Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6630457, 10 pages
https://doi.org/10.1155/2021/6630457

Research Article

Combined Effects in Singular Elliptic Problems in Punctured Domain

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Received 24 October 2020; Accepted 13 January 2021; Published 30 January 2021

Academic Editor: Stanislav Hencl

The paper deals with nonlinear elliptic differential equations subject to some boundary value conditions in a regular bounded punctured domain. By means of properties of slowly regularly varying functions at zero and the Schauder fixed-point theorem, we establish the existence of a positive continuous solution for the suggested problem. Global estimates on such solution, which could blow up at the origin, are also obtained.

1. Introduction and Statement of Main Results

Consider the problem

$$\begin{cases} -\Delta v(z) = p_1(z) v^{\alpha_1} + p_2(z) v^{\alpha_2} & \text{in } \Omega \setminus \{0\}, \\ v > 0 & \text{in } \Omega \setminus \{0\}, \\ \lim_{|z| \longrightarrow 0} |z|^{N-2} v(z) = 0, & \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^N (N \geq 3)$ containing $0, \alpha_1, \alpha_2 < 1$ and $p_1, p_2 \in C^+(\Omega \setminus \{0\})$ satisfying some adequate conditions. In this paper, we are interested in the study of existence and global asymptotic behavior of positive solutions for problem (1). In particular, as it will be seen, the solution may blow up at the origin. The main feature of this paper consists in the presence of the combined effects of singular and sublinear terms in the nonlinearity. Our approach combines properties of Karamata class and Kato class (see [1–4]) with the Schauder fixed-point theorem.

In literature, many researches have studied similar problems for both bounded and unbounded domains (see, for example, [5–13] and the references therein).

In [12], Shi and Yao studied the problem

$$\begin{cases} -\Delta v = \lambda v^r + b(z) v^{-\alpha} & \text{in } D, \\ v > 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \quad (2)$$

where $D \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain in $\mathbb{R}^N (N \geq 2)$, $r, \alpha \in (0, 1)$, and λ is a real parameter. The function b is required to be in $C^{2,\gamma}(\bar{D}), \gamma \in (0, 1)$, with $b(z) \neq 0$ in $\bar{D}$. Under various assumptions on $b(z)$, they have proved some existence results for λ belonging to a certain range.

If $b \equiv 1$, Colcite and Palmieri [14] showed that (2) has at least one solution provided that $\lambda \geq 0$ and $r \in (0, 1)$. However, if $r \geq 1$, they have also proved that there exists $\lambda^* > 0$ such that (1.2) has a solution for $\lambda \in [0, \lambda^*)$, and no solutions exist if $\lambda > \lambda^*$.

In [11], by combining monotonicity arguments with variational techniques, Radulescu and Repovš studied the competition between convex and concave nonlinearity and variable potentials. They have proved that problem (2), with $-1 < \alpha < 0$ and $r > 1$, has a solution, provided that $\lambda > 0$ is small enough.

In [7], Cîrstea et al. investigated the following bifurcation problem:

$$\begin{cases} -\Delta v = \lambda f(v) + b(z)g(v) & \text{in } D, \\ v > 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{3}$$

where $D \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain, $\lambda \in \mathbb{R}$,b is a nonnegative Hölder function, and f is positive, continuous, and nondecreasing function on $[0, \infty)$ such that the mapping $f(s)/s$ is nonincreasing in $(0, \infty)$. The nonnegative, continuous nonlinearity g is assumed to fulfill the hypotheses:

g is nonincreasing on $(0, \infty)$ with $\lim_{s \longrightarrow 0} g(s) = +\infty$.

There exist $C_0, \eta_0 > 0$ and $\alpha \in (0,1)$ so that $g(s) \leq C_0 s^{-\alpha}$, $\forall s \in (0, \eta_0)$.

They have proved the existence of a unique solution to this problem.

If $\lambda = 0$, problem (3) becomes

$$\begin{cases} -\Delta v = b(z)g(v) & \text{in } D, \\ v > 0 & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \tag{4}$$

Many physical phenomena can be described by this kind of problems, known as the Lane–Emden–Fowler equation, see, for example, [7–9, 11, 15] and the references therein.

The study of such problems has been attracted by many researches (see, for instance, [2, 16–25] and the references therein).

In [18], Ghergu et al. considered the semilinear elliptic equation

$$-\Delta v = v^{\alpha} |\log v|^{\beta} \text{ in } B \setminus \{0\}, \tag{5}$$

where B is the the unit open ball in $\mathbb{R}^N$, with $N \geq 3$, $N/N-2 < \alpha < N+2/N-2$, and $-\infty < \beta < \infty$. They have proved that the nonnegative solution $v \in C^2(B \setminus \{0\})$ of the above equation either has a removable singularity at the origin or it behaves like

$$v(z) = A(1 + o(1))|z|^{-\frac{2}{\alpha-1}} \left(\log \frac{1}{|z|} \right)^{-\frac{\beta}{\alpha-1}} \text{as } z \longrightarrow 0, \tag{6}$$

with $A = [(2/\alpha - 1)^{1-\beta}(N - 2 - 2/\alpha - 1)]^{1/\alpha - 1}$.

In [25], Zhao et al. investigated the problem

$$\begin{cases} \Delta v + \frac{1}{2} v \Delta v^2 - b_- |v|^{r-2} v + b_+ |v|^{s-2} v = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{7}$$

where $D \subset \mathbb{R}^N$,$N \geq 3$, is a bounded smooth domain, $r > 4$, $4 < s < 4N/N-2$, and $b_{\pm} \in C^+(\bar{D})$.

Using variational techniques, they have proved the existence of infinitely many solutions.

Quoirin and Umezu [20] dealt with the following concave-convex problems under Neumann boundary conditions

$$\begin{cases} -\Delta v = |v|^{r-2} v & \text{in } D, \\ \dfrac{\partial v}{\partial \mathbf{n}} = \lambda b(z) |v|^{s-2} v & \text{on } \partial D, \end{cases} \tag{8}$$

where D is a bounded and smooth domain of $\mathbb{R}^N (N \geq 2)$, $1 < s < 2 < r$,$\lambda > 0$, and $r \in C^{1+\alpha}(\partial D)$ for some $\alpha \in (0,1)$.

They have proved that $\int_{\partial D} b < 0$ is a necessary and sufficient condition for the existence of nontrivial nonnegative solutions of this problem.

In our analysis, we shall extensively use the class of slowly regularly varying functions at zero introduced by Karamata in [26] as follows.

Definition 1. A measurable function $\mathscr{M} : (0, \eta) \longrightarrow (0,\infty)$, $\eta > 0$, is said to be slowly varying at zero if $\mathscr{M}(s)$ is represented in the form

$$\mathscr{M}(s) := c \exp \left(\int_s^{\eta} \frac{w(r)}{r} dr \right), \tag{9}$$

where $c > 0$ and $w \in C([0, \eta])$ with $w(0) = 0$.

The set of slowly varying at zero (called also Karamata class) is denoted by $\mathscr{K}$. It is clear that $\mathscr{M}$ belongs to the class $\mathscr{K}$ if and only if $\mathscr{M}$ is a positive function in $C^1((0, \eta))$, for some $\eta > 0$, with

$$\lim_{s \longrightarrow 0^+} \frac{s \mathscr{M}'(s)}{\mathscr{M}(s)} = 0. \tag{10}$$

Typical examples of slowly varying functions at zero, also used as weight functions (see, [27, 28]), are

$$\prod_{j=1}^{n} \left(\log_j \left(\frac{\rho}{s} \right) \right)^{\zeta_j}, \tag{11}$$

where ρ is a positive real number, $\zeta_k \in \mathbb{R}$ and $\log_j s = \log \circ \log \circ \cdots \circ \log (s)$ (j-times). For more examples, we refer the reader to [1, 3, 4].

The Karamata class has been frequently used in describing the asymptotic analysis of solutions (see, for examples, [29–33] and the subsequent papers [5, 6, 15, 34–44]).

To simplify our statements in this paper, we need some notations.

(i) $d := \mathrm{diam}(\Omega)$ and $\delta_\Omega(z) = d(z, \partial\Omega)$ denote the Euclidean distance from z to $\partial\Omega$

(ii) η denotes a positive real number such that $\eta > d$

(iii) For $\alpha < 1, \nu \le N + (2-N)\alpha, \lambda \le 2$, and $L, \mathscr{L} \in \mathscr{K}$ defined on $(0, \eta)$, such that

$$\int_0^\eta r^{N+(2-N)\alpha-\nu-1} L(r)dr < \infty \text{ and} \int_0^\eta r^{1-\lambda}\mathscr{L}(r)dr < \infty \quad (12)$$

define the functions $\Phi_{L,\nu,\alpha}(s)$ and $\Psi_{\mathscr{L},\lambda,\alpha}(s)$, for $s \in (0, \eta)$ by

$$\Phi_{L,\nu,\alpha}(s) = \begin{cases} 1, & \text{if } \nu < 2, \\ \left(\int_s^\eta \frac{L(r)}{r}dr\right)^{\frac{1}{1-\alpha}}, & \text{if } \nu = 2, \\ (L(s))^{\frac{1}{1-\alpha}}, & \text{if } 2 < \nu < N + (2-N)\alpha, \\ \left(\int_0^s \frac{L(r)}{r}dr\right)^{\frac{1}{1-\alpha}}, & \text{if } \nu = N + (2-N)\alpha, \end{cases}$$

$$\Psi_{\mathscr{L},\lambda,\alpha}(s) = \begin{cases} 1, & \text{if } \lambda < 1 + \alpha, \\ \left(\int_s^\eta \frac{\mathscr{L}(r)}{r}dr\right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda = 1 + \alpha, \\ (\mathscr{L}(s))^{\frac{1}{1-\alpha}}, & \text{if } 1 + \alpha < \lambda < 2, \\ \left(\int_0^s \frac{\mathscr{L}(r)}{r}dr\right)^{\frac{1}{1-\alpha}}, & \text{if } \lambda = 2. \end{cases} \quad (13)$$

(iv) $\mathscr{B}(\Omega)$ (resp., $\mathscr{B}^+(\Omega)$) denotes the collection of all (resp., nonnegative) Borel measurable functions in Ω

(v) For $f, g \in \mathscr{B}^+(\Omega)$, we say $f \approx g$ in Ω, if there exists $c > 0$ such that $(1/c)f(z) \le g(z) \le cf(z)$, for all $z \in \Omega$

$$C_0(\bar{\Omega}) = \left\{ \nu \in C(\bar{\Omega}) : \lim_{z \longrightarrow \xi \in \partial\Omega} \nu(z) = 0 \right\} \quad (14)$$

(vi) G_Ω denotes Green's function of the Laplace operator in Ω with Dirichlet conditions

(vii) For $f \in \mathscr{B}^+(\Omega)$, we set

$$Vf(z) = \int_\Omega G_\Omega(z, y) f(y) dy \quad (15)$$

(viii) From ([45], Lemma 9), we know that for any function $f \in \mathscr{B}^+(\Omega)$ such that $f \in L^1_{loc}(\Omega)$ and $Vf \in L^1_{loc}(\Omega)$, we have

$$-\Delta(Vf) = f, \text{ in } \Omega \text{ (in the distributional sense)} \quad (16)$$

(ix) The letter c will denote a positive constant which may vary from line to line

In [46], the authors investigated the problem

$$\begin{cases} -\Delta \nu(z) = b(z)\nu^\alpha(z), z \in \Omega \setminus \{0\}, \\ \nu > 0, \text{ in } \Omega \setminus \{0\}, \\ \lim_{|z| \longrightarrow 0} |z|^{N-2}\nu(z) = 0, \\ \nu(z) = 0, z \in \partial\Omega, \end{cases} \quad (17)$$

where $\alpha < 1$ and b are a positive continuous function in $\Omega \setminus \{0\}$ satisfying

$$b(z) \approx |z|^{-\nu} L(|z|)(\delta_\Omega(z))^{-\lambda}\mathscr{L}(\delta_\Omega(z)), \text{for } z \in \Omega \setminus \{0\}, \quad (18)$$

where $\alpha < 1, \nu \le N + (2-N)\alpha, \lambda \le 2$, and $L, \mathscr{L} \in \mathscr{K}$ defined on $(0, \eta)$ such that (12) is fulfilled.

They have proved that problem (17) has at least one positive continuous solution ν on $\bar{\Omega} \setminus \{0\}$ satisfying

$$\nu(z) \approx |z|^{\min\left(0, \frac{2-\nu}{1-\alpha}\right)} \Phi_{L,\nu,\alpha}(|z|)(\delta_\Omega(z))^{\min\left(1, \frac{2-\lambda}{1-\alpha}\right)} \Psi_{\mathscr{L},\lambda,\alpha}(\delta_\Omega(z)). \quad (19)$$

In this paper, we aim at generalizing the results obtained in [46] to problem (1).

To this end, we make the following hypothesis:

(H) For $i \in \{1, 2\}, p_i$ is a positive continuous functions in $\Omega \setminus \{0\}$ satisfying

$$p_i(z) \approx |z|^{-\nu_i} L_i(|z|)(\delta_\Omega(z))^{-\lambda_i}\mathscr{L}_i(\delta_\Omega(z)), \quad (20)$$

where $\alpha_i < 1, \nu_i \le N + (2-N)\alpha_i, \lambda_i \le 2$, and $L_i, \mathscr{L}_i \in \mathscr{K}$ defined on $(0, \eta)$ such that

$$\int_0^\eta r^{N+(2-N)\alpha_i-\nu_i-1} L_i(r)dr < \infty \text{ and} \int_0^\eta r^{1-\lambda_i}\mathscr{L}_i(r)dr < \infty. \quad (21)$$

We may assume that

$$\frac{2-\nu_1}{1-\alpha_1} \le \frac{2-\nu_2}{1-\alpha_2} \text{ and } \frac{2-\lambda_1}{1-\alpha_1} \le \frac{2-\lambda_2}{1-\alpha_2}. \quad (22)$$

For $i \in \{1, 2\}$, we denote by

$$\sigma_i := \min\left(0, \frac{2-\nu_i}{1-\alpha_i}\right) \text{ and } \beta_i := \min\left(1, \frac{2-\lambda_i}{1-\alpha_i}\right). \quad (23)$$

Define the function θ on $\bar{\Omega} \setminus \{0\}$ as follows.

If $\sigma_1 < \sigma_2$ and $\beta_1 < \beta_2$, then

$$\theta(z) = |z|^{\sigma_1} \Phi_{L_1,\nu_1,\alpha_1}(|z|)(\delta_\Omega(z))^{\beta_1} \Psi_{\mathscr{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \quad (24)$$

If $\sigma_1 < \sigma_2$ and $\beta_1 = \beta_2$, then

$$\theta(z) = |z|^{\sigma_1}\Phi_{L_1,\nu_1,\alpha_1}(|z|)(\delta_\Omega(z))^{\beta_1}\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)). \quad (25)$$

If $\sigma_1 = \sigma_2$ and $\beta_1 < \beta_2$, then

$$\theta(z) = |z|^{\sigma_1}\left(\Phi_{L_1,\nu_1,\alpha_1}(|z|) + \Phi_{L_2,\nu_2,\alpha_2}(|z|)\right)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \quad (26)$$

If $\sigma_1 = \sigma_2$ and $\beta_1 = \beta_2$, then

$$\begin{aligned}\theta(z) = |z|^{\sigma_1}&\left(\Phi_{L_1,\nu_1,\alpha_1}(|z|) + \Phi_{L_2,\nu_2,\alpha_2}(|z|)\right)\\ &\times(\delta_\Omega(z))^{\beta_1}\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)).\end{aligned} \quad (27)$$

By using properties of slowly regularly varying functions at zero, we prove the following results.

Theorem 2. *Under hypothesis (H), we have for $z \in \bar{\Omega} \setminus \{0\}$,*

$$V(p)(z) \approx \theta(z), \quad (28)$$

where $p(z) \coloneqq p_1(z)\theta^{\alpha_1}(z) + p_2(z)\theta^{\alpha_2}(z)$.

Theorem 3. *Assume that hypothesis (H) is satisfied. Then, problem (1) admits a solution v on $C^+(\bar{\Omega} \setminus \{0\})$ satisfying*

$$v(z) \approx \theta(z). \quad (29)$$

Remark 4.

(i) Theorem 3 generalizes the main result in [46].

(ii) For $\nu_1 > 2$, we have $\lim_{|z| \longrightarrow 0} v(z) = \infty$

(iii) $\lim_{|z| \longrightarrow 0} |z|^{N-2}\theta(z) = 0$, where θ is the function given by (24), (25), (26), and (27).

The paper is organized as follows. In Section 2, we recall some fundamental properties of functions belonging to the Karamata class, and we prove Theorem 2. In Section 3, we prove Theorem 3 by means of the Schauder fixed-point theorem.

2. Karamata Class and Proof of Theorem 2

2.1. Basic Properties of Karamata Class

Lemma 5 (See [1, 3, 4]). *If $\mathscr{M}_1, \mathscr{M}_2 \in \mathscr{K}$, then*

(i) For every $c \geq 0$ and $r \in \mathbb{R}$,

$$\mathscr{M}_1 + c\mathscr{M}_2,\ \mathscr{M}_1\mathscr{M}_2,\ \text{and}\ \mathscr{M}_1^r\ \text{are also in}\ \mathscr{K}. \quad (30)$$

(ii) For every $\varepsilon > 0$,

$$\lim_{s \longrightarrow 0^+} s^{\varepsilon}\mathscr{M}_1(s) = 0\ \text{and}\ \lim_{s \longrightarrow 0^+} s^{-\varepsilon}\mathscr{M}_1(s) = \infty. \quad (31)$$

Lemma 6 (Asymptotic behavior, see [1, 3, 4]). *If $\mathscr{M} \in \mathscr{K}$, then*

(i) For every $\gamma > -1$, $\int_0^\eta r^\gamma \mathscr{M}(r)dr$ converges and

$$\int_0^s r^\gamma \mathscr{M}(r)dr \underset{s \longrightarrow 0^+}{\sim} \frac{s^{1+\gamma}\mathscr{M}(s)}{1+\gamma}. \quad (32)$$

(ii) For every $\gamma < -1$, $\int_0^\eta r^\gamma \mathscr{M}(r)dr$ diverges and

$$\int_s^\eta r^\gamma \mathscr{M}(r)dr \underset{s \longrightarrow 0^+}{\sim} -\frac{s^{1+\gamma}\mathscr{M}(s)}{1+\gamma}. \quad (33)$$

Lemma 7 (See [4, 6]). *Let $\mathscr{M} \in \mathscr{K}$, then*

(i) $s \mapsto \int_s^\eta (\mathscr{M}(r)/r)dr$ belongs to $\mathscr{K}$ and

$$\lim_{s \longrightarrow 0^+} \frac{\mathscr{M}(s)}{\int_s^\eta (\mathscr{M}(r)/r)dr} = 0. \quad (34)$$

(ii) If $\int_0^\eta (\mathscr{M}(r)/r)dr < \infty$, then $s \mapsto \int_0^s (\mathscr{M}(r)/r)dr$ belongs to $\mathscr{K}$ and

$$\lim_{s \longrightarrow 0^+} \frac{\mathscr{M}(s)}{\int_0^s (\mathscr{M}(r)/r)dr} = 0. \quad (35)$$

2.2. Proof of Theorem 2. The next Proposition, which follows from ([46], Proposition 2.12), will be used.

Proposition 8. *Let $\leq N, \zeta \leq 2$, and $\mathscr{M}_1, \mathscr{M}_2 \in \mathscr{K}$ such that*

$$\int_0^\eta r^{N-\gamma-1}\mathscr{M}_1(r)dr < \infty\ \text{and} \int_0^\eta r^{1-\zeta}\mathscr{M}_2(r)dr < \infty. \quad (36)$$

Then for $z \in \Omega \setminus \{0\}$,

$$Vh(z) \approx |z|^{\min\,(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\min\,(1,2-\zeta)}\Psi_{\mathscr{M}_2,\zeta,0}(\delta_\Omega(z)), \quad (37)$$

where $h(z) \coloneqq |z|^{-\gamma}\mathscr{M}_1(|z|)(\delta_\Omega(z))^{-\zeta}\mathscr{M}_2(\delta_\Omega(z))$.

We recall that for $i \in \{1,2\}$, $p_i \in C^+(\Omega \setminus \{0\})$ satisfying

$$p_i(z) \approx |z|^{-\nu_i}L_i(|z|)(\delta_\Omega(z))^{-\lambda_i}\mathscr{L}_i(\delta_\Omega(z)), \quad (38)$$

where $\alpha_i < 1, \nu_i \le N + (2-N)\alpha_i, \lambda_i \le 2$, and $L_i, \mathscr{L}_i \in \mathscr{K}$ defined on $(0,\eta)$ such that (21) is fulfilled.

We aim at proving that on $\Omega \setminus \{0\}$,

$$V(p)(z) \approx \theta(z), \tag{39}$$

where $p(z) := p_1(z)\theta^{\alpha_1}(z) + p_2(z)\theta^{\alpha_2}(z)$ and the function θ given by (24), (25), (26), and (27).

Throughout the proof, we will apply Lemma 5 and Lemma 7 to verify that some functions are in $\mathscr{K}$.

We distinguish the following cases:

Case 9. If $\sigma_1 < \sigma_2$ and $\beta_1 < \beta_2$, then for $z \in \Omega \setminus \{0\}$,

$$\theta(z) = |z|^{\sigma_1}\Phi_{L_1,\nu_1,\alpha_1}(|z|)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \tag{40}$$

Therefore,

$$\begin{aligned} p(z) \approx{}& |z|^{-\nu_1+\sigma_1\alpha_1}\left(L_1\Phi^{\alpha_1}_{L_1,\nu_1,\alpha_1}\right)(|z|)(\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1} \\ &\cdot \left(\mathscr{L}_1\Psi^{\alpha_1}_{\mathscr{L}_1,\lambda_1,\alpha_1}\right)(\delta_\Omega(z)) + |z|^{-\nu_2+\sigma_1\alpha_2}\left(L_2\Phi^{\alpha_2}_{L_1,\nu_1,\alpha_1}\right) \\ &\cdot (|z|)(\delta_\Omega(z))^{-\lambda_2+\beta_1\alpha_2}\left(\mathscr{L}_2\Psi^{\alpha_2}_{\mathscr{L}_1,\lambda_1,\alpha_1}\right)(\delta_\Omega(z)). \end{aligned} \tag{41}$$

Since $\sigma_1 < \sigma_2$ and $\beta_1 < \beta_2$, we deduce by Lemma 5 that

$$\begin{aligned} p(z) \approx{}& |z|^{-\nu_1+\sigma_1\alpha_1}\left(L_1\Phi^{\alpha_1}_{L_1,\nu_1,\alpha_1}\right)(|z|)(\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1} \\ &\cdot \left(\mathscr{L}_1\Psi^{\alpha_1}_{\mathscr{L}_1,\lambda_1,\alpha_1}\right)(\delta_\Omega(z)). \end{aligned} \tag{42}$$

Using Lemma 6 (i), (21), and Proposition 8 with $\gamma = \nu_1 - \sigma_1\alpha_1, \zeta = \lambda_1 - \beta_1\alpha_1, \mathscr{M}_1 = L_1\Phi^{\alpha_1}_{L_1,\nu_1,\alpha_1}$, and $\mathscr{M}_2 = \mathscr{L}_1\Psi^{\alpha_1}_{\mathscr{L}_1,\lambda_1,\alpha_1}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vp(z) \approx |z|^{\min(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta)}\Psi_{\mathscr{M}_2,\zeta,0}(\delta_\Omega(z)). \tag{43}$$

Since $\min(0, 2-\gamma) = \sigma_1$ and $\min(1, 2-\zeta) = \beta_1$, we deduce that

$$Vp(z) \approx |z|^{\sigma_1}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathscr{M}_2,\zeta,0}(\delta_\Omega(z)). \tag{44}$$

Now, by using a simple computation, we obtain for $z \in \Omega \setminus \{0\}$,

$$\Phi_{\mathscr{M}_1,\gamma,0}(|z|) \approx \Phi_{L_1,\nu_1,\alpha_1}(|z|) \text{ and } \Psi_{\mathscr{M}_2,\zeta,0}(\delta_\Omega(z)) \approx \Psi_{\mathscr{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \tag{45}$$

Combining this fact with (44), we obtain for $z \in \Omega \setminus \{0\}$,

$$Vp(z) \approx \theta(z). \tag{46}$$

Case 10. If $\sigma_1 < \sigma_2$ and $\beta_1 = \beta_2$, then for $z \in \Omega \setminus \{0\}$,

$$\theta(z) = |z|^{\sigma_1}\Phi_{L_1,\nu_1,\alpha_1}(|z|)(\delta_\Omega(z))^{\beta_1}\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)). \tag{47}$$

In this case,

$$\begin{aligned} p(z) \approx{}& |z|^{-\nu_1+\sigma_1\alpha_1}\left(L_1\Phi^{\alpha_1}_{L_1,\lambda_1,\alpha_1}\right)(|z|) \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1} \\ &\cdot \left[\mathscr{L}_1.\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_1}\right](\delta_\Omega(z)) + |z|^{-\nu_2+\sigma_1\alpha_2} \\ &\cdot \left(L_2\Phi^{\alpha_2}_{L_1,\lambda_1,\alpha_1}\right)(|z|) \times (\delta_\Omega(z))^{-\lambda_2+\beta_2\alpha_2} \\ &\cdot \left[\mathscr{L}_2.\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_2}\right](\delta_\Omega(z)). \end{aligned} \tag{48}$$

Since $\sigma_1 < \sigma_2$ and $\beta_1 = \beta_2$, we deduce that

$$\begin{aligned} p(z) \approx{}& |z|^{-\nu_1+\sigma_1\alpha_1}\left(L_1\Phi^{\alpha_1}_{L_1,\lambda_1,\alpha_1}\right)(|z|) \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1} \\ &\cdot \left[\mathscr{L}_1.\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_1}\right](\delta_\Omega(z)) + |z|^{-\nu_1+\sigma_1\alpha_1} \\ &\cdot \left(L_1\Phi^{\alpha_1}_{L_1,\lambda_1,\alpha_1}\right)(|z|) \times (\delta_\Omega(z))^{-\lambda_2+\beta_2\alpha_2} \\ &\cdot \left[\mathscr{L}_2.\left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_2}\right](\delta_\Omega(z)) \\ ={}& b_1(z) + b_2(z). \end{aligned} \tag{49}$$

Applying again Proposition 8 with $\gamma = \nu_1 - \sigma_1\alpha_1$, $\zeta_1 = \lambda_1 - \beta_1\alpha_1, \mathscr{M}_1 = L_1\Phi^{\alpha_1}_{L_1,\nu_1,\alpha_1}$ and $\mathscr{M}_2 = \mathscr{L}_1 \times (\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2})^{\alpha_1}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_1(z) \approx |z|^{\min(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_1)}\Psi_{\mathscr{M}_2,\zeta_1,0}(\delta_\Omega(z)). \tag{50}$$

Similarly, by using Proposition 8 with $\gamma = \nu_1 - \sigma_1\alpha_1, \zeta_2 = \lambda_2 - \beta_2\alpha_2, \mathscr{M}_1 = L_1\Phi^{\alpha_1}_{L_1,\nu_1,\alpha_1}$, and $\mathscr{M}_2 = \mathscr{L}_2 \times (\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2})^{\alpha_2}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_2(z) \approx |z|^{\min(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_2)}\Psi_{\mathscr{M}_2,\zeta_2,0}(\delta_\Omega(z)). \tag{51}$$

Hence,

$$\begin{aligned} Vp(z) \approx{}& Vb_1(z) + Vb_2(z) \\ \approx{}& |z|^{\min(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_1)}\Psi_{\mathscr{M}_2,\zeta_1,0} \\ &\cdot (\delta_\Omega(z)) + |z|^{\min(0,2-\gamma)}\Phi_{\mathscr{M}_1,\gamma,0}(|z|) \\ &\cdot (\delta_\Omega(z))^{\min(1,2-\zeta_2)}\Psi_{\mathscr{M}_2,\zeta_2,0}(\delta_\Omega(z)). \end{aligned} \tag{52}$$

Using the fact that $\min(0, 2-\gamma) = \sigma_1$ and $\min(1, 2-\zeta_1) = \beta_1 = \beta_2 = \min(1, 2-\zeta_2)$, we deduce that

$$Vp(z) \approx |z|^{\sigma_1}\Phi_{\mathcal{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathcal{M}_2,\zeta_1,0} \cdot (\delta_\Omega(z)) + |z|^{\sigma_1}\Phi_{\mathcal{M}_1,\gamma,0}(|z|)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathcal{M}_2,\zeta_2,0}(\delta_\Omega(z)). \tag{53}$$

Using Lemma 7, ([6], Lemmas 8 and 9) and a simple computation, we deduce that

$$\begin{cases} \Phi_{\mathcal{M}_1,\gamma,0}(|z|) \approx \Phi_{L_1,\nu_1,\alpha_1}(|z|) \\ \text{and} \\ \left(\Psi_{\mathcal{M}_2,\zeta_1,0} + \Psi_{\mathcal{M}_2,\zeta_2,0}\right)(\delta_\Omega(z)) \approx \left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)). \end{cases} \tag{54}$$

Hence,

$$Vp(z) \approx |z|^{\sigma_1}\Phi_{L_1,\nu_1,\alpha_1}(|z|)(\delta_\Omega(z))^{\beta_1}\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2}\right) \cdot (\delta_\Omega(z)) = \theta(z). \tag{55}$$

Case 11. If $\sigma_1 = \sigma_2$ and $\beta_1 < \beta_2$, then for $z \in \Omega \setminus \{0\}$,

$$\theta(z) = |z|^{\sigma_1}\left(\Phi_{L_1,\nu_1,\alpha_1}(|z|) + \Phi_{L_2,\nu_2,\alpha_2}(|z|)\right)(\delta_\Omega(z))^{\beta_1}\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \tag{56}$$

Therefore,

$$\begin{aligned} p(z) &\approx |z|^{-\nu_1+\sigma_1\alpha_1}\left[L_1.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_1}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1}\left[\mathcal{L}_1.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}\right)^{\alpha_1}\right](\delta_\Omega(z)) \\ &\quad + |z|^{-\nu_2+\sigma_1\alpha_2}\left[L_2.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_2}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_2+\beta_1\alpha_2}\left[\mathcal{L}_2.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}\right)^{\alpha_2}\right](\delta_\Omega(z)). \end{aligned} \tag{57}$$

Since $\sigma_1 = \sigma_2$ and $\beta_1 < \beta_2$, we deduce that

$$\begin{aligned} p(z) &\approx |z|^{-\nu_1+\sigma_1\alpha_1}\left[L_1.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_1}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1}\left[\mathcal{L}_1.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}\right)^{\alpha_1}\right](\delta_\Omega(z)) \\ &\quad + |z|^{-\nu_2+\sigma_2\alpha_2}\left[L_2.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_2}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1}\left[\mathcal{L}_1.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}\right)^{\alpha_1}\right](\delta_\Omega(z)) \\ &= b_1(z) + b_2(z). \end{aligned} \tag{58}$$

Using Proposition 8 with $\gamma_1 = \nu_1 - \sigma_1\alpha_1, \zeta = \lambda_1 - \beta_1\alpha_1$, $\mathcal{M}_1 = L_1.(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2})^{\alpha_1}$, and $\mathcal{M}_2 = \mathcal{L}_1.(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1})^{\alpha_1}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_1(z) \approx |z|^{\min(0,2-\gamma_1)}\Phi_{\mathcal{M}_1,\gamma_1,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_1)}\Psi_{\mathcal{M}_2,\zeta,0}(\delta_\Omega(z)). \tag{59}$$

Similarly, by using Proposition 8 with $\gamma_2 = \nu_2 - \sigma_2\alpha_2$, $\zeta = \lambda_1 - \beta_1\alpha_1, \mathcal{M}_1 = L_2.(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2})^{\alpha_2}$, and $\mathcal{M}_2 = \mathcal{L}_1.(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1})^{\alpha_1}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_2(z) \approx |z|^{\min(0,2-\gamma_2)}\Phi_{\mathcal{M}_1,\gamma_2,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta)}\Psi_{\mathcal{M}_2,\zeta,0}(\delta_\Omega(z)). \tag{60}$$

Since $\min(0, 2-\gamma_1) = \sigma_1 = \sigma_2 = \min(0, 2-\gamma_2)$ and $\min(1, 2-\zeta) = \beta_1$, we deduce that

$$\begin{aligned} Vp(z) &\approx Vb_1(z) + Vb_2(z) \\ &\approx |z|^{\sigma_1}\left(\Phi_{\mathcal{M}_1,\gamma_1,0}(|z|) + \Phi_{\mathcal{M}_1,\gamma_2,0}(|z|)\right) \\ &\quad \cdot (\delta_\Omega(z))^{\beta_1}\Psi_{\mathcal{M}_2,\zeta,0}(\delta_\Omega(z)). \end{aligned} \tag{61}$$

By using similar arguments as in the proof of Case 10, we deduce that

$$\begin{cases} \left(\Phi_{\mathcal{M}_1,\gamma_1,0} + \Phi_{\mathcal{M}_1,\gamma_2,0}\right)(|z|) \approx \left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)(|z|) \\ \text{and} \\ \Psi_{\mathcal{M}_2,\zeta,0}(\delta_\Omega(z)) \approx \Psi_{\mathcal{L}_1,\lambda_1,\alpha_1}(\delta_\Omega(z)). \end{cases} \tag{62}$$

Hence,

$$Vp(z) \approx \theta(z). \tag{63}$$

Case 12. If $\sigma_1 = \sigma_2$ and $\beta_1 = \beta_2$, then

$$\theta(z) = |z|^{\sigma_1}\left(\Phi_{L_1,\nu_1,\alpha_1}(|z|) + \Phi_{L_2,\nu_2,\alpha_2}(|z|)\right) \times (\delta_\Omega(z))^{\beta_1} \cdot \left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)). \tag{64}$$

In this case,

$$\begin{aligned} p(z) &\approx |z|^{-\nu_1+\sigma_1\alpha_1}\left[L_1.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_1}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_1+\beta_1\alpha_1}\left[\mathcal{L}_1.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_1}\right] \\ &\quad \cdot (\delta_\Omega(z)) + |z|^{-\nu_2+\sigma_2\alpha_2}\left[L_2.\left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)^{\alpha_2}\right](|z|) \\ &\quad \times (\delta_\Omega(z))^{-\lambda_2+\beta_2\alpha_2}\left[\mathcal{L}_2.\left(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2}\right)^{\alpha_2}\right](\delta_\Omega(z)) \\ &= b_1(z) + b_2(z). \end{aligned} \tag{65}$$

Now, by using Proposition 8 with $\gamma_1 = \nu_1 - \sigma_1\alpha_1$, $\zeta_1 = \lambda_1 - \beta_1\alpha_1, \mathcal{M}_1 = L_1.(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2})^{\alpha_1}$, and $\mathcal{M}_2 = \mathcal{L}_1 .(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2})^{\alpha_1}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_1(z) \approx |z|^{\min(0,2-\gamma_1)}\Phi_{\mathcal{M}_1,\gamma_1,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_1)}\Psi_{\mathcal{M}_2,\zeta_1,0}(\delta_\Omega(z)). \tag{66}$$

By applying again Proposition 8 with $\gamma_2 = \nu_2 - \sigma_2\alpha_2$, $\zeta_2 = \lambda_2 - \beta_2\alpha_2, \mathcal{M}_1 = L_2.(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2})^{\alpha_2}$, and $\mathcal{M}_2 = \mathcal{L}_2 .(\Psi_{\mathcal{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathcal{L}_2,\lambda_2,\alpha_2})^{\alpha_2}$, we obtain for $z \in \Omega \setminus \{0\}$,

$$Vb_2(z) \approx |z|^{\min(0,2-\gamma_2)}\Phi_{\mathscr{M}_1,\gamma_2,0}(|z|)(\delta_\Omega(z))^{\min(1,2-\zeta_2)}\Psi_{\mathscr{M}_2,\zeta_2,0}(\delta_\Omega(z)). \quad (67)$$

Since $\min(0, 2-\gamma_1) = \sigma_1 = \sigma_2 = \min(0, 2-\gamma_2)$ and $\min(1, 2-\zeta_1) = \beta_1 = \beta_2 = \min(1, 2-\zeta_2)$, we deduce that

$$\begin{aligned} Vp(z) &\approx |z|^{\sigma_1}(\delta_\Omega(z))^{\beta_1}\Big[\Phi_{\mathscr{M}_1,\gamma_1,0}(|z|)\Psi_{\mathscr{M}_2,\zeta_1,0}(\delta_\Omega(z)) \\ &\quad + \Phi_{\mathscr{M}_1,\gamma_2,0}(|z|)\Psi_{\mathscr{M}_2,\zeta_2,0}(\delta_\Omega(z))\Big] \\ &\approx |z|^{\sigma_1}(\delta_\Omega(z))^{\beta_1}\left(\Phi_{\mathscr{M}_1,\gamma_1,0} + \Phi_{\mathscr{M}_1,\gamma_2,0}\right)(|z|) \\ &\quad \cdot \left(\Psi_{\mathscr{M}_2,\zeta_1,0} + \Psi_{\mathscr{M}_2,\zeta_2,0}\right)(\delta_\Omega(z)). \end{aligned} \quad (68)$$

As in the proof of Case 11 and Case 10, we deduce that

$$\begin{cases} \left(\Phi_{\mathscr{M}_1,\gamma_1,0} + \Phi_{\mathscr{M}_1,\gamma_2,0}\right)(|z|) \approx \left(\Phi_{L_1,\nu_1,\alpha_1} + \Phi_{L_2,\nu_2,\alpha_2}\right)(|z|) \\ \text{and} \\ \left(\Psi_{\mathscr{M}_2,\zeta_1,0} + \Psi_{\mathscr{M}_2,\zeta_2,0}\right)(\delta_\Omega(z)) \approx \left(\Psi_{\mathscr{L}_1,\lambda_1,\alpha_1} + \Psi_{\mathscr{L}_2,\lambda_2,\alpha_2}\right)(\delta_\Omega(z)). \end{cases} \quad (69)$$

That is

$$Vp(z) \approx \theta(z). \quad (70)$$

The proof is completed.

3. Kato Class and Proof of Theorem 3

3.1. Kato Class $K(\Omega)$. From [24], we recall that Green's function $G_\Omega(z, y)$ satisfied

$$G_\Omega(z, y) \approx \frac{1}{|z-y|^{N-2}} \min\left\{1, \frac{\delta_\Omega(z)\delta_\Omega(y)}{|z-y|^2}\right\}, z, y \in \Omega. \quad (71)$$

Definition 13. A Borel measurable function q in Ω is in the Kato class $K(\Omega)$ if

$$\lim_{s\longrightarrow 0}\left(\sup_{z\in\Omega}\int_{\Omega\cap B(z,s)} \frac{\delta_\Omega(y)}{\delta_\Omega(z)} G_\Omega(z, y) \mid q(y) \mid dy\right). \quad (72)$$

Note that this class $K(\Omega)$ was introduced in [2] and properly contains the usual Kato class $K_N(\Omega)$ defined (see, [24, 47]) as

$$K_N(\Omega) = \left\{q \in \mathscr{B}(\Omega), \lim_{s\longrightarrow 0}\left(\sup_{z\in\Omega}\int_{\Omega\cap B(z,s)} \frac{1}{|z-y|^{N-2}} \mid q(y) \mid dy\right) = 0\right\}. \quad (73)$$

Proposition 14 (See [46], Proposition 2.8). *For $i \in \{1, 2\}$, let $\mathscr{M}_i \in \mathscr{K}$ and $\lambda_i \in \mathbb{R}$. The following properties are equivalent:*

(i) The function $z \mapsto |z|^{-\lambda_1}\mathscr{M}_1(|z|)(\delta_\Omega(z))^{-\lambda_2}\mathscr{M}_2(\delta_\Omega(z))$ is in $K(\Omega)$

(ii) $\int_0^\eta r^{1-\lambda_i}\mathscr{M}_i(r)dr < \infty$, for $i \in \{1, 2\}$

(iii) $\lambda_i < 2$ or $\lambda_i = 2$ with $\int_0^\eta (\mathscr{M}_i(r)/r)dr < \infty$, for $i \in \{1, 2\}$

As a consequence of the above Proposition, hypothesis (H), Lemma 5, and Lemma 6 (i), we obtain the following.

Corollary 15. *Assume that hypothesis (H) is satisfied and let*

$$p(z) := p_1(z)\theta^{\alpha_1}(z) + p_2(z)\theta^{\alpha_2}(z), \quad (74)$$

where the function θ is given by (24), (25), (26), and (27).

Then, the function $z \longrightarrow |z|^{N-2}p(z) \in K(\Omega)$.

Proposition 16 (See [46], Proposition 2.5). *Let $q \in K(\Omega)$ with $q \geq 0$. Then, the family*

$$\Lambda_q = \left\{z \longrightarrow |z|^{N-2}\int_\Omega G_\Omega(z, y)|y|^{2-N}f(y)dy, |f| \leq q\right\} \quad (75)$$

is uniformly bounded and equicontinuous in $\bar{\Omega}$. Consequently, Λ_q is relatively compact in $C_0(\bar{\Omega})$.

3.2. Proof of Theorem 3. Assume that hypothesis (H) is satisfied and let θ be the function given by (24), (25), (26), and (27).

By Theorem 2, there exists $M > 1$ such that for each $\Omega \setminus \{0\}$,

$$\frac{1}{M}Vp(z) \leq \theta(z) \leq MVp(z), \quad (76)$$

where $p(z)$ is given by (74).

Put $\alpha = \max(|\alpha_1|, |\alpha_2|), c_0 = M^{\alpha/1-\alpha}$, and consider the set

$$S = \left\{w \in C_0(\bar{\Omega}) : \frac{1}{c_0}|z|^{N-2}Vp(z) \leq w(z) \leq c_0|z|^{N-2}Vp(z)\right\}. \quad (77)$$

By Corollary 15 and Proposition 16, it follows that $z \longrightarrow |z|^{N-2}Vp(z) \in C_0(\bar{\Omega})$. So, $S \neq \varnothing$, closed, bounded, and convex set in $C_0(\bar{\Omega})$.

For $w \in S$, set

$$\begin{aligned} Fw(z) = |z|^{N-2}\int_\Omega G_\Omega(z, y)\Big[&p_1(y)|y|^{(2-N)\alpha_1}w^{\alpha_1}(y) \\ &+ p_2(y)|y|^{(2-N)\alpha_2}w^{\alpha_2}(y)\Big]dy. \end{aligned} \quad (78)$$

By using (76), for all $w \in S$, we have

$$\left|p_1(z)|z|^{(2-N)\alpha_1}w^{\alpha_1}(z) + p_2(z)|z|^{(2-N)\alpha_2}w^{\alpha_2}(z)\right| \leq c_0|z|^{2-N}q(z), \quad (79)$$

where $q(z) = |z|^{N-2}p(z)$, which belongs to $K(\Omega)$ by Corollary 15. Hence, by Proposition 16, the family of functions $\{z \longrightarrow Fw(z), w \in S\}$ is relatively compact in $C_0(\bar{\Omega})$.

Next, we aim at proving that $F(S) \subset S$.

Indeed, by using (76), for all $w \in S$, we have

$$\begin{aligned} Fw(z) &= |z|^{N-2}\int_{\Omega} G_{\Omega}(z,y)\left[p_1(y)|y|^{(2-N)\alpha_1}w^{\alpha_1}(y)\right. \\ &\quad \left.+ p_2(y)|y|^{(2-N)\alpha_2}w^{\alpha_2}(y)\right]dy \\ &\le |z|^{N-2}\int_{\Omega} G_{\Omega}(z,y)[p_1(y)c_0^{\alpha}M^{\alpha}\theta^{\alpha_1}(y) \\ &\quad + p_2(y)c_0^{\alpha}M^{\alpha}\theta^{\alpha_2}(y)]dy = c_0|z|^{N-2}Vp(z). \end{aligned} \tag{80}$$

On the other hand,

$$\begin{aligned} Fw(z) &= |z|^{N-2}\int_{\Omega} G_{\Omega}(z,y)\left[p_1(y)|y|^{(2-N)\alpha_1}w^{\alpha_1}(y)\right. \\ &\quad \left.+ p_2(y)|y|^{(2-N)\alpha_2}w^{\alpha_2}(y)\right]dy \\ &\ge |z|^{N-2}\int_{\Omega} G_{\Omega}(z,y)[p_1(y)c_0^{-\alpha}M^{-\alpha}\theta^{\alpha_1}(y) \\ &\quad + p_2(y)c_0^{-\alpha}M^{-\alpha}\theta^{\alpha_2}(y)]dy = \frac{1}{c_0}|z|^{N-2}Vp(z). \end{aligned} \tag{81}$$

Hence, $F(S) \subset S$.

Next, we shall prove the continuity of F in the supremum norm. Let $(w_k)_k$ be a sequence in S which converges to w in S. For each $z \in \Omega$, we have

$$\begin{aligned} |Fw_k(z) - Fw(z)| &\le \int_{\Omega} G_{\Omega}(z,y)\left[p_1(y)|y|^{(2-N)\alpha_1}\left|w_k^{\alpha_1}(y) - w^{\alpha_1}(y)\right|\right. \\ &\quad \left.+ p_2(y)|y|^{(2-N)\alpha_2}\left|w_k^{\alpha_2}(y) - w^{\alpha 2}(y)\right|\right]dy. \end{aligned} \tag{82}$$

Since from (79), we have

$$\begin{aligned} &\left|p_1(y)|y|^{(2-N)\alpha_1}\left|w_k^{\alpha_1}(y) - w^{\alpha_1}(y)\right|\right. \\ &\quad \left.+ p_2(y)|y|^{(2-N)\alpha_2}\left|w_k^{\alpha_2}(y) - w^{\alpha 2}(y)\right|\right| \le c_0|y|^{2-N}q(y), \end{aligned} \tag{83}$$

and we deduce by Proposition 16 and the dominated convergence theorem that for all $z \in \Omega$, $\lim_{k\longrightarrow\infty} Fw_k(z) = Fw(z)$.

Since $F(S)$ is relatively compact in $C_0(\bar{\Omega})$, then $\lim_{k\longrightarrow\infty} F w_k(z) = Fw(z)$ uniformly. Thus, we have proved that F is a compact mapping from S to itself.

So, by the Schauder fixed-point theorem, there exists a function $w \in S$ such that

$$\begin{aligned} w(z) &= |z|^{N-2}\int_{\Omega} G_{\Omega}(z,y)\left[p_1(y)|y|^{(2-N)\alpha_1}w^{\alpha_1}(y)\right. \\ &\quad \left.+ p_2(y)|y|^{(2-N)\alpha_2}w^{\alpha_2}(y)\right]dy. \end{aligned} \tag{84}$$

Put $v(z) = |z|^{2-N}w(z)$. Then, $v \in C(\bar{\Omega} \setminus \{0\})$ and satisfies the equation

$$v(z) = V(p_1v^{\alpha_1} + p_2v^{\alpha_2})(z), \text{for } z \in \bar{\Omega} \setminus \{0\}. \tag{85}$$

Since the function $y \mapsto \psi(y) \coloneqq p_1(y)v^{\alpha_1}(y) + p_2(y)v^{\alpha_2}(y) \in L^1_{loc}(\Omega \setminus \{0\})$ and from (85) the function $z \mapsto V(\psi)(z) \in L^1_{loc}(\Omega \setminus \{0\})$, we deduce by (16) that

$$-\Delta v(z) = p_1(z)v^{\alpha_1}(z) + p_2(z)v^{\alpha_2}(z), z \in \Omega \setminus \{0\}. \tag{86}$$

By (76), we deduce that

$$M^{\frac{-1}{1-\alpha}}\theta(z) \le v(z) \le M^{\frac{1}{1-\alpha}}\theta(z). \tag{87}$$

Using (87), Remark 4 (iii), and the fact that $w \in S$, we obtain

$$\lim_{|z|\longrightarrow 0} |z|^{N-2}v(z) = 0 \text{ and } \lim_{z\longrightarrow\partial\Omega} v(z) = 0. \tag{88}$$

So, v is a solution of problem (1) satisfying (29).

Example 17. Let $\alpha_1 < 0 < \alpha_2 < 1$. Consider $p_1, p_2 \in C(\Omega \setminus \{0\})$ such that

$$p_1(z) \approx |z|^{-\nu_1}\left(\log\left(\frac{4d}{|z|}\right)\right)^{-\alpha}(\delta_{\Omega}(z))^{-2}\left(\log\left(\frac{4d}{\delta_{\Omega}(z)}\right)\right)^{-2}, \tag{89}$$

$$p_2(z) \approx |z|^{-\nu_2}\left(\log\left(\frac{4d}{|z|}\right)\right)^{-\alpha}(\delta_{\Omega}(z))^{-\lambda_2}\left(\log\left(\frac{4d}{\delta_{\Omega}(z)}\right)\right)^{-\zeta}, \tag{90}$$

where $\nu_i < N + (2-N)\alpha_i$ with $2 - \nu_1/1 - \alpha_1 \le 2 - \nu_2/1 - \alpha_2$, $\lambda_2 < 2$,$\alpha \in (0,1)$, and $\zeta \in \mathbb{R}$. Then, by Theorem 3, problem (1) has at least one positive solution $v \in C(\bar{\Omega} \setminus \{0\})$ satisfying the following estimates:

(i) If $2 < \nu_1 < N + (2-N)\alpha_1$, then for $z \in \Omega \setminus \{0\}$,

$$v(z) \approx |z|^{\frac{2-\nu_1}{1-\alpha_1}}\left(\log\left(\frac{4d}{|z|}\right)\right)^{\frac{-\alpha}{1-\alpha_1}}\left(\log\left(\frac{4d}{\delta_{\Omega}(z)}\right)\right)^{\frac{-1}{1-\alpha_1}} \tag{91}$$

(ii) If $\nu_1 = \nu_2 = 2$, then for $z \in \Omega \setminus \{0\}$,

$$v(z) \approx \left(\log\left(\frac{4d}{|z|}\right)\right)^{\frac{1-\alpha}{1-\alpha_2}}\left(\log\left(\frac{4d}{\delta_{\Omega}(z)}\right)\right)^{\frac{-1}{1-\alpha_1}} \tag{92}$$

(iii) If $\nu_1 = 2$ and $\nu_2 < 2$, then for $z \in \Omega \setminus \{0\}$,

$$v(z) \approx \left(\log\left(\frac{4d}{|z|}\right)\right)^{\frac{1-\alpha}{1-\alpha_1}}\left(\log\left(\frac{4d}{\delta_{\Omega}(z)}\right)\right)^{\frac{-1}{1-\alpha}} \tag{93}$$

(iv) If $\nu_1 < 1$, then for $z \in \Omega \setminus \{0\}$,

$$\nu(z) \approx \left(\log\left(\frac{4d}{\delta_\Omega(z)}\right)\right)^{\frac{-1}{1-\alpha}} \tag{94}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research, King Saud University for funding this research group No. (RG-1435-043). The authors would like to thank the referees for their careful reading of the paper.

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