Research Article

BMO Functions Generated by $A_X(\mathbb{R}^n)$ Weights on Ball Banach Function Spaces

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Abstract

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Let $X$ be a ball Banach function space on $\mathbb{R}^n$. We introduce the class of weights $A_X(\mathbb{R}^n)$. Assuming that the Hardy-Littlewood maximal function $M$ is bounded on $X$ and $X'$, we obtain that $\text{BMO}(\mathbb{R}^n) = \{ \alpha \ln \omega : \alpha \geq 0, \omega \in A_X(\mathbb{R}^n) \}$. As a consequence, we have $\text{BMO}(\mathbb{R}^n) = \{ \alpha \ln \omega : \alpha \geq 0, \omega \in A_{L^1}(\mathbb{R}^n) \}$, where $L^1(\mathbb{R}^n)$ is the variable exponent Lebesgue space. As an application, if a linear operator $T$ is bounded on the weighted ball Banach function space $X(\omega)$ for any $\omega \in A_X(\mathbb{R}^n)$, then the commutator $[b, T]$ is bounded on $X$ with $b \in \text{BMO}(\mathbb{R}^n)$.

1. Introduction

It is well known that there is a relation between $A_{\text{loc}}(\mathbb{R}^n)$ weights and $\text{BMO}(\mathbb{R}^n)$, i.e., for any $p \in (1, \infty)$,

$$\text{BMO}(\mathbb{R}^n) = \{ \alpha \ln W : \alpha \geq 0, W \in A_p(\mathbb{R}^n) \}. \quad (1)$$

See, for instance, [1] (p. 409). The purpose of this note is to reveal the relation between $\text{BMO}(\mathbb{R}^n)$ and $A_X(\mathbb{R}^n)$ weights over the ball Banach function space $X$.

To state our results, we begin with the definition of the ball Banach function space. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}$ and

$$B := \{ B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty) \}. \quad (2)$$

Definition 1. A Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a ball Banach function space if it satisfies that

(i) $\| f \|_X = 0$ implies that $f = 0$ almost everywhere

(ii) $|g| \leq |f|$ almost everywhere implies that $\| g \|_X \leq \| f \|_X$

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\| f_m \|_X \uparrow \| f \|_X$

(iv) $B \in \mathbb{B}$ implies that $1_B \in X$, where $\mathbb{B}$ is as in (2);

(v) for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on $B$, such that, for any $f \in X$,

$$\int_B |f(x)| \, dx \leq C_{(B)} \| f \|_X \quad (3)$$

For any ball Banach function space $X$, the associate space (Köthe dual) $X'$ is defined by setting

$$X' := \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{X'} = \sup \{ \| fg \|_{L^1(\mathbb{R}^n)} : g \in X, \| g \|_X = 1 \} < \infty \}, \quad (4)$$

where $\| \cdot \|_{X'}$ is called the associate norm of $\| \cdot \|_X$ (see, for instance, [2] (Chapter 1, Definitions 2.1 and 2.3)).

Remark 2. By [3] (Proposition 2.3), we know that, if $X$ is a ball Banach function space, then its associate space $X'$ is also a ball Banach function space.
Now, we introduce the class of weights $A_X(\mathbb{R}^n)$ and recall the function space $BMO$. A weight $\omega$ is a locally integrable function such that $0 < \omega(x) < \infty$ almost everywhere $x \in \mathbb{R}^n$.

**Definition 3.** Let $X$ be a ball Banach function space. We say that a weight $\omega$ belongs to $A_X(\mathbb{R}^n)$ if
\begin{equation}
\sup_{B \subset \mathbb{R}^n} \frac{\|\omega 1_B\|_X \|\omega^{-1} 1_B\|_X'}{\|1_B\|_X^2} < \infty,
\end{equation}
here and hereafter $1_B$ is the characteristic function for $B$.

**Remark 4.**

1. There is an immediate consequence. Let $X$ be a ball Banach function space. If $\omega \in A_X(\mathbb{R}^n)$, then $\omega^{-1} \in A_X(\mathbb{R}^n)$

2. We recall that the definition of $A_p(\mathbb{R}^n)$. Let $p \in [1, \infty)$. A weight $W$ belongs to $A_p(\mathbb{R}^n)$ if
\begin{equation}
\sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{|B|} \int_B W(x) \, dx \right\} \left\{ \frac{1}{|B|} \int_B W(x)^{-p'} \, dx \right\}^{p-1} < \infty
\end{equation}

By the definition of $A_X(\mathbb{R}^n)$ and $A_p(\mathbb{R}^n)$, $W \in A_p(\mathbb{R}^n)$ if and only if $\omega = W^{1/p} \in A_p(\mathbb{R}^n)$ for any $p \in [1, \infty]$.

The classical function space $BMO(\mathbb{R}^n)$ is the collection of all locally integrable functions $f$ such that
\begin{equation}
BMO(\mathbb{R}^n) = \sup_B \left\{ \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \right\},
\end{equation}
where the supremum is taking all balls $B$ in $\mathbb{R}^n$ and $f_B$ is the mean value of the function $f$ on $B$, namely,
\begin{equation}
f_B = \frac{1}{|B|} \int_B f(y) \, dy.
\end{equation}

By the well-known John-Nirenberg inequality, John and Nirenberg [4] proved that there exists a positive constant $C$ such that
\begin{equation}
\|f\|_{BMO(\mathbb{R}^n)} \leq \|f\|_{BMO_{L^p(\mathbb{R}^n)}} \leq C\|f\|_{BMO(\mathbb{R}^n)},
\end{equation}
where $p \in [1, \infty)$ and
\begin{equation}
BMO_{L^p(\mathbb{R}^n)} := \sup_B \left\{ \frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right\}^{1/p}.
\end{equation}

We also recall that the Hardy-Littlewood maximal function $M$ is defined by setting, for any locally integrable function $f$ and $x \in \mathbb{R}^n$,
\begin{equation}
Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy.
\end{equation}

Now, we state our result as the following theorem.

**Theorem 5.** Let $X$ be ball Banach function spaces. If the Hardy-Littlewood maximal function $M$ is bounded on $X$ and $X'$, then
\begin{equation}
BMO(\mathbb{R}^n) = \{ \alpha \ln \omega : \alpha \geq 0, \omega \in A_X(\mathbb{R}^n) \}.
\end{equation}

**Remark 6.** Let $p \in (1, \infty)$, Theorem 5 goes back to the classical result for $X = L^p(\mathbb{R}^n)$.

As an example, let $\mathcal{B} = \mathcal{B}(\mathbb{R}^n)$ be the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty]$. Then, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that
\begin{equation}
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[ \frac{|f(x)|^p}{\lambda} \right] \, dx \leq 1 \right\} < \infty.
\end{equation}

Denote $p_- = \inf_{x \in \mathbb{R}^n} p(x)$ and $p_+ = \sup_{x \in \mathbb{R}^n} p(x)$. A measurable function $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is said to be globally log-Hölder continuous if there exists a $p_{\infty} \in \mathbb{R}$ such that, for any $p, q \in \mathbb{R}^n$,
\begin{equation}
|p(x) - p(y)| \leq \frac{1}{\log ((e + (1/|x - y|))},
\end{equation}
\begin{equation}
|p(x) - p_{\infty}| \leq \frac{1}{\log (e + |x|)},
\end{equation}
where the implicit positive constants are independent of $x$ and $y$.

**Definition 7** ([5], Definition 1.4.). Given an exponent function $p(\cdot) : \mathbb{R}^n \to [1, \infty]$ and a weight $\omega$, we say that $\omega \in A_{p(\cdot)}$ if there exists a constant $K$ such that for every ball $B$,
\begin{equation}
\|\omega 1_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\omega^{-1} 1_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq K|B|,
\end{equation}
where $1/p(x) + 1/p'(x) = 1$ for almost everywhere $x \in \mathbb{R}^n$.

**Remark 8.** Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. By [3] (Lemma 2.5 and Proposition 3.8.), for any ball $B \subset \mathbb{R}^n$, $|B| = \|1_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|1_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$. This shows that for $X = L^{p(\cdot)}(\mathbb{R}^n)$
\begin{equation}
A_{p(\cdot)}(\mathbb{R}^n) = A_{p(\cdot)}(\mathbb{R}^n).
\end{equation}

Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. We know that $M$ is bounded on.
bounded on $\mathbb{R}^n$ and its duality $L^{p'}(\mathbb{R}^n)$; see, for instance, [6, 7] and their references.

**Corollary 9.** Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. Then, $\text{BMO}(\mathbb{R}^n) = \{ \alpha \ln \omega : \alpha \geq 0, \omega \in A_{p(\cdot)}(\mathbb{R}^n) \}$.

**2. Proof of Theorem 5**

The following lemmas give two elementary properties of ball Banach function spaces, whose proof is similar to the one corresponding to Banach function spaces; see [2].

**Lemma 10** (Holder’s inequality). Let $X$ be a ball Banach function space with the associate space $X'$. If $f \in X$ and $g \in X'$, then $fg$ is integrable and

$$
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
$$

**Lemma 11** (G. G. Lorentz, W. A. J. Luxemburg). Every ball Banach function space $X$ coincides with its second associate space $X''$. In other words, a function $f$ belongs to $X$ if and only if it belongs to $X''$ and, in that case,

$$
\|f\|_X = \|f\|_{X''}.
$$

Under weak boundedness of the Hardy-Littlewood maximal operator $M$ on $X$, the norm $\|\cdot\|_X$ enjoys the following property; see [8] (Lemma 2.2).

**Lemma 12.** Let $X$ be a ball Banach function space and suppose that the Hardy-Littlewood maximal operator $M$ is weakly bounded on $X$ or $X'$, that is, there exists a positive constant $C$ such that

$$
\|1_{\{x \in \mathbb{R}^n : Mf(x) > \lambda\}}\|_X \leq C \lambda^{-1} \|f\|_X
$$

or

$$
\|1_{\{x \in \mathbb{R}^n : Mf(x) > \lambda\}}\|_{X'} \leq C \lambda^{-1} \|f\|_{X'}
$$

holds for all $\lambda > 0$ and all $f \in X$. Then, there exists a positive constant $C$ such that for all balls $B \in \mathcal{B}$, $\|1_B\|_X \|1_B\|_{X'} \leq C |B|$.

**Remark 13.** By Lemma 10, we have $|B| \leq \|1_B\|_X \|1_B\|_{X'}$ for any ball $B \in \mathcal{B}$.

**Lemma 14.** Let $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $X$ be a ball Banach function space. Suppose that the Hardy-Littlewood maximal operator $M$ is weakly bounded on $X$. Then, $e^{\varphi} \in A_X(\mathbb{R}^n)$ if and only if there exists a positive constant $C$ such that for any ball $B \in \mathcal{B}$

\begin{equation}
\frac{\|e^{\varphi} - \varphi\|_X}{\|1_B\|_X} \leq C,
\end{equation}

\begin{equation}
\frac{\|e^{\varphi} - \varphi\|_{X'}}{|B|} \leq C.
\end{equation}

**Proof.** We first prove the sufficiency. In fact, by the definition of $A_X(\mathbb{R}^n)$, we have

$$
\frac{\|e^{\varphi}1_B\|_X}{\|1_B\|_X} \leq \frac{\|e^{\varphi} - \varphi\|_X}{\|1_B\|_X} \leq C,
$$

$$
\frac{\|e^{\varphi}1_B\|_{X'}}{|B|} \leq \frac{\|e^{\varphi} - \varphi\|_{X'}}{|B|} \leq C.
$$

Conversely, suppose that $e^{\varphi} \in A_X(\mathbb{R}^n)$. Then by Lemmas 10 and 12

$$
\frac{\|e^{\varphi}1_B\|_X}{\|1_B\|_X} \leq \frac{1}{|B|} \int_B e^{\varphi(x)} \, dx \leq \frac{1}{|B|} \int_B e^{\varphi}\, dx \leq C.
$$

Also,

$$
\frac{\|e^{\varphi}1_B\|_{X'}}{|B|} \leq \frac{1}{|B|} \int_B e^{\varphi}\, dx \leq \frac{1}{|B|} \int_B e^{\varphi}\, dx \leq C.
$$

The John-Nirenberg inequality for ball Banach function spaces $X$ was established by Izuki et al. ([9], Theorem 3.1).

**Lemma 15.** Let $X$ be a ball Banach function space such that $M$ is bounded on $X'$ and write $C_0 := \|M\|_X^{1-\alpha}$. Then, there exists a positive constant $C_1$ such that for all balls $B, f \in \text{BM}_O(\mathbb{R}^n)$ and $\lambda \geq 0$,

$$
\|1_{\{x \in B : |f| > \lambda\}}\|_X \leq C_1 2^{\lambda(1 + \alpha/C_0)} \|f\|_{\text{BMO}} \|1_B\|_X.
$$

As a consequence of Lemma 15, we have the following inequality.

**Lemma 16.** Let $X$ be a ball Banach function space. Suppose that $M$ is bounded on $X'$. Suppose that $\varphi \in \text{BMO}$. Then for any $\alpha \in [0, (\ln 2)(2^{2\alpha^2} + 2^{2\alpha^2} + C_0)\|f\|_{\text{BMO}})$, and ball $B \in \mathcal{B}$, we have

$$
\|e^{\varphi}1_B\|_X \leq C_1 \left(1 + 2^{\alpha^2} 2^{(2\alpha^2 + C_0)\|f\|_{\text{BMO}}} + 2^{2\alpha^2 + C_0} \|f\|_{\text{BMO}}\right) \|1_B\|_X.
$$

where $C_1$ is as in Lemma 15.
Proof. By Lemma 15, we have
\[
\|e^{i\phi(x)\varphi}1_B\|_X \leq \sum_{k=0}^{\infty} \|1_{\{x \in \mathbb{R}^n : k \geq |\varphi - \varphi_x|\}} e^{i\phi(x)\varphi}1_B\|_X \\
\leq \sum_{k=0}^{\infty} e^{(k+1)} 1_{\{x \in \mathbb{R}^n : |\varphi - \varphi_x| \geq k\}} 1_B\|_X \\
\leq \sum_{k=0}^{\infty} C_1 e^{2(k+1)} \|1_{\{x \in \mathbb{R}^n : |\varphi - \varphi_x| \geq k\}} 1_B\|_X \\
= \sum_{k=0}^{\infty} C_1 e^{2(k+1)} \left(\frac{\|\varphi\|_{L^1(\mathbb{R}^n)}}{\|\varphi\|_{L^1(\mathbb{R}^n)}}\right)\|1_B\|_X \\
\leq C_1 \left(1 - 2^{\alpha/(\ln 2)} - \left(\frac{1}{(2^{\alpha}+1+2^{\alpha+1}\|\varphi\|_{L^1(\mathbb{R}^n)}}\right)\right) \cdot 2^{\alpha/(\ln 2)} \left(\frac{1}{(2^{\alpha}+1+2^{\alpha+1}\|\varphi\|_{L^1(\mathbb{R}^n)}}\right)\|1_B\|_X.
\] (27)

Lemma 17. Let X be a ball Banach function space. If \(\omega \in A_X(\mathbb{R}^n)\), then \(\ln \omega \in BMO(\mathbb{R}^n)\).

Proof. Let \(\varphi = \ln \omega\). Then, \(\omega = e^\varphi\). By Lemmas 10, 12, and 14, we obtain that
\[
\|\varphi\|_{BMO(\mathbb{R}^n)} \leq C \sup_{x \in X} \left| \frac{1}{|B|} \int_B e^{\varphi(x)\omega} dx \right| \\
\leq C \sup_{x \in X} \left| \frac{1}{|B|} \int_{\{x \in B : \varphi - \varphi_x \geq 0\}} e^{\omega(x)\varphi} dx \right| \\
+ \left| \int_{\{x \in B : \varphi - \varphi_x > 0\}} e^{-\varphi(x)\varphi} dx \right| \\
\leq C \sup_{x \in X} \left| \frac{1}{|B|} \left[ \|\varphi\|_{BMO(\mathbb{R}^n)} \right|^2 \right| \\
+ \left| \|\varphi\|_{BMO(\mathbb{R}^n)} \right|^2 < \infty.
\] (28)

Proof of Theorem 18. By Lemma 17, for any \(\omega \in A_X(\mathbb{R}^n)\) and \(\alpha \geq 0, \varphi = \alpha \ln \omega \in BMO(\mathbb{R}^n)\). Conversely, suppose that \(\varphi \in BMO(\mathbb{R}^n)\). Since M is bounded on \(X'\), by Lemma 16, we know that there exist \(\beta_1 \in [0, \ln 2/(2^{\alpha+1}+1+2^{\alpha+4}\|M\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{BMO(\mathbb{R}^n)})\) and \(C_3 \in (0, \infty)\) such that, for any ball \(B \in \mathbb{B}\),
\[
\|e^{i\beta_1(x)\varphi(x)\omega}1_B\|_{X'} \leq C_3.
\] (29)

Similarly, since M is bounded on X, by Lemmas 11 and 16, we know that there exist \(\beta_2 \in [0, \ln 2/(2^{\alpha+1}+1+2^{\alpha+4}\|M\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{BMO(\mathbb{R}^n)})\) and \(C_4 \in (0, \infty)\) such that, for any ball \(B \in \mathbb{B}\),
\[
\|e^{i\beta_2(x)\varphi(x)\omega}1_B\|_{X} \leq C_4.
\] (30)

Taking \(\alpha = \min \{\beta_1, \beta_2\}\) and \(C = \max \{C_3, C_4\}\) and applying Lemma 14, we get the desired result.

3. Applications

In this section, we will show that the boundedness of the commutator of a linear operator \(T\) on X with the BMO function can be derived from the weighted boundedness of \(T\) on \(X\). We first establish the following Minkowskity inequality.

Lemma 19. Let X be a Banach function space and F a measurable function on \(\mathbb{R}^n \times X\). If, for almost every \(x \in \mathbb{R}^n\), \(F(x, \cdot) \in L^1(\mathbb{R}^m)\) and, for almost every \(y \in \mathbb{R}^m\), \(F(\cdot, y) \in X\), then
\[
\|\int_{\mathbb{R}^m} F(\cdot, y) \|_X \leq \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X dy.
\] (31)

Proof. By Lemma 11, we have
\[
\|\int_{\mathbb{R}^m} F(\cdot, y) \|_X = \|\int_{\mathbb{R}^m} F(\cdot, y) \|_X^* \leq \sup \left\{ \left| \int_{\mathbb{R}^m} F(\cdot, y) \|g(y)\|_X \right| : g \in X' \text{ such that } \|g\|_{X'} = 1 \right\}.
\] (32)

From the Fubini theorem and Lemma 10, it follows that
\[
\left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} F(x, y) \|g(y)\|_X dx \right| \leq \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} F(x, y) \|g(y)\|_X dy dx \right| \\
\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |F(x, y)| \|g(y)\|_X dy dx \\
\leq \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X \|g\|_X, dy \\
= \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X, dy,
\] (33)

which implies the desired conclusion. This finishes the proof of Lemma 19.

Let \(T\) be a linear operator defined by
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy.
\] (34)

Given a symbol \(b\), we define the commutator \([b, T]f = bTf - T(bf)\).

Let \(\omega\) be a weight. Define \(X(\omega) = \{f \in \mathcal{M}(\mathbb{R}^n) : \omega f \in X\}\) and \(\|f\|_X = \|f\|_{X,\omega}\). We say that \(T\) is bounded on \(X(\omega)\) if there exists a positive constant \(C\) such that for all \(f \in X(\omega)\),
\[
\|(Tf)\omega\|_X \leq C\|f\|_X.
\] (35)

Theorem 20. Let X be a ball Banach function space. Suppose that M is bounded on X and X'. If, for any \(\omega \in A_X(\mathbb{R}^n)\), T is
bounded on \( X(\omega) \) then, for all \( b \in \text{BMO}(\mathbb{R}^n) \), \([b, T]\) is bounded on \( X \), i.e.,

\[
\| [b, T]f \|_X \leq C \| f \|_X, \tag{36}
\]

where \( C \) is independent of \( f \).

**Proof.** We adapt the idea from [10, 11]. Without loss of generality, we assume that \( b \neq 0 \) in \( \text{BMO}(\mathbb{R}^n) \). By Theorem 5, there exists an \( \alpha \in (0, \infty) \) such that \( e^{ab} \in A_X(\mathbb{R}^n) \). As well known, for every \( \theta \in [0, 2\pi) \), \( b \cos \theta \in \text{BMO}(\mathbb{R}^n) \) and \( \| b \cos \theta \|_{\text{BMO}(\mathbb{R}^n)} = \| b \|_{\text{BMO}(\mathbb{R}^n)} \). Thus,

\[
e^{ab \cos \theta} \in A_X(\mathbb{R}^n). \tag{37}
\]

For any \( z \in \mathbb{C} \), define \( g(z) = e^{a(b(x)-h(y))} \). Then, \( g(z) \) is analytic on \( \mathbb{C} \) and the Cauchy integral formula implies that

\[
b(x) - b(y) = \frac{g'(0)}{\alpha} = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(z)}{|z|^2} \, dz
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} e^{i\theta} |g(h(x) - b(y))| e^{-i\theta} \, d\theta. \tag{38}
\]

For any \( \theta \in [0, 2\pi) \), set \( h_\theta(x) = f(x) e^{-ab(x) e^{i\theta}} \). Since \( f \in X \), we have

\[
\| h_\theta \|_{X(e^{ab} e^{i\theta})} = \left\| f(x) e^{-ab(x) e^{i\theta}} \right\|_X = \| f \|_X. \tag{39}
\]

By this and (38), we conclude that

\[
[b, T]f(x) = \int_{\mathbb{R}^n} K(x, y) \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} |g(h(x) - b(y))| e^{-i\theta} \, d\theta \right] f(y) \, dy
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} T(h_\theta)(x) e^{ab(x) e^{i\theta}} e^{-i\theta} \, d\theta. \tag{40}
\]

Applying Lemma 19 and the weighted boundedness of \( T \), we have

\[
\| [b, T]f \|_X \leq \frac{1}{2\pi i} \int_0^{2\pi} \| T(h_\theta) \|_{X(e^{ab} e^{i\theta})} \, d\theta \leq C \| f \|_X. \tag{41}
\]

We complete the proof of Theorem 20.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**


