Research Article

BMO Functions Generated by $A_X(\mathbb{R}^n)$ Weights on Ball Banach Function Spaces

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Let X be a ball Banach function space on $\mathbb{R}^n$. We introduce the class of weights $A_X(\mathbb{R}^n)$. Assuming that the Hardy-Littlewood maximal function M is bounded on X and X' , we obtain that $\text{BMO}(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_X(\mathbb{R}^n)\}$. As a consequence, we have $\text{BMO}(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_{L^{p(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n)\}$, where $L^{p(\cdot)}(\mathbb{R}^n)$ is the variable exponent Lebesgue space. As an application, if a linear operator T is bounded on the weighted ball Banach function space $X(\omega)$ for any $\omega \in A_X(\mathbb{R}^n)$, then the commutator $[b, T]$ is bounded on X with $b \in \text{BMO}(\mathbb{R}^n)$.

1. Introduction

It is well known that there is a relation between $A_\infty(\mathbb{R}^n)$ weights and $\text{BMO}(\mathbb{R}^n)$, i.e., for any $p \in (1, \infty]$,

$$\text{BMO}(\mathbb{R}^n) = \{\alpha \ln W : \alpha \geq 0, W \in A_p(\mathbb{R}^n)\}. \quad (1)$$

See, for instance, [1] (p. 409). The purpose of this note is to reveal the relation between $\text{BMO}(\mathbb{R}^n)$ and $A_X(\mathbb{R}^n)$ weights over the ball Banach function space X .

To state our results, we begin with the definition of the ball Banach function space. Denote by the symbol $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}. \quad (2)$$

Definition 1. A Banach space $X \subset \mathcal{M}(\mathbb{R}^n)$ is called a ball Banach function space if it satisfies that

- (i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere
- (ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$

(iv) $B \in \mathbb{B}$ implies that $\mathbf{1}_B \in X$, where $\mathbb{B}$ is as in (2);

(v) for any $B \in \mathbb{B}$, there exists a positive constant $C_{(B)}$, depending on B , such that, for any $f \in X$,

$$\int_B |f(x)| dx \leq C_{(B)} \|f\|_X \quad (3)$$

For any ball Banach function space X , the *associate space* (Köthe dual) X' is defined by setting

$$X' := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup \left\{ \|fg\|_{L^1(\mathbb{R}^n)} : g \in X, \|g\|_X = 1 \right\} < \infty \right\}, \quad (4)$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$ (see, for instance, [2] (Chapter 1, Definitions 2.1 and 2.3)).

Remark 2. By [3] (Proposition 2.3), we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

Now, we introduce the class of weights $A_X(\mathbb{R}^n)$ and recall the function space BMO. A weight ω is a locally integrable function such that $0 < \omega(x) < \infty$ almost everywhere $x \in \mathbb{R}^n$.

Definition 3. Let X be a ball Banach function space. We say that a weight ω belongs to $A_X(\mathbb{R}^n)$ if

$$\sup_{B \in \mathbb{B}} \frac{\|\omega \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \frac{\|\omega^{-1} \mathbf{1}_B\|_{X'}'}{\|\mathbf{1}_B\|_{X'}'} < \infty, \quad (5)$$

here and hereafter $\mathbf{1}_B$ is the characteristic function for B .

Remark 4.

- (1) There is an immediate consequence. Let X be a ball Banach function space. If $\omega \in A_X(\mathbb{R}^n)$, then $\omega^{-1} \in A_{X'}'(\mathbb{R}^n)$
- (2) We recall that the definition of $A_p(\mathbb{R}^n)$. Let $p \in [1, \infty]$. A weight W belongs to $A_p(\mathbb{R}^n)$ if

$$\sup_{B \in \mathbb{B}} \left\{ \frac{1}{|B|} \int_B W(x) dx \right\} \left\{ \frac{1}{|B|} \int_B W(x)^{1-p'} dx \right\}^{p-1} < \infty \quad (6)$$

By the definition of $A_X(\mathbb{R}^n)$ and $A_p(\mathbb{R}^n)$, $W \in A_p(\mathbb{R}^n)$ if and only if $\omega := W^{1/p} \in A_{L^p(\mathbb{R}^n)}(\mathbb{R}^n)$ for any $p \in [1, \infty]$.

The classical function space $\text{BMO}(\mathbb{R}^n)$ is the collection of all locally integrable functions f such that

$$\text{BMO}(\mathbb{R}^n) := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx, \quad (7)$$

where the supremum is taking all balls B in $\mathbb{R}^n$ and f_B is the mean value of the function f on B , namely,

$$f_B := \frac{1}{|B|} \int_B f(y) dy. \quad (8)$$

By the well-known John-Nirenberg inequality, John and Nirenberg [4] proved that there exists a positive constant C such that

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} \leq \|f\|_{\text{BMO}_{L^p}(\mathbb{R}^n)} \leq C \|f\|_{\text{BMO}(\mathbb{R}^n)}, \quad (9)$$

where $p \in [1, \infty)$ and

$$\text{BMO}_{L^p}(\mathbb{R}^n) := \sup_B \left\{ \frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right\}^{1/p}. \quad (10)$$

We also recall that the Hardy-Littlewood maximal function M is defined by setting, for any locally integrable function f and $x \in \mathbb{R}^n$,

$$Mf(x) := \sup_{B \in \mathbb{B}} \frac{\mathbf{1}_B(x)}{|B|} \int_B |f(y)| dy. \quad (11)$$

Now, we state our result as the following theorem.

Theorem 5. Let X be ball Banach function spaces. If the Hardy-Littlewood maximal function M is bounded on X and X' , then

$$\text{BMO}(\mathbb{R}^n) := \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_X(\mathbb{R}^n)\}. \quad (12)$$

Remark 6. Let $p \in (1, \infty)$, Theorem 5 goes back to the classical result for $X := L^p(\mathbb{R}^n)$.

As an example, let $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ be the collection of all measurable functions $p(\cdot): \mathbb{R}^n \rightarrow [1, \infty]$. Then, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} dx \leq 1 \right\} < \infty. \quad (13)$$

Denote $p_- := \text{ess inf}_{x \in \mathbb{R}^n} p(x)$ and $p_+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x)$. A measurable function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is said to be *globally log-Hölder continuous* if there exists a $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |p(x) - p(y)| &\leq \frac{1}{\log(e + (1/|x - y|))}, \\ |p(x) - p_\infty| &\leq \frac{1}{\log(e + |x|)}, \end{aligned} \quad (14)$$

where the implicit positive constants are independent of x and y .

Definition 7 ([5], Definition 1.4.). Given an exponent function $p(\cdot): \mathbb{R}^n \rightarrow [1, \infty)$ and a weight ω , we say that $\omega \in A_{p(\cdot)}$ if there exists a constant K such that for every ball B ,

$$\|\omega \mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\omega^{-1} \mathbf{1}_B\|_{L^{p(\cdot)'}(\mathbb{R}^n)} \leq C|B|, \quad (15)$$

where $1/p(x) + 1/p(x)' = 1$ for almost everywhere $x \in \mathbb{R}^n$.

Remark 8. Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. By [3] (Lemma 2.5 and Proposition 3.8.), for any ball $B \subset \mathbb{R}^n$, $|B| = \|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\mathbf{1}_B\|_{L^{p(\cdot)'}(\mathbb{R}^n)}$. This shows that for $X = L^{p(\cdot)}(\mathbb{R}^n)$

$$A_{L^{p(\cdot)}(\mathbb{R}^n)}(\mathbb{R}^n) = A_{p(\cdot)}(\mathbb{R}^n). \quad (16)$$

Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. We know that M is bounded on

$L^{p(\cdot)}(\mathbb{R}^n)$ and its duality $L^{p(\cdot)'}(\mathbb{R}^n)$; see, for instance, [6, 7] and their references.

Corollary 9. *Let $p(\cdot)$ be a globally log-Hölder continuous function satisfying $1 < p_- \leq p_+ < \infty$. Then, $BMO(\mathbb{R}^n) = \{\alpha \ln \omega : \alpha \geq 0, \omega \in A_{p(\cdot)}(\mathbb{R}^n)\}$.*

2. Proof of Theorem 5

The following lemmas give two elementary properties of ball Banach function spaces, whose proof is similar to the one corresponding to Banach function spaces; see [2].

Lemma 10 (Holder's inequality). *Let X be a ball Banach function space with the associate space X' . If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq \|f\|_X \|g\|_{X'}. \quad (17)$$

Lemma 11 (G. G. Lorentz, W. A. J. Luxembour). *Every ball Banach function space X coincides with its second associate space X'' . In other words, a function f belongs to X if and only if it belongs to X'' and, in that case,*

$$\|f\|_X = \|f\|_{X''}. \quad (18)$$

Under weak boundedness of the Hardy-Littlewood maximal function M on X , the norm $\|\cdot\|_X$ enjoys the following property; see [8] (Lemma 2.2).

Lemma 12. *Let X be a ball Banach function space and suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X or X' , that is, there exists a positive constant C such that*

$$\left\| I_{\{x \in \mathbb{R}^n : Mf(x) > \lambda\}} \right\|_X \leq C\lambda^{-1} \|f\|_X \quad (19)$$

or

$$\left\| I_{\{x \in \mathbb{R}^n : Mf(x) > \lambda\}} \right\|_{X'} \leq C\lambda^{-1} \|f\|_{X'} \quad (20)$$

holds for all $\lambda > 0$ and all $f \in X$. Then, there exists a positive constant C such that for all balls $B \in \mathbb{B}$, $\|I_B\|_X \|I_B\|_{X'} \leq C|B|$.

Remark 13. By Lemma 10, we have $|B| \leq \|I_B\|_X \|I_B\|_{X'}$ for any ball $B \in \mathbb{B}$.

Lemma 14. *Let $\varphi \in L^1_{loc}(\mathbb{R}^n)$ and X be a ball Banach function space. Suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X . Then, $e^\varphi \in A_X(\mathbb{R}^n)$ if and only if there exists a positive constant C such that for any ball $B \in \mathbb{B}$*

$$\begin{aligned} \frac{\|e^{\varphi-\varphi_B} I_B\|_X}{\|I_B\|_X} &\leq C, \\ \frac{\|e^{-(\varphi-\varphi_B)} I_B\|_{X'}}{\|I_B\|_{X'}} &\leq C. \end{aligned} \quad (21)$$

Proof. We first prove the sufficiency. In fact, by the definition of $A_X(\mathbb{R}^n)$, we have

$$\frac{\|e^\varphi I_B\|_X}{\|I_B\|_X} \frac{\|e^{-\varphi} I_B\|_{X'}}{\|I_B\|_{X'}} = \frac{\|e^{\varphi-\varphi_B} I_B\|_X}{\|I_B\|_X} \frac{\|e^{-(\varphi-\varphi_B)} I_B\|_{X'}}{\|I_B\|_{X'}} \leq C. \quad (22)$$

Conversely, suppose that $e^\varphi \in A_X(\mathbb{R}^n)$. Then by Lemmas 10 and 12

$$\begin{aligned} \frac{\|e^{\varphi-\varphi_B} I_B\|_X}{\|I_B\|_X} &= e^{-\varphi_Q} \frac{\|e^\varphi I_B\|_X}{\|I_B\|_X} \leq \frac{1}{|B|} \int_B e^{-\varphi(x)} dx \frac{\|e^\varphi I_B\|_X}{\|I_B\|_X} \\ &\leq \frac{\|e^{-\varphi}\|_{X'} \|I_B\|_X}{|B|} \frac{\|e^\varphi I_B\|_X}{\|I_B\|_X} \leq C. \end{aligned} \quad (23)$$

Also,

$$\begin{aligned} \frac{\|e^{-(\varphi-\varphi_B)} I_B\|_{X'}}{\|I_B\|_{X'}} &= e^{\varphi_Q} \frac{\|e^{-\varphi} I_B\|_{X'}}{\|I_B\|_{X'}} \leq \frac{1}{|B|} \int_B e^{\varphi(x)} dx \frac{\|e^{-\varphi} I_B\|_{X'}}{\|I_B\|_{X'}} \\ &\leq \frac{\|e^\varphi\|_X \|I_B\|_{X'}}{|B|} \frac{\|e^{-\varphi} I_B\|_{X'}}{\|I_B\|_{X'}} \leq C. \end{aligned} \quad (24)$$

The John-Nirenberg inequality for ball Banach function spaces X was established by Izuki et al. ([9], Theorem 3.1).

Lemma 15. *Let X be a ball Banach function space such that M is bounded on X' and write $C_0 := \|M\|_{X' \rightarrow X'}$. Then, there exists a positive constant C_1 such that for all balls B , $f \in BM O(\mathbb{R}^n)$ and $\lambda \geq 0$,*

$$\left\| I_{\{x \in B : |f-f_B| > \lambda\}} \right\|_X \leq C_1 2^{-\left(\lambda / (2^{n+2}(1+2^{n+4}C_0)\|f\|_{BMO})\right) 2^{n+2}(1+2^{n+4}C_0)\|f\|_{BMO}} \|I_B\|_X. \quad (25)$$

As a consequence of Lemma 15, we have the following inequality.

Lemma 16. *Let X be a ball Banach function space. Suppose that M is bounded on X' . Suppose that $\varphi \in BMO$. Then for any $\alpha \in [0, (\ln 2 / (2^{n+2}(1+2^{n+4}C_0)\|f\|_{BMO}))]$ and ball $B \in \mathbb{B}$, we have*

$$\|e^{\alpha|\varphi-\varphi_B|} I_B\|_X \leq C_1 \left(1 - 2^{(\alpha \ln 2) - (1/(2^{n+2}(1+2^{n+4}C_0)\|f\|_{BMO}))}\right) 2^{1/(2^{n+2}(1+2^{n+4}C_0)\|f\|_{BMO})} \|I_B\|_X, \quad (26)$$

where C_1 is as in Lemma 15.

Proof. By Lemma 15, we have

$$\begin{aligned}
\|e^{\alpha|\varphi-\varphi_B|}\|_X &\leq \sum_{k=0}^{\infty} \left\| \mathbf{1}_{\{x \in B: k+1 \geq |\varphi-\varphi_B| > k\}} e^{\alpha|\varphi-\varphi_B|} \right\|_X \\
&\leq \sum_{k=0}^{\infty} e^{\alpha(k+1)} \left\| \mathbf{1}_{\{x \in B: |\varphi-\varphi_B| > k\}} \right\|_X \\
&\leq \sum_{k=0}^{\infty} C_1 e^{\alpha(k+1)} 2^{-k/2} (2^{n+2} (1+2^{n+4} C_0) \|f\|_{\text{BMO}}) \|\mathbf{1}_B\|_X \\
&= \sum_{k=0}^{\infty} C_1 e^{\alpha k} 2^{((\alpha k/\ln 2) - (k/2)(2^{n+2} (1+2^{n+4} C_0) \|f\|_{\text{BMO}}))} \|\mathbf{1}_B\|_X \\
&\leq C_1 \left(1 - 2^{(\alpha/\ln 2) - (1/2)(2^{n+2} (1+2^{n+4} C_0) \|f\|_{\text{BMO}})} \right) \\
&\quad \cdot 2^{1/2(2^{n+2} (1+2^{n+4} C_0) \|f\|_{\text{BMO}})} \|\mathbf{1}_B\|_X.
\end{aligned} \tag{27}$$

Lemma 17. Let X be a ball Banach function space. If $\omega \in A_X(\mathbb{R}^n)$, then $\ln \omega \in \text{BMO}(\mathbb{R}^n)$.

Proof. Let $\varphi := \ln \omega$. Then, $\omega = e^\varphi$. By Lemmas 10, 12, and 14, we obtain that

$$\begin{aligned}
\|\varphi\|_{\text{BMO}(\mathbb{R}^n)} &\leq C \sup_{B \in \mathbb{B}} \frac{1}{|B|} \int_B e^{|\varphi(x)-\varphi_B|} dx \\
&\leq C \sup_{B \in \mathbb{B}} \frac{1}{|B|} \left[\int_{\{x \in B: \varphi(x)-\varphi_B \geq 0\}} e^{\varphi(x)-\varphi_B} dx \right. \\
&\quad \left. + \int_{\{x \in B: \varphi(x)-\varphi_B < 0\}} e^{-(\varphi(x)-\varphi_B)} dx \right] \\
&\leq C \sup_{B \in \mathbb{B}} \frac{1}{|B|} \left[\|e^{\varphi-\varphi_B} \mathbf{1}_B\|_X \|\mathbf{1}_B\|_{X'}' \right. \\
&\quad \left. + \|e^{-(\varphi-\varphi_B)} \mathbf{1}_B\|_{X'}' \|\mathbf{1}_B\|_X \right] < \infty.
\end{aligned} \tag{28}$$

Proof of Theorem 18. By Lemma 17, for any $\omega \in A_X(\mathbb{R}^n)$ and $\alpha \geq 0$, $\varphi := \alpha \ln \omega \in \text{BMO}(\mathbb{R}^n)$. Conversely, suppose that $\varphi \in \text{BMO}(\mathbb{R}^n)$. Since M is bounded on X' , by Lemma 16, we know that there exist $\beta_1 \in [0, \ln 2/(2^{n+2}(1+2^{n+4}\|M\|_{X' \rightarrow X'})\|\varphi\|_{\text{BMO}}))$ and $C_3 \in (0, \infty)$ such that, for any ball $B \in \mathbb{B}$,

$$\frac{\|e^{\beta_1|\varphi-\varphi_B|} \mathbf{1}_B\|_X}{\|\mathbf{1}_B\|_X} \leq C_3. \tag{29}$$

Similarly, since M is bounded on X , by Lemmas 11 and 16, we know that there exist $\beta_2 \in [0, \ln 2/(2^{n+2}(1+2^{n+4}\|M\|_{X \rightarrow X})\|\varphi\|_{\text{BMO}}))$ and $C_4 \in (0, \infty)$ such that, for any ball $B \in \mathbb{B}$,

$$\frac{\|e^{\beta_2|\varphi-\varphi_B|} \mathbf{1}_B\|_{X'}'}{\|\mathbf{1}_B\|_{X'}'} \leq C_4. \tag{30}$$

Taking $\alpha = \min \{\beta_1, \beta_2\}$ and $C = \max \{C_3, C_4\}$ and applying Lemma 14, we get the desired result.

3. Applications

In this section, we will show that the boundedness of the commutator of a linear operator T on X with the BMO function can be derived from the weighted boundedness of T on X . We first establish the following Minkowski-type inequality.

Lemma 19. Let X be a Banach function space and F a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$. If, for almost every $x \in \mathbb{R}^n$, $F(x, \cdot) \in L^1(\mathbb{R}^m)$ and, for almost every $y \in \mathbb{R}^m$, $F(\cdot, y) \in X$, then

$$\left\| \int_{\mathbb{R}^m} |F(\cdot, y)| dy \right\|_X \leq \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X dy. \tag{31}$$

Proof. By Lemma 11, we have

$$\begin{aligned}
\left\| \int_{\mathbb{R}^m} |F(\cdot, y)| dy \right\|_X &= \left\| \int_{\mathbb{R}^m} |F(\cdot, y)| dy \right\|_X'' \\
&= \sup \left\{ \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y)| dy g(x) dx \right| : g \in X' \text{ such that } \|g\|_{X'}' = 1 \right\}.
\end{aligned} \tag{32}$$

From the Fubini theorem and Lemma 10, it follows that

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y)| dy g(x) dx \right| &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |F(x, y)| |g(x)| dy dx \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |F(x, y)| |g(x)| dx dy \\
&\leq \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X \|g\|_{X'}' dy \\
&= \int_{\mathbb{R}^m} \|F(\cdot, y)\|_X dy,
\end{aligned} \tag{33}$$

which implies the desired conclusion. This finishes the proof of Lemma 19.

Let T be a linear operator defined by

$$Tf(x) := \int_{\mathbb{R}^n} K(x, y) f(y) dy. \tag{34}$$

Given a symbol b , we define the commutator $[b, T]f = bTf - T(bf)$.

Let ω be a weight. Define $X(\omega) := \{f \in \mathcal{M}(\mathbb{R}^n): f\omega \in X\}$ and $\|f\|_{X(\omega)} := \|f\omega\|_X$. We say that T is bounded on $X(\omega)$ if there exists a positive constant C such that for all $f \in X(\omega)$,

$$\|(Tf)\omega\|_X \leq C \|f\omega\|_X. \tag{35}$$

Theorem 20. Let X be a ball Banach function space. Suppose that M is bounded on X and X' . If, for any $\omega \in A_X(\mathbb{R}^n)$, T is

bounded on $X(\omega)$ then, for all $b \in \text{BMO}(\mathbb{R}^n)$, $[b, T]$ is bounded on X , i.e.,

$$\|[b, T]f\|_X \leq C\|f\|_X, \quad (36)$$

where C is independent of f .

Proof. We adapt the idea from [10, 11]. Without loss of generality, we assume that $b \neq 0$ in $\text{BMO}(\mathbb{R}^n)$. By Theorem 5, there exists a $\alpha \in (0, \infty)$ such that $e^{\alpha b} \in A_X(\mathbb{R}^n)$. As well known, for every $\theta \in [0, 2\pi]$, $b \cos \theta \in \text{BMO}(\mathbb{R}^n)$ and $\|b \cos \theta\|_{\text{BMO}(\mathbb{R}^n)} = \|b\|_{\text{BMO}(\mathbb{R}^n)}$. Thus,

$$e^{\alpha b \cos \theta} \in A_X(\mathbb{R}^n). \quad (37)$$

For any $z \in \mathbb{C}$, define $g(z) := e^{z\alpha[b(x)-b(y)]}$. Then, $g(z)$ is analytic on $\mathbb{C}$ and the Cauchy integral formula implies that

$$\begin{aligned} b(x) - b(y) &= \frac{g'(0)}{\alpha} = \frac{1}{2\alpha\pi i} \int_{|z|=1} \frac{g(z)}{|z|^2} dz \\ &= \frac{1}{2\alpha\pi} \int_0^{2\pi} e^{i\theta\alpha[b(x)-b(y)]} e^{-i\theta} d\theta. \end{aligned} \quad (38)$$

For any $\theta \in [0, 2\pi]$, set $h_\theta(x) := f(x)e^{-\alpha b(x)e^{i\theta}}$. Since $f \in X$, we have

$$\|h_\theta\|_{X(e^{\alpha b \cos \theta})} = \left\| f(x)e^{-\alpha b(x)e^{i\theta}} e^{\alpha b \cos \theta} \right\|_X = \|f\|_X. \quad (39)$$

By this and (38), we conclude that

$$\begin{aligned} [b, T]f(x) &= \int_{\mathbb{R}^n} K(x, y) \left[\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta\alpha[b(x)-b(y)]} e^{-i\theta} d\theta \right] f(y) dy \\ &= \frac{1}{2\pi} \int_0^{2\pi} T(h_\theta)(x) e^{\alpha b(x)e^{i\theta}} e^{-i\theta} d\theta. \end{aligned} \quad (40)$$

Applying Lemma 19 and the weighted boundedness of T , we have

$$\|[b, T]f\|_X \leq \frac{1}{2\alpha\pi} \int_0^{2\pi} \|T(h_\theta)\|_{X(e^{\alpha b \cos \theta})} d\theta \leq C\|f\|_X. \quad (41)$$

We complete the proof of Theorem 20.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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