Research Article

Lie Symmetry Analysis, Exact Solutions, and Conservation Laws for the Generalized Time-Fractional KdV-Like Equation

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In this paper, the problem of constructing the Lie point symmetries group of the nonlinear partial differential equation appeared in mathematical physics known as the generalized KdV-Like equation is discussed. By using the Lie symmetry method for the generalized KdV-Like equation, the point symmetry operators are constructed and are used to reduce the equation to another fractional ordinary differential equation based on Erdélyi-Kober differential operator. The symmetries of this equation are also used to construct the conservation Laws by applying the new conservation theorem introduced by Ibragimov. Furthermore, another type of solutions is given by means of power series method and the convergence of the solutions is provided; also, some graphics of solutions are plotted in 3D.

1. Introduction

Fractional calculus is the generalization of the ordinary differentiation and integration to noninteger (arbitrary) order; the subject is as old as calculus of differentiation and integration and goes back to the time when Leibniz, Newton, and Gauss invented this kind of calculation. Additionally, it is considered as one of the most interesting topics in several fields, especially mathematics and physics, due to its applications in modelization of physical process related to its historical states (nonlocal property), which can be effectively described by using the theory of derivatives and integrals of fractional order, that make the models described by fractional order more realistic than those described by integer order. The most important part of fractional calculus is devoted to the fractional differential equations (FDEs); in the literature, there are diverse definitions for fractional derivative including Riemann-Liouville derivative, Caputo derivative, and Conformable derivative, but the most popular one is Riemann-Liouville derivative. The fractional derivative has attracted the attention of many researchers in different areas such as viscoelasticity, vibration, economic, biology, and fluid mechanics (see [1–11]). Unfortunately, it is almost

difficult to solving and detecting all solutions of nonlinear partial differential equations (PDEs) which renders it a challenging problem, because of this, an interesting advance has been made, and some methods for solving this type of equations have been discussed; among them are subequation method, homotopy perturbation method, the first integral method, and Lie group method (see [12–18]). Lie symmetry analysis was introduced by Sophus Lie (1842, 1899), a Norwegian mathematician who made significant contributions to the theories of algebraic invariants and differential equations. It can be said that the Lie symmetry method is the most important approach for constructing analytical solutions of nonlinear PDEs. It is based to study the invariance of differential equations (DEs) under a one-parameter group of transformations which transforms a solution to another new solution and is also used to reduce the order such as the number of variables of DEs; moreover, the conservation laws (CLs) can be constructed by using the symmetries of the DEs (see [19–24]). A short time ago, the Lie symmetry analysis is also used for FDEs; in [25], Gazizov et al. showed us how the prolongation formulae for fractional derivatives is formulated; by this work, the Lie group method becomes Valid for FDEs; after that, many researches are devoted for

studying the FDEs by using Lie symmetry analysis method, for more details see ([26–29]). The CLs are very important, are a mathematical description of the statement that the total amount of a certain physical quantity including such energy, linear momentum, angular momentum, and charge, and remains unchanged during the evolution of a physical system. It is also the first step towards finding a solution; furthermore, the concept of integrability will be possible if the equation has conservation laws, and the strategy of constructing the CLs of FPDEs is given by the combination of two works of Ibragimov [30] and Lukashchuk et al. [31] and has shown how the CLs of FDEs can be constructed even those equations without fractional Lagrangians. For the first time in 1895, the Korteweg-de Vries (in short words KdV) equation

$$\frac{\partial u}{\partial t} + 6uu_x + u_{xxx} = 0, \quad (1)$$

emerged as an evolution equation representing the waves of surface gravity propagation in a water shallow canal (see, [32]) and largely used by the mathematicians and physicists to model a variety of different physical phenomena as hydro-magnetic collision-free waves, ion-acoustic waves, acoustic solitons in plasmas, lattice dynamics, stratified waves interior, internal gravity waves, and so on (see, e.g., [33–36]). Such equations have been studied extensively, especially, for the soliton solutions, solitary wave solutions, and periodic wave solutions. In order to find other properties that can be difficult for the standard type or for finding some solutions in common, a different equation is founded known as KdV-Like equations, for more details (see [37–42]). In [43], an inquiry was undertaken to increase the reliability and precision of a genetic programming-based method to deduce model equations from a proven analytical solution, especially by using the solitary wave solution; the program, instead of giving the Eq.(1), surprisingly gave the following generalized KdV-Like equation

$$\frac{\partial u}{\partial t} + \left(3(1-\delta)u + (\delta+1)\frac{u_{xx}}{u}\right)u_x - \delta u_{xxx} = 0. \quad (2)$$

The benefit of using fractional derivatives in Eq.(2) while modeling the real-world problems is the nonlocal property, and this implies that the next state of the system relies not only upon its present state but also upon all of its historical states (see [44, 45]). Because of that, we will study in this paper the generalized KdV-Like equation with the fractional order derivative presented as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \left(3(1-\delta)u + (\delta+1)\frac{u_{xx}}{u}\right)u_x - \delta u_{xxx} = 0, 0 < \alpha \leq 1, \quad (3)$$

where $\partial_t^\alpha u$ is the Riemann-Liouville (R-L) fractional derivative of order α with respect to t and δ is an arbitrary constant. For $\alpha = 1$, we have two cases; the first is $\delta = -1$; here, the Eq. (3) is reduced to the classical KdV equation which are considered by many authors (see [46–49]), and the second case

is $\delta = 1$, so the Eq. (3) is reduced to the classical KdV-Like equation which are studied by using Lie symmetry method and other approaches in [37, 38, 43]. The paper is structured as follows. In Sec. 2, we present some main results of Lie symmetries analysis for FPDEs in the general form. In Sec. 3, we construct the Lie point symmetries and similarity reduction of generalized fractional KdV-Like equation. By means of Ibragimov's theorem, the conservation laws of Eq. (3) are given in Sec. 4. In Sec. 5, we suggest an extra type of solutions in the form of power series by using the power series method. Finally, some conclusions are given in Section 6.

2. Some Main Results in Lie Symmetry Analysis

In this section, we present a brief introduction of Lie symmetry analysis for fractional partial differential equations (FPDEs), and we will give some results which will be used throughout this study, so let us consider a general form of the FPDE introduced by

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = F(x, t, u, u_x, u_{xx}, u_{xxx}), \quad (4)$$

where the subscripts indicate the partial derivatives and $\partial^\alpha / \partial t^\alpha$ is R-L fractional derivative operator given by

$$\frac{\partial^\alpha}{\partial t^\alpha} f(x, t) = \begin{cases} \frac{\partial^n}{\partial t^n} f(x, t), & \text{if } \alpha = n, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} f(x, s) ds, & 0 \leq n-1 < \alpha < n, \end{cases} \quad n \in \mathbb{N}^+. \quad (5)$$

Now, assuming that Eq. (4) is invariant under the following one-parameter Lie group of transformations expressed as

$$\hat{t} = t + \varepsilon \tau(x, t, u) + O(\varepsilon), \quad (6)$$

$$\hat{x} = x + \varepsilon \xi(x, t, u) + O(\varepsilon), \quad (7)$$

$$\hat{u} = u + \varepsilon \eta(x, t, u) + O(\varepsilon), \quad (8)$$

$$\hat{u}_t^\alpha = u_t^\alpha + \varepsilon \eta_t^\alpha(x, t, u) + O(\varepsilon), \quad (9)$$

$$\hat{u}_{\hat{x}} = u_x + \varepsilon \eta_x^\alpha(x, t, u) + O(\varepsilon), \quad (10)$$

$$\hat{u}_{\hat{x}\hat{x}} = u_{xx} + \varepsilon \eta_{xx}^\alpha(x, t, u) + O(\varepsilon), \quad (11)$$

$$\hat{u}_{\hat{x}\hat{x}\hat{x}} = u_{xxx} + \varepsilon \eta_{xxx}^\alpha(x, t, u) + O(\varepsilon), \quad (12)$$

where ε is the group parameter and ξ, τ, η are the infinitesimals and their corresponding extended infinitesimals of order 1, 2, and 3 are the functions $\eta^x, \eta^{xx}, \eta^{xxx}$, and α presented by

$$\begin{aligned} \eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ \eta^{xx} &= D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau), \\ \eta_t^\alpha &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) \\ &\quad - D_t^{\alpha-1}(\tau u) + \tau D_t^{\alpha+1}(u), \end{aligned} \quad (13)$$

where D_x is the total derivative operator with respect to x written as

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots, \quad (14)$$

the explicit form of the extended infinitesimal η_t^α of order α is given by

$$\begin{aligned} \eta_t^\alpha &= \partial_t^\alpha(\eta) + (\eta_u - \alpha D_t(\tau)) \partial_t^\alpha u - u \partial_t^\alpha(\eta_u) + \mu \\ &+ \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] \partial_t^{\alpha-n} u \\ &- \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) \partial_t^{\alpha-n}(u_x), \end{aligned} \quad (15)$$

where D_t^n is the operator of differentiation of integer order n and

$$\begin{aligned} \mu &= \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{r}{k} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ &\times [-u]^r \frac{d^m}{dt^m} [u^{k-r}] \frac{\partial^{n-m+k} \eta_u}{\partial t^{n-m} \partial u^k}. \end{aligned} \quad (16)$$

The expression of μ vanishes when the infinitesimal $\eta(x, t, u)$ is linear in the variable u , which means

$$\eta(x, t, u) = u(x, t)f(x, t) + h(x, t) \Rightarrow \mu = 0. \quad (17)$$

The generator of the one-parameter Lie group (6) or the infinitesimal operator is the differential operator defined as

$$X = \tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u}. \quad (18)$$

The corresponding prolonged generator $X^{(\alpha,3)}$ of order $(\alpha, 3)$ is

$$X^{(\alpha,3)} = X + \eta_t^\alpha \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}}. \quad (19)$$

Theorem 1 (Infinitesimal criterion of invariance). *The vector field $X = \tau(x, t, u)(\partial/\partial t) + \xi(x, t, u)(\partial/\partial x) + \eta(x, t, u)(\partial/\partial u)$, is a point symmetry of Eq. (4) if*

$$X^{\alpha,3}(\Delta)|_{(\Delta=0)} = 0, \Delta = \partial_t^\alpha u - F. \quad (20)$$

Remark 2 (Invariance condition). In the Eq. (4), the lower limit of the integral must be invariant under (6), which means

$$X(t)|_{(t=0)} = 0 \Rightarrow \tau(x, t, u)|_{t=0} = 0. \quad (21)$$

Definition 3. A solution $u = f(x, t)$ is an invariant solution of (4) if satisfies the following conditions

(i) $u = f(x, t)$ is an invariant surface of (18), which is equivalent to

$$Xf = 0 \Rightarrow \left[\tau(x, t, u) \frac{\partial}{\partial t} + \xi(x, t, u) \frac{\partial}{\partial x} + \eta(x, t, u) \frac{\partial}{\partial u} \right] f = 0, \quad (22)$$

(ii) $u = f(x, t)$ satisfies Eq. (4)

3. Lie Symmetry Analysis for the Generalized KdV-Like Equation

In this section, the Lie symmetry group of the generalized KdV-Like equation is performed.

The symmetry group of Eq. (3) is generated by the vector field (18), so applying the third prolongation $X^{(\alpha,3)}$ to the Eq. (3), we obtain the infinitesimal criterion of invariance corresponding Eq. (3), expressed as

$$\begin{aligned} \eta_t^\alpha + \eta \left[3(1-\delta) - (\delta+1) \frac{u_{xx}}{u^2} \right] u_x + \eta_x \left[3(1-\delta)u + (\delta+1) \frac{u_{xx}}{u} \right] \\ + \eta_{xx}(\delta+1) \frac{u_x}{u} - \eta_{xxx}\delta = 0. \end{aligned} \quad (23)$$

Substituting the explicit expressions $\eta^x, \eta^{xx}, \eta^{xxx}$, and η_t^α into (23) and equating powers of derivatives up to zero, we get an overdetermined system of linear partial differential equations; after resolving this system, the infinitesimals functions are given by

$$\begin{aligned} \tau(t, x, u) &= 3C_1 t, \\ \xi(t, x, u) &= C_1 \alpha x + C_2, \\ \eta(t, x, u) &= -2C_1 \alpha u, \end{aligned} \quad (24)$$

where C_1, C_2 are arbitrary constants. The corresponding Lie algebra is given by

$$X = (C_1 \alpha x - C_2) \frac{\partial}{\partial x} + 3C_1 t \frac{\partial}{\partial t} - 2C_1 \alpha u \frac{\partial}{\partial u}. \quad (25)$$

If we set

$$X_1 = \frac{\partial}{\partial x}, X_2 = 3t \frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} - 2\alpha u \frac{\partial}{\partial u}, \quad (26)$$

we can see that the vector fields X_1, X_2 is closed under Lie bracket defined by $[X_i, X_j] = X_i X_j - X_j X_i$; therefore, the Lie algebra X is generated by the vectors fields X_i (1,2), which means

$$X = C_1 X_1 + C_2 X_2. \quad (27)$$

FIGURE 1: The solution $u_1(x, t)$ for Eq. (1) for $K_1 = 1$ and $(\alpha = 0.9, \alpha = 0.75, \alpha = 0.5, \alpha = 0.25)$.

Now, by solving the following characteristic equation

$$\frac{dt}{\tau(x, t, u)} = \frac{dx}{\xi(x, t, u)} = \frac{du}{\eta(x, t, u)}, \quad (28)$$

corresponding X_1 and X_2 , we can obtain the reduced forms of generalized KdV-Like equation in the form of FODE.

Case 1. Reduction with $X_1 = \partial/\partial x$:
solving the characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}. \quad (29)$$

Leads to the following similarity variable and similarity transformation

$$\zeta = t, u = f(t), \quad (30)$$

with $f(t)$ satisfies

$$D_t^\alpha f(t) = 0. \quad (31)$$

Therefore, the group invariant solution corresponding to X_1 , is presented as

$$u_1 = k_1 t^{\alpha-1}, \quad (32)$$

where k_1 is an arbitrary constant. Figure 1 presents the graph of solution $u_1(x, t)$ for some different values of α ,

Case 2. Reduction with $X_2 = 3t(\partial/\partial t) + \alpha x(\partial/\partial x) - 2\alpha u(\partial/\partial u)$.

The similarity variable ζ and similarity transformation $f(\zeta)$ corresponding to the infinitesimal generator X_2 is obtained by solving the associated characteristic equation given by

$$\frac{dt}{3t} = \frac{dx}{\alpha x} = \frac{du}{-2\alpha u}. \quad (33)$$

Thus,

$$\zeta = xt^{\frac{-\alpha}{3}}, f(\zeta) = t^{\frac{2\alpha}{3}} u. \quad (34)$$

Therefore,

$$u = t^{\frac{-2\alpha}{3}} f\left(xt^{\frac{-\alpha}{3}}\right). \quad (35)$$

Theorem 3. By using the similarity transformation (34) in (3), the generalized KdV-Like equation is transformed into a non-linear FODE given by

$$\left(P_{\frac{3}{\alpha}}^{\frac{-5\alpha}{3}+1, \alpha} f\right)(\zeta) + \left(3(1-\delta)f + (\delta+1)\frac{f_{\zeta\zeta}}{f}\right)f_{\zeta} - \delta f_{\zeta\zeta\zeta} = 0, \quad (36)$$

with $(P_{\lambda}^{\delta, \alpha} f)(\zeta)$ is the Erdélyi-Kober differential operator given by

$$\begin{aligned} (P_{\lambda}^{\delta, \alpha} f)(\zeta) &= \prod_{i=0}^{n-1} \left(\delta + i - \frac{1}{\lambda} \zeta \frac{d}{d\zeta} \right) (K^{\delta+\alpha, n-\alpha} f)(\zeta), \\ m &= \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \end{aligned} \quad (37)$$

where $K_{\lambda}^{\delta, \alpha}$ is Erdélyi-Kober fractional integral operator introduced by

$$\begin{cases} (K_{\lambda}^{\delta, \alpha} f)(\zeta) = \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (z-1)^{\alpha-1} z^{-(\delta+\alpha)} f\left(\zeta z^{\frac{1}{\lambda}}\right) dz, & \alpha > 0, \\ f(\zeta), & \alpha = 0. \end{cases} \quad (38)$$

Proof. By using the Riemann-Liouville fractional derivative definition for the similarity transformation

$$u = t^{\frac{-2\alpha}{3}} f\left(xt^{\frac{-\alpha}{3}}\right), \quad (39)$$

we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{\frac{-2\alpha}{3}} f\left(xs^{\frac{-\alpha}{3}}\right) ds \right]. \quad (40)$$

Let $v = t/s$, we have $ds = -t/v^2$, so the above expression can be expressed as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n} \left[t^{n-\frac{5\alpha}{3}} \frac{1}{\Gamma(n-\alpha)} \int_1^\infty (\nu-1)^{n-\alpha-1} \nu^{-(n+1-\frac{5\alpha}{3})} f\left(\zeta \nu^{\frac{\alpha}{3}}\right) d\nu \right], \\ &= \frac{\partial^n}{\partial t^n} \left[t^{n-\frac{5\alpha}{3}} \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3}, n-\alpha} f \right) (\zeta) \right].\end{aligned}\quad (41)$$

We have

$$t \frac{\partial}{\partial t} \phi(\zeta) = -tx \frac{\alpha}{3} t^{\frac{-\alpha}{3}-1} \phi'(\zeta) = -\frac{\alpha}{3} \zeta \phi'(\zeta). \quad (42)$$

Then,

$$\begin{aligned}\frac{\partial^n u}{\partial t^n} &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[\frac{\partial}{\partial t} \left(t^{n-\frac{5\alpha}{3}} \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3}, n-\alpha} f \right) (\zeta) \right) \right], \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-\frac{5\alpha}{3}-1} \left(n - \frac{5\alpha}{3} - \frac{\alpha}{3} \zeta \frac{\partial}{\partial \zeta} \right) \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3}, n-\alpha} f \right) (\zeta) \right].\end{aligned}\quad (43)$$

By repeating this procedure $n-1$ times, we get

$$\begin{aligned}\frac{\partial^n u}{\partial t^n} &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-\frac{5\alpha}{3}-1} \left(n - \frac{5\alpha}{3} - \frac{\alpha}{3} \zeta \frac{\partial}{\partial \zeta} \right) \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3}, n-\alpha} f \right) (\zeta) \right], \\ &\quad \vdots \\ &= t^{-\frac{5\alpha}{3}} \prod_{j=0}^{n-1} \left(1 - \frac{5\alpha}{3} + j - \frac{\alpha}{3} \zeta \frac{\partial}{\partial \zeta} \right) \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3}, n-\alpha} f \right) (\zeta), \\ &= t^{-\frac{5\alpha}{3}} \left(P_{\frac{3}{\alpha}}^{-\frac{5\alpha}{3}+1, \alpha} f \right) (\zeta).\end{aligned}\quad (44)$$

Continuing further by calculating u_x , u_{xx} , and u_{xxx} for (35) and replacing in Eq. (3), we find that time-fractional generalized KdV-Like equation is reduced into a FODE given by

$$\left(P_{\frac{3}{\alpha}}^{-\frac{5\alpha}{3}+1, \alpha} f \right) (\zeta) + \left(3(1-\delta)f + (\delta+1) \frac{f \zeta \zeta'}{f} \right) f_\zeta - \delta f_{\zeta \zeta} = 0. \quad (45)$$

So the proof becomes complete.

4. Conservation Laws

In this section, based on its Lie point symmetry and by using the new conservation theorem [30], the conservation laws of the generalized KdV-Like equation are constructed.

All vectors $C = (C_t, C_x)$ verify the following conservation equation

$$D_t(C_t) + D_x(C_x) = 0. \quad (46)$$

In all solutions of Eq. (3), it is called conservation law of Eq. (3), with D_t and D_x are the total derivative operators with respect to t and x , respectively.

The formal Lagrangian of Eq. (3) is given by

$$\mathfrak{L} = \theta \left[\frac{\partial^\alpha u}{\partial t^\alpha} + \left(3(1-\delta)u + (\delta+1) \frac{u_{xx}}{u} \right) u_x - \delta u_{xxx} \right], \quad (47)$$

where $\theta(x, t)$ is a new dependent variable. The adjoint equation of generalized KdV-Like equation is written as

$$\frac{\delta \mathfrak{L}}{\delta u} = 0, \quad (48)$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial (D_t^\alpha u)} - D_x \frac{\partial}{\partial (u_x)} + D_x^2 \frac{\partial}{\partial (u_{xx})} - D_x^3 \frac{\partial}{\partial (u_{xxx})} + \dots, \quad (49)$$

$(D_t^\alpha)^*$ is the adjoint operator of D_t^α

$$(D_t^\alpha)^* = (-1)^n P_T^{n-\alpha} (D_T^n) = {}_t^c D_t^\alpha, \quad (50)$$

with

$$P_t^{n-\alpha} f(x, t) = \frac{1}{\Gamma(n-\alpha)} \int_t^S \frac{f(x, s)}{(s-t)^{1+\alpha-n}} ds, \quad n = [\alpha] + 1. \quad (51)$$

and $(D_t^\alpha)^*$ is the right-side Caputo operator. Now, the construction of CLs for FPDEs is similar to PDEs. So, the fundamental Neoter identity is given by

$$X^{(\alpha, 3)} + D_t(\tau) + D_x(\xi) = W_i \frac{\delta}{\delta u} + D_t(N_t) + D_x(N_x), \quad (52)$$

where N_x, N_t are Neoter operators, $X^{(\alpha, 3)}$ is given by (19), and W_i are the characteristic functions written as

$$W_i = \eta_i - \tau_i u_t - \xi_i u_x. \quad (53)$$

The x -component of Neoter operator is introduced by N_x

$$\begin{aligned}N_x &= \xi + W_i \left[\frac{\partial}{\partial u_x} - D_x \left(\frac{\partial}{\partial u_{xx}} \right) + D_{xx} \left(\frac{\partial}{\partial u_{xxx}} \right) \right] \\ &\quad + D_x(W_i) \left[\frac{\partial}{\partial u_{xx}} - D_x \left(\frac{\partial}{\partial u_{xxx}} \right) \right] + D_{xx}(W_i) \frac{\partial}{\partial u_{xxx}}.\end{aligned}\quad (54)$$

For R-L time-fractional derivative, N_t is determined by

$$N_t = \tau + \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k} (W_i) D_t^k \frac{\partial}{\partial D_t^\alpha u} - (-1)^m I \left(W_i, D_t^m \frac{\partial}{\partial (D_t^\alpha)} \right), \quad (55)$$

where I is defined as

$$I(f, h) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau, x) h(\phi, x)}{(\phi-\tau)^{\alpha+1-m}} d\phi d\tau. \quad (56)$$

By applying (52) on (47) for $X_i(1, 2)$ and for all solutions, we deduce that $X^{(\alpha, 3)} + D_t(\tau)\mathfrak{Z} + D_x(\xi)\mathfrak{Z} = 0$, and $\delta\mathfrak{Z}/\delta u = 0$; therefore,

$$D_t(N_t\mathfrak{Z}) + D_x(N_x\mathfrak{Z}) = 0, \quad (57)$$

Observing that (57) satisfies (46), so the components of the conserved vector rewritten as

$$\begin{aligned} C_t &= N_t\mathfrak{Z} = \tau\mathfrak{Z} + \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i) D_t^k \frac{\partial\mathfrak{Z}}{\partial D_t^\alpha u} \\ &\quad - (-1)^m I \left(W_i, D_t^m \frac{\partial\mathfrak{Z}}{\partial(D_t^\alpha)} \right), \\ &= \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i) D_t^k \frac{\partial\mathfrak{Z}}{\partial D_t^\alpha u} \\ &\quad - (-1)^m I \left(W_i, D_t^m \frac{\partial\mathfrak{Z}}{\partial(D_t^\alpha)} \right), \\ C_x &= N_x\mathfrak{Z} = \xi\mathfrak{Z} + W_i \left[\frac{\partial\mathfrak{Z}}{\partial u_x} - D_x \left(\frac{\partial\mathfrak{Z}}{\partial u_{xx}} \right) + D_{xx} \left(\frac{\partial\mathfrak{Z}}{\partial u_{xxx}} \right) \right] \\ &\quad + D_x(W_i) \left[\frac{\partial\mathfrak{Z}}{\partial u_{xx}} - D_x \left(\frac{\partial\mathfrak{Z}}{\partial u_{xxx}} \right) \right] + D_{xx}(W_i) \frac{\partial\mathfrak{Z}}{\partial u_{xxx}}, \\ &= W_i \left[\frac{\partial\mathfrak{Z}}{\partial u_x} - D_x \left(\frac{\partial\mathfrak{Z}}{\partial u_{xx}} \right) + D_{xx} \left(\frac{\partial\mathfrak{Z}}{\partial u_{xxx}} \right) \right] \\ &\quad + D_x(W_i) \left[\frac{\partial\mathfrak{Z}}{\partial u_{xx}} - D_x \left(\frac{\partial\mathfrak{Z}}{\partial u_{xxx}} \right) \right] + D_{xx}(W_i) \frac{\partial\mathfrak{Z}}{\partial u_{xxx}}. \end{aligned} \quad (58)$$

Now, by using the definitions and the results described above, we can construct the conservation laws of Eq. (3). According to the previous section, the time-fractional generalized KdV-Like equation accepts two infinitesimal generators defined in Section 3 by

$$X_1 = \frac{\partial}{\partial x}, X_2 = 3t \frac{\partial}{\partial t} + \alpha x \frac{\partial}{\partial x} - 2\alpha u \frac{\partial}{\partial u}. \quad (59)$$

The characteristic functions corresponding of each generator are presented by

$$W_1 = -u_x, W_2 = -2\alpha u - 3tu_t - \alpha xu_x.$$

$$C_t^1 = \theta D_t^{\alpha-1}(-u_x) + I(-u_x, \theta_t),$$

$$\begin{aligned} C_x^1 &= -u_x \left[\left(3(1-\delta)u + \frac{u_{xx}}{u} \right) \theta - D_x \left(\frac{u_x}{u} \theta \right) + D_{xx}(-\delta\theta) \right] \\ &\quad - D_x(u_x) \left[\frac{u_x}{u} \theta - D_x(-\delta\theta) \right], \end{aligned}$$

$$+ D_{xx}(u_x)(\delta\theta),$$

$$\begin{aligned} &= \delta\theta_{xx}u_x - 3(1-\delta)\theta uu_x + \theta_x \frac{u_x^2}{u} - \theta \frac{u_x^3}{u^2} - \delta\theta_x u_{xx} \\ &\quad - \theta \frac{u_x u_{xx}}{u} + \delta\theta u_{xxx}. \end{aligned}$$

$$C_t^2 = \theta D_t^{\alpha-1}(-2\alpha u - 3tu_t - \alpha xu_x) + I(-2\alpha u - 3tu_t - \alpha xu_x, \theta_t),$$

$$\begin{aligned} C_x^2 &= -(2\alpha u + 3tu_t + \alpha xu_x) \\ &\quad \cdot \left[\left(3(1-\delta)u + \frac{u_{xx}}{u} \right) \theta - D_x \left(\frac{u_x}{u} \theta \right) + D_{xx}(-\delta\theta) \right] \\ &\quad - D_x(2\alpha u + 3tu_t + \alpha xu_x) \left[\frac{u_x}{u} \theta - D_x(-\delta\theta) \right] \\ &\quad + D_{xx}(2\alpha u + 3tu_t + \alpha xu_x)(\delta\theta), \end{aligned}$$

$$\begin{aligned} &= -(2\alpha u + 3tu_t + \alpha xu_x) \left[3(1-\delta)\theta u - \theta_x \frac{u_x}{u} + \theta \frac{u_x^2}{u^2} - \delta\theta_{xx} \right] \\ &\quad - (3\alpha u_x + \alpha xu_{xx} + 3tu_{xt}) \left(\theta \frac{u_x}{u} + \delta\theta_x \right) \\ &\quad + (4\alpha u_{xx} + \alpha xu_{xxx} + 3tu_{xxt})(\delta\theta). \end{aligned}$$

(60)

5. Power Series Solution

In this section, an additional exact solution is extracted by applying the power series method; also, the convergence of power series solutions is proved.

Firstly, let us use the special fractional complex transformation introduced by

$$u(x, t) = u(\zeta), \zeta = ax - \frac{bt^\alpha}{\Gamma(1+\alpha)}, \quad (61)$$

where a and b are two arbitrary constants, so substituting (61) into Eq. (3), we get

$$-bu_\zeta + \left(3(1-\delta)u + (\delta+1)a^2 \frac{u_{\zeta\zeta}}{u} \right) au_\zeta - a^3 \delta u_{\zeta\zeta\zeta} = 0. \quad (62)$$

Hence, (62) leads to

$$-bu_\zeta u + \left(3(1-\delta)u^2 + (\delta+1)a^2 u_{\zeta\zeta} \right) au_\zeta - a^3 \delta uu_{\zeta\zeta\zeta} = 0. \quad (63)$$

Now, we integrate (63) with respect to ζ ; we obtain

$$-\frac{1}{2}bu^2 + a(1-\delta)u^3 + \left(\delta + \frac{1}{2} \right) a^3 u_\zeta^2 - a^3 \delta uu_{\zeta\zeta} + c = 0, \quad (64)$$

where c is an integration constant. Now, we seek a solution of (64) in power series form

$$u(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \sum_{n=0}^{\infty} a_n \left(ax - \frac{bt^\alpha}{\Gamma(1+\alpha)} \right)^n. \quad (65)$$

We have

$$u_\zeta = \sum_{n=0}^{\infty} (n+1)a_{n+1}\zeta^n, \quad (66)$$

$$u_{\zeta\zeta} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\zeta^n. \quad (67)$$

Substituting (65), (66), and (67) into (64), we obtain

$$\begin{aligned} & -\frac{1}{2}b \left(\sum_{n=0}^{\infty} a_n \zeta^n \right)^2 + a(1-\delta) \left(\sum_{n=0}^{\infty} a_n \zeta^n \right)^3 + \left(\delta + \frac{1}{2} \right) a^3 \\ & \cdot \left(\sum_{n=0}^{\infty} (n+1)a_{n+1}\zeta^n \right)^2 - \delta a^3 \left(\sum_{n=0}^{\infty} a_n \zeta^n \right) \\ & \cdot \left(\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\zeta^n \right) + c = 0. \end{aligned} \quad (68)$$

Therefore,

$$\begin{aligned} & -\frac{1}{2}b \sum_{n=0}^{\infty} \sum_{i=0}^n a_{n-i}a_i \zeta^n + a(1-\delta) \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^i a_{n-i}a_{i-j}a_j \zeta^n \\ & + \left(\delta + \frac{1}{2} \right) a^3 \sum_{n=0}^{\infty} \sum_{i=0}^n (n-i+1)(i+1)a_{n-i+1}a_{i+1} \zeta^n \\ & - \delta a^3 \sum_{n=0}^{\infty} \sum_{i=0}^n (n-i+2)(n-i+1)a_{n-i+2}a_i \zeta^n + c = 0. \end{aligned} \quad (69)$$

Comparing the coefficient of ζ , for $n=0$, we obtain

$$a_2 = \frac{-ba_0^2 + 2a(1-\delta)a_0^3 + (2\delta+1)a^3a_1^2 + 2c}{4\delta a^3 a_0}. \quad (70)$$

The second case is when $n \geq 0$

$$\begin{aligned} a_{n+2} = & \frac{1}{\delta a^3 a_0 (n+2)(n+1)} \left[-\frac{1}{2}b \sum_{i=0}^n a_{n-i}a_i \right. \\ & + a(1-\delta) \sum_{i=0}^n \sum_{j=0}^i a_{n-i}a_{i-j}a_j + \left(\delta + \frac{1}{2} \right) a^3 \sum_{i=0}^n \\ & \cdot (n-i+1)(i+1)a_{n-i+1}a_{i+1} \left. \right], n=1, 2, \dots \end{aligned} \quad (71)$$

Hence, the power series solution of Eq. (3) becomes

$$\begin{aligned} u(\zeta) = & a_0 + a_1\zeta + a_2\zeta^2 + \sum_{n=1}^{\infty} a_{n+2}\zeta^{n+2}, = a_0 + a_1 \left(ax - \frac{bt^\alpha}{\Gamma(1+\alpha)} \right) \\ & + \left(\frac{-ba_0^2 + 2a(1-\delta)a_0^3 + (2\delta+1)a^3a_1^2 + 2c}{4\delta a^3 a_0} \right) \\ & \cdot \left(ax - \frac{bt^\alpha}{\Gamma(1+\alpha)} \right)^2 + \sum_{n=1}^{\infty} \left[\frac{1}{\delta a^3 a_0 (n+2)(n+1)} \right. \\ & \cdot \left[-\frac{1}{2}b \sum_{i=0}^n a_{n-i}a_i + a(1-\delta) \sum_{i=0}^n \sum_{j=0}^i a_{n-i}a_{i-j}a_j \right. \\ & + \left(\delta + \frac{1}{2} \right) a^3 \sum_{i=0}^n (n-i+1)(i+1)a_{n-i+1}a_{i+1} \left. \right] \\ & \times \left(ax - \frac{bt^\alpha}{\Gamma(1+\alpha)} \right)^{n+2}, \end{aligned} \quad (72)$$

where a_0 and a_1 are the arbitrary constants, and by using (70) and (71), all the coefficients of sequence $\{a_n\}_3^\infty$ can be calculated. Now, we can prove the convergence of the power series solution.

Observing that

$$|a_{n+2}| \leq C \left[\sum_{i=0}^n a_{n-i}a_i + \sum_{i=0}^n \sum_{j=0}^i a_{n-i}a_{i-j}a_j + \sum_{i=0}^n (n-i+1)(i+1)a_{n-i+1}a_{i+1} \right], \quad (73)$$

where $C = \max \{ |-b/2\delta a^3 a_0|, |a(1-\delta)/\delta a^3 a_0|, |(2\delta+1)a^3/2\delta a^3 a_0| \}$. Now, by using another power series introduced by

$$M = M(\zeta) = \sum_{n=0}^{\infty} m_n \zeta^n, \quad (74)$$

with

$$m_0 = |a_0|, m_1 = |a_1|, m_2 = |a_2|$$

$$\begin{aligned} m_{n+1} = & C \left[\sum_{i=0}^n |a_{n-i}||a_i| + \sum_{i=0}^n \sum_{j=0}^i |a_{n-i}||a_{i-j}||a_j| \right. \\ & + \left. \sum_{i=0}^n |a_{n-i+1}||a_{i+1}| \right], n=1, 2, \dots, \end{aligned} \quad (75)$$

we can see easily that $|a_n| \leq m_n$, for $n=0, 1, 2, \dots$; then, $M(\zeta) = \sum_{n=0}^{\infty} m_n \zeta^n$ is a majorant series of (65). It remains to show that $M(\zeta)$ has a positive radius of convergence.

$$\begin{aligned} M(\zeta) = & m_0 + m_1\zeta + m_2\zeta^2 + C \left[\sum_{n=1}^{\infty} \sum_{i=1}^n m_{n-i}m_i \zeta^{n+2} \right. \\ & + \left. \sum_{n=1}^{\infty} \sum_{i=0}^n \sum_{j=0}^i m_{n-i}m_{i-j}m_j \zeta^{n+2} + \sum_{n=1}^{\infty} \sum_{i=1}^n m_{n-i+1}m_{i+1} \zeta^{n+2} \right], \end{aligned}$$

FIGURE 2: The solution $u_2(x, t)$ for Eq. (3) when $a_0 = 2, a_1 = 3, \delta = 3, c = 1, a = 2, b = 5$.

FIGURE 3: The solution $u_2(x, t)$ for Eq.(3) when $a_0 = 2, a_1 = 3, \delta = 3, c = 1, a = 2, b = 5$.

$$\begin{aligned}
 &= m_0 + m_1 \zeta + m_2 \zeta^2 + C \left[(M^2 - m_0^2) \zeta^2 + (M^3 - m_0^3) \zeta^2 \right. \\
 &\quad \left. + (M - m_0)^2 - m_1^2 \zeta^2 \right].
 \end{aligned}
 \tag{76}$$

After that, we define an implicit functional equation with respect to the independent variable ζ by the following form

$$\begin{aligned}
 R(\zeta, r) = r - m_0 - m_1 \zeta - m_2 \zeta^2 - C \left[(M^2 - m_0^2) \zeta^2 \right. \\
 \left. + (M^3 - m_0^3) \zeta^2 + (M - m_0)^2 - m_1^2 \zeta^2 \right].
 \end{aligned}
 \tag{77}$$

Then, from (77), it is clear that $R(\zeta, r)$ is analytic in the neighborhood of $(0, m_0)$, with

$$R(0, m_0) = 0, R'_m(0, m_0) = 1 \neq 0.
 \tag{78}$$

With the aid of the implicit function theorem given in [50, 51], $M(\zeta)$ is analytic in a neighborhood of $(0, m_0)$ with a positive radius; therefore, the power series $M = M(\zeta) = \sum_{n=0}^{\infty} m_n \zeta^n$ converges in a neighborhood of $(0, m_0)$; consequently, (65) is convergent in a neighborhood of $(0, m_0)$, so the proof is complete.

Finally, the graphics of the exact power series solutions (72) are plotted in Figures 2 and 3 by choosing the appropriate parameters for different values of α .

Case 1: $\alpha = 0.25$.

Case 2: $\alpha = 0.75$.

6. Conclusion

In this paper, Lie point symmetry and similarity reduction of generalized KdV-Like equation are investigated by using the Lie symmetry analysis. With the aid of similarity transformation, the equation is reduced into a FODE with Erdélyi-Kober the fractional differential operator. The CLs are also constructed by using Ibragimov's approach. Finally, by means of the power series method, other kinds of solutions are presented; moreover, we investigated the convergence analysis for the obtained explicit solution, and we presented some of the 3D graphics of power series solutions. Our outcomes show that the fractional-Lie symmetry analysis approach and the power series method provide the useful and powerful mathematical tools to study other FDEs in mathematical physic and engineering. No more of that, the Lie analysis of FODEs which are related to the Erdélyi-Kober is not yet explored in the area of fractional calculus. Hence, it will be eligible as a future subject works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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