Research Article

Existence of Multiple Solutions for Second-Order Problem with Stieltjes Integral Boundary Condition

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In this paper, we consider the existence of multiple solutions for second-order equation with Stieltjes integral boundary condition using the three-critical-point theorem and variational method. Firstly, a novel space is established and proved to be Hilbert one. Secondly, based on the above work, we obtain the existence of multiple solutions for our problem. Finally, in order to illustrate the effectiveness of our problem better, the example is listed.

1. Introduction

We are concerned with the following general second-order differential equation

$$\begin{cases} -\omega''(t) = \beta g(t, \omega(t)), \\ \omega(0) = 0, \omega(1) = \int_0^1 \omega(s) d\gamma(s), \end{cases} \quad (1)$$

where β is a positive parameter, $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $\gamma(s)$ is a nondecreasing function, $0 < \int_0^1 s d\gamma(s) < 1$, $0 < \int_0^1 s^2 d\gamma(s) < 1$.

In the past few decades, the boundary value problems have appealed to many scholars in the mathematical field. Generally speaking, the boundary value problems mostly involve in two-point [1–5], three-point [6–8], and multi-point [9–11]. Many physical phenomena were formulated as nonlocal mathematical models with integral boundary conditions [12–21], such as fractional differential equation [22–30], nonlinear singular parabolic equation [31], and general second-order equation [19, 32–40].

For the integral boundary value problems such as (1), many researchers studied mainly by the method of topologi-

cal degree. For examples, using the fixed point theorem, Ma [39] studied ordinary second-order equation as below

$$\begin{cases} \chi'' + a(t)g(\chi) = 0, 0 < t < 1, \\ \chi(0) = 0, \chi(1) = \int_{\xi}^{\eta} \gamma(z)\chi(z)dz, \end{cases} \quad (2)$$

where $0 < \xi < \eta < 1$ and $g(\chi)$ are either superlinear or sublinear and obtained the existence of positive solutions; Karakostas et al. [19] studied the existence of three positive solutions of the following problem

$$\begin{cases} \chi'' + \chi(t)g(\chi) = 0, 0 \leq t \leq 1, \\ \chi(0) = 0, \chi(1) = \int_{\xi}^{\eta} \chi(z)d\gamma(z) \end{cases} \quad (3)$$

by the theory of fixed point index on a cone; Benchohra et al. [33] investigated the following second-order equation

$$\begin{cases} \chi'' + g(t, \chi) = 0, 0 \leq t \leq 1, \\ \chi(0) = 0, \chi(1) = \int_0^1 \chi(z)\gamma(z)dz \end{cases} \quad (4)$$

via contraction principle and Leray-Schauder alternative theorem; Galvis [41] considered the nonlinear second-order problem

$$\begin{cases} \chi'' + \chi(t)g(t, \chi) = 0, 0 \leq t \leq r, \\ \chi(0) = 0, \chi(r) = \beta \int_0^a \chi(z) dz \end{cases} \quad (5)$$

by Schauder's fixed point theorem. Some other works on fractional equation with integral boundary conditions can be found in [26, 28, 29].

To the best of my knowledge, no one use the theory of critical point and variational method to deal with the existence of solution of problem (1).

In order to discuss problem (1), we introduce a new space as follows:

$$\mathcal{W}(0, 1) = \left\{ \omega \in W^{1,2}(0, 1): \omega(0) = 0, \omega(1) = \int_0^1 \omega(s) d\gamma(s) \right\}, \quad (6)$$

endowed with the norm $\|\omega\| = \left[\int_0^1 |\omega'|^2 dt \right]^{1/2}$.

The paper is organized as follows. In Section 2, we introduce some concepts related to solve problem (1) more conveniently. In Section 3, we prove that constructed space is a Hilbert. Section 4 demonstrates the existence of at least three solutions for problem (1) mostly via a three-critical-point theorem. Section 5 gives an example to explain efficacy of our method. Our method is different from those in [19, 33, 39, 41], and the nonlinear term g is neither superlinear nor sublinear in our paper. Some ideas of our proof come from [42, 43].

2. Preliminaries

Definition 1. Let $(\bar{E}, \varsigma)$ and $(\bar{E}_1, \varsigma_1)$ be metric spaces, and the operator $\tilde{\phi} : \bar{E} \longrightarrow \bar{E}_1$ meets the conditions as follows:

- (1) The mapping $\tilde{\phi}$ is surjective
- (2) $\varsigma(\omega, \nu) = \varsigma_1(\tilde{\phi}\omega, \tilde{\phi}\nu)$, for any $\omega, \nu \in \bar{E}$

Then, $(\bar{E}, \varsigma)$ is isometric isomorphic to $(\bar{E}_1, \varsigma_1)$; moreover, $\tilde{\phi}$ is named as an isometric isomorphic mapping.

Lemma 2. Let $\bar{E}$ be linear normed space. Suppose that for any $\tilde{\varepsilon} \in (0, 2)$, there exist $\tilde{\delta} = \tilde{\delta}(\tilde{\varepsilon}) > 0$, $\|\omega\| + \|\nu\| = 1$, $\|\omega - \nu\| \geq \tilde{\varepsilon}$ satisfying

$$\left\| \frac{\omega + \nu}{2} \right\| \leq 1 - \tilde{\delta}(\tilde{\varepsilon}); \quad (7)$$

then, space $\bar{E}$ is uniformly convex.

Lemma 3 (Milman theorem [44]). A uniformly convex Banach space is reflexive.

Lemma 4 (Clarkson inequality [44]). For $\omega, \nu \in L^p((0, 1))$, $2 \leq p < +\infty$, one has

$$\left\| \frac{\omega + \nu}{2} \right\|^p + \left\| \frac{\omega - \nu}{2} \right\|^p \leq \frac{\|\omega\|^p + \|\nu\|^p}{2}. \quad (8)$$

Lemma 5 ([45]). Let $\mathcal{X}$ be a nonempty set. Y and Φ are two real functions on $\mathcal{X}$. Assume that there are $\alpha > 0$ and $\omega_0, \omega_1 \in \mathcal{X}$ such that

$$\begin{aligned} Y(\omega_0) &= \Phi(\omega_0) = 0, \\ Y(\omega_1) &> \alpha, \end{aligned} \quad (9)$$

$$\sup_{\omega \in Y^{-1}\tilde{\mathcal{E}}(-\infty, \alpha] \cap \mathcal{E}^{\oplus}} \Phi(x) < \alpha \frac{\Phi(\omega_1)}{Y(\omega_1)};$$

then, for all ρ , satisfying

$$\sup_{\omega \in Y^{-1}\tilde{\mathcal{E}}(-\infty, \alpha] \cap \mathcal{E}^{\oplus}} \Phi(x) < \rho < \alpha \frac{\Phi(\omega_1)}{Y(\omega_1)}, \quad (10)$$

one has

$$\sup_{\beta \geq 0} \inf_{\omega \in \mathcal{X}} (Y(\omega) + \beta(\rho - \Phi(\omega))) < \inf_{\omega \in \mathcal{X}} \sup_{\beta \geq 0} (Y(\omega) + \beta(\rho - \Phi(\omega))). \quad (11)$$

Theorem 6 (three-critical-point theorem [46]). Let $\mathcal{X}$ be a separable and reflexive Banach space, $Y : \mathcal{X} \longrightarrow \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative generates a continuous inverse on $\mathcal{X}^*$, and $\Phi : \mathcal{X} \longrightarrow \mathbb{R}$ is a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume

- (1) for all $\beta \in [0, +\infty)$,

$$\lim_{\|\omega\| \rightarrow +\infty} Y(\omega) + \beta\Phi(\omega) = +\infty, \quad (12)$$

- (2) there exists a continuous concave function $H : [0, +\infty) \longrightarrow \mathbb{R}$ such that

$$\sup_{\beta \geq 0} \inf_{\omega \in \mathcal{X}} (Y(\omega) + \beta\Phi(\omega) + H(\beta)) < \inf_{\omega \in \mathcal{X}} \sup_{\beta \geq 0} (Y(\omega) + \beta\Phi(\omega) + H(\beta)). \quad (13)$$

Then, there exist an open interval $\Omega \subseteq (0, +\infty)$ and a positive real number p such that, for each $\beta \in \Omega$, the equation

$$Y'(\omega) + \beta\Phi'(\omega) = 0 \quad (14)$$

has at least three solutions in $\mathcal{X}$ whose norms are less than p .

3. Separability and Reflexivity for Space $\mathcal{W}$

In this section, we illustrate that $\mathcal{W}$ is a separable and reflexive real Banach space to guarantee our main results. The following theorem is given.

Theorem 7. $\mathcal{W}$ is a separable and reflexive real Banach space.

Proof. The proof is divided into three parts.

Part 1. Space $\mathcal{W}$ is a Banach space.

It is well known that $\mathcal{W}$ is a normed linear space. Let $\{\omega_n(x)\}$ be an arbitrary cauchy sequence of space $\mathcal{W}$. According to Morrey's inequality [47], one has

$$\|\omega_n(x) - \omega(x)\|_{\mathcal{C}^{0,r}} \leq M_0 \|\omega_n(x) - \omega(x)\|_{\mathcal{W}}, \quad (15)$$

where M_0 is a constant, $r \in (0, 1)$. So, for any n , $\omega_n(x)$ is uniform convergence in space $\mathcal{W}$, which means

$$\begin{cases} \omega_n(0) = \omega(0) = 0, \\ \lim_{n \rightarrow +\infty} \omega_n(1) = \omega(1). \end{cases} \quad (16)$$

By Lebesgue control convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \omega_n(1) = \lim_{n \rightarrow +\infty} \int_0^1 \omega_n(s) d\gamma(s) = \int_0^1 \lim_{n \rightarrow +\infty} \omega_n(s) d\gamma(s) = \int_0^1 \omega(s) d\gamma(s). \quad (17)$$

From the second of (16) and (17), we obtain $\omega(1) = \int_0^1 \omega(s) d\gamma(s)$. Therefore, *Part 1.* holds.

Part 2. Space $\mathcal{W}$ is separable.

Let $\Lambda \subset W^{1,2}(0, 1)$ be an enumerable subset. Due to the separability of space $W^{1,2}(0, 1)$, there exists $\{\omega_n\} \subset \Lambda$ with $\lim_{n \rightarrow +\infty} \omega_n = \omega$, for any $\omega \in W^{1,2}(0, 1)$.

Define mapping $\mathcal{T} : \Lambda \rightarrow \mathcal{W}$ satisfying

$$\begin{aligned} (\mathcal{T}\omega)(t) &= \omega(t) + \frac{1}{M_1} (-\hat{A}t^2 + t) \left[\int_0^1 \omega(s) d\gamma(s) - \gamma(1) \right] \\ &\quad - (M_2t^2 + M_3t + 1)\omega(0), \end{aligned} \quad (18)$$

where $\omega \in \Lambda$. Constants $\hat{A}, M_1, M_2$, and M_3 in (3) are denote by

$$\begin{cases} \hat{A} = \frac{1 - \int_0^1 t d\gamma(t)}{1 - \int_0^1 t^2 d\gamma(t)}, \\ M_1 = \frac{\int_0^1 t d\gamma(t) - \int_0^1 t^2 d\gamma(t)}{1 - \int_0^1 t^2 d\gamma(t)}, \\ M_2 = \frac{\int_0^1 d\gamma(t) - \int_0^1 t d\gamma(t)}{\int_0^1 t d\gamma(t) - \int_0^1 t^2 d\gamma(t)}, \\ M_3 = \frac{\int_0^1 t^2 d\gamma(t) - \int_0^1 d\gamma(t)}{\int_0^1 t d\gamma(t) - \int_0^1 t^2 d\gamma(t)}. \end{cases} \quad (19)$$

Let $\Theta = \mathcal{T}(\omega)$, and it is easy to obtain $\Theta \in \mathcal{W}$. Consider set $\hat{\Lambda} = \{M = \mathcal{T}(\omega) : \omega \in \Lambda\}$; so, $\hat{\Lambda} \subset \mathcal{W}$ is an enumerable subset.

Next, let us demonstrate that there exists $M_n \subset \hat{\Lambda}$ with $\lim_{n \rightarrow +\infty} M_n = \omega$, for any $\omega \in \mathcal{W}$. Owing to $\omega \in \mathcal{W}$, there exists sequence $\{\omega_n\} \subset \Lambda$ such that $\lim_{n \rightarrow +\infty} \omega_n = \omega$. Let $M_n = \mathcal{T}(\omega_n)$ and write

$$\begin{aligned} \|M_n - \omega_n\| &= \left[\int_0^1 |(M_n - \omega_n)'|^2 dt \right]^{1/2} \\ &= \left[\int_0^1 \left(\frac{1}{M_1} (-2\hat{A}t + 1) \left(\int_0^1 \omega_n(s) d\gamma(s) - \omega_n(1) \right) \right. \right. \\ &\quad \left. \left. - (2M_2t + M_3)\omega_n(0) \right)^2 dt \right]^{1/2} \\ &= \left[\int_0^1 \left(\frac{1}{M_1^2} (-2\hat{A}t + 1)^2 \left(\int_0^1 \omega_n(s) d\gamma(s) - \omega_n(1) \right)^2 \right. \right. \\ &\quad \left. \left. + (2M_2t + M_3)^2 \omega_n^2(0) - 2 \frac{1}{M_1} (-2\hat{A}t + 1) \right. \right. \\ &\quad \left. \left. \cdot (2M_2t + M_3) \left(\int_0^1 \omega_n(s) d\gamma(s) - \omega_n(1) \right) \omega_n(0) \right) dt \right]^{1/2} \\ &= \left[\left(\frac{4\hat{A}^2}{3} + 1 - 2\hat{A} \right) \left(\int_0^1 \omega_n(s) d\gamma(s) - \omega_n(1) \right)^2 \right. \\ &\quad \left. + \left(\frac{4M_2^2}{3} + M_3^2 + 2M_2M_3 \right) \omega_n^2(0) - \frac{2}{M_1} \left(-\frac{4\hat{A}M_2}{3} \right. \right. \\ &\quad \left. \left. - \hat{A}M_3 + M_2 + M_3 \right) \left(\int_0^1 \omega_n(s) d\gamma(s) - \omega_n(1) \right) \omega_n(0) \right]^{1/2} \\ &\rightarrow \left[\left(\frac{4\hat{A}^2}{3} + 1 - 2\hat{A} \right) \left(\int_0^1 \omega(s) d\gamma(s) - \omega(1) \right)^2 \right. \\ &\quad \left. + \left(\frac{4M_2^2}{3} + M_3^2 + 2M_2M_3 \right) \omega^2(0) - \frac{2}{M_1} \left(-\frac{4\hat{A}M_2}{3} \right. \right. \\ &\quad \left. \left. - \hat{A}M_3 + M_2 + M_3 \right) \left(\int_0^1 \omega(s) d\gamma(s) - \omega(1) \right) \omega(0) \right]^{1/2} \\ &= 0, \quad \text{as } n \rightarrow +\infty. \end{aligned} \quad (20)$$

Hence, *Part 2.* holds.

Part 3. $\mathcal{W}$ is reflexive.

Define operator $\mathcal{P}$:

$$\mathcal{W} \longrightarrow L^2(0, 1). \quad (21)$$

That is to say, $\mathcal{P} : \omega \longrightarrow \omega' \in L^2(0, 1)$, for all $\omega \in \mathcal{W}$. Let set

$$Y = \{\omega' : \omega \in \mathcal{W}\} \quad (22)$$

endowed with the norm

$$\|\omega'\| = \sqrt{\int_0^1 |\omega'|^2 dt}, \quad (23)$$

we obtain

$$Y \subset L^2(0, 1). \quad (24)$$

Consequently,

$$Y \subset L^2(0, 1), \quad \omega' = v. \quad (25)$$

Next, we will prove operator $\mathcal{P}$ is an isometric isomorphic mapping. Evidently, mapping $\mathcal{P}$ is surjective. By Definition 1, we only illustrate $\mathcal{P}$ as isometric. According to mapping $\mathcal{P} : \mathcal{W} \longrightarrow Y$, we deduce

$$\|\omega\| = \|v\| = \|\mathcal{P}\omega\|. \quad (26)$$

Hence, operator $\mathcal{P}$ is an isometric isomorphic mapping.

Then, we demonstrate Y that is uniformly convex. For $\forall \tilde{\varepsilon} > 0$, $\omega, v \in Y$, such that $\|\omega\| = \|v\| = 1$, $\|\omega - v\| \geq \tilde{\varepsilon}$, choosing $\tilde{\delta}(\tilde{\varepsilon}) = 1 - \sqrt{1 - \tilde{\varepsilon}^2/4} > 0$. On account of Lemma 4, particularly $p = 2$, we deduce

$$\left\| \frac{\omega + v}{2} \right\| \leq \sqrt{\frac{1}{2} (\|\omega\|^2 + \|v\|^2) - \left\| \frac{\omega - v}{2} \right\|^2} \leq \sqrt{1 - \frac{\tilde{\varepsilon}^2}{4}} = 1 - \tilde{\delta}(\tilde{\varepsilon}). \quad (27)$$

By Lemma 2, Y is uniformly convex. Owing to $\mathcal{P}$ is an isometric isomorphic mapping, and $\mathcal{W}$ is also uniformly convex. Part 3. holds by Lemma 3.

We complete the proof of the theorem.

4. Main Results

In this section, we will show there are at least three solutions for problem (1) mainly by Theorem 6. Define function

$$\rho(t, \eta) = \int_0^\eta g(t, s) ds \quad (28)$$

for any $(t, \eta) \in [0, 1] \times \mathbb{R}$, and $g : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. For our main results, the following lemma is first given.

Lemma 8. Suppose that there exist six positive constants K, P, L, A_1, A_2 and $\hat{A}$ with $P < L\sqrt{K}/8\hat{A}$ and $0 < A_1 < A_2 < 1$, such that

- (1) $\rho(t, \eta) \geq 0$, $(t, \eta) \in [0, A_1] \cup [A_2, 1] \times [0, (L/4\hat{A})]$;
- (2) $\max_{(t, \eta) \in [0, 1] \times [-P, P]} \rho(t, \eta) < 64\hat{A}^2 P^2 / KL^2 \int_{A_1}^{A_2} \rho(t, (L/4\hat{A})) dt$,

then, there exist $\alpha > 0$, $\mu \in \mathcal{W}$ such that $2\alpha < \|\mu\|^2$ and

$$\max_{(t, \eta) \in [0, 1] \times [-P, P]} \rho(t, \eta) < 2\alpha \frac{\int_0^1 \rho(t, \mu(t)) dt}{\|\mu\|^2}. \quad (29)$$

Proof. Define

$$\mu(t) = L \left(t - \frac{1 - \int_0^1 t d\gamma(t)}{1 - \int_0^1 t^2 d\gamma(t)} t^2 \right). \quad (30)$$

Notice that space $\mathcal{W}$ is normed $\|\mu\| = \{\int_0^1 |\mu'|^2 dt\}^{1/2}$. It is readily found that $\mu(t) \in \mathcal{W}$. In a simple calculation, we have

$$\|\mu\|^2 = \int_0^1 |\mu'|^2 dt = L^2 \left(1 - \frac{10\hat{A}}{3} \right)^2, \quad (31)$$

where

$$\hat{A} = \frac{1 - \int_0^1 t d\lambda(t)}{1 - \int_0^1 t^2 d\lambda(t)}. \quad (32)$$

By the hypotheses $P < L\sqrt{K}/8\hat{A}$, choosing $\alpha = 2P^2$, $0 < A_1 < 1/2\hat{A} < A_2 < 1$, any number and $K = 16(3 - 10\hat{A})^2 \hat{A}^2/9$, we have $2\alpha < \|\mu\|^2$. According to assumption (a), one has

$$\begin{aligned} 2\alpha \frac{\int_0^1 \rho(t, \mu(t)) dt}{\|\mu\|^2} &\geq \frac{4P^2}{\|\mu\|^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt \\ &= \frac{4P^2}{L^2(1 - 10\hat{A}/3)^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt \\ &= \frac{64\hat{A}^2 P^2}{KL^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt \\ &> \max_{(t, \eta) \in [0, 1] \times [-P, P]} \rho(t, \eta). \end{aligned} \quad (33)$$

Thus, inequality (29) holds.

Theorem 9. Assume that there exist eight positive constants $K, P, L, A_1, A_2, c, \kappa$, and $\hat{A}$ with $P < L\sqrt{K}/8\hat{A}$, $0 < A_1 < A_2 < 1$ and $\kappa < 2$, such that

- (1) $\rho(t, \eta) \geq 0$, $(t, \eta) \in [0, A_1] \cup [A_2, 1] \times [0, (L/4\hat{A})]$;
- (2) $\max_{(t, \eta) \in [0, 1] \times [-P, P]} \rho(t, \eta) < 64\hat{A}^2 P^2 / KL^2 \int_{A_1}^{A_2} \rho(t, (L/4\hat{A})) dt$;
- (3) $\rho(t, \eta) \leq c(1 + |\eta|^\kappa)$, for $t \in [0, 1]$ and $\eta \in \mathbb{R}$

Then, there exists an open interval $\Omega \subseteq (0, +\infty)$ and a positive real number p such that for each $\beta \in \Omega$, problem (1) has at least three solutions belonging to $\mathcal{W}$ whose norms are less than p .

Proof. Consider the following functions.

$$\begin{aligned} 2\alpha \frac{\int_0^1 \rho(t, \mu(t)) dt}{\|\omega\|^2} &\geq \frac{4P^2}{\|\mu\|^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt \\ &= \frac{4P^2}{L^2(1-10\hat{A}/3)^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt \\ &= \frac{64\hat{A}^2 P^2}{KL^2} \int_{A_1}^{A_2} \rho\left(t, \frac{L}{4\hat{A}}\right) dt > \max_{(t,\eta) \in [0,1] \times [-P,P]} \rho(t, \eta). \end{aligned} \quad (34)$$

for any $\omega \in \mathcal{W}$.

Notice that the critical points of Γ are the generalized solutions of problem (1). Therefore, we just validate that Y and Φ accord with the conditions of Theorem 6. It is obvious that Y is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative yields a continuous inverse on $\mathcal{W}^*$, and Φ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.

In addition, by condition (c) and Poincaré' inequality, one readily has

$$\lim_{\|\omega\| \rightarrow +\infty} \Gamma(\omega) = \lim_{\|\omega\| \rightarrow +\infty} Y(\omega) + \beta\Phi(\omega) = +\infty, \quad (35)$$

for any $\beta \geq 0$.

Next, we prove that there exist $\alpha > 0$, $\omega \in \mathcal{W}$ satisfying

$$\sup_{\omega \in \Phi^{-1} \mathcal{E}(-\infty, \alpha] \cap \mathcal{E}^0} (-\Phi(\omega)) < -\alpha \frac{\Phi(\omega)}{Y(\omega)}. \quad (36)$$

From Lemma 8, $\omega(t) = \mu(t)$, one has $\max_{t \in [0,1]} |\omega(t)| = L/4\hat{A}$; on the other hand, $\|\omega\| = |L(1-10\hat{A}/3)|$. Thus, $\max_{t \in [0,1]} |\omega| \leq \|\omega\|$, for all $\omega \in \mathcal{W}$. Thus,

$$Y^{-1}(-\infty, \alpha] \subseteq \left\{ \omega \in \mathcal{W} : \|\omega\| \leq \sqrt{2\alpha}, 0 \leq t \leq 1 \right\}, \quad (37)$$

for any $\alpha > 0$. Moreover,

$$\sup_{\omega \in Y^{-1}(-\infty, \alpha]} (-\Phi(\omega)) \leq \sup_{\|\omega\|^2 \leq 2\alpha} \int_0^1 \rho(t, \omega) dt \leq \max_{(t,\eta) \in [0,1] \times [-\sqrt{\frac{\alpha}{2}}, \sqrt{\frac{\alpha}{2}}]} \rho(t, \eta). \quad (38)$$

From Lemma 8, one has

$$\max_{(t,\eta) \in [0,1] \times [-\sqrt{\frac{\alpha}{2}}, \sqrt{\frac{\alpha}{2}}]} \rho(t, \eta) < 2\alpha \frac{\int_0^1 \rho(t, \omega(t)) dt}{\|\omega\|^2} = -\alpha \frac{\Phi(\omega)}{Y(\omega)}. \quad (39)$$

So, inequality (36) holds.

By Lemma 5, selecting $H(\beta) = \rho\beta$, we have

$$\sup_{\beta \geq 0} \inf_{\omega \in \mathcal{W}} (Y(\omega) + \beta(\Phi(\omega) + \rho)) < \inf_{\omega \in \mathcal{W}} \sup_{\beta \geq 0} (Y(\omega) + \beta(\Phi(\omega) + \rho)). \quad (40)$$

Therefore, the proof is complete by Theorem 6.

For the purpose of explaining the validity of our results, an example is given as follows.

Example. Consider the special second-order problem as below.

$$\begin{cases} -\omega''(t) = \beta t f(\omega), \\ \omega(0) = 0, \omega(1) = \int_0^1 \omega(s) ds, \end{cases} \quad (41)$$

where $g(t, \omega) = t f(\omega)$, $\gamma(s) = s$. Choosing $L = 30$, $f(\omega)$ defined by

$$f(\omega) = \begin{cases} e^\omega, & \text{if } \omega \leq \frac{L}{3}, \\ \omega + e^{L/3} - \frac{L}{3}, & \text{if } \omega > \frac{L}{3}. \end{cases} \quad (42)$$

Proof. According to the integrability of $f(\omega)$ and $f(\omega) \geq 0$ for $\omega \in [0, \eta]$, where $\eta \in [0, (L/4\hat{A})]$, we have

$$\int_0^\eta f(y) dy \geq 0, \quad (43)$$

which means $t \int_0^\eta f(y) dy \geq 0$, for $t \in [0, A_1] \cup [A_2, 1]$. So, the assumption (a) of Theorem 9 is satisfied.

Define

$$F(\eta) = \int_0^\eta f(y) dy. \quad (44)$$

Thus,

$$F(\eta) = \begin{cases} e^\eta - 1, & \text{if } \eta \leq \frac{L}{3}, \\ \eta \left(e^{L/3} - \frac{L}{3} \right) + e^{L/3} \left(1 - \frac{L}{3} \right) + \frac{L^2}{18} + \frac{\eta^2}{2} - 1, & \text{if } \eta > \frac{L}{3}. \end{cases} \quad (45)$$

Owing to $g(t, \omega) = t f(\omega)$, one has

$$\max_{(t,\eta) \in [0,1] \times [-P,P]} \rho(t, \eta) \leq \max_{(t,\eta) \in [0,1] \times [-P,P]} \int_0^\eta g(t, y) dy = \max_{t \in [0,1]} t F(P). \quad (46)$$

Choosing $\kappa = 3/2$, $c = e^{L/3}$, $P = 1$, $K = 9$, $A_2 = 1/2$, and $A_2 = 3/4$, we guarantee that conditions (b) and (c) of Theorem 9 are satisfied.

Data Availability

No data were used to support the findings of study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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