a bounded operator T on $\mathcal{F}^p_\phi$.

Proof. Since $Pf = f$ for $f \in \mathcal{F}^p_\phi$, then $\|T\|_e = \|TP\|_e$, and we always assume that $\|TP\|_e > 0$. Thus, Lemma 21 shows there is some $r > 0$ satisfying

$$\left\|TP - \left(\sum_{j=1}^{\infty}M_{\chi_{F_j}}TPM_{\chi_{G_j}}\right)\right\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi} < \frac{1}{2}\|TP\|_e. \tag{125}$$

Because of $\sum_{j\leq m}M_{\chi_{F_j}}TPM_{\chi_{G_j}}$ is a compact operator where $m \in \mathbb{N}$, then

$$\begin{aligned}\|TP\|_e &\leq \left\|TP - \left(\sum_{j=1}^{m}M_{\chi_{F_j}}TPM_{\chi_{G_j}}\right)\right\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi}\\ &\leq \left\|TP - \left(\sum_{j=1}^{\infty}M_{\chi_{F_j}}TPM_{\chi_{G_j}}\right)\right\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi}\\ &\quad+\|T_m\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi} < \frac{1}{2}\|TP\|_e + \|T_m\|_{\mathcal{F}^p_\phi\longrightarrow\mathcal{F}^p_\phi},\end{aligned} \tag{126}$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. For the rest of the task, we show

$$\limsup_{m \longrightarrow \infty} \| T_m \|_{\mathscr{F}_\phi^p} \lesssim C \limsup_{z \longrightarrow \infty} \sup_{w \in B^r(z)} |\langle Tk_{p,z}, k_{q,w} \rangle_{\mathscr{F}_\phi^2}| + \frac{1}{4} \| TP \|_e. \tag{127}$$

Let $f \in \mathscr{F}_\phi^p$ and $\|f\|_{\mathscr{F}_\phi^p} \leq 1$. Note that $F_j \cap F_k = \varnothing (j \neq k)$, hence

$$\begin{aligned} \|T_m f\|_{\mathscr{F}_\phi^p}^p &= \sum_{j>m} \left\| M_{\chi_{F_j}} TPM_{\chi_{G_j}} f \right\|_{\mathscr{F}_\phi^p}^p \\ &= \sum_{j>m} \left\| M_{\chi_{F_j}} Tl_j \right\|_{\mathscr{F}_\phi^p}^p \left\| M_{\chi_{G_j}} f \right\|_{\mathscr{F}_\phi^p}^p \\ &\leq N \left(\sup_{j>m} \left\| M_{\chi_{F_j}} Tl_j \right\|_{\mathscr{F}_\phi^p}^p \right) \|f\|_{\mathscr{F}_\phi^p}^p, \end{aligned} \tag{128}$$

where $l_j := PM_{\chi_{G_j}} f / \|M_{\chi_{G_j}} f\|_{\mathscr{F}_\phi^p}$. It follows that

$$\|T_m\|_{\mathscr{F}_\phi^p} \lesssim \sup_{j>m} \sup_{\|f\|_{\mathscr{F}_\phi^p} \leq 1} \left\{ \left\| M_{\chi_{F_j}} Tl_j \right\|_{\mathscr{F}_\phi^p} : l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{\mathscr{F}_\phi^p}} \right\}. \tag{129}$$

Furthermore, we have

$$\limsup_{m \longrightarrow \infty} \|T_m\|_{\mathscr{F}_\phi^p} \lesssim \limsup_{j \longrightarrow \infty} \sup_{\|f\|_{\mathscr{F}_\phi^p} \leq 1} \left\{ \left\| M_{\chi_{F_j}} Tl_j \right\|_{\mathscr{F}_\phi^p} : l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{\mathscr{F}_\phi^p}} \right\}. \tag{130}$$

Let $\{f_j\}$ be a sequence with $\|f_j\|_{\mathscr{F}_\phi^p} \leq 1$ such that

$$\begin{aligned} &\limsup_{j \longrightarrow \infty} \sup_{\|f\|_{\mathscr{F}_\phi^p} \leq 1} \left\{ \left\| M_{\chi_{F_j}} Tl_j \right\|_{\mathscr{F}_\phi^p} : l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{\mathscr{F}_\phi^p}} \right\} - \frac{1}{4} \|TP\|_e \\ &\leq \limsup_{j \longrightarrow \infty} \left\| M_{\chi_{F_j}} Tg_j \right\|_{\mathscr{F}_\phi^p}, \end{aligned} \tag{131}$$

where $g_j := PM_{\chi_{G_j}} f_j / \|M_{\chi_{G_j}} f_j\|_{\mathscr{F}_\phi^p}$. Fix j, for $z \in F_j \subset G_j$, and there is a $b = b(r) > 0$ such that $G_j \subset B^{br}(z)$ by (16). By joining the proof of Lemma 22 and Hölder's inequality, we deduce

$$\begin{aligned} &\limsup_{j \longrightarrow \infty} \left\| M_{\chi_{F_j}} Tg_j \right\|_{\mathscr{F}_\phi^p}^p \\ &\leq \limsup_{j \longrightarrow \infty} \int_{F_j} \left(\int_{G_j} \frac{|f(w)| |TK_w(z)| e^{-2\phi(w)}}{\left\| \chi_{G_j} f \right\|_{p,\phi}} dv(w) \right)^p e^{-p\phi(z)} dv(z) \\ &\simeq \limsup_{j \longrightarrow \infty} \int_{F_j} \left(\int_{G_j} \frac{|f(w)| e^{-\phi(w)}}{\left\| \chi_{G_j} f \right\|_{p,\phi}} \left| \langle Tk_{p,w}, k_{q,z} \rangle_{\mathscr{F}_\phi^2} \right| \rho(z)^{\frac{2n}{q} - 2n} \rho(w)^{\frac{2n}{p} - 2n} dv(w) \right)^p dv(z) \\ &\lesssim \limsup_{j \longrightarrow \infty} \int_{F_j} \int_{G_j} \frac{|f(w)|^p e^{-p\phi(w)}}{\left\| \chi_{G_j} f \right\|_{p,\phi}^p} \left| \langle Tk_{p,w}, k_{q,z} \rangle_{\mathscr{F}_\phi^2} \right| \rho(z)^{-2n} dv(w) \\ &\cdot \left(\int_{G_j} \rho(w)^{-2n} dv(w) \right)^{\frac{p}{q}} dv(z) \\ &\lesssim \limsup_{j \longrightarrow \infty} \int_{F_j} \int_{G_j} \frac{|f(w)|^p e^{-p\phi(w)}}{\left\| \chi_{G_j} f \right\|_{p,\phi}^p} \left| \langle Tk_{p,w}, k_{q,z} \rangle_{\mathscr{F}_\phi^2} \right| \rho(z)^{-2n} dv(w) dv(z) \\ &\lesssim \limsup_{z \longrightarrow \infty} \sup_{w \in B^{br}(z)} \left| \langle Tk_{p,z}, k_{q,w} \rangle_{\mathscr{F}_\phi^2} \right|^p \left(\sup_j \int_{G_j} \rho(z)^{-2n} \int_{G_j} \frac{|f(w)|^p e^{-p\phi(w)}}{\left\| \chi_{G_j} f \right\|_{p,\phi}^p} dv(w) dv(z) \right) \\ &\leq C(r) \limsup_{z \longrightarrow \infty} \left(\sup_{w \in B^{br}(z)} \left| \langle Tk_{p,z}, k_{q,w} \rangle_{\mathscr{F}_\phi^2} \right| \right)^p, \end{aligned} \tag{132}$$

where $C(r)$ is independent of j. This finishes the proof.

Theorem 26. *Suppose $0 < p < \infty$ and $T \in WL_p^\phi$. Then, $T \in \mathscr{K}(\mathscr{F}_\phi^p)$ if and only if $\lim_{z \longrightarrow \infty} \|Tk_{p,z}\|_{p,\phi} = 0$.*

Proof. ($\Leftarrow$) For any $\varepsilon > 0$, by Lemma 20, we get

$$\left\| T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p} < \varepsilon. \tag{133}$$

Consider $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$, then $P(\sum_{j=1}^m M_{\chi_{F_j}} TP M_{\chi_{G_j}})$ is a compact operator on $\mathscr{F}_\phi^p$, where m is any positive integer. Hence,

$$\begin{aligned} \|T\|_e^p &\leq \left\| T - P \left(\sum_{j=1}^{m} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p}^p \\ &\leq \left(\left\| T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p} + \|PT_m\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p} \right)^p \\ &< \left(\varepsilon + \|PT_m\|_{\mathscr{F}_\phi^p \longrightarrow \mathscr{F}_\phi^p} \right)^p. \end{aligned} \tag{134}$$

With our assumption, there is an $R > 0$ such that $\|Tk_{p,z}\|_{p,\phi} < \varepsilon$ for $|z| > R$. Since $\bigcup_{j>m} (G_j)_r^+ \subset B(0, R)^c$ whenever m is large enough, then Lemma 22 indicates $\|T\|_e = 0$ for $0 < p \leq 1$. Also, when $p > 1$, $\|T\|_e = 0$ follows immediately from Theorems 23 and 25.

($\Rightarrow$) The case $1 < p < \infty$ is similar to the following discussion of $0 < p \leq 1$.

Consider $0 < p \leq 1$ and $r > 0$. With the help of Theorem 23, we will finish the proof if $\sup_{w \in B^r(z)} |\langle Tk_{p,z}, k_{2n-2n/p,w} \rangle_{\mathscr{F}_\phi^2}|$

$\longrightarrow 0$ as $z \longrightarrow \infty$. Recall first that, for $f \in \mathscr{F}^p_\phi$, by (25), we have

$$|f(z)| \le Ce^{\phi(z)}\rho(z)^{-\frac{2n}{p}}\|f\|_{p,\phi}. \tag{135}$$

Since $Tk_{p,z} \in \mathscr{F}^p_\phi$, estimate (18), $|K_w(u)| = |K_u(w)|$, and Proposition 12, then

$$\begin{aligned}
&\left|\left\langle PM_{\chi_{B(0,R)}}Tk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right| \\
&= \left|\int_{B(0,R)} Tk_{p,z}(u)\overline{k_{2n\frac{2n}{p},w}}(u)e^{-2\phi(u)}dv(u)\right| \\
&\le \int_{B(0,R)} \left|Tk_{p,z}(u)\right|\left|\overline{k_{2n\frac{2n}{p},w}}(u)\right|e^{-2\phi(u)}dv(u) \\
&\lesssim \left\|Tk_{p,z}\right\|_{p,\phi}\int_{B(0,R)}\left|\overline{k_{2n\frac{2n}{p},w}}(u)\right|e^{-\phi(u)}\rho(u)^{-\frac{2n}{p}}dv(u) \\
&\lesssim \left\|Tk_{p,z}\right\|_{p,\phi}\int_{B(0,R)}|K_w(u)|e^{-\phi(u)-\phi(w)}\rho(w)^{\frac{2n}{p}}\rho(u)^{-\frac{2n}{p}}dv(u) \\
&\lesssim \left\|Tk_{p,z}\right\|_{p,\phi}\rho(w)^{\frac{2n}{p}-n}\int_{B(0,R)}\rho(u)^{-n-\frac{2n}{p}}e^{-\epsilon\left(\frac{|w|-|u|}{\rho(u)}\right)^\alpha}dv(u) \\
&\lesssim R^{2n}\left\|Tk_{p,z}\right\|_{p,\phi}\left(\rho(w)^{\frac{2n}{p}-n}e^{-g(w)^\epsilon}\right)
\end{aligned} \tag{136}$$

goes to 0 as $w \longrightarrow \infty$, where $g(w) = (|w| - R)/\max\{\rho(u): u \in B(\bar{0}, R)\}$.

On the other hand, Theorems 17 and 24 conclude that

$$\begin{aligned}
&\left|\left\langle\left(PM_{\chi_{B(0,R)^c}}T\right)k_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right| \\
&\simeq \left|\left\langle\left(T - PM_{\chi_{B(0,R)}}T\right)k_{p,z}, K_w\right\rangle_{\mathscr{F}^2_\phi}e^{-\phi(w)}\rho(w)^{\frac{2n}{p}}\right| \\
&\lesssim \left\|\left(T - PM_{\chi_{B(0,R)}}T\right)k_{p,z}\right\|_{p,\phi} \\
&\le \left\|T - PM_{\chi_{B(0,R)}}T\right\|_{p,\phi} \longrightarrow 0
\end{aligned} \tag{137}$$

as $R \longrightarrow \infty$. Altogether gives that

$$\begin{aligned}
\left|\left\langle Tk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right| \le{}& \left|\left\langle PM_{\chi_{B(0,R)}}Tk_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right| \\
&+ \left|\left\langle\left(PM_{\chi_{B(0,R)^c}}T\right)k_{p,z}, k_{2n-\frac{2n}{p},w}\right\rangle_{\mathscr{F}^2_\phi}\right|,
\end{aligned} \tag{138}$$

which ends the proof since $\sup_{w\in B^r(z)} |\langle Tk_{p,z}, k_{2n-2n/p,w}\rangle_{\mathscr{F}^2_\phi}| \longrightarrow 0$ as $z \longrightarrow \infty$.

Data Availability

No data are used.

Conflicts of Interest

The authors declare that they have no competing interests.

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