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Research Article

Quantum Integral Inequalities with Respect to Raina's Function via Coordinated Generalized Ψ-Convex Functions with Applications

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In accordance with the quantum calculus, we introduced the two variable forms of Hermite-Hadamard- ($\mathscr{H}\mathscr{H}$-) type inequality over finite rectangles for generalized Ψ-convex functions. This novel framework is the convolution of quantum calculus, convexity, and special functions. Taking into account the $\widehat{q}_1\widehat{q}_2$-integral identity, we demonstrate the novel generalizations of the $\mathscr{H}\mathscr{H}$-type inequality for $\widehat{q}_1\widehat{q}_2$-differentiable function by acquainting Raina's functions. Additionally, we present a different approach that can be used to characterize $\mathscr{H}\mathscr{H}$-type variants with respect to Raina's function of coordinated generalized Ψ-convex functions within the quantum techniques. This new study has the ability to generate certain novel bounds and some well-known consequences in the relative literature. As application viewpoint, the proposed study for changing parametric values associated with Raina's functions exhibits interesting results in order to show the applicability and supremacy of the obtained results. It is expected that this method which is very useful, accurate, and versatile will open a new venue for the real-world phenomena of special relativity and quantum theory.

1. Introduction

Recently, a nonrestricted analysis is recognized as quantum calculus (in short, $\widehat{q}$-calculus) and has initiated numerous $\widehat{q}$-mathematical formulation as $\widehat{q} \mapsto 1^-$. In 1707–1783, Euler proposed $\widehat{q}$-calculus theory. Accordingly, Jackson [1] explored the investigation of $\widehat{q}$-integrals efficiently. The previously mentioned outcomes prompted an escalated presentation on quantum theory in the 20th Century. As an application perspective, the concept of $\widehat{q}$-calculus has been potentially utilized in quantum mechanics, special relativity theory, anomalous diffusion equations, orthogonal polynomials, fractional calculus, and henceforth. In [2, 3], authors contemplated the $\widehat{q}$-derivatives on finite intervals of real line and amplified several new generalizations of classical convexity, $\widehat{q}$-version of Grüss, $\widehat{q}$-Cebyšev's, and $\widehat{q}$-Pólya-Szegö type inequalities. Over the most recent couple of years, the subject of $\widehat{q}$-theory has become a fascinating theme for several researchers, and new developments have been investigated in the relative literature (see [4–6]).

Within the framework of $\widehat{q}$-calculus, mechanothermodynamics, translimiting states, analysis, and generalization of experimental data, several special approaches are being developed to assess the quantum calculus in terms of a generalized energy states (see [7, 8]).

Convex functions have potential applications in many intriguing and captivating fields of research and furthermore played a remarkable role in numerous areas, such as coding theory, optimization, physics, information theory, engineering, and inequality theory. Several new classes of classical convexity have been proposed in the literature (see [9–14]). Mathematical inequalities are viewed as the prominent framework for assembling the qualitative and quantitative characterization in the area of applied analysis. A persistent development of intrigue has emerged to address the prerequisites of issue for rich utilization of these variants. Numerous generalizations were investigated by several scientists who thus utilized different procedures for introducing and proposing these bounds [15–17]. Additionally, many authors demonstrated various forms of inequalities such as Ostrowski, Lyenger, Opial, Hardy, and Olsen, and the most distinguished one is the Hermite-Hadamard inequality. Here, we intend to find the novel version of $\mathscr{H}\mathscr{H}$-type inequality in the frame of $\widehat{q}_1\widehat{q}_2$-integral on coordinated generalized Ψ-convex functions that correlates with Raina's function. Also, we shall represent the application of our findings in the Mittag-Leffler and hypergeometric functions which show the applicability of the suggested scheme.

Let $\mathscr{G} : \mathscr{I} \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a convex function such that $\varphi_1 < \varphi_2$. Then,

$$\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2}\right) \leq \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} \mathscr{G}(z)dz \leq \frac{\mathscr{G}(\varphi_1)+\mathscr{G}(\varphi_2)}{2}. \quad (1)$$

The inequality (1) is a well-known paramount in related literature and plays a pivotal role in optimization, coding, and fractional calculus theory [18, 19].

In [20], Dragomir proposed the two-variable version of the $\mathscr{H}\mathscr{H}$-type inequality for convex functions as follows:

Theorem 1. (see [20]). *Let $\mathscr{G} : \Delta \mapsto \mathbb{R}$ be the coordinated convex on Δ. Then, the following inequalities hold:*

$$\begin{aligned}\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2},\frac{\phi_1+\phi_2}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}\left(\mu,\frac{\phi_1+\phi_2}{2}\right)d\mu\right.\\ &\quad\left.+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}\left(\frac{\varphi_1+\varphi_2}{2},\nu\right)d\nu\right]\\ &\leq \frac{1}{(\varphi_2-\varphi_1)(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_2}\int_{\phi_1}^{\phi_2}\mathscr{G}(\mu,\nu)d\mu d\nu\\ &\leq \frac{1}{4}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_1)d\mu+\frac{1}{\varphi_2-\varphi_1}\right.\\ &\quad\cdot\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_2)d\mu+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_1,\nu)d\nu\\ &\quad\left.+\frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_2,\nu)d\nu\right]\\ &\leq \frac{\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_1,\phi_2)+\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{4}.\end{aligned} \quad (2)$$

In [21], Kunt et al. established the $\widehat{q}$-$\mathscr{H}\mathscr{H}$-type inequality for functions of two variables utilizing convexity on rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Let $\mathscr{G} : \Delta = [\varphi_1, \varphi_2] \times [\phi_1, \phi_2] \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$ be convex on the coordinates on Δ with $0 < \widehat{q}_1, \widehat{q}_2 < 1$ and $\varphi_1 < \varphi_2$, $\phi_1 < \phi_2$. Then, one has the following inequalities:*

$$\begin{aligned}\mathscr{G}\left(\frac{\widehat{q}_1\varphi_1+\varphi_2}{\widehat{q}_1+1},\frac{\widehat{q}_2\phi_1+\phi_2}{\widehat{q}_2+1}\right) &\leq \frac{1}{2}\left[\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}\mathscr{G}\left(\mu,\frac{\widehat{q}_2\phi_1+\phi_2}{\widehat{q}_2+1}\right)_{\varphi_1}d_{\widehat{q}_1}\mu+\frac{1}{\phi_2-\phi_1}\right.\\ &\quad\left.\cdot\int_{\phi_1}^{\phi_2}\mathscr{G}\left(\frac{\widehat{q}_1\varphi_1+\varphi_2}{\widehat{q}_1+1},\nu\right)_{\phi_1}d_{\widehat{q}_2}\nu\right]\\ &\leq \frac{1}{(\varphi_2-\varphi_1)(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_2}\int_{\phi_1}^{\phi_2}\mathscr{G}(\mu,\nu)_{\varphi_1}d_{\widehat{q}_1}\mu_{\phi_1}d_{\widehat{q}_2}\nu\\ &\leq \frac{1}{2}\left[\frac{\widehat{q}_2}{(1+q2)(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right.\\ &\quad+\frac{\widehat{q}_2}{(1+\widehat{q}_2)(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_2}\mathscr{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\\ &\quad+\frac{\widehat{q}_1}{(1+\widehat{q}_1)(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\\ &\quad\left.+\frac{\widehat{q}_1}{(1+\widehat{q}_1)(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_2}\mathscr{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right]\\ &\leq \frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\widehat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\widehat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.\end{aligned} \quad (3)$$

For many useful consequences on the coordinates on rectangle from the plane $\mathbb{R}^2$ with the various sorts of variants for mappings that hold numerous types of convex mappings, see [22–24] and the references cited therein.

Owing to the above-mentioned work, this research is aimed at exploring the novel generalizations of $\mathscr{H}\mathscr{H}$-type inequalities on the coordinates by the use of generalized Ψ-convex functions which are elaborated. An auxiliary identity is derived with respect to the $\widehat{q}_1\widehat{q}_2$-derivative by the correlation of Raina's function. Considering this new approach, we derive certain novel quantum bounds of $\mathscr{H}\mathscr{H}$-type variants for coordinated generalized Ψ-convex mappings. Meanwhile, we recapture remarkable cases in the relative literature. For the change of parameter in Raina's function, we generate numerous new outcomes depending on hypergeometric and Mittag-Leffler functions. This new study may stimulate further investigation in this dynamic field of inequality theory.

2. Prelude

This segment evokes certain earlier ideas and necessary details related to the notion of a coordinated generalized Ψ-convex set and coordinated generalized Ψ-convex function by considering Raina's function.

Assume that a finite interval of real numbers $\mathscr{I}$, and we say that a mapping $\mathscr{G} : \mathscr{I} \mapsto \mathbb{R}$ is known to be convex if

$$\mathscr{G}(\zeta x+(1-\zeta)y) \leq \zeta\mathscr{G}(x)+(1-\zeta)\mathscr{G}(y), x, y \in \mathscr{I}, \zeta \in [0,1]. \quad (4)$$

In [20], Dragomir introduced a new term in convexity theory, which is known as the coordinated convex function described as follows:

Definition 3. Let a mapping $\mathscr{G} : \nabla^{\sim} \to \mathbb{R}$ be said to be convex on the coordinates, for all $\zeta, \theta \in [0, 1]$ with $(x, y), (u, v) \in \tilde{\nabla}$, if the partial functions

$$\begin{aligned}\mathcal{G}(\zeta x+(1-\zeta)u,\theta y+(1-\theta)v)\\ \leq \zeta\theta\mathcal{G}(x,y)+\zeta(1-\theta)\mathcal{G}(x,v)\\ +(1-\zeta)\theta\mathcal{G}(u,y)+(1-\zeta)(1-\theta)\mathcal{G}(u,v),\end{aligned} \tag{5}$$

holds for all $\zeta,\theta\in 0,1]$ and $(x,y),(u,v)\in\tilde{\nabla}$.

In [25], Raina contemplated the subsequent class of function

$$\mathcal{F}^{\lambda}_{\gamma,\rho}(t)=\mathcal{F}^{\lambda(0),\lambda(1),\cdots}_{\gamma,\rho}(t)=\sum_{p=0}^{\infty}\frac{\lambda(p)}{\Gamma(\gamma p+\rho)}t^{p}, \tag{6}$$

where $\gamma,\rho>0,|t|<\mathbb{R}$ and

$$\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots), \tag{7}$$

is a bounded sequence of $\mathbb{R}^{+}$. Also, setting $\gamma=1$, $\rho=0$ in (6) and

$$\lambda(p)=\frac{(\vartheta_1)_p(\vartheta_2)_p}{(\vartheta_3)_p}\quad \text{for } p=0,1,2,3\cdots, \tag{8}$$

where the parameters ϑ_i, $(i=1,2,3)$ are assumed to be real or complex (provided that $\vartheta_3=0,-1,-2,\cdots$) and the symbol $(z)_p$ mentions the value

$$(z)_p=\frac{\Gamma(z+p)}{\Gamma(z)}=z(z+1)\cdots(z+p-1),\quad p=0,1,2,\cdots, \tag{9}$$

and its domain is restricted as $|t|\leq 1$ (with $t\in\mathbb{C}$), then we attain the subsequent hypergeometric function,

$$\mathcal{F}^{\lambda}_{\gamma,\rho}(t)=F(\vartheta_1;\vartheta_2;\vartheta_3;t)=\sum_{p=0}^{\infty}\frac{(\vartheta_1)_p(\vartheta_2)_p}{p!(\vartheta_3)_p}t^{p}. \tag{10}$$

Furthermore, if $\lambda=(1,1,\cdots)$ with $\gamma=\vartheta_1,(\Re(\vartheta_1)>0)$, $\lambda=1$ and its domain is restricted as $t\in\mathbb{C}$ in (6), then we attain the subsequent Mittag-Leffler function

$$E_{\vartheta_1}(t)=\sum_{p=0}^{\infty}\frac{1}{\Gamma(1+\vartheta_1 p)}t^{p}. \tag{11}$$

Next, we mention a novel concept that reunites the coordinated convex function and Raina's function as mentioned above.

Definition 4. For $\gamma,\lambda>0$ and $\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots)$ is assumed to be a bounded sequence of $\mathbb{R}^{+}$. A nonempty set $\tilde{\Delta}$ is known to be a coordinated generalized Ψ -convex set

$$\mathcal{G}\left(z+\zeta\mathcal{F}^{\lambda}_{\gamma,\rho}(x-z),w+\theta\mathcal{F}^{\lambda}_{\sigma,\rho}(y-w)\right)\in\tilde{\Delta}, \tag{12}$$

holds for all $\zeta\theta\in 0,1]$, $(x,y),(z,w)\in\tilde{\Delta}$, and $\mathcal{F}^{\lambda}_{\gamma,\rho}(.)$ denotes Raina's function.

Definition 5. For $\gamma,\lambda>0$ and $\lambda=(\lambda(0),\lambda(1),\cdots\lambda(p),\cdots)$ is assumed to be a bounded sequence of $\mathbb{R}^{+}$. A mapping $\mathcal{G}:\tilde{\Delta}\rightarrow\mathbb{R}$ is said to be a coordinated generalized Ψ -convex, if

$$\begin{aligned}\mathcal{G}\left(z+\zeta\mathcal{F}^{\lambda}_{\gamma,\rho}(x-z),w+\theta\mathcal{F}^{\lambda}_{\gamma,\rho}(y-w)\right)\\ \leq\zeta\theta\mathcal{G}(x,y)+\zeta(1-\theta)\mathcal{G}(x,w)\\ +(1-\zeta)\theta\mathcal{G}(z,y)+(1-\zeta)(1-\theta)\mathcal{G}(z,w),\end{aligned} \tag{13}$$

holds for all $\zeta,\theta\in 0,1]$ and $(x,y),(z,w)\in\tilde{\Delta}$.

Remark 6. Setting $\mathcal{F}^{\lambda}_{\gamma,\rho}(x-\varphi_1)=x-\varphi_1>0$ and $\mathcal{F}^{\lambda}_{\gamma,\rho}(y-\phi_1)=y-\phi_1>0$ in Definition 5, we get Definition 3.

Furthermore, we demonstrate some essential ideas and preliminaries in $\hat{q}$-analog for a single and two-variable senses.

Let $\mathcal{J}=[\varrho_1,\varrho_2]\subseteq\mathbb{R}$, and let $\mathcal{U}=[\varrho_1,\varrho_2]\times\varrho_3,\varrho_4]\subseteq\mathbb{R}^2$ with constants $\hat{q},\hat{q}_k\in(0,1)$, $k=1,2$.

Tariboon and Ntouyas [2, 3] studied the concept of $\hat{q}$ -derivative, $\hat{q}$-integral, and characteristics for finite interval, which has been shown as

Definition 7. Assume that a continuous mapping $\mathcal{G}:\mathcal{J}\rightarrow\mathbb{R}$ and $t\in\mathcal{J}$. Then, one has $\hat{q}$ -derivative of $\mathcal{G}$ on $\mathcal{J}$ at t which is stated as

$${}_{\varrho_1}\mathcal{D}_{\hat{q}}\mathcal{G}(t)=\frac{\mathcal{G}(t)-\mathcal{G}(qt+(1-q)\varrho_1)}{(1-q)(t-\varrho_1)},\quad t\neq\varrho_1. \tag{14}$$

Clearly, we see that

$$\lim_{t\rightarrow\varrho_1}{}_{\varrho_1}\mathcal{D}_{\hat{q}}\mathcal{G}(t)={}_{\varrho_1}\mathcal{D}_{\hat{q}}\mathcal{G}(\varrho_1). \tag{15}$$

We say that the mapping $\mathcal{G}$ is $\hat{q}$-differentiable over $\mathcal{J}$, also ${}_{\rho_1}\mathcal{D}_{\hat{q}}\mathcal{G}(t)$ exists $\forall t\in\mathcal{J}$.

Observe that if $\varrho_1=0$ in (14), then ${}_0\mathcal{D}_{\hat{q}}\mathcal{G}=\mathcal{D}_{\hat{q}}\mathcal{G}$, where $\mathcal{D}_{\hat{q}}\mathcal{G}$ is a well-defined $\hat{q}$-derivative of $\mathcal{G}(t)$, i.e, it is mentioned as

$$\mathcal{D}_{\hat{q}}\mathcal{G}(t)=\frac{\mathcal{G}(t)-\mathcal{G}(qt)}{(1-q)(t)}. \tag{16}$$

Definition 8. Assume that a continuous mapping $\mathcal{G}:\mathcal{J}\rightarrow\mathbb{R}$ is symbolized as ${}_{\varrho_1}\mathcal{D}^2_{\hat{q}}\mathcal{G}$, given that ${}_{\varrho_1}\mathcal{D}^2_{\hat{q}}\mathcal{G}$ is $\hat{q}$ -differentiable from $\mathcal{J}\rightarrow\mathbb{R}$ defined by

$${}_{\varrho_1}\mathcal{D}^2_{\hat{q}}\mathcal{G}={}_{\varrho_1}\mathcal{D}_{\hat{q}}\left({}_{\varrho_1}\mathcal{D}_{\hat{q}}\mathcal{G}\right). \tag{17}$$

Therefore, the higher order $\hat{q}$-differentiable is defined as ${}_{\varrho_1}\mathcal{D}^n_{\hat{q}}\mathcal{G}:\mathcal{J}\rightarrow\mathbb{R}$.

Definition 9. Assume that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ and the $\hat{q}$ -integral on $\mathscr{J}$ is stated as

$$\int_{\varrho_1}^{t} \mathscr{G}(z)_{\varrho_1} d_{\hat{q}} z = (1-\hat{q})(t-\varrho_1) \sum_{n=0}^{\infty} q\wedge^n \mathscr{G}(q\wedge^n t + (1-q\wedge^n)\varrho_1), \quad \forall t \in \mathscr{J}. \tag{18}$$

Next, if $\varrho_1 = 0$ in (18), then we have a new formulation of $\hat{q}$-integral, which is pointed out as

$$\int_{0}^{t} \mathscr{G}(z)_0 d_{\hat{q}} z = (1-\hat{q}) t \sum_{n=0}^{\infty} q\wedge^n \mathscr{G}(q\wedge^n t). \tag{19}$$

Theorem 10. *Assuming that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$, the following assumptions hold:*

$$\begin{gathered} {}_{\varrho_1}D_{\hat{q}} \int_{\varrho_1}^{t} G(z)_{\varrho_1} d_{\hat{q}} z = G(t), \\ \int_{\varrho_1}^{t} {}_{\varrho_1}\mathscr{D}_{\hat{q}} \mathscr{G}(z)_{\varrho_1} d_{\hat{q}} z = \mathscr{G}(t), \\ \int_{\varrho_2}^{t} {}_{\varrho_1}\mathscr{D}_{\hat{q}} \mathscr{G}(z)_{\varrho_1} d_{\hat{q}} z = \mathscr{G}(t) - \mathscr{G}(\varrho_2), \quad \varrho_2 \in (\varrho_1, t). \end{gathered} \tag{20}$$

Theorem 11. *Assuming that a continuous mapping $\mathscr{G} : \mathscr{J} \to \mathbb{R}$ and $a \in \mathbb{R}$, then the following assumptions hold:*

$$\begin{gathered} \int_{\varrho_1}^{t} [\mathscr{G}_1(z) + \mathscr{G}_2(z)]_{\varrho_1} d_{\hat{q}} z = \int_{\varrho_1}^{t} \mathscr{G}_1(z)_{\varrho_1} d_{\hat{q}} z + \int_{\varrho_1}^{t} \mathscr{G}_2(z)_{\varrho_1} d_{\hat{q}} z, \\ \int_{\varrho_1}^{t} (a\mathscr{G}_1(z))_{\varrho_1} d_{\hat{q}} z = a \int_{\varrho_1}^{t} \mathscr{G}_1(z)_{\varrho_1} d_{\hat{q}} z. \end{gathered} \tag{21}$$

In [26], Kalsoom et al. introduced the quantum integral identities in a two-variable sense as follows:

Definition 12. Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{U} \to \mathbb{R}$, then the partial $\hat{q}_1$ -derivative, $\hat{q}_2$ -derivative, and $\hat{q}_1\hat{q}_2$ -derivative at $(z, w) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$ are, respectively, stated as

$$\begin{aligned} \frac{{}_{\varrho_1}\partial_{\hat{q}_1} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\hat{q}_1} z} &= \frac{\mathscr{G}(z,w) - \mathscr{G}(\hat{q}_1 z + (1-\hat{q}_1)\varrho_1, w)}{(1-\hat{q}_1)(z-\varrho_1)}, \quad z \neq \varrho_1, \\ \frac{{}_{\varrho_3}\partial_{\hat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_3}\partial_{\hat{q}_2} w} &= \frac{\mathscr{G}(z,w) - \mathscr{G}(z, \hat{q}_2 w + (1-\hat{q}_2)\varrho_3)}{(1-\hat{q}_2)(w-\varrho_3)}, \quad w \neq \varrho_3, \\ \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\hat{q}_1,\hat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\hat{q}_1} z \, {}_{\varrho_3}\partial_{\hat{q}_2} w} &= \frac{1}{(1-\hat{q}_1)(1-\hat{q}_2)(z-\varrho_1)(w-\varrho_3)} \\ &\quad \times [\mathscr{G}(\hat{q}_1 z + (1-\hat{q}_1)\varrho_1, \hat{q}_2 w + (1-\hat{q}_2)\varrho_3) \\ &\quad - \mathscr{G}(\hat{q}_1 z + (1-\hat{q}_1)\varrho_1, w) \\ &\quad - \mathscr{G}(z, \hat{q}_2 w + (1-\hat{q}_2)\varrho_3) + \mathscr{G}(z,w)], \quad z \neq \varrho_1, w \neq \varrho_3. \end{aligned} \tag{22}$$

Definition 13. Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{U} \to \mathbb{R}$, then the definite $\hat{q}_1\hat{q}_2$ -integral on $[\varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$ is stated as

$$\int_{\varrho_3}^{t} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w = (1-\hat{q}_1)(1-\hat{q}_2)(t-\varrho_1)(t_1-\varrho_3) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{q}_1^n \hat{q}_2^m \mathscr{G}(\hat{q}_1^n t + (1-\hat{q}_1^n)\varrho_1, \hat{q}_2^m t_1 + (1-\hat{q}_2^m)\varrho_3), \tag{23}$$

for $(t, t_1) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$.

Theorem 14. *Consider a continuous mapping in two-variable sense$\mathscr{G} : \mathscr{B} \to \mathbb{R}$, then the following assumptions hold:*

$$\begin{gathered} \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\hat{q}_1,\hat{q}_2}}{{}_{\varrho_1}\partial_{\hat{q}_1} t \, {}_{\varrho_3}\partial_{\hat{q}_2} t_1} \int_{\varrho_4}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w = \mathscr{G}(t, t_1), \\ \int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\hat{q}_1,\hat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\hat{q}_1} z_{\varrho_3} \partial_{\hat{q}_2} w} \varrho_1 d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w = \mathscr{G}(t, t_1), \\ \int_{t_2}^{t_1} \int_{y_1}^{t} \frac{{}_{\varrho_1,\varrho_3}\partial^2_{\hat{q}_1,\hat{q}_2} \mathscr{G}(z,w)}{{}_{\varrho_1}\partial_{\hat{q}_1} z_{\varrho_3} \partial_{\hat{q}_2} w} \varrho_1 d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w; \\ = \mathscr{G}(t, t_1) - \mathscr{G}(t, t_2) - \mathscr{G}(y_1, t_1) \\ + \mathscr{G}(y_1, t_2), (y_1, t_2) \in (\varrho_1, t) \times (\varrho_4, t_1). \end{gathered} \tag{24}$$

Theorem 15. *Suppose that $\mathscr{G}_1, \mathscr{G}_2 : \mathscr{U} \to \mathbb{R}$ are continuous mappings of two variables. Then, the following properties hold for $(t, t_1) \in \varrho_1, \varrho_2] \times \varrho_3, \varrho_4]$,*

$$\begin{aligned} &\int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} [\mathscr{G}_1(z,w) + \mathscr{G}_2(z,w)]_{\varrho_1} d_{\hat{q}_1} z_{\varrho_4} d_{\hat{q}_2} w \\ &\quad = \int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}_1(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w \\ &\quad + \int_{\varrho_3}^{t} \int_{\varrho_1}^{t} \mathscr{G}_2(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w, \\ &\int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} a\mathscr{G}(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w \\ &\quad = a \int_{\varrho_3}^{t_1} \int_{\varrho_1}^{t} \mathscr{G}(z,w)_{\varrho_1} d_{\hat{q}_1} z_{\varrho_3} d_{\hat{q}_2} w. \end{aligned} \tag{25}$$

3. Quantum $\mathscr{H}\mathscr{H}$-Type Inequality for Generalized Ψ-Convex on the Coordinates

This section addresses the $\widehat{q}_1$-$\mathscr{H}\mathscr{H}$-type inequality on the coordinates via generalized Ψ-convex functions.

Theorem 16. *For $\gamma, \rho > 0$ with $\lambda = (\lambda(0), \cdots, \lambda(p))$ as the bounded sequence of positive real numbers and let $\mathscr{G} : \Delta \mapsto \mathbb{R}$ be the coordinated generalized Ψ -convex and partially differentiable function on Δ° with $0 < \widehat{q}_1, \widehat{q}_2 < 1$, then the following inequalities hold:*

$$\begin{aligned}
&\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\le \frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\cdot\left(\mu, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu\\
&\quad+\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\cdot\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_2}\nu\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu\\
&\le \frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)_{\varphi_1} d_{\widehat{q}_1}\mu\right)\\
&\quad+\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_{\varphi_1} d_{\widehat{q}_1}\mu\right)\\
&\quad+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)_{\phi_1} d_{\widehat{q}_2}\mu\right)\\
&\quad+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_{\phi_1} d_{\widehat{q}_2}\nu\right)\\
&\le \frac{\widehat{q}_1\widehat{q}_2\mathscr{G}(\varphi_1,\phi_1)+\widehat{q}_1\mathscr{G}(\varphi_1,\phi_2)+\widehat{q}_2\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.
\end{aligned} \tag{26}$$

Proof. Since $\mathscr{G}$ is the coordinated generalized Ψ-convex on Δ and partially differentiable mappings on Δ°, clearly, we see that the mapping $\mathscr{G}_\mu : [\phi_1, \phi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)] \mapsto \mathbb{R}$, $\mathscr{G}_\mu(\nu) \coloneqq \mathscr{G}(\mu, \nu)$ is a generalized Ψ-convex on $[\phi_1, \phi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)]$ and a differentiable function on $(\phi_1, \phi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1))$ for all $\mu \in [\varphi_1, \varphi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)]$. Then, by using the $\widehat{q}_1$-$\mathscr{H}\mathscr{H}$-type inequality, we obtain

$$\begin{aligned}
&\mathscr{G}_\mu\left(\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}_\mu(\nu)_\mu d_{\widehat{q}_2}\nu\\
&\le \frac{\widehat{q}_2\mathscr{G}_\mu(\phi_1)+\mathscr{G}_{\phi_1}(\phi_2)}{1+\widehat{q}_2}, \left(\mu \in \left[\varphi_1, \varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\right]\right),
\end{aligned} \tag{27}$$

which can be written as

$$\begin{aligned}
&\mathscr{G}\left(\mu, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu\\
&\le \frac{\widehat{q}_2\mathscr{G}(\mu,\phi_1)+\mathscr{G}(\mu,\phi_2)}{1+\widehat{q}_2}, \left(\mu \in \left[\varphi_1, \varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\right]\right).
\end{aligned} \tag{28}$$

Applying $\widehat{q}_1$-inegration on the above inequalities over $[\varphi_1, \varphi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)]$, we have

$$\begin{aligned}
&\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu, \frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu\\
&\le \frac{1}{1+\widehat{q}_2}\left[\frac{\widehat{q}_2}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)_{\varphi_1} d_{\widehat{q}_1}\mu\right.\\
&\quad\left.+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_{\varphi_1} d_{\widehat{q}_1}\mu\right].
\end{aligned} \tag{29}$$

Adopting the same procedure for the mapping $\mathscr{G}_\nu : [\varphi_1, \varphi_1 + \mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)] \mapsto \mathbb{R}, \mathscr{G}_\nu(\mu) \coloneqq \mathscr{G}(\mu, \nu)$, we have

$$\begin{aligned}
&\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\cdot\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_1}\mu\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)_{\phi_1} d_{\widehat{q}_2}\nu_{\varphi_1} d_{\widehat{q}_1}\mu\\
&\le \frac{1}{1+\widehat{q}_1}\left[\frac{\widehat{q}_1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)_{\phi_1} d_{\widehat{q}_2}\nu\right.\\
&\quad\left.+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_{\phi_1} d_{\widehat{q}_2}\nu\right].
\end{aligned} \tag{30}$$

Adding (29) and (30), yields

$$\begin{aligned}
&\frac{1}{2\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\\
&\quad\cdot\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\quad+\frac{1}{2\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\\
&\quad\cdot\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)_{\phi_1}d_{\widehat{q}_2}\nu+\\
&\le\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\\
&\quad\cdot\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\mu,\nu)_{\phi_1}d_{\widehat{q}_2}\nu_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\le\left[\frac{\widehat{q}_2}{2(1+\widehat{q}_2)\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right.\\
&\quad+\frac{1}{2(1+\widehat{q}_2)\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\quad+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\\
&\quad\left.+\frac{1}{2(1+\widehat{q}_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right].
\end{aligned}\tag{31}$$

Also, by considering the $\widehat{q}$-$\mathcal{H}\mathcal{H}$-type inequality, we have

$$\begin{aligned}
&\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\quad\le\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1}d_{\widehat{q}_1}\mu,
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\quad\le\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)_{\phi_1}d_{\widehat{q}_2}\nu.
\end{aligned}\tag{33}$$

Adding the inequalities (32) and (33), we have the following inequality:

$$\begin{aligned}
&\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)\\
&\quad\le\frac{1}{2\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1}d_{\widehat{q}_1}\mu\\
&\quad+\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right)_{\phi_1}d_{\widehat{q}_2}\nu.
\end{aligned}\tag{34}$$

Consequently, we have

$$\begin{aligned}
&\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right)\\
&\quad\le\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{\widehat{q}_1\mathcal{G}(\varphi_1,\phi_1)+\mathcal{G}(\varphi_2,\phi_1)}{1+\widehat{q}_1}\right),\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\\
&\quad\cdot\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\right)\\
&\quad\le\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{\widehat{q}_1\mathcal{G}(\varphi_1,\phi_2)+\mathcal{G}(\varphi_2,\phi_2)}{1+\widehat{q}_1}\right),\\
&\quad\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\mu\right)\\
&\quad\le\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{\widehat{q}_2\mathcal{G}(\varphi_1,\phi_1)+\mathcal{G}(\varphi_1,\phi_2)}{1+\widehat{q}_2}\right),\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\\
&\quad\cdot\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right)\\
&\quad\le\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{\widehat{q}_2\mathcal{G}(\varphi_2,\phi_1)+\mathcal{G}(\varphi_2,\phi_2)}{1+\widehat{q}_2}\right).
\end{aligned}\tag{35}$$

Adding the above inequalities yields

$$\begin{aligned}
&\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right)\\
&\quad+\frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\right)\\
&\quad+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\mu\right)\\
&\quad+\frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right)\\
&\le\frac{\widehat{q}_1\widehat{q}_2\mathcal{G}(\varphi_1,\phi_1)+\widehat{q}_1\mathcal{G}(\varphi_1,\phi_2)+\widehat{q}_2\mathcal{G}(\varphi_2,\phi_1)+\mathcal{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.
\end{aligned}\tag{36}$$

A combination of (31), (34), and (36) gives (36). This completes the proof.

Corollary 17. *In Theorem 16, if we choose $\hat{q}_1, \hat{q}_2 \mapsto 1^-$, we have the following new double inequality:*

$$\begin{aligned}
&\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\\
&\le \frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu\\
&\quad+\frac{1}{2\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1},\nu\right)d\nu\\
&\le \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,d\nu\, d\mu\\
&\le \frac{1}{4}\left\{\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_1)\,{}_{\varphi_1}d_{\hat{q}_1}\mu\right)\right.\\
&\quad+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_{\varphi_1}d_{\hat{q}_1}\mu\right)\\
&\quad+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_1,\nu)\,{}_{\phi_1}d_{\hat{q}_2}\mu\right)\\
&\quad\left.+\left(\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)\,{}_{\phi_1}d_{\hat{q}_2}\nu\right)\right\}\\
&\le \frac{\mathscr{G}(\varphi_1,\phi_1)+\mathscr{G}(\varphi_1,\phi_2)+\mathscr{G}(\varphi_2,\phi_1)+\mathscr{G}(\varphi_2,\phi_2)}{4}.
\end{aligned} \tag{37}$$

Remark 18. In Theorem 16,

(i) letting $\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\hat{q}_1,\hat{q}_2\mapsto 1^-$, then we attain Theorem 1 in [20]

(ii) letting $\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 4 in [21]

4. Quantum Integral Identity for Coordinated Generalized Ψ-Convex Functions

The following identity plays a significant role in inaugurating the main consequences of this paper. The identification is expressed as follows.

Lemma 19. *For $\gamma,\rho>0$ with $\lambda=(\lambda(0),\cdots,\lambda(p))$ as the bounded sequence of positive real numbers and let a twice partially $\hat{q}_1\hat{q}_2$ -differentiable mapping $\mathscr{G}:\Delta\mapsto\mathbb{R}$ be defined on Δ° (the interior of Δ). If the second-order partial $\hat{q}_1\hat{q}_2$ -derivatives are continuous and integrable over Δ with $0<\hat{q}_1,\hat{q}_2<1$, then the following equality holds:*

$$\begin{aligned}
&Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\\
&\coloneqq \mathscr{G}\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1},\frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right)\\
&\quad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{(\hat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\hat{q}_2}\right){}_{\varphi_1}d_{\hat{q}_1}\mu\\
&\quad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\hat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\hat{q}_1},\nu\right){}_{\phi_1}d_{\hat{q}_2}\nu\\
&\quad+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,{}_{\phi_1}d_{\hat{q}_2}\nu\,{}_{\varphi_1}d_{\hat{q}_1}\mu,
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
&Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\\
&\coloneqq \hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta,
\end{aligned}$$

$$\mathscr{A}(\zeta,\theta)=\begin{cases}
\zeta\theta, & (\zeta,\theta)\in\left[0,\dfrac{1}{1+\hat{q}_1}\right]\times\left[0,\dfrac{1}{1+\hat{q}_2}\right],\\[2mm]
\zeta\left(\theta-\dfrac{1}{\hat{q}_2}\right), & (\zeta,\theta)\in\left[0,\dfrac{1}{1+\hat{q}_1}\right]\times\left(\dfrac{1}{1+\hat{q}_2},1\right],\\[2mm]
\theta\left(\zeta-\dfrac{1}{\hat{q}_1}\right), & (\zeta,\theta)\in\left(\dfrac{1}{1+\hat{q}_1},1\right]\times\left[0,\dfrac{1}{1+\hat{q}_2}\right],\\[2mm]
\left(\zeta-\dfrac{1}{\hat{q}_1}\right)\left(\theta-\dfrac{1}{\hat{q}_2}\right), & (\zeta,\theta)\in\left(\dfrac{1}{1+\hat{q}_1},1\right]\times\left(\dfrac{1}{1+\hat{q}_2},1\right].
\end{cases} \tag{39}$$

Proof. Consider

$$\begin{aligned}
&\hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\\
&=\hat{q}_1\hat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\times\left\{\int_0^{1/(1+\hat{q}_1)}\int_0^{1/(1+\hat{q}_2)}\zeta\theta\right.\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\\
&\quad+\int_{\frac{1}{1+\hat{q}_1}}^1\int_0^{1/(1+\hat{q}_2)}\theta\left(\zeta-\frac{1}{\hat{q}_1}\right)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\\
&\quad+\int_0^{1/(1+\hat{q}_1)}\int_{1/(1+\hat{q}_2)}^1\zeta\left(\theta-\frac{1}{\hat{q}_2}\right)\\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\\
&\quad+\int_{1/(1+\hat{q}_1)}^1\int_{1/(1+\hat{q}_2)}^1\left(\zeta-\frac{1}{\hat{q}_1}\right)\left(\theta-\frac{1}{\hat{q}_2}\right)\\
&\quad\left.\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\,{}_{\phi_1}\partial_{\hat{q}_2}\theta}\,{}_0d_{\hat{q}_2}\theta\,{}_0d_{\hat{q}_1}\zeta\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\Bigg\{\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_2)}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\int_{1/(1+\widehat{q}_1)}^{1}\int_0^{1/(1+\widehat{q}_2)}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\int_0^{1/(1+\widehat{q}_1)}\int_{1/(1+\widehat{q}_2)}^{1}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\int_{1/(1+\widehat{q}_1)}^{1}\int_{1/(1+\widehat{q}_2)}^{1}\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\frac{1}{\widehat{q}_1}\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_2)}\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\frac{1}{\widehat{q}_2}\int_{1/(1+\widehat{q}_1)}^{1}\int_0^{1/(1+\widehat{q}_2)}\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\frac{1}{\widehat{q}_2}\int_0^{1/(1+\widehat{q}_1)}\int_{1/(1+\widehat{q}_2)}^{1}\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad\cdot\frac{1}{\widehat{q}_1\widehat{q}_2}\int_{1/(1+\widehat{q}_1)}^{1}\int_{1/(1+\widehat{q}_2)1/(1+\widehat{q}_2)}^{1} \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{e}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\Bigg\} \\
&= \Bigg\{\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)}\zeta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1 \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1 \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad-\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)} \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad+\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^{\frac{1}{1+\widehat{q}_2}} \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\Bigg\}.
\end{aligned} \tag{40}
$$

In view of Definition 12 and Definition 13, we conclude the following identities with the aid of the last nine integrals appearing in the aforementioned identities as follows:

$$
\begin{aligned}
&\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1} \\
&\quad= -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)_0 d_{\widehat{q}_1}\mu \\
&\quad\quad-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_0 d_{\widehat{q}_2}\nu \\
&\quad\quad+\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \\
&\quad\quad\cdot\mathscr{G}(\mu,\nu)_{\phi_1}d_{\widehat{q}_2}\nu_{\varphi_1}d_{\widehat{q}_1}\mu,
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
&\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\theta \\
&\quad\cdot\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta \\
&\quad= -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\varphi_2,\nu)_0 d_{\widehat{q}_2}\nu,
\end{aligned} \tag{42}
$$

$$\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\zeta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}(\varphi_2,\phi_2)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}(\mu,\phi_2)\,{}_0d_{\widehat{q}_1}\nu, \tag{43}$$

$$\widehat{q}_2\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1\theta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\phi_2\right)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\nu\right){}_0d_{\widehat{q}_1}\nu, \tag{44}$$

$$\widehat{q}_1\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)}\zeta \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\varphi_2,\frac{(\widehat{q}_1+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_1}\right)-\frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)} \cdot \int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right){}_0d_{\widehat{q}_1}\mu, \tag{45}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}(\varphi_2,\phi_2), \tag{46}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^1 \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\phi_2\right), \tag{47}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^{1/(1+\widehat{q}_1)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\varphi_2,\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right), \tag{48}$$

$$\left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_2)} \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta\,{}_{\phi_1}\partial_{\widehat{q}_2}\theta}\,{}_0d_{\widehat{q}_2}\theta\,{}_0d_{\widehat{q}_1}\zeta = -\mathscr{G}\left(\frac{(\widehat{q}_1+1)\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{1+\widehat{q}_1},\frac{(\widehat{q}_2+1)\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right). \tag{49}$$

Combining (42), (43), (44), (45), (46), (47), (48), and (49), we have the identity (38). This is the proof of Lemma 19.

Corollary 20. *In Lemma 19, if we choose $\widehat{q}_1,\widehat{q}_2 \mapsto 1^-$, we have the following new identity:*

$$\Lambda(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G}) := \mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right) - \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G} \cdot \left(\mu,\frac{2\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu - \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu + \frac{1}{\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)} \cdot \int_{\phi_1}^{\phi_1+\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)\,d\nu\,d\mu, \tag{50}$$

where

$$\Lambda(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G}) := \left(\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\int_0^1\int_0^1\mathscr{A}(\zeta,\theta) \cdot \frac{\partial^2\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)}{\partial\zeta\partial\theta}\,d\theta\,d\zeta,$$

$$A(\zeta,\theta)\begin{cases}\zeta\theta, & (\zeta-\theta)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right],\\ \zeta(\theta-1), & (\zeta-\theta)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right],\\ \theta(\zeta-1), & (\zeta-\theta)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right],\\ (\zeta-1)(\theta-1), & (\zeta-\theta)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right].\end{cases} \tag{51}$$

5. Certain New $\widehat{q}_1\widehat{q}_2$-Integral Estimates for Generalized Ψ-Convex Functions

The following results exhibit some practice related to Lemma 19 on quantum calculus for generalized Ψ-convex on coordinates.

Theorem 21. *For $\gamma,\rho>0$ with $\lambda=(\lambda(0),\cdots,\lambda(p))$ as the bounded sequence of positive real numbers and let a mapping $\mathscr{G}:\Delta\mapsto\mathbb{R}$ be a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable on Δ° such that continuous partial $\widehat{q}_1\widehat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ is integrable on Δ with $0<\widehat{q}_1,\widehat{q}_2<1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on Δ for $\sigma\ge 1$, where $\sigma^{-1}+\beta^{-1}=1$. Then, the following inequality holds:*

$$\begin{aligned}\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\right| &\le \widehat{q}_1\widehat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot(\mathbb{B}_1(q\wedge_1,q\wedge_2))^{1-(1/\sigma)}\times\left[\mathbb{B}_2(q\wedge_1,q\wedge_2)\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_3(q\wedge_1,q\wedge_2)\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_4(q\wedge_1,q\wedge_2)\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_5(q\wedge_1,q\wedge_2)\\ &\cdot\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},\end{aligned} \tag{52}$$

where

$$\mathbb{B}_1(\widehat{q}_1,\widehat{q}_2):=\frac{4}{(1+q\wedge_1)^3(1+q\wedge_2)^3}, \tag{53}$$

$$\mathbb{B}_2(\widehat{q}_1,\widehat{q}_2):=\frac{9}{(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)}, \tag{54}$$

$$\mathbb{B}_3(\widehat{q}_1,\widehat{q}_2):=\frac{1+\widehat{q}_1+\widehat{q}_2+\widehat{q}_2^2-3\widehat{q}_1\widehat{q}_2-\widehat{q}_1\widehat{q}_2^3}{\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)}, \tag{55}$$

$$\mathbb{B}_4(\widehat{q}_1,\widehat{q}_2):=\frac{1+\widehat{q}_1+\widehat{q}_2+\widehat{q}_1^2+\widehat{q}_2^2-3\widehat{q}_1\widehat{q}_2-\widehat{q}_2\widehat{q}_1^3-\widehat{q}_1^3\widehat{q}_2^2+6\widehat{q}_1\widehat{q}_2^2-\widehat{q}_1^2\widehat{q}_2^3-\widehat{q}_1^3\widehat{q}_2^3+5\widehat{q}_1\widehat{q}_2^3}{\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)} \tag{56}$$

$$\begin{aligned}\mathbb{B}_5(\widehat{q}_1,\widehat{q}_2):=&\left(-2\widehat{q}_1^5-6\widehat{q}_1^4+2\widehat{q}_1^4\widehat{q}_2^3+2\widehat{q}_1^3\widehat{q}_2^4-4\widehat{q}_1^4\widehat{q}_2^2-4\widehat{q}_1^4\widehat{q}_2-2\widehat{q}_1\widehat{q}_2^5\right.\\ &-2\widehat{q}_1^5\widehat{q}_2-2\widehat{q}_1^5\widehat{q}_2^3+16\widehat{q}_1^3\widehat{q}_2^3-4\widehat{q}_1^2+2\widehat{q}_1^3\widehat{q}_2-2\widehat{q}_1^2\widehat{q}_2^5-4\widehat{q}_1^2\widehat{q}_2^4\\ &+4\widehat{q}_1^2\widehat{q}_2+6\widehat{q}_1^3+8\widehat{q}_1^2\widehat{q}_2^2+10\widehat{q}_1^3\widehat{q}_2^2+10\widehat{q}_1^2\widehat{q}_2^3-6\widehat{q}_2^4-6\widehat{q}_2^3\\ &\left.-4\widehat{q}_2^2-4\widehat{q}_1\widehat{q}_2^4+2\widehat{q}_1\widehat{q}_2^3+4\widehat{q}_1\widehat{q}_2^2+9\widehat{q}_1\widehat{q}_2-2\widehat{q}_2^5\right)\\ &/\widehat{q}_1\widehat{q}_2(1+q\wedge_1)^3(1+q\wedge_2)^3\left(1+\widehat{q}_1+\widehat{q}_1^2\right)\left(1+\widehat{q}_2+\widehat{q}_2^2\right)\end{aligned} \tag{57}$$

Proof. Taking into consideration the $\widehat{q}_1\widehat{q}_2$-integral power mean inequality, the generalized Ψ-convexity of $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ on the coordinates on Δ with the aid of Lemma 19, we have

$$\begin{aligned}&\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\right|\\ &\le\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\times\left\{\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta}\right|_0\\ &\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta}\right|\cdot d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\right\}\\ &\le\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|_0 d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1-(1/\sigma)}\\ &\times\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|\right.\\ &\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\\ &\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\cdot d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1/\sigma}\\ &=\widehat{q}_1\widehat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot\left[\int_0^{1/(1+\widehat{q}_1)}\int_0^{1/(1+\widehat{q}_1)}\zeta\theta_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta+\int_0^{1/(1+\widehat{q}_1)}\int_{1/(1+\widehat{q}_2)}^1\zeta\right.\\ &\quad\cdot\left(\frac{1}{\widehat{q}_2}-\theta\right)_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\end{aligned}$$

$$+\int_{1/(1+\widehat{q}_1)}^{1}\int_{0}^{1/(1+\widehat{q}_2)}\theta\left(\frac{1}{\widehat{q}_1}-\zeta\right)_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta$$

$$+\int_{1/(1+\widehat{q}_1)}^{1}\int_{1/(1+\widehat{q}_2)}^{1}\left(\frac{1}{\widehat{q}_1}-\zeta\right)\left(\frac{1}{\widehat{q}_2}-\theta\right)_0 d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta\Bigg]$$

$$\times\left[\int_0^{1/(1+q\wedge_1)}\int_0^{1/(1+q\wedge_1)}\zeta\theta\left[\begin{array}{c}\zeta\theta\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0\right.$$

$$\cdot d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\quad+\int_0^{1/(1+q\wedge_1)}\int_{1/(1+q\wedge_2)}^{1}\zeta\left(\frac{1}{q\wedge_2}-\theta\right)$$

$$\cdot\left[\begin{array}{c}\zeta\theta\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0$$

$$\cdot d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\quad+\int_{1/(1+q\wedge_1)}^{1}\int_0^{1/(1+q\wedge_2)}\theta\left(\frac{1}{q\wedge_1}-\zeta\right)$$

$$\cdot\left[\begin{array}{c}\zeta\theta\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0$$

$$\cdot d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\quad+\int_{1/(1+q\wedge_1)}^{1}\int_{1/(1+q\wedge_2)}^{1}\left(\frac{1}{q\wedge_1}-\zeta\right)\left(\frac{1}{q\wedge_2}-\theta\right)$$

$$\left.\times\left[\begin{array}{c}\zeta\theta\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \theta(1-\zeta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ \zeta(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\\ (1-\zeta)(1-\theta)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\end{array}\right]_0 d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right]^{1/\sigma}$$

$$=\widehat{q}_1\widehat{q}_2\left(\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\mathbb{B}_1(q\wedge_1,q\wedge_2)\right)^{1-1/\sigma}$$
$$\times\left[\mathbb{B}_2(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_3(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right.$$
$$\left.+\mathbb{B}_4(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma+\mathbb{B}_5(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma}. \tag{58}$$

This completes the proof of Theorem 21.

Corollary 22. *In Theorem 21, if we choose $\widehat{q}_1,\widehat{q}_2\mapsto 1^-$, we have the following new inequality:*

$$\left|\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\right.$$
$$-\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu,\frac{2\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu$$
$$-\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{2\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu$$
$$\left.-\frac{1}{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}\mathcal{G}(\mu,\nu)d\nu d\mu\right|$$
$$\leq\frac{\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)}{16}$$
$$\cdot\left\{\frac{\left|\partial^2\mathcal{G}(\varphi_2,\phi_2)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_1,\phi_2)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_2,\phi_1)/\partial\zeta\partial\theta\right|^\sigma+\left|\partial^2\mathcal{G}(\varphi_1,\phi_1)/\partial\zeta\partial\theta\right|^\sigma}{4}\right\}. \tag{59}$$

Remark 23. In Theorem 21,

(i) letting $\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 5 in [21]

(ii) letting $\mathcal{F}^{\lambda}_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathcal{F}^{\lambda}_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\widehat{q}_1,\widehat{q}_2\mapsto 1^-$, then we attain Corollary 1 in [21] and Theorem 4 in [27], respectively

Theorem 24. *For $\gamma,\rho>0$ with $\lambda=(\lambda(0),\cdots,\lambda(1))$ as the bounded sequence of positive real numbers and let a mapping $\mathcal{G}:\Delta\mapsto\mathbb{R}$ be a twice partially $\widehat{q}_1\widehat{q}_2$ -differentiable on Δ°(the* *interior of Δ) such that continuous partial $\hat{q}_1\hat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}/{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta$ is integrable on Δ with $0<\hat{q}_1,\hat{q}_2<1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on Δ for $\sigma>1$ where $\sigma^{-1}+\beta^{-1}=1$. Then, the following inequality holds*

$$\begin{aligned}&\left|Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\right| \le \hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\cdot\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\ &\times\left[\left(\left|{}_{\varphi_1,\varphi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_2)/{}_{\phi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\ &+q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\ &+q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\ &\left.+q\wedge_1q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)\\ &\left./((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma},\end{aligned} \tag{60}$$

where $\mathscr{A}(\zeta,\theta)$ is defined as in (38).

Proof. Taking into consideration the $\hat{q}_1\hat{q}_2$-Hölder integral inequality, the generalized Ψ-convexity of $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ on the coordinates on Δ with the aid of Lemma 19, we have

$$\begin{aligned}&\left|Y_{\hat{q}_1,\hat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathscr{G})\right| \le \hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\times\left\{\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\sigma_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta}\right|_0\right.\\ &\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta}\right|\cdot d_{\hat{q}_2}\theta_0d_{\hat{q}_1}\zeta\right\}\\ &\le\hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\\ &\times\left[\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\right.\\ &\times\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\right.\\ &\left.\left.\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}\left(\varphi_1+\zeta\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1),\phi_1+\theta\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma_0\cdot d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\sigma}\right]\\ &=\hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\ &\times\left[\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\zeta\theta_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right.\\ &+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\theta(1-\zeta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\\ &+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1\zeta(1-\theta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\\ &\left.+\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\int_0^1\int_0^1(1-\zeta)(1-\theta)_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right]^{1/\sigma}\\ &=\hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)\left(\int_0^1\int_0^1|\mathscr{A}(\zeta,\theta)|^\beta{}_0d_{q\wedge_2}\theta_0d_{q\wedge_1}\zeta\right)^{1/\beta}\\ &\times\left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma+q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\ &+q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\ &\left.+q\wedge_1q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)\\ &\left./((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma}.\end{aligned} \tag{61}$$

This completes the proof of Theorem 21.

Corollary 25. *In Theorem 21, if we choose $\hat{q}_1,\hat{q}_2\mapsto 1^-$, we have the following new inequality:*

$$\begin{aligned}&\left|\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\frac{2\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)\right.\\ &-\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\mathscr{G}\left(\mu,\frac{2\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}{2}\right)d\mu\\ &-\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}\left(\frac{2\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}{2},\nu\right)d\nu\\ &-\frac{1}{\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)}\\ &\left.\cdot\int_{\phi_1}^{\phi_1+\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)}\mathscr{G}(\mu,\nu)d\nu d\mu\right|\le\frac{\hat{q}_1\hat{q}_2\left(\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)\right)}{4(\beta+1)^{2/\beta}}\\ &\times\left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right.\\ &+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\ &+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\\ &\left.\left.+\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)/(4)\right]^{1/\sigma}.\end{aligned} \tag{62}$$

Remark 26. In Theorem 21,

(i) letting $\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$, then we attain Theorem 6 in [21]

(ii) letting $\mathscr{F}^\lambda_{\gamma,\rho}(\varphi_2-\varphi_1)=\varphi_2-\varphi_1$ and $\mathscr{F}^\lambda_{\gamma,\rho}(\phi_2-\phi_1)=\phi_2-\phi_1$ along with $\hat{q}_1\hat{q}_2\mapsto 1^-$, then we attain Theorem 3 in [27]

6. Applications

This section contains some useful utilities of our findings derived in the previous sections. For appropriate and suitable selections of parameters γ, ρ, and λ in the special functions stated in (6), (10), and (11). Taking into account Raina's function (6), we shall derive outcomes for the hypergeometric function and Mittag-Leffler function as particular cases.

6.1. Hypergeometric Function. Letting $\gamma = 1$ and $\rho = 0$, and

$$\lambda(p) = \frac{(\vartheta_1)_p(\vartheta_2)_p}{(\vartheta_3)_p}, \quad \text{for}\, p = 0, 1, 2, \cdots, \tag{63}$$

then for Theorem 16, Lemma 19, and Theorems 21–24, the following results hold.

Theorem 27. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let $\mathcal{G} : O = [\varphi_1, \varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ is the coordinated generalized Ψ -convex and partially differentiable function on O° with $0 < \hat{q}_1, \hat{q}_2 < 1$, then the following inequalities hold:*

$$\begin{aligned}
&\mathcal{G}\left(\frac{(\hat{q}_1 + 1)\varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \hat{q}_1}, \frac{(\hat{q}_2 + 1)\phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \hat{q}_2}\right) \\
&\leq \frac{1}{2\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathcal{G} \\
&\quad \cdot \left(\mu, \frac{(\hat{q}_2 + 1)\phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \hat{q}_2}\right)_{\varphi_1} d_{\hat{q}_1}\mu \\
&\quad + \frac{1}{2\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G} \\
&\quad \cdot \left(\frac{(\hat{q}_1 + 1)\varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \hat{q}_1}, \nu\right)_{\phi_1} d_{\hat{q}_2}\nu \\
&\leq \frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G}(\mu, \nu)_{\phi_1} d_{\hat{q}_2}\nu_{\varphi_1} d_{\hat{q}_1} \\
&\leq \frac{\hat{q}_2}{2(1 + \hat{q}_2)}\left(\frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathcal{G}(\mu, \phi_1)_{\varphi_1} d_{\hat{q}_1}\mu\right) \\
&\quad + \frac{\hat{q}_2}{2(1 + \hat{q}_2)}\left(\frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathcal{G}(\mu, \phi_2)_{\varphi_1} d_{\hat{q}_1}\mu\right) \\
&\quad + \frac{\hat{q}_1}{2(1 + \hat{q}_1)}\left(\frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G}(\varphi_1, \nu)_{\phi_1} d_{\hat{q}_2}\mu\right) \\
&\quad + \frac{\hat{q}_1}{2(1 + \hat{q}_1)}\left(\frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G}(\varphi_2, \nu)_{\phi_1} d_{\hat{q}_2}\nu\right) \\
&\leq \frac{\hat{q}_1\hat{q}_2\mathcal{G}(\varphi_1, \phi_1) + \hat{q}_1\mathcal{G}(\varphi_1, \phi_2) + \hat{q}_2\mathcal{G}(\varphi_2, \phi_1) + \mathcal{G}(\varphi_2, \phi_2)}{(1 + \hat{q}_1)(1 + \hat{q}_2)}.
\end{aligned} \tag{64}$$

Lemma 28. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ be the bounded sequence of positive real numbers and let a twice partially $\hat{q}_1 \hat{q}_2$ -differentiable mapping $\mathcal{G} : O = [\varphi_1, \varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ defined on O° (the interior of O). If the second-order partial $\hat{q}_1\hat{q}_2$ -derivatives are continuous and integrable over O with $0 < \hat{q}_1, \hat{q}_2 < 1$, then the following equality holds:*

$$\begin{aligned}
&\tilde{Y}_{\hat{q}_1,\hat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathcal{G}) \\
&:= \mathcal{G}\left(\frac{(\hat{q}_1 + 1)\varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \hat{q}_1}, \frac{(\hat{q}_2 + 1)\phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 -_c 1)}{1 + \hat{q}_2}\right) \\
&\quad - \frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \mathcal{G} \\
&\quad \cdot \left(\mu, \frac{(\hat{q}_2 + 1)\phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)}{1 + \hat{q}_2}\right)_{\varphi_1} d_{\hat{q}_1}\mu \quad - \frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G}\left(\frac{(\hat{q}_1 + 1)\varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)}{1 + \hat{q}_1}, \nu\right)_{\phi_1} d_{\hat{q}_2}\nu \\
&\quad + \frac{1}{\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)} \int_{\varphi_1}^{\varphi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\varphi_2-\varphi_1)} \\
&\quad \cdot \int_{\phi_1}^{\phi_1 + \mathcal{F}(\vartheta_1;\vartheta_2;\vartheta_3,\phi_2-\phi_1)} \mathcal{G}(\mu, \nu)_{\phi_1} d_{\hat{q}_2}\nu_{\varphi_1} d_{\hat{q}_1}\mu,
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
&\tilde{Y}_{\hat{q}_1,\hat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathcal{G}) := \hat{q}_1\hat{q}_2(\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\quad \cdot \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)) \times \int_0^1 \int_0^1 \mathcal{A}(\zeta, \theta) \\
&\quad \cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}(\varphi_1 + \zeta\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1), \phi_1 + \theta\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))}{{}_{\varphi_1}\partial_{\hat{q}_1}\zeta\, {}_{\phi_1}\partial_{\hat{q}_2}\theta}{}_0 \\
&\quad \cdot d_{\hat{q}_2}\theta_0 d_{\hat{q}_1}\zeta f\nu,
\end{aligned} \tag{66}$$

and $\mathcal{A}(\zeta, \theta)$ given in (38).

Theorem 29. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let a mapping $\mathcal{G} : O = [\varphi_1, \varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ be a twice partially $\hat{q}_1\hat{q}_2$ -differentiable on O° such that continuous partial $\hat{q}_1\hat{q}_2$ -derivatives ${}_{\varphi_1,\phi_1}\partial^2_{\hat{q}_1,\hat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\hat{q}_1}\zeta_{\phi_1}\partial_{\hat{q}_2}\theta$ is integrable on O with $0 < \hat{q}_1, \hat{q}_2 < 1$. If $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ is a generalized Ψ -convex on the coordinates on O for $\sigma \geq 1$ where $\sigma^{-1} + \beta^{-1} = 1$. Then, the following inequality holds:*

$$\begin{aligned}
&\left|\tilde{Y}_{\hat{q}_1,\hat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathcal{G})\right| \leq \hat{q}_1\hat{q}_2(\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\quad \cdot \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))(\mathbb{B}_1(q\wedge_1, q\wedge_2))^{1-1/\sigma} \\
&\quad \times \left[\mathbb{B}_2(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_3(q\wedge_1, q\wedge_2)\right. \\
&\quad \cdot \left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1, \phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_4(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma \\
&\quad \left. + \mathbb{B}_5(q\wedge_1, q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1, \phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},
\end{aligned} \tag{67}$$

where $\mathbb{B}_1(\hat{q}_1, \hat{q}_2), \mathbb{B}_2(\hat{q}_1, \hat{q}_2), \mathbb{B}_3(\hat{q}_1, \hat{q}_2), \mathbb{B}_4(\hat{q}_1, \hat{q}_2)$, and $\mathbb{B}_5(\hat{q}_1, \hat{q}_2)$ are given in (53), (54), (55), (56), and (57), respectively.

Theorem 30. *Suppose $\lambda = (\lambda(0), \cdots, \lambda(p))$ is the bounded sequence of positive real numbers and let a mapping $\mathcal{G} : O$*

$= [\varphi_1, \varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable on* O° *such that continuous partial* $\widehat{q}_1\widehat{q}_2$ *-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ *is integrable on* O *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on* O *for* $\sigma > 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|Y_{\widehat{q}_1,\widehat{q}_2}(\varphi_1, \varphi_2, \phi_1, \phi_2)(\mathcal{G})\right| \le \widehat{q}_1\widehat{q}_2(\mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1) \\
&\cdot \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1))\left(\int_0^1\int_0^1 |\mathcal{A}(\zeta,\theta)|^\beta {}_0d_{q\wedge_2}\theta_0 d_{q\wedge_1}\zeta\right)^{1/\beta} \\
&\times \left[\left(\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right.\right. \\
&+ q\wedge_1\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma \\
&+ q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma \\
&\left.+ q\wedge_1 q\wedge_2\left|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta\right|^\sigma\right)/ \\
&\left.((1+q\wedge_1)(1+q\wedge_2))\right]^{1/\sigma},
\end{aligned} \tag{68}$$

where $\mathcal{A}(\zeta,\theta)$ *is defined as in (38).*

6.2. Mittag-Leffler Function. Setting $\nu = (1, 1, \cdots)$ having $\gamma = \vartheta_1, \mathfrak{R}(\vartheta_1) > 0$ and $\rho = 1$, then from Theorem 16, Lemma 19, and Theorems 21–24, the following results hold.

Theorem 31. *Let* $\mathcal{G} : \mathcal{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be the coordinated generalized* Ψ*-convex and partially differentiable function on* $\mathcal{S}^\circ$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$, *then the following inequalities hold:*

$$\begin{aligned}
&\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right) \\
&\le \frac{1}{2E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathcal{G} \\
&\cdot\left(\mu, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} d_{\widehat{q}_1}\mu + \frac{1}{2E_{\vartheta_1}(\phi_2-\phi_1)} \\
&\cdot\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\phi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} d_{\widehat{q}_2}\nu \\
&\le \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G} \\
&\cdot(\mu,\nu)_{\phi_1}d_{\widehat{q}_2}\nu_{\varphi_1}d_{\widehat{q}_1}\mu \le \frac{\widehat{q}_2}{2(1+\widehat{q}_2)} \\
&\cdot\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_1)_{\varphi_1}d_{\widehat{q}_1}\mu\right) \\
&+ \frac{\widehat{q}_2}{2(1+\widehat{q}_2)}\left(\frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathcal{G}(\mu,\phi_2)_{\varphi_1}d_{\widehat{q}_1}\mu\right) \\
&+ \frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G}(\varphi_1,\nu)_{\phi_1}d_{\widehat{q}_2}\mu\right) \\
&+ \frac{\widehat{q}_1}{2(1+\widehat{q}_1)}\left(\frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G}(\varphi_2,\nu)_{\phi_1}d_{\widehat{q}_2}\nu\right) \\
&\le \frac{\widehat{q}_1\widehat{q}_2\mathcal{G}(\varphi_1,\phi_1) + \widehat{q}_1\mathcal{G}(\varphi_1,\phi_2) + \widehat{q}_2\mathcal{G}(\varphi_2,\phi_1) + \mathcal{G}(\varphi_2,\phi_2)}{(1+\widehat{q}_1)(1+\widehat{q}_2)}.
\end{aligned} \tag{69}$$

Lemma 32. *Let a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable mapping* $\mathcal{G} : \mathcal{S} = [\varphi_1, \varphi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + \mathcal{F}(\vartheta_1 ; \vartheta_2 ; \vartheta_3, \phi_2 - \phi_1)] \mapsto \mathbb{R}$ *defined on* $\mathcal{S}^\circ$*(the interior of* $\mathcal{S}$*). If the second-order partial* $\widehat{q}_1\widehat{q}_2$*-derivatives are continuous and integrable over* $\mathcal{S}$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$, *then the following equality holds:*

$$\begin{aligned}
&\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G}) \\
&\coloneqq \mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2 -_c 1)}{1+\widehat{q}_2}\right) \\
&- \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)}\mathcal{G}\left(\mu, \frac{(\widehat{q}_2+1)\phi_1 + E_{\vartheta_1}(\phi_2-\phi_1)}{1+\widehat{q}_2}\right)_{\varphi_1} \\
&\cdot d_{\widehat{q}_1}\mu - \frac{1}{E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G}\left(\frac{(\widehat{q}_1+1)\varphi_1 + E_{\vartheta_1}(\varphi_2-\varphi_1)}{1+\widehat{q}_1}, \nu\right)_{\phi_1} \\
&\cdot d_{\widehat{q}_2}\nu + \frac{1}{E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)}\int_{\varphi_1}^{\varphi_1+E_{\vartheta_1}(\varphi_2-\varphi_1)} \\
&\cdot\int_{\phi_1}^{\phi_1+E_{\vartheta_1}(\phi_2-\phi_1)}\mathcal{G}(\mu,\nu)_{\phi_1}d_{\widehat{q}_2}\nu_{\varphi_1}d_{\widehat{q}_1}\mu,
\end{aligned} \tag{70}$$

where

$$\begin{aligned}
&\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G}) \\
&\coloneqq \widehat{q}_1\widehat{q}_2\left(E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)\right) \times \int_0^1\int_0^1 \mathcal{A}(\zeta,\theta) \\
&\cdot \frac{{}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}\left(\varphi_1 + \zeta E_{\vartheta_1}(\varphi_2-\varphi_1), \phi_1 + \theta E_{\vartheta_1}(\phi_2-\phi_1)\right)}{\varphi_1\partial_{\widehat{q}_1}\zeta\,\phi_1\partial_{\widehat{q}_2}\theta}{}_0 \\
&\cdot d_{\widehat{q}_2}\theta_0 d_{\widehat{q}_1}\zeta,
\end{aligned} \tag{71}$$

and $\mathcal{A}(\zeta,\theta)$ *given in (38).*

Theorem 33. *Let a mapping* $\mathcal{G} : \mathcal{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable on* $\mathcal{S}^\circ$ *such that continuous partial* $\widehat{q}_1\widehat{q}_2$*-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathcal{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ *is integrable on* $\mathcal{S}$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on* $\mathcal{S}$ *for* $\sigma \ge 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|\widetilde{\widetilde{Y}}_{\widehat{q}_1,\widehat{q}_2}(\varphi_1,\varphi_2,\phi_1,\phi_2)(\mathcal{G})\right| \\
&\le \widehat{q}_1\widehat{q}_2\left(E_{\vartheta_1}(\varphi_2-\varphi_1)E_{\vartheta_1}(\phi_2-\phi_1)\right)(\mathbb{B}_1(q\wedge_1, q\wedge_2))^{1-1/\sigma} \\
&\times\left[\mathbb{B}_2(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_3(q\wedge_1,q\wedge_2)\right. \\
&\cdot\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_2)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma + \mathbb{B}_4(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_2,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma \\
&\left.+ \mathbb{B}_5(q\wedge_1,q\wedge_2)\left|\frac{{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathcal{G}(\varphi_1,\phi_1)}{{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta}\right|^\sigma\right]^{1/\sigma},
\end{aligned} \tag{72}$$

where $\mathbb{B}_1(\widehat{q}_1,\widehat{q}_2)$, $\mathbb{B}_2(\widehat{q}_1,\widehat{q}_2)$, $\mathbb{B}_3(\widehat{q}_1,\widehat{q}_2)$, $\mathbb{B}_4(\widehat{q}_1,\widehat{q}_2)$, *and* $\mathbb{B}_5(\widehat{q}_1,\widehat{q}_2)$ *are given in (53), (54), (55), (56), and (57), respectively.*

Theorem 34. *Let a mapping* $\mathscr{G} : \mathscr{S} = [\varphi_1, \varphi_1 + E_{\vartheta_1}(\varphi_2 - \varphi_1)] \times [\phi_1, \phi_1 + E_{\vartheta_1}(\phi_2 - \phi_1)] \mapsto \mathbb{R}$ *be a twice partially* $\widehat{q}_1\widehat{q}_2$ *-differentiable on* $\mathscr{S}^\circ$ *such that continuous partial* $\widehat{q}_1\widehat{q}_2$ *-derivatives* ${}_{\varphi_1,\phi_1}\partial^2_{\widehat{q}_1,\widehat{q}_2}\mathscr{G}/{}_{\varphi_1}\partial_{\widehat{q}_1}\zeta_{\phi_1}\partial_{\widehat{q}_2}\theta$ *is integrable on* $\mathscr{S}$ *with* $0 < \widehat{q}_1, \widehat{q}_2 < 1$. *If* $|{}_{\varphi_1,\phi_1}\partial^2_{q\wedge_1,q\wedge_2}\mathscr{G}/{}_{\varphi_1}\partial_{q\wedge_1}\zeta_{\phi_1}\partial_{q\wedge_2}\theta|^\sigma$ *is a generalized* Ψ *-convex on the coordinates on* $\mathscr{S}$ *for* $\sigma > 1$ *where* $\sigma^{-1} + \beta^{-1} = 1$. *Then, the following inequality holds:*

where $\mathscr{A}(\zeta,\theta)$ *is defined as in (38).*

7. Conclusion

The main objective of this paper will be a motivation source for future studies. An auxiliary result in $\widehat{q}_1\widehat{q}_2$-integrals has been derived. We established some new generalizations for the $\mathscr{H}\mathscr{H}$-type inequality pertaining to $\widehat{q}_1\widehat{q}_2$-differentiable mappings for generalized Ψ-convex functions on coordinates in the special Raina's function sense that correlates with the $\widehat{q}_1\widehat{q}_2$-identity. Some useful applications of our findings have been illustrated with the association of the well-known special functions (hypergeometric and Mittag-Leffler function). Moreover, our findings are essentially applicable for obtaining the solution of integral equations that interact with$n$ bodies subject to mixed boundary conditions (see [7, 8]). For further potential investigation, we left the details for futuristic research. Every aspect of the suggested scheme is versatile and simple to execute. We apprehended noteworthy special cases for varying the parametric values in the involvement of special functions. This new study is explicit and viable and can be effectively utilized in inequality theory, special relativity theory, and quantum mechanics.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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