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Research Article

An Extension of the Picard Theorem to Fractional Differential Equations with a Caputo-Fabrizio Derivative

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In this paper, we consider fractional differential equations with the new fractional derivative involving a nonsingular kernel, namely, the Caputo-Fabrizio fractional derivative. Using a successive approximation method, we prove an extension of the Picard-Lindelöf existence and uniqueness theorem for fractional differential equations with this derivative, which gives a set of conditions, under which a fractional initial value problem has a unique solution.

1. Introduction

Due to the demonstrated applications of fractional operators in various and widespread fields of many sciences, such as mathematics, physics, chemistry, engineering, and statistics [1–4], various operators of a fractional calculus have been found to be remarkably popular for modelling of numerous varied problems in these sciences. We mention here some of these definitions, such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Weyl, Riesz, Erdélyi-Kober, and Caputo. Compared with an integer order, a significant feature of a fractional order differential operator appeared in its hereditary property. In other words, when we describe a process by a fractional operator, we predict the future state by its current as well as its past states. Therefore, the memory and hereditary properties of materials and systems can be intervened in the modeling of a process by making use of differential equations of an arbitrary order. So, in recent years, fractional differential equations have been paid a great interest and also have appeared in new areas for applications of initial and boundary value problems of such equations. The Riemann-Liouville definition for the fractional derivative is one of the most widely used definitions and has many applications. But this definition had its drawbacks, such as the fact that the derivative of a constant function is not zero, and in practical examples, we need the value of fractional derivatives as initial values. The Caputo fractional derivative does not have the above weaknesses and is believed to be one of the most efficient definitions of fractional derivative applied in many areas of science and engineering.

However, the new definition suggested by Caputo and Fabrizio [5], which has all the characteristics of the old definitions, assumes two different representations for the temporal and spatial variables. In fact, they claimed that the classical definition given by Caputo appears to be particularly convenient for mechanical phenomena, related with plasticity, fatigue, damage, and with electromagnetic hysteresis. When these effects are not present, it seems more appropriate to use the new Caputo-Fabrizio operator.

The main advantage of the Caputo-Fabrizio approach is that the boundary conditions of the fractional differential equations with Caputo-Fabrizio derivatives admit the same form as for the integer-order differential equations. On the other hand, the Caputo-Fabrizio fractional derivative has many significant properties, such as its ability in describing matter heterogeneities and configurations with different scales [6–8]. Therefore, there are some certain phenomena that cannot be well-modeled using the Riemann-Liouville,

Caputo, or other standard fractional operators [5, 9–13]. For an example, in issues related to material heterogeneities, we encounter some problems that are not well described by the above fractional operators. Also, later, some other definitions with a nonsingular kernel, such as the Atangana-Baleanu [6] fractional derivative, were defined.

Many researchers have shared their contributions to obtain properties of many models with new and old definitions of fractional derivatives. In [14], we have the analytic solutions of a viscous fluid with the Caputo and Caputo-Fabrizio fractional derivatives. In [15], the authors used the fractional derivative with a nonsingular kernel to model a Maxwell fluid and found semianalytical solutions. In [16], we found a comparison approach of two latest fractional derivatives models, namely, Atangana-Baleanu and Caputo-Fabrizio, for a generalized Casson fluid and obtained exact solutions. In [17–19], the authors also used the Caputo-Fabrizio fractional derivative to model some important examples.

Due to the abovementioned applications, the existence of solutions for nonlinear differential equations is an attractive research topic and has been studied using different techniques of nonlinear analysis [20–23]. One of the most important theorems in ordinary differential equations is Picard's existence and uniqueness theorem. This theorem, which is applied on first-order ordinary differential equations, can be generalized to establish existence and uniqueness results for both higher-order ordinary differential equations and for systems of differential equations. This theorem is a good introduction to the broad class of existence and uniqueness theorems that are based on fixed-point techniques [24–30].

In this paper, we obtain an extension of Picard's theorem for differential equations with the Caputo-Fabrizio fractional derivative. This theorem provides conditions for which a fractional initial value problem involving the Caputo-Fabrizio derivative has a unique solution. On the other hand, the proof of this extension of Picard's theorem provides a way of constructing successive approximations to the solution.

2. Preliminaries

In this section, we recall some notations and definitions which are needed throughout this paper. Further, some lemmas and theorems are stated as preparations for the main results. First, in the following, we provide some basic concepts and definitions in connection with the new Caputo-Fabrizio derivative.

The well-known left-sided Caputo fractional derivative ${}^{C}D^{\alpha}_{0^+,t}$ of a function $f(x) \in H^1(0,b)$ with $0 < \alpha < 1$, is defined by

$$ {}^{C}D^{\alpha}_{0^+,t} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t f'(s)(t-s)^{-\alpha} ds. \tag{1} $$

In [5], Caputo and Fabrizio proposed the new operator by replacing the singular kernel $(x-t)^{-\alpha}$ with $e^{-\alpha(x-t)/(1-\alpha)}$ and $1/\Gamma(1-\alpha)$ with $N(\alpha)/(1-\alpha)$ in the Caputo definition to obtain the following definition.

Definition 1. Let g be a given function in $H^1(a,b)$. The Caputo-Fabrizio derivative of fractional order $\alpha \in [0,1]$ is defined as

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \left(\frac{N(\alpha)}{1-\alpha}\right) \int_a^t g'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx, \tag{2} $$

where $N(\alpha)$ is a normalization function [5]. Also, if a certain function g does not satisfy in the restriction $g \in H^1(a,b)$, then its fractional derivative is redefined as

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \frac{\alpha N(\alpha)}{1-\alpha} \int_a^t (g(t)-g(x)) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx. \tag{3} $$

Clearly, as mentioned in [5], if one sets $\sigma = (1-\alpha)/\alpha \in [0,\infty]$ and $\alpha = 1/(1+\sigma) \in [0,1]$, then the Caputo-Fabrizio definition becomes

$$ {}^{CF}_{a}D^{\alpha}_{t}(g(t)) = \frac{N(\sigma)}{\sigma} \int_a^t g'(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \tag{4} $$

where $N(0) = N(\infty) = 1$, and

$$ \lim_{\sigma \to 0} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \tag{5} $$

Also, the fractional derivative of order $(n+\alpha)$ when $n \geq 1$ and $\alpha \in [0,1]$ is defined by the following

$$ {}^{CF}_{a}D^{(\alpha+n)}_{t}(g(t)) = {}^{CF}_{a}D^{(\alpha)}_{t}\left(D^{(n)}_{t} g(t)\right). \tag{6} $$

Definition 2. Let $g \in H^1(a,b)$, then its fractional integral of an arbitrary order is defined as follows:

$$ {}^{CF}_{a}I^{\alpha}_{t}(g(t)) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t) + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_a^t g(s) ds, \quad t \geq 0. \tag{7} $$

It is clear, in view of the above definition, that the αth Caputo-Fabrizio derivative of a function g is average between g and its first-order integral. Therefore,

$$ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} + \frac{2\alpha}{(2-\alpha)N(\alpha)} = 1. \tag{8} $$

So, we arrive at the following

$$ N(\alpha) = \frac{2}{2-\alpha}, \quad 0 \leq \alpha \leq 1. \tag{9} $$

The Laplace transform of the Caputo-Fabrizio derivative

is

$$L\left\{{}^{CF}D_t^{\alpha}(g(t))\right\} = \frac{sL\{g(t)\} - g(0)}{(1-\alpha)s+\alpha}. \tag{10}$$

Theorem 3 (Picard theorem [31]). *Let D be an open set in (t, x)-space. Let $(t_0, x^0) \in D$ and a and b be positive constants such that the set*

$$R = \{(t, x) \mid |t - t_0| \le a, |x - x_0| \le b\}, \tag{11}$$

is contained in D. Suppose that the function g is defined, continuous on D, and satisfies a Lipschitz condition with respect to x in R. Let

$$\begin{aligned} M &= \max_{(t,x)\in R} |g(t, x)|, \\ A &= \min\left\{a, \frac{b}{M}\right\}. \end{aligned} \tag{12}$$

Then, the following initial value problem

$$x' = g(t, x), x(t_0) = x^0, \tag{13}$$

has a unique solution, $x(t)$, on the interval $(t_0 - A, t_0 + A)$. For this solution in the domain $(t_0 - A, t_0 + A)$, we have

$$|x(t) - x^0| \le MA. \tag{14}$$

Note that by the mean-value theorem, the Lipschitz condition will be satisfied if we have $|(\partial/\partial x)g(t, x)| \le K$.

3. Extension of Picard Theorem

Picard's Theorem 3 guarantees the existence and uniqueness of the solution of the following initial value problem of first-order differential equations:

$$\frac{dy}{dt} = f(t, y(t))\ t \ge t_0, \tag{15}$$

$$y(t_0) = y_0. \tag{16}$$

In proving this theorem, the solution is obtained by the well-known successive approximations method (Picard-Lindelöf method) [31]. In this method, the approximate solution for solving (15) is defined by

$$y_{k+1} = y_0 + \int_{t_0}^{t} f(s, y_k(s))ds, \quad k \in N. \tag{17}$$

By continuing this process, when $k \to \infty$, the exact solution is obtained. In practice, the exact solution is approximated for a sufficient large k by y_k.

In this section, we consider the following differential equation

$${}_0^{CF}D_t^{\alpha}u(t) = g(t, u), \tag{18}$$

such that $t \in J = [0, 1]$, with the initial condition $u(0) = u_0$, where ${}_0^{CF}D_t^{\alpha}$ denotes the fractional Caputo-Fabrizio derivative. We extend Picard's theorem to this problem, and by the successive approximation method, an iterative process is provided to obtain the solution. We state the following generalized Picard existence and uniqueness theorem.

Theorem 4. *Suppose that the function g is defined, continuous on an open set Ω in (t, u)-space, and satisfies*

$$|g(t, u) - g(t, v)| \le k|u - v|, \quad 0 < k < 1. \tag{19}$$

Let $M = \max_{t\in J} |g(t, u)|$. Then, the fractional differential equation (18) has a unique solution such that $u(0) = u_0$.

To prove the theorem, first, we need to establish the following lemma.

Lemma 5. *The function $u(t)$ is the solution of (18) under the initial condition $u(0) = u_0$ if and only if it satisfies the following integral equation:*

$$\begin{aligned} u(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u(s))ds. \end{aligned} \tag{20}$$

Proof. If $u(t)$ is a solution of (18), then taking the fractional integral of order α, we obtain (20). The second part of the theorem comes from differentiating equation (20).

In the reminder of the proof, using the successive approximation method, we show that the sequence defined by

$$\begin{aligned} u_0(t) &= u_0, \\ u_1(t) &= u_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_0) \\ &\quad + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u_0)ds, \\ &\vdots \\ u_m(t) &= u_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_{m-1}(t)) \\ &\quad + \frac{2\alpha}{(2-\alpha)N(\alpha)} \int_0^t g(s, u_{m-1}(s))ds, \end{aligned} \tag{21}$$

converges to a function, which is a solution of (20), and then we show that this solution is unique.

Lemma 6. *For each m, the function $u_m(t)$ is defined, continuous on J and satisfies*

$$|u_m(t) - u_0| \le M. \tag{22}$$

Proof. We prove the lemma by induction. Since

$$\begin{aligned}|u_1(t)-u_0(t)| &\le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|g(t,u_0(t))| \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |g(s,u_0(s))|ds \\ &\le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M + \frac{2\alpha}{(2-\alpha)N(\alpha)}M\int_0^t \\ &\cdot ds \le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}M = M,\end{aligned} \tag{23}$$

the result is obviously true for $m=0$. Let us suppose that for $t\in J$,

$$|u_m(t)-u_0|\le M. \tag{24}$$

This yields that $f(t,u_m(t))$ is defined on J, and since $f(t,u_m(t))$ is continuous at t, one asserts that

$$\begin{aligned}u_{m+1}(t) &= u_0 + \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}g(t,u_m(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t g(s,u_m(s))ds,\end{aligned} \tag{25}$$

is defined and continuous. Indeed, we have

$$\begin{aligned}|u_{m+1}(t)-u_0| &\le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|g(t,u_m(t))| \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |g(s,u_m(s))|ds \le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}M \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}M\int_0^t ds = M.\end{aligned} \tag{26}$$

Lemma 7. *The sequence $\{u_m(t)\}$ converges uniformly on J to a continuous function $u(t)$.*

Proof. It is obvious that the convergence of the series

$$u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)], \tag{27}$$

yields convergence of the sequence $\{u_m(t)\}$. For $t\in J$, let us denote

$$\begin{aligned}d_n(t) &= |u_{n+1}(t)-u_n(t)|, \\ F_n(t) &= g(t,u_{n+1}(t)) - g(t,u_n(t)).\end{aligned} \tag{28}$$

Then, for each n, one has

$$\begin{aligned}d_n(t) &\le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}|F_n(t)| \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t |F_n(s)|ds \\ &\le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}k|u_n(t)-u_{n-1}(t)| \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_0^t |u_n(s)-u_{n-1}(s)|ds \\ &= \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kd_{n-1}(t) + \frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_{t_0}^t \\ &\cdot d_{n-1}(s)ds = k_0^{CF}I_t^{\alpha}d_{n-1}(t),\end{aligned} \tag{29}$$

where k is the Lipschitz constant of g and $0<k<1$. Now, we show that for each n, we have

$$d_n(t)\le Mk^n. \tag{30}$$

From Lemma 6, we have

$$d_0(t) = |u_1(t)-u_0(t)|\le M. \tag{31}$$

By induction, let $d_n(t)\le Mk^n$. Then, from (29) and (8), one writes

$$\begin{aligned}d_{n+1}(t) &\le k_0^{CF}I_t^{\alpha}d_n(t) = \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kd_n(t) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}k\int_{t_0}^t d_n(s)ds \le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}kMk^n \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}kMk^n\int_0^t ds \le \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)}Mk^{n+1} \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}Mk^{n+1} = Mk^{n+1}.\end{aligned} \tag{32}$$

Therefore,

$$\sum_{n=0}^{\infty} d_n(t) \le M\sum_{n=0}^{\infty}k^n. \tag{33}$$

Since $0<k<1$, the uniform convergence of (27) follows from the Weierstrass test or by a simple comparison test.

Lemma 8. *The function $u(t)$ is satisfied in (18), and we have $u(0)=u_0$.*

Proof. First, let us show that $|u(t)-u_0|$ is bounded. That is,

$$|u(t)-u_0|<B, \tag{34}$$

for some constant B. We can deduce that $g(t,u(t))$ is defined for $t\in J$. For $t\in J$ and $\varepsilon>0$ and for a sufficiently large m, one

has

$$|u(t) - u_0| \le |u(t) - u_m(t)| + |u_m(t) - u_0| \le \varepsilon + M < B. \quad (35)$$

Then, by the Lipschitz condition of g, we have

$$\begin{aligned} \left|\int_0^t g(s, u(s)) - g(s, u_m(s))ds\right| &\le \int_0^t |g(s, u(s)) \\ &- g(s, u_m(s))|ds \le k\int_0^t |u(s) - u_m(s)|ds \le k\varepsilon. \end{aligned} \quad (36)$$

Therefore, $\lim_{m\to\infty}\int_0^t g(s, u_m(s)) = \int_0^t g(s, u(s))ds$. Now, by taking the limit with respect to m on both sides of the following equation

$$\begin{aligned} u_m(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u_{m-1}(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t g(s, u_{m-1}(s))ds, \end{aligned} \quad (37)$$

we obtain

$$\begin{aligned} u(t) = u_0 &+ \frac{2(1-\alpha)}{(2-\alpha)N(\alpha)} g(t, u(t)) \\ &+ \frac{2\alpha}{(2-\alpha)N(\alpha)}\int_0^t g(s, u(s))ds. \end{aligned} \quad (38)$$

Now, we prove the uniqueness of the solution.

Lemma 9. *The solution $u(t)$ of the integral equation (7) satisfying the condition $u(t_0) = u_0$, is the unique solution of (18) with this initial condition.*

Proof. Suppose that there exist two solutions $u_1(t)$ and $u_2(t)$ of the integral equation (7) on J subject to the condition $u_1(t_0) = u_2(t_0) = u_0$. First, since $u_1(t)$ and $u_2(t)$ are continuous functions, there exists a constant $B > 0$ such that in the closed interval J, we have

$$|u_1(t) - u_2(t)| < B. \quad (39)$$

Let us suppose that for each positive integer m,

$$|u_1(t) - u_2(t)| < k^m B. \quad (40)$$

Then, from (7), we have $|u_1(t) - u_2(t)| < k^{m+1}B$. Therefore, by induction, $|u_1(t) - u_2(t)|$ is less than each term of the convergent geometric series of $B/(1-k)$. This yields that for each ε, $|u_1(t) - u_2(t)| < \varepsilon$, and therefore, we have $u_1(t) = u_2(t)$.

By proving the above lemma, the proof of Theorem 3 is completed. Note that the iterative process (21) provides a constructive approach to obtain the solution. We describe the following simple example where the hypotheses of Theorem (4) hold:

$$\begin{aligned} {}_0^{CF}D_t^{\alpha} u(t) &= \frac{1}{u+1}, \\ u(0) &= 0. \end{aligned} \quad (41)$$

By assuming $C = 2(1-\alpha)/(2-\alpha)N(\alpha)$ and $D = 2\alpha/(2-\alpha)N(\alpha)$, the results of using (21) are as follows:

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= C + Dt, \\ u_2(t) &= \frac{C}{Dt + C + 1} + \ln(Dt + C + 1), \\ u_3(t) &= \frac{C}{C/(Dt + C + 1) + \ln(Dt + C + 1) + 1} \\ &\quad + D\int_0^t \frac{1}{C/(Ds + C + 1) + \ln(Ds + C + 1) + 1}. \\ &\vdots \end{aligned} \quad (42)$$

To ensure the results, let us choose $\alpha = 1$. In this case, it is easy to show that the obtained sequence $0, t, \ln(t+1), \cdots$ converges to the exact solution $u(t) = \sqrt{2t+1} - 1$.

4. Conclusion

By Picard's theorem, we can study the existence and uniqueness of a solution of first-order differential equations. Also, this theorem can be applied to ensure the existence of a unique solution of higher-order ordinary differential equations and for systems of differential equations. On the other hand, this theorem is an essential tool in fixed-point theory. Therefore, a generalization of this theorem for fractional differential equations would be interesting. In this paper, we proved an extension of this theorem to the initial value problems of fractional ordinary differential equations with the Caputo-Fabrizio derivative, and by the successive approximation method, an iterative process was provided to obtain the solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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