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Research Article

Existence Theorems on Advanced Contractions with Applications

Sang Og Kim[1] **and Muhammad Nazam**[2]

[1]*School of Data Science, Hallym University, Chuncheon 24252, Republic of Korea*
[2]*Department of Mathematics, Allama Iqbal Open University, H-8 Islamabad, Pakistan*

Correspondence should be addressed to Muhammad Nazam; muhammad.nazam@aiou.edu.pk

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In this research article, by introducing a mapping φ defined on $[0,\infty)^4$, with some axioms, we define two generalized contractions called $F^{\varphi}_{H^+}$-contractions and $\varphi H^+_{\mathscr{P}}$-contractions. We investigate their mutual relation and establish an existence theorem addressing $F^{\varphi}_{H^+}$-contractions with some applications.

1. Introduction

Frechet gave an abstraction to the notion of distance in Euclidean spaces by introducing metric spaces. Partial metrics (denoted by $\mathscr{P}$) were introduced in [1] as a generalization of the notion of metric to allow nonzero self-distance for the purpose of modeling partial objects in reasoning about data flow networks. The self-distance $\mathscr{P}(\alpha, \alpha)$ is to be understood as a quantification of the extent to which α is unknown. Matthews [1] proved an analogue of Banach's fixed point theorem in partial metric spaces. This remarkable fixed point theorem led many researchers to investigate fixed points of self-mappings in partial metric spaces (see [2–7]).

The investigation of fixed points of multivalued or set-valued mappings was started by Nadler [8]. For this purpose, Nadler introduced a metric function H to measure distance between two nonempty closed and bounded sets. This metric function is also known as the Hausdorff metric in literature. Aydi et al. [5] generalized the Hausdorff metric to the partial Hausdorff metric and hence generalized the Nadler fixed point theorem. Nazam et al. [7] established various fixed point results using the partial Hausdorff metric. Recently, Pathak et al. [9] introduced another metric function H^+ to measure the distance between two nonempty closed and bounded sets and hence proved some fixed point results. Nashine et al. [10] also proved some fixed points theorems on H^+-multivalued contractions and their application to homotopy theory. For recent research in this direction, see [11–13].

In 1922, Banach introduced the Banach Contraction Principle in his PhD thesis. Since then, there has been a trend to generalize and apply it to show the existence of the solutions to various mathematical models (both linear and nonlinear). A large number of research articles contain many useful generalizations of Banach Contraction Principle. In one such attempt, Wardowski [14] introduced F-contractions, where F represents the class of nonlinear real-valued functions satisfying three axioms (F_1, F_2, F_3). The concept of F-contractions proved to be a useful addition in fixed point theory (see for instant [15–18] and references therein). The advancement in the study of F-contraction is in progress, and in this direction recently, Abbas et al. [19] introduced the Presic-type F-contraction and established a fixed point theorem for such kind of mappings. Tomar et al. [20] provided an existence theorem for six self-mappings under the notion of F-contraction. Durmaz et al. [17] studied F-contraction under the effect of a partial order. Sgroi et al. [21] extended the notion of F-contraction to multivalued F-contraction by combining the ideas of Wardowski and Nadler. Durmaz et al. [22] generalized the results given in [16, 17, 21] by introducing (α, F)-contraction. Similarly, Piri et al. [23] proved some theorems on the F-Suzuki type inequalities under some weaker conditions, and Shukla et al. [24] established a common fixed point theorem for weak F-contraction under 0-complete partial metric spaces. Recently, Karapinar et al. [25] presented a survey paper which encompasses almost all the results addressing F-contractions.

The motivation to write this article is the contents of the article [26]. In [26], authors introduced a function (called auxiliary function) defined on $[0,\infty)^4$ satisfying some axioms and used it to establish a fixed point theorem. It was then shown that the Banach fixed point theorem, Kannan fixed point theorem, Chatterjea fixed point theorem, Reich fixed point theorem, Hardy and Rogers fixed point theorem, and Círíc type fixed point theorems are particular cases of this fixed point theorem. Since all the above mentioned fixed point theorems have been generalized using the notion of F-contraction both in metric spaces and partial metric spaces (see [25]), we develop a general fixed point theorem, representing them all, in the Hausdorff partial metric spaces.

This article is organized as follows. In Section 2, some basic notions are given. In Section 3, we give highlights of Hausdorff p-ms. In Section 4, we introduce the $F_{H^+}^{\varphi}$-contractions and $\varphi H_{\mathscr{P}}^{+}$-contractions and investigate the relations between them. We also study the existence theorem and its consequences. And in Section 5, we derive two results regarding applications by applying the existence theorem given in Section 4. The presented existence theorem generalizes, improves, and extends the results established by Pathak et al. [9].

2. Basic Notions

Let partial metric spaces be denoted by p-m-s.

Matthews [1], while working on networking topologies, noticed the nonzero self-distance (loop is the best example to understand his point). The self-distance played a key role in introduction of p-m-s. Matthews [1] defined the p-m-s as follows: let $\mathfrak{J}$ be a nonempty set, and the function $\mathscr{P}:\mathfrak{J}\times\mathfrak{J}\to[0,\infty)$ is said to be a partial metric (p-m) on $\mathfrak{J}$ if for all $\alpha,\beta,\gamma\in\mathfrak{J}$, the axioms ($p_1$)-($p_4$) are satisfied.

(p_1) $\alpha=\beta\Leftrightarrow\mathscr{P}(\alpha,\alpha)=\mathscr{P}(\alpha,\beta)=\mathscr{P}(\beta,\beta)$

(p_2) $\mathscr{P}(\alpha,\alpha)\le\mathscr{P}(\alpha,\beta)$

(p_3) $\mathscr{P}(\alpha,\beta)=\mathscr{P}(\beta,\alpha)$

(p_4) $\mathscr{P}(\alpha,\gamma)\le\mathscr{P}(\alpha,\beta)+\mathscr{P}(\beta,\gamma)-\mathscr{P}(\beta,\beta)$.

Some examples of $(\mathfrak{J},\mathscr{P})$ are as follows. The function $\mathscr{P}:\mathfrak{J}^2\to[0,\infty)$ defined by

(1) $\mathscr{P}(\alpha,\beta)=|\alpha-\beta|+C\,;C\ge0$ for all $\alpha,\beta\in\mathfrak{J}$ is a $(\mathfrak{J},\mathscr{P})$

(2) $\mathscr{P}(\alpha,\beta)=\max\{\alpha,\beta\}$, is a $(\mathfrak{J},\mathscr{P})$

(3) $\mathscr{P}(\alpha,\beta)=e^{|\alpha-\beta|}+\max\{\alpha,\beta\}$, is a $(\mathfrak{J},\mathscr{P})$.

It is noted that $\mathscr{P}(\alpha,\beta)=0$ implies $\alpha=\beta$. The p-m function $\mathscr{P}$ is continuous. If $\mathscr{P}$ is a p-m then the function $d_{\mathscr{P}}:\mathfrak{J}\times\mathfrak{J}\to[0,\infty)$ defined by

$$d_{\mathscr{P}}(\alpha,\beta)=2\mathscr{P}(\alpha,\beta)-[\mathscr{P}(\alpha,\alpha)+\mathscr{P}(\beta,\beta)]\text{ for all }\alpha,\beta\in\mathfrak{J}\tag{1}$$

defines a metric on $\mathfrak{J}$. A T_0 topology can be defined on $(\mathfrak{J},\mathscr{P})$ with $\mathscr{P}$-open balls being its elements. The $\mathscr{P}$-open ball centered at σ_0 having radius ε is defined by $O_{\mathscr{P}}(\sigma_0,\varepsilon)=\{\sigma\in\mathfrak{J}:\mathscr{P}(\sigma_0,\sigma)<\mathscr{P}(\sigma_0,\sigma_0)+\varepsilon\}$. A set G is said to be bounded in $(\mathfrak{J},\mathscr{P})$ if there exist $\sigma_0\in\mathfrak{J}$ and $\Delta\ge0$ such that $\mathscr{P}(\sigma_0,\eta)<\mathscr{P}(\eta,\eta)+\Delta$ for all $\eta\in G$. Also it is easy to write $\eta\in\bar{G}$ (closure of G) $\Leftrightarrow\mathscr{P}(\eta,G)=\mathscr{P}(\eta,\eta)$ and G is closed in $(\mathfrak{J},\mathscr{P})$ if and only if $G=\bar{G}$. If $\mathscr{P}(\sigma,\sigma)=\lim_{n\to\infty}\mathscr{P}(\sigma,\sigma_n)$; then we say that $\{\sigma_n\}$ converges to σ and conversely. If $\lim_{n,m\to\infty}\mathscr{P}(\sigma_n,\sigma_m)$ is finite, then the sequence $\{\sigma_n\}$ is said to be Cauchy, and in particular, if this Cauchy sequence converges in $(\mathfrak{J},\mathscr{P})$, then we say that the p-m-s $(\mathfrak{J},\mathscr{P})$ is complete. Lemma 1 provides fundamental rules to work in the p-m-s.

Lemma 1 [1].

(1) If the sequence σ_n is Cauchy sequence in $(\mathfrak{J},\mathscr{P})$, then it is Cauchy sequence in the metric space $(\mathfrak{J},d_{\mathscr{P}})$ and conversely

(2) The completeness of $(\mathfrak{J},\mathscr{P})$ implies the completeness of $(\mathfrak{J},d_{\mathscr{P}})$ and conversely

(3) $\lim_{n\to\infty}d_{\mathscr{P}}(\sigma,\sigma_n)=0\Leftrightarrow\lim_{n\to\infty}\mathscr{P}(\sigma,\sigma_n)=\mathscr{P}(\sigma,\sigma)=\lim_{n,m\to\infty}\mathscr{P}(\sigma_n,\sigma_m)$, provided $(\mathfrak{J},\mathscr{P})$ is complete.

Remark 1. There are sequences which converge in p-m-s but not in metric spaces. Indeed, for the sequence $\{1/n:n\in\mathbb{N}\}$ in $\mathfrak{J}=[0,1]$ and p-m $\mathscr{P}$ defined by $\mathscr{P}(\rho,\varsigma)=|\rho-\varsigma|+C$ $(C\ge0)$ $\forall\rho,\varsigma\in\mathfrak{J}$, it is easy to check that the sequence $\{1/n\}$ converges to 0 with respect to $\mathscr{P}$ but does not converge to 0 with respect to metric d defined by $d(\rho,\varsigma)=\mathscr{P}(\rho,\varsigma)$ if $\rho\ne\varsigma$ and 0 otherwise.

3. Hausdorff Partial Metric

Let the set of nonempty closed and bounded subsets of $(\mathfrak{J},\mathscr{P})$ be denoted by $CB_{\mathscr{P}}(\mathfrak{J})$. Let $\mathscr{P}(\sigma,A)=\inf\{\mathscr{P}(\sigma,a):a\in A\}$, $A\in CB_{\mathscr{P}}(\mathfrak{J})$. Let $\Delta_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{J})\times CB_{\mathscr{P}}(\mathfrak{J})\to[0,\infty)$ be defined by $\Delta_{\mathscr{P}}(X,Y)=\sup\{\mathscr{P}(a,Y):a\in X\}$. Let $H_{\mathscr{P}}:CB_{\mathscr{P}}(\mathfrak{J})\times CB_{\mathscr{P}}(\mathfrak{J})\to[0,\infty)$ be defined by

$$H_{\mathscr{P}}(X,Y)=\max\{\Delta_{\mathscr{P}}(X,Y),\Delta_{\mathscr{P}}(Y,X)\}.\tag{2}$$

Let $H_{\mathscr{P}}^{+}:CB_{\mathscr{P}}(\mathfrak{J})\times CB_{\mathscr{P}}(\mathfrak{J})\to[0,\infty)$ be defined by

$$H_{\mathscr{P}}^{+}(X,Y)=\frac{1}{2}\{\Delta_{\mathscr{P}}(X,Y)+\Delta_{\mathscr{P}}(Y,X)\}.\tag{3}$$

Since $\max\{\sigma,\varsigma\}\ge1/2(\sigma+\varsigma)$, $H_{\mathscr{P}}(X,Y)\ge H_{\mathscr{P}}^{+}(X,Y)$ for all $X,Y\in CB_{\mathscr{P}}(\mathfrak{J})$. A comprehensive study of the distance $H^{+}(X,Y)$ with reference to metric d was presented by Pathak et al. in [9]. We claim that

(a) $H_{\mathscr{P}}^{+}(X,Y)$ and $H_{\mathscr{P}}(X,Y)$ are topological equivalent

(b) the mapping $H_{\mathscr{P}}^{+}:CB_{\mathscr{P}}(\mathfrak{J})\times CB_{\mathscr{P}}(\mathfrak{J})\to[0,\infty)$ defines a p-m on $CB_{\mathscr{P}}(\mathfrak{J})$

(c) if the p-m-s $(\mathfrak{J}, \mathscr{P})$ is complete then $(CB_{\mathscr{P}}(\mathfrak{J}), H^{+}_{\mathscr{P}})$ is also complete and vice versa

(d) the mapping $H^{+}_{\mathscr{P}} : CB_{\mathscr{P}}(\mathfrak{J}) \times CB_{\mathscr{P}}(\mathfrak{J}) \to [0,\infty)$ is continuous.

Proposition 1 [7]. *Let $(\mathfrak{J}, \mathscr{P})$ be p-m-s. For any $J, K, L \in CB_{\mathscr{P}}(\mathfrak{J})$, we have the following:*

(1) $\Delta_{\mathscr{P}}(J, J) = \sup \{\mathscr{P}(u, v): u, v \in J\}$

(2) $\Delta_{\mathscr{P}}(J, K) = \Delta_{\mathscr{P}}(K, J)$

(3) $\Delta_{\mathscr{P}}(J, K) = 0 \Rightarrow J \subseteq K$

(4) $\Delta_{\mathscr{P}}(J, L) \leq \Delta_{\mathscr{P}}(J, K) + \Delta_{\mathscr{P}}(K, L) - \inf_{k\in K} \mathscr{P}(k, k)$.

Proposition 2. *Let $(\mathfrak{J}, \mathscr{P})$ be p-m-s. For any $J, K, L \in CB_{\mathscr{P}}(\mathfrak{J})$, we have the following:*

(1) $H^{+}_{\mathscr{P}}(J, K) = 0$ implies $J = K$

(2) $H^{+}_{\mathscr{P}}(J, J) \leq H^{+}_{\mathscr{P}}(J, K)$

(3) $H^{+}_{\mathscr{P}}(J, K) = H^{+}_{\mathscr{P}}(K, J)$

(4) $H^{+}_{\mathscr{P}}(J, L) \leq H^{+}_{\mathscr{P}}(J, K) + H^{+}_{\mathscr{P}}(K, L) - \inf_{k\in K} \mathscr{P}(k, k)$.

Proof. Following the arguments given in ([5], Proposition 2.2 and Proposition 2.3), we get the result. We omit its details.

$H_{\mathscr{P}}$-**contraction**: Let $(\mathfrak{J}, \mathscr{P})$ be a p-m-s, the mapping $T : \mathfrak{J} \to CB_{\mathscr{P}}(\mathfrak{J})$ is called an $H_{\mathscr{P}}$-contraction, if there exists $k < 1$ such that $H_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \leq k\mathscr{P}(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathfrak{J}$ (see [7]).

$H^{+}_{\mathscr{P}}$-**contraction**: Let $(\mathfrak{J}, \mathscr{P})$ be a p-m-s; the mapping $T : \mathfrak{J} \to CB_{\mathscr{P}}(\mathfrak{J})$ is called an $H^{+}_{\mathscr{P}}$-contraction, if (1) there exists $k < 1$ such that $H^{+}_{\mathscr{P}}(T(\sigma) \setminus \{\sigma\}, T(\varsigma) \setminus \{\varsigma\}) \leq k\mathscr{P}(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathfrak{J}$; (2) for all $\sigma \in \mathfrak{J}$, $\{\varsigma\} \in T(\sigma)$, $\varepsilon > 0$ there exists $\{\xi\} \in T(\varsigma)$ such that $\mathscr{P}(\varsigma, \xi) \leq H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon$ (see [27]).

Since $H_{\mathscr{P}}(X, Y) \geq H^{+}_{\mathscr{P}}(X, Y)$ for all $X, Y \in CB_{\mathscr{P}}(\mathfrak{J})$, $H_{\mathscr{P}}$-contraction implies $H^{+}_{\mathscr{P}}$-contraction but not conversely (see Example 1).

Example 1. Let $\mathfrak{J} = \{0, 1/7, 1\}$. *Define the function* $\mathscr{P} : \mathfrak{J} \times \mathfrak{J} \to [0,\infty)$ *by*

$$\mathscr{P}(\sigma, \varsigma) = \max \{\sigma, \varsigma\} \text{ for all } \sigma, \varsigma \in \mathfrak{J}. \quad (4)$$

Then $(\mathfrak{J}, \mathscr{P})$ is a p-m-s. Let $T : \mathfrak{J} \to CB_{\mathscr{P}}(\mathfrak{J})$ be defined by

$$T(\sigma) = \begin{cases} \{0\} & \text{if } \sigma = 0 \\ \left\{0, \dfrac{1}{7}\right\} & \text{if } \sigma = \dfrac{1}{7} \\ \{0, 1\} & \text{if } \sigma = 1. \end{cases} \quad (5)$$

We have three cases (Case 1: $\sigma = 0, \varsigma = 1/7$, Case 2: $\sigma = 0, \varsigma = 1$, and Case 3: $\sigma = 1/7, \varsigma = 1$).

Case 1. If $\sigma = 0, \varsigma = 1/7$, then $\mathscr{P}(\sigma, \varsigma) = 1/7$, $H_{\mathscr{P}}(T(0), R(1/7)) = 1/7$, and $H^{+}_{\mathscr{P}}(T(0), R(1/7)) = 1/14$. This clearly shows that

$$H^{+}_{\mathscr{P}}\left(T(0), R\left(\frac{1}{7}\right)\right) \leq L\mathscr{P}\left(0, \frac{1}{7}\right) \text{holds for all } L \geq \frac{1}{2}, \quad (6)$$

whereas

$$H_{\mathscr{P}}\left(T(0), R\left(\frac{1}{7}\right)\right) > L\mathscr{P}\left(0, \frac{1}{7}\right) \text{for any } L < 1. \quad (7)$$

Case 2. If $\sigma = 0, \varsigma = 1$, then $\mathscr{P}(\sigma, \varsigma) = 1$, $H_{\mathscr{P}}(T(0), R(1)) = 1$, and $H^{+}_{\mathscr{P}}(T(0), R(1)) = 1/2$. This clearly shows that

$$H^{+}_{\mathscr{P}}(T(0), R(1)) \leq L\mathscr{P}(0, 1) \text{ holds for all } L \geq \frac{1}{2}, \quad (8)$$

whereas

$$H_{\mathscr{P}}(T(0), R(1)) > L\mathscr{P}(0, 1) \text{ for any } L < 1. \quad (9)$$

Case 3. If $\sigma = 1/7, \varsigma = 1$, then $\mathscr{P}(\sigma, \varsigma) = 1$, $H_{\mathscr{P}}(R(1/7), R(1)) = 1$, and $H^{+}_{\mathscr{P}}(R(1/7), R(1)) = 4/7$. This clearly shows that

$$H^{+}_{\mathscr{P}}\left(R\left(\frac{1}{7}\right), R(1)\right) \leq L\mathscr{P}\left(\frac{1}{7}, 1\right) \text{ holds for all } L \geq \frac{1}{2}, \quad (10)$$

whereas

$$H_{\mathscr{P}}\left(R\left(\frac{1}{7}\right), R(1)\right) > L\mathscr{P}\left(\frac{1}{7}, 1\right) \text{ for any } L < 1. \quad (11)$$

Note: the inequality $\mathscr{P}(\varsigma, \xi) \leq \mathrm{H}^{+}_{\mathscr{P}}(\mathrm{T}(\sigma), \mathrm{T}(\varsigma)) + \varepsilon$ also holds for each case, and for all $\sigma \in \mathfrak{J}$, $\varsigma \in \mathrm{T}(\sigma)$, $\xi \in \mathrm{T}(\varsigma)$.

4. Fixed Points of $F^{\varphi}_{H^{+}}$-Contraction

Let $T : \mathfrak{J} \to \mathfrak{J}$ be a self-mapping defined on nonempty set $\mathfrak{J}$. The problem "to find $\sigma^* \in \mathfrak{J}$ such that $\sigma^* = T(\sigma^*)$" is called fixed point problem. If $T : \mathfrak{J} \to CB(\mathfrak{J})$, then the fixed point problem turns into the form "to find $\sigma^* \in \mathfrak{J}$ such that $\sigma^* \in T(\sigma^*)$." For the solution of fixed point problem, generally, a Picard iterative sequence ($\{\sigma_n\}$such that $\sigma_{n+1} = T(\sigma_n)$) is proved to be a Cauchy sequence subject to contractive condition and completeness of the underlying abstract metric space leads to such σ^*. In this section, at first, we introduce and compare $F^{\varphi}_{H^{+}}$-contraction and $\varphi H^{+}_{\mathscr{P}}$-contraction, and secondly, we obtain a theorem assuring unique fixed point of $F^{\varphi}_{H^{+}}$-contraction. We proceed with definitions of functions F and φ associated with some axioms.

Wardowski [14] considered a nonlinear function $F : (0, \infty) \to \mathbb{R}$ with the following axioms: (F_1): F is strictly increasing. (F_2): For each sequence $\{\sigma_n\}$ of positive numbers, $\lim_{n\to\infty}\sigma_n = 0$ if and only if $\lim_{n\to\infty}F(\sigma_n) = -\infty$. (F_3): For each sequence $\{\sigma_n\}$ of positive numbers $\lim_{n\to\infty}\sigma_n = 0$, there exists $\theta \in (0, 1)$ such that $\lim_{\sigma_n\to 0^+}(\sigma_n)^{\theta}F(\sigma_n) = 0$. Let $\mathscr{F} = \{F : (0,\infty) \to \mathbb{R} | F \text{ satisfies} (F_1) - (F_3)\}$.

The collection $\mathcal{F}$ is nonempty: $f(\sigma)=\ln(\sigma)$, $g(\sigma)=\sigma+\ln(\sigma)$, $h(\sigma)=\ln(\sigma^2+\sigma)$, and $k(\sigma)=-1/\sqrt{\sigma}$ are members of this collection.

Let us consider the function $\varphi:[0,\infty)^4\rightarrow[0,\infty)$ satisfying the following axioms:

(C_1) φ is continuous and non-decreasing in each coordinate

(C_2) if there exist $\sigma,\varsigma\in[0,\infty)$ such that $\sigma<\varsigma$ then $\varphi(\varsigma,\varsigma,\sigma,\varsigma)\leq\varsigma$

(C_3) if there exists $\sigma\in[0,\infty)$ such that $\sigma\leq\varphi(0,0,\sigma,(\sigma/2))$ then $\varphi(0,0,\sigma,(\sigma/2))=\sigma$.

Let $\mathscr{C}_\varphi=\{\varphi:[0,\infty)^4\rightarrow[0,\infty)|_\varphi \text{ satisfies } (C_1)-(C_3)\}$. The following examples show that the set $\mathscr{C}_\varphi$ is nonempty:

(1) $\varphi_a(\sigma,\varsigma,\omega,\theta)=\max\{\sigma,\varsigma,\omega,\theta\}$

(2) $\varphi_b(\sigma,\varsigma,\omega,\theta)=\theta$

(3) $\varphi_c(\sigma,\varsigma,\omega,\theta)=\max\{\sigma,\varsigma,\omega\}$

(4) $\varphi_d(\sigma,\varsigma,\omega,\theta)=\max\{\varsigma,\omega\}$

(5) $\varphi_e(\sigma,\varsigma,\omega,\theta)=\sigma$

(6) $\varphi_f(\sigma,\varsigma,\omega,\theta)=1/2(\varsigma+\omega)$

(7) $\varphi_g(\sigma,\varsigma,\omega,\theta)=\max\{\sigma,(\varsigma+\omega/2),\theta\}$

(8) $\varphi_Z(\sigma,\varsigma,\omega,\theta)=a\sigma+b(\varsigma+\omega)+2c\theta$, $a+2b+2c=1$

(9) $\varphi_i(\sigma,\varsigma,\omega,\theta)=a\sigma+b\varsigma+c\omega$, $a+b+c=1$.

Definition 1. Let $T:\mathfrak{J}\rightarrow P(\mathfrak{J})$ *and* $\alpha:\mathfrak{J}\times\mathfrak{J}\rightarrow[0,\infty)$ be two functions. A mapping T is said to be strictly α-admissible if for each $\sigma\in\mathfrak{J}$ *and* $\varsigma\in T(\sigma)$ with $\alpha(\sigma,\varsigma)>1$, there exists $\omega\in T(\varsigma)$ such that $\alpha(\varsigma,\omega)>1$.

Definition 2. Let $(\mathfrak{J},\mathscr{P})$ be a p-m-s and let $\alpha:\mathfrak{J}\times\mathfrak{J}\rightarrow[0,\infty)$ be a function. The space $(\mathfrak{J},\mathscr{P})$ is said to be strictly α-regular if for any sequence $\{\sigma_n\}\subset\mathfrak{J}$ such that $\alpha(\sigma_n,\sigma_{n+1})>1$ for all $n\in\mathbb{N}$ and $\sigma_n\rightarrow\sigma$ *as* $n\rightarrow\infty$, we have $\alpha(\sigma_n,\sigma)>1$ for all $n\in\mathbb{N}$.

Definition 3. Let $(\mathfrak{J},\mathscr{P})$ be a p-m-s. A mapping $T:\mathfrak{J}\rightarrow CB_{\mathscr{P}}(\mathfrak{J})$ is said to be a $\varphi H^+_{\mathscr{P}}$-contraction if there exist $k\in[0,1)$ and $\varphi\in\mathscr{C}_\varphi$ such that

$$\alpha(\sigma,\varsigma)H^+_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})\leq k\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right), \tag{12}$$

for all $\sigma,\varsigma\in\mathfrak{J}$.

Let $\mathscr{A}^*=\{(\sigma,\varsigma)\in\mathfrak{J}^2\mid\alpha(\sigma,\varsigma)\geq1 \text{and} H^+(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})>0\}$.

Definition 4. Let $(\mathfrak{J},\mathscr{P})$ be a p-m-s. A mapping $T:\mathfrak{J}\rightarrow CB_{\mathscr{P}}(\mathfrak{J})$ is said to be an $F^\varphi_{H^+}$-contraction if

(a) there exist $\varphi\in\mathscr{C}_\varphi$, $\tau>0$ and $F\in\mathcal{F}$ such that

$$\tau+F(\alpha(\sigma,\varsigma)H^+_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right)\right), \tag{13}$$

for all $\sigma,\varsigma\in\mathscr{A}^*$.

(b) For every $\varepsilon>0$, $\sigma\in\mathfrak{J}$, and $\varsigma\in T(\sigma)$, there exists $\xi\in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi)\leq H^+_{\mathscr{P}}(T(\sigma),T(\varsigma))+\varepsilon. \tag{14}$$

Remark 2. In particular if $\mathscr{P}(\sigma,\sigma)=0$, then for $\varphi_e\in\mathscr{C}_\varphi$, the inequality (13) turns into H^+ -contraction [9] for $F(\sigma)=\ln(\sigma)$.

Proposition 3. *Every $\varphi H^+_{\mathscr{P}}$-contraction is an $F^\varphi_{H^+}$-contraction, but the converse may not be true.*

Proof. Let $T:\mathfrak{J}\rightarrow CB_{\mathscr{P}}(\mathfrak{J})$ be a $\varphi H^+_{\mathscr{P}}$-contraction defined on $(\mathfrak{J},\mathscr{P})$; then for all $\sigma,\varsigma\in\mathfrak{J}$ there exist $k\in[0,1)$ and $\varphi\in\mathscr{C}_\varphi$ such that

$$\begin{aligned}&\alpha(\sigma,\varsigma)H^+_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})\\&\leq k\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\right.\\&\left.\quad\dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right).\end{aligned} \tag{15}$$

This can be written as

$$\begin{aligned}&\ln\left(\frac{1}{k}\right)+\ln(\alpha(\sigma,\varsigma)H^+_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\}))\\&\leq\ln\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\right.\\&\left.\quad\dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right)\\&\quad\cdot\left(\varphi(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\right.\\&\left.\quad\dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2})\right).\end{aligned} \tag{16}$$

Let $F\in\mathcal{F}$ be defined by $F(\sigma)=\ln(\sigma)$ for all $\sigma>0$ and put $\tau=\ln(1/k)$. The inequality (16) leads to

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right)\right). \tag{17}$$

The following example (Example 2) shows that an $F^{\varphi}_{H^{+}}$-contraction needs not to be a $\varphi H^{+}_{\mathscr{P}}$-contraction.

Example 2. Let $\varphi_a \in \mathscr{C}_{\varphi}\tau = 1$ and $F \in \mathscr{F}$ defined by $F(\sigma) = \ln(\sigma) + \sigma$ where $\varphi_a = \varphi_a(u,r,s,t)$: $(u = \mathscr{P}(\sigma,\varsigma)$, $r = \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\})$, $s = \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\})$, $t = \mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})/4)$. Let $\mathfrak{J} = \{0,1,2,\cdots\}$ equipped with p-m $\mathscr{P}: \mathfrak{J}\times\mathfrak{J}\to[0,\infty)$ defined by

$$\mathscr{P}(\sigma,\varsigma) = \sigma + \varsigma \text{ for all } \sigma \leq \varsigma. \tag{18}$$

Then, $(\mathfrak{J},\mathscr{P})$ is a p-m-s. Define the mapping $T : \mathfrak{J} \to 2^{\mathfrak{J}}$ by

$$R(\sigma) = \begin{cases}\{0\} & \text{if } \sigma \in \{0,1\};\\ \{0,\sigma-1\} & \text{if } \sigma \geq 2,\end{cases}$$
$$\alpha(\sigma,\varsigma) = \begin{cases}0 & \text{if } \sigma,\varsigma \in (-\infty,0);\\ e^{\mathscr{P}(\sigma,\varsigma)} & \text{if } \sigma,\varsigma \in \{0,1,2,\cdots\}.\end{cases} \tag{19}$$

The mapping T is α-admissible, closed, and bounded. We show that this mapping satisfies inequality (13) for all $\sigma,\varsigma \in \mathfrak{J}$. We observe that $H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) > 0$ if and only if $\sigma \geq 2$ and $\varsigma > 0$. Also for all $\sigma,\varsigma \in \mathfrak{J}$ with $\varsigma \in T(\sigma)$ and taking $\zeta = 0 \in T(\varsigma)$, we have

$$\begin{aligned}\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) &= \alpha(\sigma,\varsigma)\mathscr{P}(\varsigma,\zeta)\\ &= \alpha(\sigma,\varsigma)\varsigma < \alpha(\sigma,\varsigma)(\sigma+\varsigma) = \alpha(\sigma,\varsigma)\mathscr{P}(\sigma,\varsigma),\end{aligned} \tag{20}$$

and thus

$$\begin{aligned}&\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) - \varphi_a(u,r,s,t)\\ &\leq \alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) - \mathscr{P}(\sigma,\varsigma) \leq -2.\end{aligned} \tag{21}$$

Consequently,

$$\frac{\alpha(\sigma,\varsigma)H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma))}{\varphi_a(u,r,s,t)} e^{\alpha(\sigma,\varsigma)H^{+}(T(\sigma),T(\varsigma)) - \varphi_a(u,r,s,t)} \leq e^{-1}. \tag{22}$$

Hence,

$$1 + F(\alpha(\varsigma,\omega)H^{+}_{\mathscr{P}}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right)\right). \tag{23}$$

Similarly, for every member of $\mathscr{C}_{\varphi}$, the mapping T is $F^{\varphi}_{H^{+}}$-contraction. However, the mapping T is not $\varphi H^{+}_{\mathscr{P}}$-contraction: for $\varphi_e \in \mathscr{C}_{\varphi}$ and $\sigma \neq \varsigma = 0$, we have

$$\alpha(\sigma,0)H^{+}_{\mathscr{P}}(T(\sigma),T(0)) \leq k\varphi_e(u,r,s,t) \Rightarrow e^{\sigma}(\sigma-1) \leq k\sigma, \tag{24}$$

which then gives $e^{\sigma}(\sigma-1)/\sigma \leq k$, and $\lim_{\sigma\to\infty} e^{\sigma}(\sigma-1)/\sigma \leq k$ implies $k \geq \infty$, a contradiction. Hence, T is not $\varphi H^{+}_{\mathscr{P}}$-contraction for this particular member of $\mathscr{C}_{\varphi}$. Similarly, for $\varphi_b(u,r,s,t) = t \in \mathscr{C}_{\varphi}$ and $\sigma \neq \varsigma = 1$, we have

$$\alpha(\sigma,1)H^{+}_{\mathscr{P}}(T(\sigma),T(1)) \leq k\varphi_b(u,r,s,t) \text{ does not exist.} \tag{25}$$

Hence, T is not $\varphi H^{+}_{\mathscr{P}}$-contraction for this member of $\mathscr{C}_{\varphi}$. The mapping T has similar nature for other members of $\mathscr{C}_{\varphi}$.

The following theorem (Theorem 1) gives the proof of all particular problems corresponding to members of $\mathscr{C}_{\varphi}$ in one attempt.

Theorem 1. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{J} \to BC_{\mathscr{P}}(\mathfrak{J})$ be an $F^{\varphi}_{H^{+}}$-contraction such that*

(1) T is a strictly α-admissible mapping

(2) $\exists\sigma_0$ and $\sigma_1 \in T(\sigma_0)$ in $\mathfrak{J}$ such that $\alpha(\sigma_0,\sigma_1) > 1$

(3) $\mathfrak{J}$ is a strictly α-regular space

(4) F is continuous.

Then, there exists $x^* \in \mathfrak{J}$ such that $x^* \in T(x^*)$.

Proof. By assumption (2), there exist σ_0 and $\sigma_1 \in T(\sigma_0)$ in $\mathscr{A}$ such that $\alpha(\sigma_0,\sigma_1) > 1$. Note that if $\sigma_0 \in T(\sigma_0)$, then σ_0 is a fixed point of T, and if $\sigma_1 \in T(\sigma_1)$, then σ_1 is a fixed point of T as required. We proceed by assuming $\sigma_0 \notin T(\sigma_0)$ and $\sigma_1 \notin T(\sigma_1)$; thus, $\sigma_0,\sigma_1 \in \mathscr{A}^*$. Given $\alpha(\sigma_0,\sigma_1) > 1$ and $T(\sigma_0), T(\sigma_1)$ are nonempty, closed, and bounded sets, so, by Definition 4(b), there exists $\sigma_2 \in T(\sigma_1)$ such that

$$\mathscr{P}(\sigma_1,\sigma_2) \leq H^{+}_{\mathscr{P}}(T(\sigma_0),T(\sigma_1)) + \varepsilon. \tag{26}$$

Letting $\varepsilon = (\alpha(\sigma_0, \sigma_1) - 1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1))$, we have

$$\begin{aligned}\mathscr{P}(\sigma_1, \sigma_2) &\le H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)) \\ &\quad + (\alpha(\sigma_0, \sigma_1) - 1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)) \\ &= \alpha(\sigma_0, \sigma_1)H^+_{\mathscr{P}}(T(\sigma_0), T(\sigma_1)).\end{aligned} \tag{27}$$

By (F_1), (13) and (C_1), we have

$$\begin{aligned}F(\mathscr{P}(\sigma_1, \sigma_2)) &\le F(\alpha(\sigma_0, \sigma_1)H^+(T(\sigma_0), T(\sigma_1))) \\ &\le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_0, T(\sigma_0)), \mathscr{P}(\sigma_1, T(\sigma_1)), \\ \\ \frac{\mathscr{P}(\sigma_1, T(\sigma_0)) + \mathscr{P}(\sigma_0, T(\sigma_1))}{2}\end{array}\right)\right) \\ &\quad - \tau, \le F\left(\varphi\left(\mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \frac{\mathscr{P}(\sigma_1, \sigma_1) + \mathscr{P}(\sigma_0, \sigma_2)}{2}\right)\right) - \tau.\end{aligned} \tag{28}$$

By the triangular inequality, we have

$$\mathscr{P}(\sigma_0, \sigma_2) + \mathscr{P}(\sigma_1, \sigma_1) \le \mathscr{P}(\sigma_0, \sigma_1) + \mathscr{P}(\sigma_1, \sigma_2). \tag{29}$$

We claim that $\mathscr{P}(\sigma_1, \sigma_2) < \mathscr{P}(\sigma_0, \sigma_1)$. On the contrary, if $\mathscr{P}(\sigma_1, \sigma_2) \ge \mathscr{P}(\sigma_0, \sigma_1)$, then due to (29), we get $\mathscr{P}(\sigma_0, \sigma_2) \le 2\mathscr{P}(\sigma_1, \sigma_2)$. The inequality (28) implies

$$F(\mathscr{P}(\sigma_1, \sigma_2)) < F(\varphi(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2))). \tag{30}$$

By (C_2), we have $\varphi(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_0, \sigma_1), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2)) \le \mathscr{P}(\sigma_1, \sigma_2)$, and by axiom (F_1), the inequality (30) reduces to

$$F(\mathscr{P}(\sigma_1, \sigma_2)) < F(\mathscr{P}(\sigma_1, \sigma_2)). \tag{31}$$

This is an absurdity. This indicates that our claim is valid. Thus, $\mathscr{P}(\sigma_1, \sigma_2) < \mathscr{P}(\sigma_0, \sigma_1)$. Let $\mathscr{P}_n = \mathscr{P}(\sigma_n, \sigma_{n+1})$ for all positive integers n, and by inequality (28) we obtain

$$F(\mathscr{P}_1) \le F(\varphi(\mathscr{P}_0, \mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_0)) - \tau. \tag{32}$$

Applying (C_2) and (F_1) consecutively, we have

$$F(\mathscr{P}_1) \le F(\mathscr{P}_0) - \tau. \tag{33}$$

Since T is a strictly α-admissible mapping, $\alpha(\sigma_0, \sigma_1) > 1$ implies $\alpha(\sigma_1, \sigma_2) > 1$; thus, $\sigma_1, \sigma_2 \in \mathscr{A}^*$ (assume $\sigma_2 \notin T(\sigma_2)$). Since, $T(\sigma_1), T(\sigma_2)$ are nonempty, closed, and bounded sets. By Definition 4(b), there exists $\sigma_3 \in T(\sigma_2)$ such that

$$\mathscr{P}(\sigma_2, \sigma_3) \le H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) + \varepsilon. \tag{34}$$

Letting $\varepsilon = (\alpha(\sigma_1, \sigma_2) - 1)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2))$, we have

$$\begin{aligned}\mathscr{P}(\sigma_2, \sigma_3) &\le H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) \\ &\quad + (\alpha(\sigma_1, \sigma_2) - 1)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)) \\ &= \alpha(\sigma_1, \sigma_2)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2)).\end{aligned} \tag{35}$$

By (F_1), (13) and (C_1), we have

$$\begin{aligned}F(\mathscr{P}(\sigma_2, \sigma_3)) &\le F(\alpha(\sigma_1, \sigma_2)H^+_{\mathscr{P}}(T(\sigma_1), T(\sigma_2))) \\ &\le F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, T(\sigma_1)), \mathscr{P}(\sigma_2, T(\sigma_2)), \\ \\ \frac{\mathscr{P}(\sigma_2, T(\sigma_1)) + \mathscr{P}(\sigma_1, T(\sigma_2))}{2}\end{array}\right)\right) \\ &\quad - \tau, \le F\left(\varphi\left(\mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \frac{\mathscr{P}(\sigma_2, \sigma_2) + \mathscr{P}(\sigma_1, \sigma_3)}{2}\right)\right) - \tau.\end{aligned} \tag{36}$$

By the triangular inequality, we have

$$\mathscr{P}(\sigma_1, \sigma_3) + \mathscr{P}(\sigma_2, \sigma_2) \le \mathscr{P}(\sigma_1, \sigma_2) + \mathscr{P}(\sigma_2, \sigma_3). \tag{37}$$

We claim that $\mathscr{P}(\sigma_2, \sigma_3) < \mathscr{P}(\sigma_1, \sigma_2)$. On the contrary, if $\mathscr{P}(\sigma_2, \sigma_3) \ge \mathscr{P}(\sigma_1, \sigma_2)$, then by (37), we get $\mathscr{P}(\sigma_1, \sigma_3) \le 2\mathscr{P}(\sigma_2, \sigma_3)$. The inequality (36) implies

$$F(\mathscr{P}(\sigma_2, \sigma_3)) < F(\varphi(\mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_2, \sigma_3))) - \tau. \tag{38}$$

By (C_2), $\varphi(\mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_1, \sigma_2), \mathscr{P}(\sigma_2, \sigma_3), \mathscr{P}(\sigma_2, \sigma_3)) \le \mathscr{P}(\sigma_2, \sigma_3)$. By (F_1) and (38), we have

$$F(\mathscr{P}(\sigma_2, \sigma_3)) < F(\mathscr{P}(\sigma_2, \sigma_3)). \tag{39}$$

This is an absurdity. Thus, $\mathscr{P}(\sigma_2, \sigma_3) < \mathscr{P}(\sigma_1, \sigma_2)$. By (36), we obtain

$$F(\mathscr{P}_2) \le F(\varphi(\mathscr{P}_1, \mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_1)) - \tau. \tag{40}$$

Again applying the condition (C_2) followed by (F_1), we have

$$F(\mathscr{P}_2) \le F(\mathscr{P}_1) - \tau \le F(\mathscr{P}_0) - 2\tau. \tag{41}$$

Similarly, there exists $\sigma_4 \in T(\sigma_3)(\sigma_3 \notin T(\sigma_3))$, such that

$$F(\mathscr{P}_3) \le F(\mathscr{P}_2) - \tau \le F(\mathscr{P}_0) - 3\tau. \tag{42}$$

Thus, we are able to construct an iterative sequence $\{\sigma_n\} \subset X$ such that

$$\begin{aligned}&\sigma_n \in T(\sigma_{n-1}), \sigma_{n-1} \notin T(\sigma_{n-1}), \alpha(\sigma_{n-1}, \sigma_n) > 1, \\ &\quad \mathscr{P}_n < \mathscr{P}_{n-1} \text{ for all } n \in \mathbb{N} \text{ and}\end{aligned} \tag{43}$$

$$F(\mathscr{P}_n)) \le F(\mathscr{P}_0)) - n\tau. \tag{44}$$

By (44), we obtain $\lim_{n\to\infty} F(\mathscr{P}_n) = -\infty$, by (F_2) we have $\lim_{n\to\infty} \mathscr{P}_n = 0$, and by (F_3), there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} ((\mathscr{P}_n)^\kappa F(\mathscr{P}_n)) = 0. \tag{45}$$

Following (44), for all $n \in \mathbb{N}$, we obtain

$$(\mathscr{P}_n)^\kappa (F(\mathscr{P}_n) - F(\mathscr{P}_0)) \le -(\mathscr{P}_n)^\kappa n\tau \le 0. \tag{46}$$

Letting $n \to \infty$, in (46), we have $\lim_{n\to\infty}(n(\mathscr{P}_n)^\kappa) = 0$; thus, there exists $n_1 \in \mathbb{N}$, such that $n(\mathscr{P}_n)^\kappa \le 1$ for all $n \ge n_1$, that is $\mathscr{P}_n \le (1/n^{1/\kappa})$forall$n \ge n_1$.

For $m > n \ge n_1$,

$$\begin{aligned}\mathscr{P}(\sigma_n, \sigma_m) &\le \mathscr{P}(\sigma_n, \sigma_{n+1}) + \mathscr{P}(\sigma_{n+1}, \sigma_{n+2}) + \mathscr{P}(\sigma_{n+2}, \sigma_{n+3}) \\ &\quad +\cdots+\mathscr{P}(\sigma_{m-1}, \sigma_m) \\ &\le \sum_{i=n}^{m-1} \mathscr{P}(\sigma_i, \sigma_{i+1}) \le \sum_{i=n}^{\infty} \mathscr{P}(\sigma_i, \sigma_{i+1}) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.\end{aligned} \tag{47}$$

Since the series $\sum_{i=n}^{\infty}(1/i^{1/\kappa})$ on the right hand side is convergent and by definition of metric $d_{\mathscr{P}}$ defined on $\mathfrak{J}$, we know that $d_{\mathscr{P}}(\sigma_n, \sigma_m) \le 2\mathscr{P}(\sigma_n, \sigma_m)$; thus, $\lim_{n,m\to\infty} d_{\mathscr{P}}(\sigma_n, \sigma_m) = 0$. This implies $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{J}, d_{\mathscr{P}})$. Since $(\mathfrak{J}, \mathscr{P})$ is complete, so by Lemma 1(2), the metric space $(\mathfrak{J}, d_{\mathscr{P}})$ is complete. Thus, there exists $x^* \in \mathfrak{J}$ such that $\sigma_n \to x^*$ as $n \to \infty$ with respect to metric $d_{\mathscr{P}}$. Then Lemma 1(3) implies

$$\lim_{n\to\infty} \mathscr{P}(x^*, \sigma_n) = \mathscr{P}(x^*, x^*) = \lim_{n,m\to\infty} \mathscr{P}(\sigma_n, \sigma_m). \tag{48}$$

This shows that $\{\sigma_n\}$ is a Cauchy sequence in $(\mathfrak{J}, \mathscr{P})$. Now, we show that $x^* \in T(x^*)$, and to do so, we claim that $\mathscr{P}(x^*, T(x^*)) = 0$. If on the other hand $\mathscr{P}(x^*, T(x^*)) > 0$, then there exists $n_1 \in \mathbb{N}$ such that $\mathscr{P}(\sigma_n, T(x^*)) > 0$ for each $n \ge n_1$. By assumption (3), $\alpha(\sigma_n, x^*) > 1$. By (13),

$$\begin{aligned}F(\mathscr{P}(\sigma_{n+1}, T(x^*))) &\le F(\alpha(\sigma_n, x^*) H^+(T(\sigma_n), T(x^*))) \\ &\le F\Bigg(\varphi\Bigg(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, T(\sigma_n)), \mathscr{P}(x^*, T(x^*)), \\ &\quad \frac{\mathscr{P}(x^*, T(\sigma_n)) + \mathscr{P}(\sigma_n, T(x^*))}{2}\Bigg)\Bigg) \\ &\quad - \tau \le F\Bigg(\varphi\Bigg(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, \sigma_{n+1}), \\ &\quad \mathscr{P}(x^*, T(x^*)), \frac{\mathscr{P}(x^*, \sigma_{n+1}) + \mathscr{P}(\sigma_n, T(x^*))}{2}\Bigg)\Bigg) - \tau.\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}F(\mathscr{P}(\sigma_{n+1}, T(x^*))) &< F\Bigg(\varphi\Bigg(\mathscr{P}(\sigma_n, x^*), \mathscr{P}(\sigma_n, \sigma_{n+1}), \mathscr{P}(x^*, T(x^*)), \\ &\quad \frac{\mathscr{P}(x^*, \sigma_{n+1}) + \mathscr{P}(\sigma_n, T(x^*))}{2}\Bigg)\Bigg).\end{aligned} \tag{50}$$

Since φ is a coordinate-wise continuous function, letting $n \to \infty$ in the above inequality, we obtain

$$F(\mathscr{P}(x^*, T(x^*))) < F\left(\varphi\left(0, 0, \mathscr{P}(x^*, T(x^*)), \frac{\mathscr{P}(x^*, T(x^*))}{2}\right)\right). \tag{51}$$

By (C_3), we have

$$F(\mathscr{P}(x^*, T(x^*))) < F(\mathscr{P}(x^*, T(x^*))). \tag{52}$$

This is an absurdity and consequently $\mathscr{P}(x^*, T(x^*)) = 0$; thus, we have $\mathscr{P}(x^*, T(x^*)) = \mathscr{P}(x^*, x^*)$ which implies that $x^* \in \overline{T(x^*)} = T(x^*)$. Hence, x^* is a fixed point of T.

The following example explains Theorem 1.

Example 3. Consistent with ([28], Example 3.3), let $\varphi_a \in \mathscr{C}_\varphi$ where

$$\begin{aligned}\varphi_a = \varphi_a(u, r, s, t)&: \Bigg(u = \mathscr{P}(\sigma, \varsigma), r = \mathscr{P}(\sigma, T(\sigma)), s = \mathscr{P}(\varsigma, T(\varsigma)), \\ t &= \frac{\mathscr{P}(\varsigma, T(\sigma)) + \mathscr{P}(\sigma, T(\varsigma))}{2}\Bigg),\end{aligned} \tag{53}$$

$\tau = 1$ and $F \in \mathscr{F}$ defined by $F(\sigma) = \ln(\sigma) + \sigma$. Let $\mathfrak{J} = \{0, 1, 2, \cdots\}$ equipped with p-m $\mathscr{P} : \mathfrak{J} \times \mathfrak{J} \to [0,\infty)$ defined by

$$\mathscr{P}(\sigma, \varsigma) = \sigma + \varsigma \text{forall} \sigma \ne \varsigma. \tag{54}$$

Then, $(\mathfrak{J}, \mathscr{P})$ is a complete p-m-s. Define the mapping $T : \mathfrak{J} \to 2^{\mathfrak{J}}$ by

$$\begin{aligned}R(\sigma) &= \begin{cases}\{0\} & \text{if } \sigma \in \{0, 1\}; \\ \{0, \sigma - 1\} & \text{if } \sigma \ge 2,\end{cases} \\ \alpha(\sigma, \varsigma) &= \begin{cases}0 & \text{if } \sigma, \varsigma \in (-\infty, 0); \\ e^{\mathscr{P}(\sigma,\varsigma)} & \text{if } \sigma, \varsigma \in \{0, 1, 2, \cdots\}.\end{cases}\end{aligned} \tag{55}$$

The mapping T is strict α-admissible, closed, and bounded. We show that T is $F_{H^+}^{\varphi}$-contraction. We observe that $H^+(T(\sigma), T(\varsigma)) > 0$ if and only if $\sigma \ge 2$ and $\varsigma > 0$. Also for all $\sigma, \varsigma \in \mathfrak{J}$ with $\varsigma \in T(\sigma)$ and taking $\zeta = 0 \in T(\varsigma)$, we have

$$\begin{aligned}\alpha(\sigma, \varsigma) H^+(T(\sigma), T(\varsigma)) &= e^{\mathscr{P}(\sigma,\varsigma)} \mathscr{P}(\varsigma, \zeta) \\ &= e^{\mathscr{P}(\sigma,\varsigma)} \varsigma < e^{\mathscr{P}(\sigma,\varsigma)} (\sigma + \varsigma) \\ &= e^{\mathscr{P}(\sigma,\varsigma)} \mathscr{P}(\sigma, \varsigma), \text{ and thus,}\end{aligned}$$

$$\begin{aligned}&e^{\mathscr{P}(\sigma,\varsigma)} H^+(T(\sigma), T(\varsigma)) - \varphi_a(u, r, s, t) \\ &\quad \le e^{\mathscr{P}(\sigma,\varsigma)} H^+(T(\sigma), T(\varsigma)) - \mathscr{P}(\sigma, \varsigma) \le -2.\end{aligned} \tag{56}$$

Consequently,

$$\frac{e^{\mathscr{P}(\sigma,\varsigma)} H^+(T(\sigma), T(\varsigma))}{\varphi_a(u, r, s, t)} e^{e^{\mathscr{P}(\sigma,\varsigma)} H^+(T(\sigma), T(\varsigma)) - \varphi_a(u,r,s,t)} \le e^{-1}. \tag{57}$$

Hence,

$$1 + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma),T(\varsigma))) \leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)),\mathscr{P}(\varsigma,T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\end{array}\right)\right). \tag{58}$$

Similarly, for every member of $\mathscr{C}_{\varphi}$, the mapping T satisfies all assumptions in Theorem 1. As it is clear from Proposition 3 that $F^{\varphi}_{H^{+}}$-contraction needs not to be $\varphi H^{+}_{\mathscr{P}}$-contraction, and hence, it is not $H^{+}_{\mathscr{P}}$-contraction. Consequently, $F^{\varphi}_{H^{+}}$-contraction needs not to be H^{+}-contraction. Thus, the results in [9, 10, 27] are not applicable in this case.

Remark 3. In the following section, we obtain the corollaries of Theorem 1. To simplify the expression of the corollaries, we consider the three conditions below.

Let

(A1) there exist σ_0 in $\mathfrak{I}$ such that $\alpha(\sigma_0, T(\sigma_0)) > 1$

(A2) $\mathfrak{I}$ be a strictly α-regular space

(A3) F be continuous.

Corollary 1. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admisible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F(\mathscr{P}(\sigma,\varsigma)), \tag{59}$$

for all $\sigma,\varsigma \in \mathscr{A}^{*}$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \leq H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{60}$$

Then, the mapping T has a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right) = \mathscr{P}(\sigma,\varsigma) \tag{61}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 2. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admisible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F(\max\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}), \tag{62}$$

for all $\sigma,\varsigma \in \mathscr{A}^{*}$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, $\exists \xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \leq H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{63}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right) = \max\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\} \tag{64}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 3. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admissible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F(\max\{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\}), \tag{65}$$

for all $\sigma,\varsigma \in \mathscr{A}^{*}$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \leq H^{+}_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{66}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\varphi\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right) = \max\{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})\} \tag{67}$$

and following the steps given in the proof of Theorem 1, we obtain the result.

Corollary 4. *Let $(\mathfrak{I},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ be a strictly α-admissible mapping. Assume that*

$$\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\},T(\varsigma)\setminus\{\varsigma\})) \leq F\left(\max\left\{\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}),\mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\})+\mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{array}\right\}\right), \tag{68}$$

for all $\sigma,\varsigma \in \mathscr{A}^{*}$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{I}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^+_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{69}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}
&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \\
&\quad \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg) \\
&= \max\Bigg\{\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \\
&\quad \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg\},
\end{aligned} \tag{70}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 5. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{J} \to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admissible mapping. Assume that*

$$\begin{aligned}
&\tau + F(\alpha(\sigma,\varsigma)H^+(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \\
&\quad \le F\left(\frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right),
\end{aligned} \tag{71}$$

for all $\sigma,\varsigma \in \mathscr{A}^*$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{J}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^+_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{72}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}
&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \\
&\quad \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg) \\
&= \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}
\end{aligned} \tag{73}$$

in the proof of Theorem 1, we get the result.

Corollary 6. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{J} \to BC_{\mathscr{P}}(\mathfrak{J})$ be a strictly α-admissible mapping. Assume that*

$$\begin{aligned}
&\tau + F(\alpha(\sigma,\varsigma)H^+(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \\
&\quad \le F\left(\frac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2}\right),
\end{aligned} \tag{74}$$

for all $\sigma,\varsigma \in \mathscr{A}^*$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{J}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^+_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{75}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}
&\varphi\Bigg(\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\}), \\
&\quad \frac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg) \\
&= \frac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2}
\end{aligned} \tag{76}$$

in the proof of Theorem 1, we get the result.

Corollary 7. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{J} \to BC_{\mathscr{P}}(\mathfrak{J})$ be strictly α-admissible mapping. Assume that*

$$\begin{aligned}
&\tau + F(\alpha(\sigma,\varsigma)H^+(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \\
&\le F\left(\max\left\{\begin{matrix}\mathscr{P}(\sigma,\varsigma), \dfrac{\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})}{2}, \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\})}{2}\end{matrix}\right\}\right),
\end{aligned} \tag{77}$$

for all $\sigma,\varsigma \in \mathscr{A}^*$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{J}$, and $\varsigma \in T(\sigma)$, there exists $\xi \in T(\varsigma)$ such that

$$\mathscr{P}(\varsigma,\xi) \le H^+_{\mathscr{P}}(T(\sigma),T(\varsigma)) + \varepsilon. \tag{78}$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}
&\varphi\left(\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma,T(\sigma)), \mathscr{P}(\varsigma,T(\varsigma)), \frac{\mathscr{P}(\varsigma,T(\sigma)) + \mathscr{P}(\sigma,T(\varsigma))}{2}\right) \\
&= \max\left\{\mathscr{P}(\sigma,\varsigma), \frac{\mathscr{P}(\sigma,T(\sigma)) + \mathscr{P}(\varsigma,T(\varsigma))}{2}, \frac{\mathscr{P}(\varsigma,T(\sigma)) + \mathscr{P}(\sigma,T(\varsigma))}{2}\right\}
\end{aligned} \tag{79}$$

and following the proof of Theorem 1, we obtain the result.

Corollary 8. *Let $(\mathfrak{J},\mathscr{P})$ be a complete p-m-s and $T : \mathfrak{J} \to BC_{\mathscr{P}}(\mathfrak{J})$ be strictly α-admisible mapping. Assume that there exist $a \ge 0$, $b \ge 0$, $c \ge 0$ satisfying $a + 2b + 2c = 1$, such that*

$$\begin{aligned}
&\tau + F(\alpha(\sigma,\varsigma)H^+(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\})) \\
&\le F\left(\begin{matrix}a\mathscr{P}(\sigma,\varsigma) + b(\mathscr{P}(\sigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma,T(\varsigma)\setminus\{\varsigma\})) \\ +c(\mathscr{P}(\varsigma,T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma,T(\varsigma)\setminus\{\varsigma\}))\end{matrix}\right),
\end{aligned} \tag{80}$$

for all $\sigma,\varsigma \in \mathscr{A}^*$, and for every $\varepsilon > 0$, $\sigma \in \mathfrak{J}$, and $\varsigma \in T(\sigma)$, *there exists* $\xi \in T(\varsigma)$ *such that*

$$\mathscr{P}(\varsigma, \xi) \leq H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon. \quad (81)$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. Defining $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}\varphi\Bigg(&\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ &\frac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\ &= a\mathscr{P}(\sigma,\varsigma) + b(\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\})\\ &\quad + 2c\frac{\mathscr{P}(\varsigma, T(\sigma)\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\{\varsigma\})}{2},\end{aligned} \quad (82)$$

in the proof of Theorem 1, we obtain the result.

Corollary 9. *Let* $(\mathfrak{I}, \mathscr{P})$ *be a complete p-m-s and* $T : \mathfrak{I} \to B C_{\mathscr{P}}(\mathfrak{I})$ *be a strictly α-admissible mapping. Assume that there exist* $a \geq 0, b \geq 0, c \geq 0$ *satisfying* $a + b + c = 1$*, such that*

$$\begin{aligned}&\tau + F(\alpha(\sigma,\varsigma)H^{+}(T(\sigma)\setminus\{\sigma\}, T(\varsigma)\setminus\{\varsigma\}))\\ &\quad \leq F(a\mathscr{P}(\sigma,\varsigma) + b\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + c\mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\})),\end{aligned} \quad (83)$$

for all $\sigma, \varsigma \in \mathscr{A}^*$*, and for every* $\varepsilon > 0$*,* $\sigma \in \mathfrak{I}$*, and* $\varsigma \in T(\sigma)$*, there exists* $\xi \in T(\varsigma)$ *such that*

$$\mathscr{P}(\varsigma, \xi) \leq H^{+}_{\mathscr{P}}(T(\sigma), T(\varsigma)) + \varepsilon. \quad (84)$$

Then, T admits a fixed point provided (A1)-(A3) hold.

Proof. If we define $\varphi : [0,\infty)^4 \to [0,\infty)$ by

$$\begin{aligned}\varphi\Bigg(&\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}), \mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\\ &\frac{\mathscr{P}(\varsigma, T(\sigma)\setminus\{\sigma\}) + \mathscr{P}(\sigma, T(\varsigma)\setminus\{\varsigma\})}{2}\Bigg)\\ &= a\mathscr{P}(\sigma,\varsigma) + b\mathscr{P}(\sigma, T(\sigma)\setminus\{\sigma\}) + c\mathscr{P}(\varsigma, T(\varsigma)\setminus\{\varsigma\}),\end{aligned} \quad (85)$$

in the proof of Theorem 1, then the result follows.
Let

$$\mathfrak{I}^* = \left\{(\sigma,\varsigma) \in \mathfrak{I}^2 \mid \alpha(\sigma,\varsigma) > 1 \text{and} \mathscr{P}(T(\sigma), T(\varsigma)) > 0\right\}. \quad (86)$$

For a single-valued self-mapping, Theorem 1 can be stated as follows:

Theorem 2. *Let* $(\mathfrak{I}, \mathscr{P})$ *be a complete p-m-s and* $T : \mathfrak{I} \to \mathfrak{I}$ *be a* φF*-contraction, that is, there exist* $\varphi \in \mathscr{C}_{\varphi}$ *and* $F \in \mathscr{F}$ *such that*

$$\begin{aligned}&\tau + F(\alpha(\sigma,\varsigma)\mathscr{P}(T(\sigma), T(\varsigma)))\\ &\leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma), \mathscr{P}(\sigma, T(\sigma)), \mathscr{P}(\varsigma, T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma, T(\sigma)) + \mathscr{P}(\sigma, T(\varsigma))}{2}\end{array}\right)\right),\end{aligned} \quad (87)$$

for all $\sigma, \varsigma \in \mathfrak{I}^*$ *and*

(1) T *is a strictly α-admissible mapping*

(2) there exists σ_0 *in* $\mathfrak{I}$ *such that* $\alpha(\sigma_0, T(\sigma_0)) > 1$

(3) $\mathfrak{I}$ *is a strictly α-regular space*

(4) F *is continuous.*

Then, T admits a fixed point.

We omit its proof as it is a mere repetition of the proof of Theorem 1 with some minor modifications.

5. Applications of Theorem 2

5.1. Applications to Fractional Differential Equations. Lacroix (1819) introduced and investigated several applicable properties of fractional differentials. Recently, various new models involving Caputo-Fabrizio derivative (CFD) were discovered and analyzed in [29–31]. We investigate one of these models in p-m-s. We introduce some notations as follows:

Let $\mathscr{C}_{0,1} = \{f \mid f : [0,1] \to \mathbb{R} \text{ and } f \text{ is continuous}\}$. Define the metric function $d : \mathscr{C}_{0,1} \times \mathscr{C}_{0,1} \to [0,\infty)$ by

$$d(f,g) = \|f - g\|_{\infty} = \max_{\nu\in[0,1]} |f(\nu) - g(\nu)|, \text{for all } f, g \in \mathscr{C}_{0,1}. \quad (88)$$

Then, the space $(\mathscr{C}_{0,1}, d)$ is a complete metric space. The function $\alpha : \mathscr{C}_{0,1} \times \mathscr{C}_{0,1} \to (1,\infty)$ by

$$\alpha(r,t) = e^{\|r+t\|_{\infty}} \text{ for all } r, t \in \mathscr{C}_{0,1}. \quad (89)$$

Let $K_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We shall investigate the following CFDE:

$${}^{C}D^{\beta} f(\nu) = K_1(\nu, f(\nu)), \quad (90)$$

with boundary conditions

$$\sigma(0) = 0, I\sigma(1) = \sigma'(0). \quad (91)$$

Here, ${}^{C}D^{\beta}$ denotes CFD of order β defined by

$${}^{C}D^{\beta} K_1(\nu) = \frac{1}{\Gamma(n-\beta)} \int_0^{\nu} (\nu - \eta)^{n-\beta-1} K_1^n(\eta)\Big)\, d\eta, \quad (92)$$

where

$$n-1<\beta<n\text{and}n=[\beta]+1, \tag{93}$$

and $I^{\beta}K_1$ is given by

$$I^{\beta}K_1(\nu)=\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta)\,d\eta,\text{with}\beta>0. \tag{94}$$

Then, the equation (90) can be modified to

$$\begin{aligned} f(\nu)=&\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,f(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,f(u))\,du d\eta, \end{aligned} \tag{95}$$

Theorem 3. *Equation (90) admits a solution in $\mathscr{C}_{0,1}$ provided*

(I) there exists $\tau>0$ such that for all $\sigma,\varsigma\in\mathscr{C}_{0,1}$, we have

$$|K_1(\eta,\sigma(\eta))-K_1(\eta,\varsigma(\eta))|\le\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}|\sigma(\eta)-\varsigma(\eta)| \tag{96}$$

(II) there exists $\sigma_0\in\mathscr{C}_{0,1}$ such that for all $\nu\in[0,1]$, we have

$$\begin{aligned} \sigma_0(\nu)\le&\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma_0(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma_0(u))\,du d\eta. \end{aligned} \tag{97}$$

Proof. Consistent with the notations introduced above and defining the mapping $R:\mathscr{C}_{0,1}\to\mathscr{C}_{0,1}$ by

$$\begin{aligned} R(\sigma(\nu))=&\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\,d\eta\\ &+\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma(u))\,du d\eta. \end{aligned} \tag{98}$$

By (II), there exists $\sigma_0\in\mathscr{C}_{0,1}$ such that $\sigma_n=t^n(\sigma_0))$. The continuity of function K_1 leads to the continuity of mapping t on $\mathscr{C}_{0,1}$. It is easy to verify the assumptions (1)-(4) in Theorem 2. In the following, we verify the contractive condition (87) of Theorem 2.

$$|R(\sigma(\nu))-R(\varsigma(\nu))|=\left|\begin{array}{l}\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\,d\eta\\ -\frac{1}{\Gamma(\beta)}\int_0^{\nu}(\nu-\eta)^{\beta-1}K_1(\eta,\varsigma(\eta))\,d\eta\\ +\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\sigma(u))\,du d\eta\\ -\frac{2\nu}{\Gamma(\beta)}\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}K_1(u,\varsigma(u))\,du d\eta\end{array}\right|\text{implies}$$

$$\begin{aligned} |R(\sigma(\nu))-R(\varsigma(\nu))|\le&\left|\int_0^{\nu}\left(\frac{1}{\Gamma(\beta)}(\nu-\eta)^{\beta-1}K_1(\eta,\sigma(\eta))\right.\right.\\ &\left.\left.-\frac{1}{\Gamma(\beta)}(\nu-\eta)^{\beta-1}K_1(\eta,\varsigma(\eta))\right)d\eta\right|\\ &+\left|\int_0^1\int_0^{\eta}\left(\frac{2}{\Gamma(\beta)}(\eta-u)^{\beta-1}K_1(\eta,\sigma(\eta))\right.\right.\\ &\left.\left.-\frac{2}{\Gamma(\beta)}(\eta-u)^{\beta-1}K_1(u,\varsigma(u))\right)du d\eta\right|\\ \le&\frac{1}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot\int_0^{\nu}(\nu-\eta)^{\beta-1}(\sigma(\eta)-\varsigma(\eta))\,d\eta\\ &+\frac{2}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}(\varsigma(u)-\sigma(u))\,du d\eta\\ \le&\frac{1}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)}\cdot d(\sigma,\varsigma)\cdot\int_0^{\nu}(\nu-\eta)^{\beta-1}\,d\eta\\ &+\frac{2}{\Gamma(\beta)}\frac{e^{-\tau}\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\alpha)\cdot\Gamma(\beta+1)}\cdot d(\sigma,\varsigma)\\ &\cdot\int_0^1\int_0^{\eta}(\eta-u)^{\beta-1}\,du d\eta\le\left(\frac{e^{-\tau}\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\beta)\cdot\Gamma(\beta+1)}\right)\\ &\cdot d(\sigma,\varsigma)+2e^{-\tau}B(\beta+1,1)\frac{\Gamma(\beta)\cdot\Gamma(\beta+1)}{4\alpha(\sigma,\varsigma)\Gamma(\beta)\cdot\Gamma(\beta+1)}\\ &\cdot d(\sigma,\varsigma)\le\frac{e^{-\tau}}{4\alpha(\sigma,\varsigma)}d(\sigma,\varsigma)+\frac{e^{-\tau}}{2\alpha(\sigma,\varsigma)}d(\sigma,\varsigma), \end{aligned} \tag{99}$$

where B is the beta function. The last inequality can be written by that

$$\alpha(\sigma,\varsigma)d(R(\sigma),R(\varsigma))\le e^{-\tau}d(\sigma,\varsigma). \tag{100}$$

Let us define the metric d on $\mathscr{C}_{0,1}$ by

$$d(\sigma,\varsigma)=\begin{cases}\mathscr{P}(\sigma,\varsigma)=\|\sigma-\varsigma\|_{\infty}+l(l\ge 0) & \text{if }\sigma\ne\varsigma\\ 0 & \text{if }\sigma=\varsigma.\end{cases} \tag{101}$$

Thus, (100) can be written as

$$\alpha(\sigma,\varsigma)\mathscr{P}(R(\sigma),R(\varsigma))\le e^{-\tau}\mathscr{P}(\sigma,\varsigma). \tag{102}$$

Define the functions $\varphi\in\mathscr{C}_{\varphi}$ and F by

$$\varphi_a\left(\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,t(\sigma)),\mathscr{P}(\varsigma,t(\varsigma)),\frac{\mathscr{P}(\varsigma,t(\sigma))+\mathscr{P}(\sigma,t(\varsigma))}{2}\right)$$
$$=\max\left\{\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,t(\sigma)),\mathscr{P}(\varsigma,t(\varsigma)),\frac{\mathscr{P}(\varsigma,t(\sigma))+\mathscr{P}(\sigma,t(\varsigma))}{2}\right\}, \tag{103}$$

$F(\sigma(\nu))=\ln(\sigma(\nu))$ for all $\sigma,\varsigma\in\mathscr{C}_{0,1}$. Under these definitions, the inequality (102) gets the form

$$\tau+F(\alpha(\sigma,\varsigma)\mathscr{P}(R(\sigma),R(\varsigma)))\leq F\left(\varphi\left(\begin{array}{c}\mathscr{P}(\sigma,\varsigma),\mathscr{P}(\sigma,T(\sigma)),\mathscr{P}(\varsigma,T(\varsigma)),\\ \\ \dfrac{\mathscr{P}(\varsigma,T(\sigma))+\mathscr{P}(\sigma,T(\varsigma))}{2}\end{array}\right)\right). \tag{104}$$

Hence, by Theorem 2, the self-mapping t admits a fixed point, and hence, the equation (90) has a solution.

5.2. Applications to the Matrix Equations. In this section, by Theorem 2, we shall investigate study the existence of the solutions to

$$X=\mathbb{D}+\frac{1}{m+\theta}\left(\sum_{i=1}^{m}\mathbb{W}_i^*X\mathbb{W}_i+\sum_{i=1}^{m}\mathbb{G}_i^*X\mathbb{G}_i\right), \tag{105}$$

where $\theta\in(0,1)$, $\mathbb{D}\in\mathscr{P}^{(m)}$ (set of $m\times m$ positive definite matrices), and $\mathbb{W}_i,\mathbb{G}_i$ are arbitrary $m\times m$ matrices for each i and are entries of block matrices given by

$$\mathbb{W}=\begin{bmatrix}\mathbb{W}_1\\ \mathbb{W}_2\\ \mathbb{W}_3\\ \vdots\\ \mathbb{W}_m\end{bmatrix}\mathbb{G}=\begin{bmatrix}\mathbb{G}_1\\ \mathbb{G}_2\\ \mathbb{G}_3\\ \vdots\\ \mathbb{G}_m\end{bmatrix}. \tag{106}$$

Let $\mathbb{W}_Z\in\mathscr{Z}^{(m)}$ (set of $m\times m$ Hermitian matrices) be an arbitrary matrix; then, its eigenvalues $e_1,e_2,e_3,\cdots e_m$ are real. Moreover, if $\mathbb{W}_Z\in\mathscr{Z}_+^{(m)}$, then the eigenvalues are nonnegative. Let the functional $\|.\|_{tr}:\mathscr{Z}^{(m)}\to\mathbb{R}$ be defined by

$$\|\mathbb{W}_Z\|_{tr}=\sum_{i=1}^{m}|e_i|. \tag{107}$$

Let $X\in\mathscr{P}^{(m)}$ be arbitrary and define $\|\mathbb{W}_Z\|_{tr,X}=\|X^{1/2}\mathbb{W}_ZX^{1/2}\|_{tr}$. By ([32], Theorem IX.2.2), $(\mathscr{Z}^{(m)},\|.\|_{tr,X})$ is a Banach space (see also [33–35]). Hence, $(\mathscr{Z}^{(m)},d)$ is a complete metric space. The induced metric $d:\mathscr{Z}^{(m)}\times\mathscr{Z}^{(m)}\to\mathbb{R}$ is defined by

$$d(\mathbb{W}_Z,\mathbb{G}_Z)=\|\mathbb{W}_Z-\mathbb{G}_Z\|_{tr,X}\text{forall}\mathbb{W}_Z,\mathbb{G}_Z\in\mathscr{Z}^{(m)}. \tag{108}$$

To establish the existence result we need the following lemma.

Lemma 2 [35]. *If $\mathbb{W}_Z,\mathbb{G}_Z\in\mathscr{Z}_+^{(m)}$, then*

$$0\leq Tr(\mathbb{W}_Z\mathbb{G}_Z)\leq\|\mathbb{W}_Z\|Tr(\mathbb{G}_Z). \tag{109}$$

Define the operator $\mathscr{E}:\mathscr{Z}^{(m)}\to\mathscr{Z}^{(m)}$ by

$$\mathscr{E}(U)=\mathbb{G}+\frac{1}{m+\theta}\left(\sum_{i=1}^{m}\mathbb{W}_i^*U\mathbb{W}_i+\sum_{i=1}^{m}\mathbb{G}_i^*U\mathbb{G}_i\right),\text{forall}U\in\mathscr{Z}^{(m)}. \tag{110}$$

Remark 4. Since $\mathscr{E}(U)-\mathbb{G}\in\mathscr{P}^{(m)}$ for all $U\in\mathscr{Z}^{(m)}$, in particular, we have $\mathscr{E}(\mathbb{G})-\mathbb{G}\in\mathscr{P}^{(m)}$. The operator $\mathscr{E}$ is continuous on $\mathscr{Z}^{(m)}$.

The solution of the matrix equation (105) is the fixed point of the operator $\mathscr{E}$.

Theorem 4. *Let X and Y be two positive definite matrices such that $\sum_{i=1}^{m}\mathbb{W}_i^*X\mathbb{W}_i\prec 1/2X$ and $\sum_{i=1}^{m}\mathbb{G}_i^*X\mathbb{G}_i\prec 1/2X$. Then, the operator $\mathscr{E}$ has a fixed point in $\mathscr{Z}^{(m)}$.*

Proof. Let U and V be any matrices in $\mathscr{P}^{(m)}$. We observe that the operator $\mathscr{E}$ and the space $(\mathscr{Z}^{(m)},\|.\|_{tr,X})$ fulfill the assumptions (1)-(4) in Theorem 2. To prove that $\mathscr{E}$ is an φ F-contraction, we proceed with

$$\begin{aligned}
&\|\mathscr{E}(V)-\mathscr{E}(U)\|_{tr,X}=tr\left(X^{1/2}(\mathscr{E}(V)-\mathscr{E}(U))X^{1/2}\right)\\
&=tr\left(\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}(\mathbb{W}_i^*(V-U)\mathbb{W}_i+\mathbb{G}_i^*(V-U)\mathbb{G}_i)X^{1/2}\right\}\right)\\
&=tr\left(\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}\left(\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right\}+\frac{1}{m+\theta}\sum_{i=1}^{m}\left\{X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_i\right)X^{1/2}\right\}\right)\\
&=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}+X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)=\frac{1}{m+\theta}\sum_{i=1}^{m}\\
&\quad\cdot\left\{tr\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right)+tr\left(X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)\right\}=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\quad\cdot\left(X^{1/2}\mathbb{W}_i^*(V-U)\mathbb{W}_iX^{1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\left(X^{1/2}\mathbb{G}_i^*(V-U)\mathbb{G}_iX^{1/2}\right)\\
&=\frac{1}{m+\theta}\sum_{i=1}^{m}tr(\mathbb{W}_iX\mathbb{W}_i^*(V-U))+\frac{1}{m+\theta}\sum_{i=1}^{m}tr(\mathbb{G}_iX\mathbb{G}_i^*(V-U))=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\quad\cdot\left(\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}X^{-1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\quad\cdot\left(\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}X^{-1/2}\right)=\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\quad\cdot\left(X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)+\frac{1}{m+\theta}\sum_{i=1}^{m}tr\\
&\quad\cdot\left(X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)=\frac{1}{m+\theta}tr\\
&\quad\cdot\left(\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)+\frac{1}{m+\theta}tr\\
&\quad\cdot\left(\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}X^{1/2}(V-U)X^{1/2}\right)\leq\frac{1}{m+\theta}\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}\right\|\\
&\quad\cdot\|V-U\|_{tr,X}+\frac{1}{m+\theta}\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}\right\|\|V-U\|_{tr,X}\text{ by Lemma2}\\
&\quad\cdot=\frac{1}{m+\theta}\left(\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{W}_iX\mathbb{W}_i^*X^{-1/2}\right\|+\left\|\sum_{i=1}^{m}X^{-1/2}\mathbb{G}_iX\mathbb{G}_i^*X^{-1/2}\right\|\right)\|V-U\|_{tr,X}.
\end{aligned} \tag{111}$$

Given $\sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec 1/2X$, $\sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec 1/2X$, and letting K be a number such that

$$K = \left\| \sum_{i=1}^{m} X^{-1/2} \mathbb{W}_i X \mathbb{W}_i^* X^{-1/2} \right\| + \left\| \sum_{i=1}^{m} X^{-1/2} \mathbb{G}_i X \mathbb{G}_i^* X^{-1/2} \right\| < 1, \text{ we have}$$

$$\| \mathscr{E}(V) - \mathscr{E}(U) \|_{tr,X} \leq \frac{K}{m+\theta} \| V - U \|_{tr,X}. \tag{112}$$

Thus,

$$\frac{m+\theta}{K} d(\mathscr{E}(V), \mathscr{E}(U)) \leq K d(V, U). \tag{113}$$

We define $\alpha : \mathscr{Z}^{(m)} \times \mathscr{Z}^{(m)} \to (1, \infty)$ by

$$\alpha(U, V) = m + \theta \text{ for all } U, V \in \mathscr{Z}^{(m)} \text{ and } \theta \in (0,1), \tag{114}$$

and the metric d on $\mathscr{Z}^{(m)}$ by

$$d(\mathbb{W}_i, \mathbb{G}_i) = \begin{cases} \mathscr{P}(\mathbb{W}_i, \mathbb{G}_i) & \text{if } \mathbb{W}_i \neq \mathbb{G}_i;\ \sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec \frac{1}{2} X \text{ and } \sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec \frac{1}{2} X \\ 0 & \text{if } \mathbb{W}_i = \mathbb{G}_i, \end{cases} \tag{115}$$

In view of the metric defined above, the inequality (113) can be written as

$$\begin{gathered} \frac{(m+\theta) \mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))}{K} \leq \mathscr{P}(V, U) \\ \ln \left(\frac{\alpha(U, V) \mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))}{K} \right) \leq \ln (\mathscr{P}(V, U)). \end{gathered} \tag{116}$$

Define the functions $\varphi \in \mathscr{C}_\varphi$ and F by

$$\begin{aligned} &\varphi_a \left(\mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, \mathscr{E}(V))}{2} \right) \\ &= \max \left\{ \mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, T(V))}{2} \right\}, \end{aligned} \tag{117}$$

$F(\sigma) = \ln(\sigma)$ for all $\sigma \in (0, \infty)$, respectively. Under these definitions, we have

$$\begin{aligned} &\tau + F(\alpha(U, V) \mathscr{P}(\mathscr{E}(V), \mathscr{E}(U))) \leq F(\mathscr{P}(V, U)) \text{put} \tau = \ln \left(K^{-1} \right) \\ &\leq F \left(\max \left\{ \mathscr{P}(U, V), \mathscr{P}(U, \mathscr{E}(U)), \mathscr{P}(V, \mathscr{E}(V)), \frac{\mathscr{P}(V, \mathscr{E}(U)) + \mathscr{P}(U, \mathscr{E}(V))}{2} \right\} \right). \end{aligned} \tag{118}$$

By Theorem 2, the operator $\mathscr{E}$ has a fixed point, and hence, the matrix equation (105) has a solution.

Remark 5. The numerical explanation of the conditions $\sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i \prec 1/2X$ and $\sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \prec 1/2X$ imposed in Theorem 4 for $i = 2$ and taking 4×4 matrices is as follows:

$$\text{let} \mathbb{W}_1 = \begin{bmatrix} 0.1 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.1 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.1 & 0.05 \\ 0.05 & 0.05 & 0.05 & 0.1 \end{bmatrix} \mathbb{W}_2 = \begin{bmatrix} 0.5 & -0.02 & -0.02 & -0.02 \\ -0.02 & 0.5 & -0.02 & -0.02 \\ -0.02 & -0.02 & 0.5 & -0.02 \\ -0.02 & -0.02 & -0.02 & 0.5 \end{bmatrix}. \tag{119}$$

Then, for a matrix

$$X = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}, \tag{120}$$

we have

$$\sum_{i=1}^{2} \mathbb{W}_i^* X \mathbb{W}_i = \begin{bmatrix} 0.2662 & 0.0479 & 0.0479 & 0.0479 \\ 0.0479 & 0.2662 & 0.0479 & 0.0479 \\ 0.0479 & 0.0479 & 0.2662 & 0.0479 \\ 0.0479 & 0.0479 & 0.0479 & 0.2662 \end{bmatrix} \prec \frac{1}{2} X.$$

$$\text{Similarly, let } \mathbb{G}_1 = \begin{bmatrix} 0.01 & 0.001 & 0.01 & 0.01 \\ 0.001 & 0.01 & 0.01 & 0.001 \\ 0.01 & 0.001 & 0.001 & 0.01 \\ 0.001 & 0.01 & 0.001 & 0.001 \end{bmatrix} \mathbb{G}_2 = \begin{bmatrix} 0.1413 & 0.008294 & 0.1413 & 0.1413 \\ 0.008294 & 0.0997 & 0.008294 & 0.1413 \\ 0.1413 & 0.008294 & 0.1413 & 0.0997 \\ 0.1109 & 0.1413 & 0.008294 & 0.0997 \end{bmatrix}. \tag{121}$$

Then, for a matrix

$$X = \begin{bmatrix} 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 \end{bmatrix}, \tag{122}$$

we have

$$\sum_{i=1}^{2} \mathbb{G}_i^* X \mathbb{G}_i = \begin{bmatrix} 0.0744 & 0.0359 & 0.0570 & 0.0.0760 \\ 0.0359 & 0.0376 & 0.0191 & 0.0491 \\ 0.0570 & 0.0191 & 0.0502 & 0.0579 \\ 0.0760 & 0.0491 & 0.0579 & 0.0.0946 \end{bmatrix} \prec \frac{1}{2} X. \tag{123}$$

6. Conclusion

The introduced contractions encompass the F-contractions and multivalued contractions and hence the Banach contractions, Kannan contractions, Chatterjea contractions, Reich contractions, Hardy-Rogers contractions, and Ciric-type contractions (both metric and p-m versions). It is a real generalization of Matthews contractions and F-contractions. The theorems give general criteria for the existence of the uniqueness of the fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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References

[1] S. G. Matthews, "Partial metric topology," *Annals of the New York Academy of Sciences*, vol. 728, 1 General Topol, pp. 183–197, 1994.

[2] T. Abdeljawad, E. Karapinar, and K. Tas, "Existence and uniqueness of a common fixed point on partial metric spaces," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1900–1904, 2011.

[3] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology and its Applications*, vol. 157, no. 18, pp. 2778–2785, 2010.

[4] H. Aydi, E. Karapinar, and P. Kumam, "A note on Modified proof of Caristi's fixed point theorem on partial metric spaces," *Journal of Inequalities and Applications*, vol. 2013, Article ID 355, 2013.

[5] H. Aydi, M. Abbas, and C. Vetro, "Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces," *Topology and its Applications*, vol. 159, no. 14, pp. 3234–3242, 2012.

[6] K. P. Chi, E. Karapinar, and T. D. Thanh, "A generalized contraction principle in partial metric spaces," *Mathematical and Computer Modelling*, vol. 55, no. 5-6, pp. 1673–1681, 2012.

[7] M. Nazam and Ö. Acar, "Fixed points of (α, ψ)-contractions in Hausdorff partial metric spaces," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 16, pp. 5159–5173, 2019.

[8] S. B. Nadler, "Multivalued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[9] H. K. Pathak and N. Shahzad, "A generalization of Nadler's fixed point theorem and its application to nonconvex integral inclusions," *Topological Methods in Nonlinear Analysis*, vol. 41, no. 1, pp. 207–227, 2013.

[10] H. K. Nashine, R. P. Agarwal, and Z. Kadelburg, "H^+-multivalued contractions and their application to homotopy theory," *Fixed Point Theory and Applications*, vol. 19, no. 4, pp. 2309–2325, 2017.

[11] H. Aydi, M. A. Barakat, Z. D. Mitrović, and V. Šešum-Čavić, "A Suzuki type multivalued contraction on weak partial metric spaces and applications," *Journal of Inequalities and Applications*, vol. 2018, no. 1, Article ID 270, 2018.

[12] H. Aydi, M. Abbas, and C. Vetro, "Common fixed points for multivalued generalized contractions on partial metric spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 108, no. 2, pp. 483–501, 2014.

[13] P. Patle, D. Patel, H. Aydi, and S. Radenovic, "ON H^+ type multivalued contractions and applications in symmetric and probabilistic spaces," *Mathematics*, vol. 7, no. 2, p. 144, 2019.

[14] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, Article ID 94, 2012.

[15] O. Acar and I. Altun, "Multivalued F-contractive mappings with a graph and some fixed point results," *Publicationes Mathematicae Debrecen*, vol. 88, no. 3-4, pp. 305–317, 2016.

[16] M. U. Ali and T. Kamran, "Multivalued F-contractions and related fixed point theorems with an application," *Filomat*, vol. 30, no. 14, pp. 3779–3793, 2016.

[17] G. Mınak, G. Durmaz, and I. Altun, "Fixed points of ordered F-contractions," *Hacettepe Journal of Mathematics and Statistics*, vol. 1, no. 45, pp. 15–21, 2016.

[18] M. Nazam, M. Arshad, and M. Abbas, "Existence of common fixed points of improved F-contraction on partial metric spaces," *Applied General Topology*, vol. 18, no. 2, pp. 277–287, 2017.

[19] M. Abbas, M. Berzig, T. Nazir, and E. Karapmar, "Iterative approximation of fixed points for presic type F-contraction operators," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 78, no. 2, pp. 147–160, 2016.

[20] A. Tomar, Giniswamy, C. Jeyanthi, and P. G. Maheshwari, "On coincidence and common fixed point of six maps satisfying F-contractions," *TWMS Journal of Applied and Engineering Mathematics*, vol. 6, no. 2, pp. 224–231, 2016.

[21] M. Sgroi and C. Vetro, "Multi-valued F-contractions and the solutions of certain functional and integral equations," *Filomat*, vol. 27, no. 7, pp. 1259–1268, 2013.

[22] G. Durmaz and I. Altun, "Fixed point results for α-admissible multivalued F-contractions," *Miskolc Mathematical Notes*, vol. 17, no. 1, pp. 187–199, 2016.

[23] H. Piri and P. Kumam, "Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, Article ID 210, 2014.

[24] S. Shukla and S. Radenović, "Some common fixed point theorems for F-contraction type mappings in 0-complete partial

metric spaces," *Journal of Mathematics*, vol. 2013, Article ID 878730, 7 pages, 2013.

[25] E. Karapınar, A. Fulga, and R. P. Agarwal, "A survey: F-contractions with related fixed point results," *Journal of Fixed Point Theory and Applications*, vol. 22, no. 3, p. 69, 2020.

[26] M. U. Ali and C. Vetro, "*Fixed point theorems for multivalued maps via new auxiliary function*," *Nonlinear Analysis: Modelling and Control*, vol. 2017, no. 1, pp. 84–98, 2017.

[27] I. Beg and H. K. Pathak, "A variant of Nadler's theorem on partial metric spaces with application to a homotopy result," *Vietnam Journal of Mathematics*, vol. 46, no. 3, pp. 693–706, 2018.

[28] M. Cosentino and P. Vetroa, "Fixed point results for F-contractive mappings of Hardy-Rogers-Type," *Filomat*, vol. 28, no. 4, pp. 715–722, 2014.

[29] S. Alizadeh, D. Baleanu, and S. Rezapour, "Analyzing transient response of the parallel RCL circuit by using the Caputo-Fabrizio fractional derivative," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 55, 2020.

[30] D. Baleanu, A. Jajarmi, H. Mohammadi, and S. Rezapour, "A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative," *Chaos, Solitons & Fractals*, vol. 134, article 109705, 2020.

[31] D. Baleanu, H. Mohammadi, and S. Rezapour, "Analysis of the model of HIV-1 infection of CD^{4+} T-cell with a new approach of fractional derivative," *Advances in Difference Equations*, vol. 2020, no. 1, Article ID 71, 2020.

[32] I. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhä user, Boston, MA, 1981, MR 83b:47001.

[33] M. Berzig and B. Samet, "Solving systems of nonlinear matrix equations involving Lipschitzian mappings," *Fixed Point Theory and Applications*, vol. 2011, no. 1, Article ID 89, 2011.

[34] M. Berzig, "Solving a class of matrix equations via the Bhaskar-Lakshmikantham coupled fixed point theorem," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1638–1643, 2012.

[35] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2003.