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Research Article

Phillips-Type q-Bernstein Operators on Triangles

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The purpose of the paper is to introduce a new analogue of Phillips-type Bernstein operators $(\mathscr{B}^u_{m,q}f)(u,v)$ and $(\mathscr{B}^v_{n,q}f)(u,v)$, their products $(\mathscr{P}_{mn,q}f)(u,v)$ and $(\mathscr{Q}_{nm,q}f)(u,v)$, their Boolean sums $(\mathscr{S}_{mn,q}f)(u,v)$ and $(\mathscr{T}_{nm,q}f)(u,v)$ on triangle $\mathscr{T}_h$, which interpolate a given function on the edges, respectively, at the vertices of triangle using quantum analogue. Based on Peano's theorem and using modulus of continuity, the remainders of the approximation formula of corresponding operators are evaluated. Graphical representations are added to demonstrate consistency to theoretical findings. It has been shown that parameter q provides flexibility for approximation and reduces to its classical case for $q = 1$.

1. Introduction and Essential Preliminaries

In 1912, Bernstein constructed polynomials to provide a constructive proof of the Weierstrass approximation theorem [1, 2] using probabilistic interpolation, which is now known as Bernstein polynomials in approximation theory. In computer-aided geometric design (CAGD), the basis of Bernstein polynomials plays a significant role to preserve the shape of the curves and surfaces.

Further, with the development of q-calculus (quantum analogue), the first q-analogue of Bernstein operators (rational) was constructed by Lupas in [3]. In 1997, Phillips [4] initiated another generalization of Bernstein polynomials based on the q-integers (quantum analogue) called q-Bernstein polynomials. The q-Bernstein polynomials attracted a lot of attention and were studied broadly by several researchers. One can find a survey of the obtained results and references on the subject in [5].

Computer-aided geometric design (CAGD) is a discipline which deals with computational aspects of geometric objects. It emphasizes on the mathematical development of curves and surfaces such that it becomes compatible with computers. Popular programs, like Adobe's Illustrator and Flash, and font imaging systems, such as Postscript, utilize Bernstein polynomials to form what are known as Bézier curves [6–9].

The approximating operators on triangles and their basis have important applications in finite element analysis and computer-aided geometric design [10] etc. Starting with the paper [11] of Barnhill et al., the blending interpolation operators were considered in the papers [12–14].

In this paper, we construct new operators based on quantum analogue of Phillips. Bernstein-type operators also interpolate the value of a given function on the boundary of the triangle. Also, we will discuss some particular cases. Using modulus of continuity and Peano's theorem, the remainders of the corresponding approximation formulas are evaluated. The accuracy of the approximation is also illustrated by graphics of given functions with suitable Bernstein-type approximation. For more information regarding such operators, their properties and their remainders one can refer to [15–28].

In this paper, we would like to draw attention to the Phillips q-analogue of the Bernstein operators and obtain new results using q-analogue on triangles. To present results by Phillips, we recall the following definitions. For other relevant works, one can see [29].

Let $q > 0$. For any $m = 0, 1, 2, \cdots$, the q-integer $[m]_q$ is defined by

$$[m]_q \coloneqq 1 + q + \cdots + q^{m-1}, \quad m = 1, 2, \cdots, [0]_q \coloneqq 0, \tag{1}$$

and the q-factorial $[m]_q!$ by

$$[m]_q! := [1]_q [2]_q \cdots [m]_q, \quad m = 1, 2, \cdots, [0]_q! = 1. \tag{2}$$

For integers $0 \le i \le m$, the q-binomial or the Gaussian coefficient is defined by

$$\begin{bmatrix} m \\ i \end{bmatrix}_q := \frac{[m]_q!}{[i]_q! [m-i]_q!}. \tag{3}$$

Clearly, for $q = 1$,

$$[m]_1 = m, [m]_1! = m!, \begin{bmatrix} m \\ i \end{bmatrix}_1 = \binom{m}{i}. \tag{4}$$

The q-binomial coefficients are involved in Cauchy's q-binomial theorem (cf. [30], Chapter 10, Section 10.2). The first one is a q-analogue as an extension to Newton's binomial formula:

$$(au + bv)_q^m := \sum_{i=0}^{m} q^{(i(i-1))/2} \begin{bmatrix} m \\ i \end{bmatrix}_q a^{m-i} b^i u^{m-i} v^i, \tag{5}$$

$$(1+u)(1+qu) \cdot \cdots \cdot (1 + q^{m-1} u) = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_q q^{i(i-1)/2} u^i. \tag{6}$$

Following Phillips, we denote

$$b_{m,i}(u, v) = \begin{bmatrix} m \\ i \end{bmatrix}_q \prod_{s=0}^{m-i-1} (1 - q^s u). \tag{7}$$

It follows from (6) that

$$\sum_{i=0}^{m} b_{m,i}(q; u) = 1, \quad u \in [0, 1], \tag{8}$$

for integers $k \ge i \ge 0$. These recurrence relations are satisfied by q-binomial coefficients

$$\begin{gathered} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q = q^{k-i+1} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q + \begin{bmatrix} k \\ i \end{bmatrix}_q, \\ \begin{bmatrix} k+1 \\ i \end{bmatrix}_q = \begin{bmatrix} k \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} k \\ i \end{bmatrix}_q, \end{gathered} \tag{9}$$

when $q = 1$, both the relations reduce to the Pascal identity. In the next section, we construct quantum analogue of operators studied in [31] on triangles.

2. Construction of New Univariate Operators on Triangle

In [31], the authors considered only the standard triangle sufficient due to affine invariance as

$$\mathcal{T}_h = \left\{ (u, v) \in \mathbb{R}^2 \mid u \ge 0, v \ge 0, u + v \le h \right\}, \quad \text{for } h > 0. \tag{10}$$

Let $\Delta_m^u = \{ i((h-v)/m), i = \overline{0, m} \}$ and $\Delta_n^v = \{ j((h-u)/n), j = \overline{0, n} \}$ be uniform partitions of the intervals $[0, h - v]$ and $[0, h - u]$, respectively.

In 2009, they [31] constructed some univariant Bernstein-type operators on triangle $\mathcal{T}_h$ as follows:

$$\begin{aligned} (\mathscr{B}_m^u f)(u, v) &= \sum_{i=0}^{m} p_{m,i}(u, v) f\left(\frac{i}{m}(h - v), v \right), \\ (\mathscr{B}_n^v f)(u, v) &= \sum_{j=0}^{n} q_{n,j}(u, v) f\left(u, \frac{j}{n}(h - u) \right), \end{aligned} \tag{11}$$

where

$$p_{m,i}(u, v) = \frac{\binom{m}{i} u^i (h - u - v)^{m-i}}{(h - v)^m}, \quad 0 \le u + v \le h, \tag{12}$$

$$q_{n,j}(u, v) = \frac{\binom{n}{j} v^j (h - u - v)^{n-j}}{(h - u)^n}, \quad 0 \le u + v \le h, \tag{13}$$

respectively.

Consider a real-valued function f defined on $\mathcal{T}_h$ as done in [31]. Through the point $(u, v) \in \mathcal{T}_h$, one considers the parallel lines to the coordinate axes which intersect the edges Γ_i, $i = 1, 2, 3$, of the triangle at the points $(0, v)$ and $(h - v, v)$, respectively $(u, 0)$ and $(u, h - u)$ ([31], Figure 1).

Let $\Delta_m^u = \{ i((h-v)/m), i = \overline{0, m} \}$ and $\Delta_n^v = \{ j((h-u)/n), j = \overline{0, n} \}$ be uniform partitions of the intervals $[0, h - v]$ and $[0, h - u]$, respectively.

We define the new Phillips-type Bernstein operators $\mathscr{B}_{m,q}^u$ and $\mathscr{B}_{n,q}^v$ on triangle by using quantum calculus as follows:

$$\begin{aligned} \left(\mathscr{B}_{m,q}^u f \right)(u, v) &= \begin{cases} \displaystyle\sum_{i=0}^{m} \tilde{p}_{m,i}(u, v) f\left(\frac{[i]_q}{[m]_q}(h - v), v \right), & (u, v) \in \mathcal{T}_h \setminus (0, h), \\ f(0, h), & (0, h) \in \mathcal{T}_h, \end{cases} \\ \left(\mathscr{B}_{n,q}^v f \right)(u, v) &= \begin{cases} \displaystyle\sum_{j=0}^{n} \tilde{q}_{n,j}(u, v) f\left(u, \frac{[j]_q}{[n]_q}(h - u) \right), & (u, v) \in \mathcal{T}_h \setminus (h, 0), \\ f(h, 0), & (h, 0) \in \mathcal{T}_h, \end{cases} \end{aligned} \tag{14}$$

where

$$\tilde{p}_{m,i}(u,v)=\frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^s u)}{(h-v)^m},\quad 0\le u+v\le h\,(\text{except the point}(0,h)),\tag{15}$$

$$\tilde{q}_{n,j}(u,v)=\frac{\begin{bmatrix} n \\ j \end{bmatrix}_q v^j\prod_{t=0}^{n-j-1}(h-u-q^t v)}{(h-u)^n},\quad 0\le u+v\le h\,(\text{except the point}(0,h)),\tag{16}$$

respectively. These operators reduce to Phillips-type operator on $[0,1]$. One can note that the bases (15) and (16) of the operators constructed using quantum calculus are different from the bases (12) and (13) of the operators constructed by Blaga and Coman [31]. In case $q=1$, corresponding operators reduce to its classical case on triangles. Now, we generalize various results of [31] in quantum calculus frame.

For the sake of convenience, we use the following notation onwards:

$$\begin{aligned}(h-v)^m &:= \sum_{i=0}^{m}\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^s u),\\ (h-u)^n &:= \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q v^i\prod_{s=0}^{n-i-1}(h-u-q^s v).\end{aligned}\tag{17}$$

Theorem 1. *If f is a real-valued function defined on $\mathcal{T}_h$, then*

(i) $\mathcal{B}^u_{m,q}f=f$ *on* $\Gamma_2\cup\Gamma_3$

(ii) $(\mathcal{B}^u_{m,q}e_{i0})(u,v)=u^i$, $i=0,1$ $(dex(\mathcal{B}^u_{m,q})=1)$

(iii) $(\mathcal{B}^u_{m,q}e_{20})(u,v)=u^2+((u(h-u-v))/[m]_q)$

$$\left(\mathcal{B}^u_{m,q}e_{ij}\right)(u,v)=\begin{cases} v^ju^i, & i=0,1,\ j\in N,\\ v^j\left(u^2+\dfrac{u(h-u-v)}{[m]_q}\right), & i=2,\ j\in\mathbb{N},\end{cases}\tag{18}$$

where $e_{ij}(u,v)=u^iv^j$ and dex $(\mathcal{B}^u_{m,q})$ is the degree of exactness of the operator $\mathcal{B}^u_{m,q}$.

Proof. By definition, $(B^u_{m,q}f)(0,h)=f(0,h)$. So we will calculate the moments only on $\mathcal{T}_h\setminus(0,h)$. The interpolation property (i) follows from the relations

$$\begin{aligned}\tilde{p}_{m,i}(0,v)&=\begin{cases}1, & \text{if } i=0,\\ 0, & i\ne 0,\end{cases}\\ \tilde{p}_{m,i}(h-v,v)&=\begin{cases}1, & \text{if } i=m,\\ 0, & i\ne m.\end{cases}\end{aligned}\tag{19}$$

Regarding the property (ii), we have

$$\left(\mathcal{B}^u_{m,q}e_{00}\right)(u,v)=\sum_{i=0}^{m}\frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^su)}{(h-v)^m}=\frac{(h-v)^m}{(h-v)^m}=1,$$

$$\begin{aligned}\left(\mathcal{B}^u_{m,q}e_{10}\right)(u,v)&=\sum_{i=0}^{m}\frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^su)}{(h-v)^m}\frac{[i]_q}{[m]_q}(h-v)\\ &=\sum_{i=0}^{m}\frac{\left([i]_q/[m]_q\right)\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^su)}{(h-v)^{m-1}}\\ &=\sum_{i=0}^{m-1}\frac{\begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^{i+1}\prod_{s=0}^{m-i-2}(h-v-q^su)}{(h-v)^{m-1}}\\ &=u\sum_{i=0}^{m-1}\frac{\begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^i\prod_{s=0}^{(m-1)-i-1}(h-v-q^su)}{(h-v)^{m-1}}=u,\end{aligned}$$

$$\begin{aligned}\left(\mathcal{B}^u_{m,q}e_{20}\right)(u,v)&=\sum_{i=0}^{m}\frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-1}(h-v-q^su)}{(h-v)^m}\frac{[i]_q^2}{[m]_q^2}(h-v)^2\\ &=(h-v)^2\sum_{i=0}^{m-1}\frac{([i+1]q/[m]q)\begin{bmatrix} m \\ i \end{bmatrix}_q u^{i+1}\prod_{s=0}^{m-i-2}(h-v-q^su)}{(h-v)^m}\\ &=(h-v)^2u\sum_{i=0}^{m-1}\frac{((1+q[i]q)/[m]q)\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-2}(h-v-q^su)}{(h-v)^m}\\ &=(h-v)\frac{u}{[m]_q}\sum_{i=0}^{m-1}\frac{\begin{bmatrix} m \\ i \end{bmatrix}_q u^i\prod_{s=0}^{(m-1)-i-1}(h-v-q^su)}{(h-v)^{m-1}}+(h-v)^2u\\ &\quad\cdot\sum_{i=0}^{m-1}\frac{\left(q[m-1]_q/[m]_q\right)\left([i]_q/[m-1]_q\right)\begin{bmatrix} m-1 \\ i \end{bmatrix}_q u^i\prod_{s=0}^{m-i-2}(h-v-q^su)}{(h-v)^m}\\ &=(h-v)\frac{u}{[m]_q}+\frac{q[m-1]_qu^2}{[m]_q}\sum_{i=0}^{m-2}\frac{\begin{bmatrix} m-2 \\ i \end{bmatrix}_q u^i\prod_{s=0}^{(m-2)-i-1}(h-v-q^su)}{(h-v)^{m-2}},\end{aligned}$$

$$\left(\mathcal{B}^u_{m,q}e_{20}\right)(u,v)=(h-v)\frac{u}{[m]_q}+\frac{q[m-1]_qu^2}{[m]_q},\tag{20}$$

or equivalently,

$$\left(\mathcal{B}^u_{m,q}e_{20}\right)(u,v)=(h-v)\frac{u}{[m]_q}+u^2\left(1-\frac{1}{[m]_q}\right)=u^2+\frac{u(h-u-v)}{[m]_q}.\tag{21}$$

Remark 2. In the same way, it can be proved that if f is a real-valued function defined on $\mathcal{T}_h$, then

(i) $\mathcal{B}^{v}_{n,q}f = f$ on $\Gamma_1 \cup \Gamma_3$

(ii) $(\mathcal{B}^{v}_{n,q}e_{0j})(u,v) = v^j$, $j = 0, 1 (\operatorname{dex}(\mathcal{B}^{v}_{n,q}) = 1)$

(iii) $(\mathcal{B}^{v}_{n,q}e_{02})(u,v) = v^2 + ((v(h-u-v))/[n]_q)$

$$\left(\mathcal{B}^{v}_{n,q}e_{ij}\right)(u,v) = \begin{cases} u^i v^j, & j = 0, 1, i \in N, \\ u^i\left(v^2 + \dfrac{v(h-u-v)}{[n]_q}\right), & j = 2, i \in \mathbb{N}. \end{cases} \tag{22}$$

Based on the following approximation formula

$$f = \mathcal{B}^{u}_{m,q}f + \mathcal{R}^{u}_{m,q}f, \tag{23}$$

we present the following results.

Theorem 3. *If $f(.,v) \in C[0, h-v]$, then*

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2\delta\sqrt{[m]_q}}\right) w(f(.,v);\delta), v \in [0,h], \tag{24}$$

where modulus of continuity of the function f with respect to the variable u is denoted by $w(f(.,v);\delta)$.

Further, if $\delta = 1/\sqrt{[m]_q}$, then

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2}\right) w\left(f(.,v); \frac{1}{\sqrt{[m]_q}}\right), v \in [0,h]. \tag{25}$$

Proof. Since by definition, $(B^{u}_{m,q}f)(0,h) = f(0,h)$ and hence remainder will be zero at $(0,h)$ due to interpolation. We have

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \sum_{i=0}^{m} \tilde{p}_{m,i}(u,v) \left| f(u,v) - f\left(\frac{[i]_q(h-v)}{[m]_q}, v\right)\right|. \tag{26}$$

Since

$$|f(u,v) - f\left(\frac{[i]_q(h-v)}{[m]_q}, v\right)| \le \left(\frac{1}{\delta} \left| u - \frac{[i]_q(h-v)}{[m]_q}\right| + 1\right) w(f(.,v);\delta), \tag{27}$$

one obtains

$$\begin{aligned} \left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| &\le \sum_{i=0}^{m} \tilde{p}_{m,i}(u,v) \left(\frac{1}{\delta} \left| u - \frac{[i]_q(h-v)}{[m]_q}\right| + 1\right) w(f(.,v);\delta) \\ &\le \left[1 + \frac{1}{\delta}\left(\sum_{i=0}^{m} \tilde{p}_{m,i}(u,v)\left(u - \frac{[i]_q(h-v)}{[m]_q}\right)^2\right)^{1/2}\right] w(f(.,v);\delta) \\ &= \left[1 + \frac{1}{\delta}\sqrt{\frac{u(h-u-v)}{[m]_q}}\right] w(f(.,v);\delta). \end{aligned} \tag{28}$$

As

$$\max_{\mathcal{T}_h}[u(h-u-v)] = \frac{h^2}{4}, \tag{29}$$

it follows that

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2\delta\sqrt{[m]_q}}\right) w(f(.,v);\delta). \tag{30}$$

For $\delta = 1/\sqrt{[m]_q}$, we obtain

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2}\right) w\left(f(.,v); \frac{1}{\sqrt{[m]_q}}\right). \tag{31}$$

Theorem 4. *If $f(.,v) \in C^2[0,h]$, then*

$$\left(\mathcal{R}^{u}_{m,q}f\right)(u,v) = -\frac{u(h-u-v)}{2[m]_q} f^{(2,0)}(\xi, y), \xi \in [0, h-v], \tag{32}$$

$$\left|\left(\mathcal{R}^{u}_{m,q}f\right)(u,v)\right| \le \frac{h^2}{8[m]_q} \mathcal{M}_{20}f, (u,v) \in \mathcal{T}_h, \tag{33}$$

where

$$\mathcal{M}_{ij}f = \max_{\mathcal{T}_h}\left| f^{(i,j)}(u,v)\right|. \tag{34}$$

Proof. As $\operatorname{dex}(\mathcal{B}^{u}_{m,q}) = 1$, by Peano's theorem, one obtains

$$\left(\mathcal{R}^{u}_{m,q}f\right)(u,v) = \int_0^{h-v} \mathcal{K}_{20}(u,v;t) f^{(2,0)}(t,v)\,dt, \tag{35}$$

where the kernel

$$\begin{aligned}\mathscr{K}_{20}(u,v;t) &:= \mathscr{R}^u_{m,q}\left[(u-t)_+\right] \\ &= (u-t)_+ - \sum_{i=0}^{m} \tilde{p}_{m,i}(u,v)\left([i]_q \frac{h-v}{[m]_q} - t\right)_+\end{aligned} \tag{36}$$

does not change the sign($\mathscr{K}_{20}(u,v;t) \le 0, u \in [0,h-v]$). By the Mean Value Theorem, it follows that

$$\left(\mathscr{R}^u_{m,q}f\right)(u,v) = f^{(2,0)}(\xi,v)\int_0^{h-v} \mathscr{K}_{20}(u,v;t)dt, \quad \xi \in [0,h-v]. \tag{37}$$

After an easy calculation, we get

$$\left(\mathscr{R}^u_{m,q}f\right)(u,v) = -\frac{u(h-u-v)}{2[m]_q} f^{(2,0)}(\xi,v), \tag{38}$$

where $\xi \in [0,h-v]$.

By using it in Equation (32), we get

$$\left|\left(\mathscr{R}^u_{m,q}f\right)(u,v)\right| \le \frac{h^2}{8[m]_q}\mathscr{M}_{20}f, \quad (u,v) \in \mathscr{T}_h. \tag{39}$$

Remark 5. From (32), it follows that

(i) if $f(.,v)$ is a concave function, then $(\mathscr{R}^u_{m,q}f)(u,v) \ge 0$, i.e.,

$$\left(\mathscr{B}^u_{m,q}f\right)(u,v) \le f(u,v), \tag{40}$$

(ii) if $f(.,v)$ is a convex function, then $(\mathscr{R}^u_{m,q}f)(u,v) \le 0$, i.e.,

$$\left(\mathscr{B}^u_{m,q}f\right)(u,v) \ge f(u,v), \tag{41}$$

for $u \in [0,h-v]$ and $v \in [0,h]$.

Remark 6. For the remainder $\mathscr{R}^v_{n,q}f$ of the approximation formula

$$f = \mathscr{B}^v_{n,q}f + \mathscr{R}^v_{n,q}f. \tag{42}$$

We also have the following:

(A) If $f(u,.) \in C[0,h-u]$, then

$$\left|\left(\mathscr{R}^v_{n,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2\delta\sqrt{[n]_q}}\right) w(f(u,.);\delta), \quad u \in [0,h]. \tag{43}$$

And for $\delta = 1/\sqrt{[n]_q}$,

$$\left|\left(\mathscr{R}^v_{n,q}f\right)(u,v)\right| \le \left(1 + \frac{h}{2}\right) w\left(f(u,.);\frac{1}{\sqrt{[n]_q}}\right), \quad u \in [0,h]. \tag{44}$$

(B) If $f(u,.) \in C^2[0,h]$, then

$$\begin{aligned}&\left(\mathscr{R}^v_{n,q}f\right)(u,v) = -\frac{v(h-u-v)}{2[n]_q} f^{(0,2)}(u,\eta), \quad \eta \in [0,h-u], \\ &\left|\left(\mathscr{R}^v_{n,q}f\right)(u,v)\right| \le \frac{h^2}{8[n]_q}\mathscr{M}_{02}f, \quad (u,v) \in \mathscr{T}_h,\end{aligned} \tag{45}$$

where

$$\mathscr{M}_{ij}f = \max_{\mathscr{T}_h}\left|f^{(i,j)}(u,v)\right|. \tag{46}$$

3. Product Operators

Let $\mathscr{P}_{mn,q} = \mathscr{B}^u_{m,q}\mathscr{B}^v_{n,q}$ and $\mathscr{Q}_{mn,q} = \mathscr{B}^v_{n,q}\mathscr{B}^u_{m,q}$ be the products of operators $\mathscr{B}^u_{m,q}$ and $\mathscr{B}^v_{n,q}$.

We have

$$\left(\mathscr{P}_{mn,q}f\right)(u,v) = \begin{cases} \displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} \tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right) f\left([i]_q\frac{(h-v)}{[m]_q}, [j]_q\frac{\left([m]_q - [i]_q\right)h + [i]_q v}{[m]_q[n]_q}\right), & (u,v) \in \mathscr{T}_h \setminus \{(0,h),(h,0)\}, \\ f(0,h), & (0,h) \in \mathscr{T}_h, \\ f(h,0), & (h,0) \in \mathscr{T}_h. \end{cases} \tag{47}$$

Remark 7. The nodes of the operator $\mathscr{P}_{mn,q}$ are the q-analogue of the nodes, which are given in [31], Figure 2, for $i = \overline{0, m}$; $j = \overline{0, n}$, and $v \in [0, h]$.

Theorem 8. *The product operator $\mathscr{P}_{mn,q}$ satisfies the following relations:*

(i) $(\mathscr{P}_{mn,q}f)(u, 0) = (\mathscr{B}^u_{m,q}f)(u, 0)$

(ii) $(\mathscr{P}_{mn,q}f)(0, v) = (\mathscr{B}^v_{n,q}f)(0, v)$

(iii) $(\mathscr{P}_{mn,q}f)(u, h-u) = f(u, h-u),\ u, v \in [0, h]$

The above proofs follow from some simple computation.

The property (i) or (ii) implies that $(\mathscr{P}_{mn,q}f)(0, 0) = f(0, 0)$.

Remark 9. The product operator $\mathscr{P}_{mn,q}$ interpolates the function f at the vertex $(0, 0)$ and on the hypotenuse $u + v = h$ of the triangle $\mathscr{T}_h$.

The product operator $\mathscr{Q}_{mn,q}$, given by

$$(\mathscr{Q}_{nm,q}f)(u,v) = \begin{cases} \displaystyle\sum_{i=0}^{m}\sum_{j=0}^{n} \tilde{p}_{m,i}\left(u, [j]_q \frac{(h-u)}{[n]_q}\right)\tilde{q}_{n,j}(u,v) f\left([i]_q \frac{\left([n]_q - [j]_q\right)h + [j]_q u}{[m]_q[n]_q}, [j]_q \frac{(h-u)}{[n]_q}\right), & (u,v) \in \mathscr{T}_h \setminus \{(0,h),(h,0)\}, \\ f(0,h), \quad (0,h) \in \mathscr{T}_h, & \\ f(h,0), \quad (h,0) \in \mathscr{T}_h, & \end{cases} \tag{48}$$

has the nodes, which are q-analogue of nodes given in [31], Figure 3, for $i = \overline{0, m}$, $j = \overline{0, n}$, $u \in [0, h]$, and the properties:

(i) $(\mathscr{Q}_{nm,q}f)(u, 0) = (\mathscr{B}^u_{m,q}f)(u, 0)$

(ii) $(\mathscr{Q}_{nm,q}f)(0, v) = (\mathscr{B}^v_{n,q}f)(0, v)$

(iii) $(\mathscr{Q}_{nm,q}f)(h-v, v) = f(h-v, v),\ u, v \in [0, h]$

Let us consider the approximation formula

$$f = \mathscr{P}_{mn,q}f + \mathscr{R}^{\mathscr{P}}_{mn,q}f. \tag{49}$$

Theorem 10. *If $f \in C(\mathscr{T}_h)$ and $0 < q \le 1$, then*

$$\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right| \le (1+h)\,w\left(f; \frac{1}{\sqrt{[m]_q}}, \frac{1}{\sqrt{[n]_q}}\right), \quad (u,v) \in \mathscr{T}_h. \tag{50}$$

Proof. We have

$$\begin{aligned} &\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right| \\ &\le \left[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right)\left|u - [i]_q\frac{(h-v)}{[m]_q}\right|\right. \\ &\quad + \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right)\left|v - [j]_q\frac{\left([m]_q - [i]_q\right)h + [i]_q v}{[m]_q[n]_q}\right| \\ &\quad \left. + \sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right)\right] w(f; \delta_1, \delta_2). \end{aligned} \tag{51}$$

After some transformations, one obtains

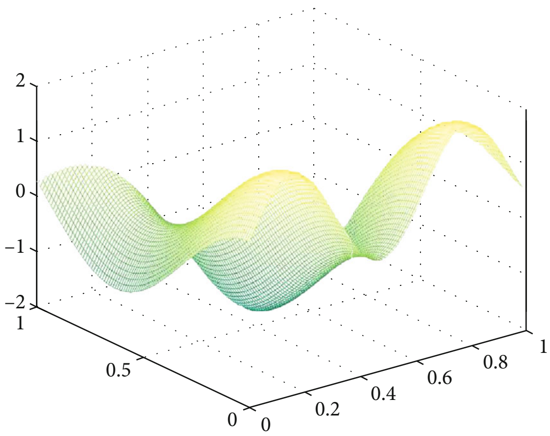
$$\begin{aligned} &\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right)\left|u - [i]_q\frac{(h-v)}{[m]_q}\right| \le \sqrt{\frac{u(h-u-v)}{[m]_q}}, \\ &\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right)\left|v - [j]_q\frac{\left([m]_q - [i]_q\right)h + [i]_q v}{[m]_q[n]_q}\right| \le \sqrt{\frac{v(h-u-v)}{[n]_q}}, \end{aligned} \tag{52}$$

while

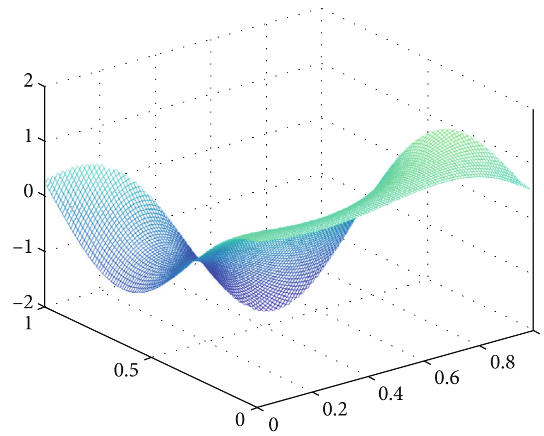
$$\sum_{i=0}^{m}\sum_{j=0}^{n}\tilde{p}_{m,i}(u,v)\tilde{q}_{n,j}\left([i]_q\frac{(h-v)}{[m]_q}, v\right) = 1. \tag{53}$$

It follows

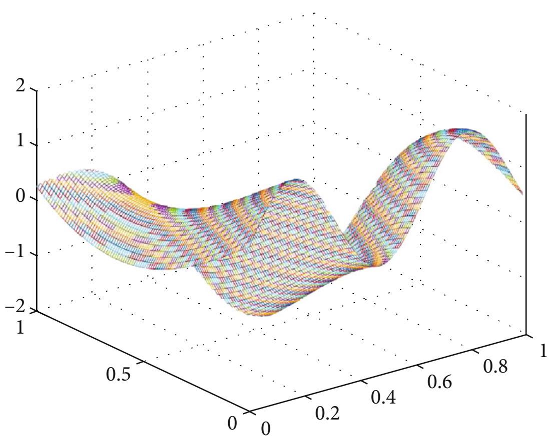
$$\left|\left(\mathscr{R}^{\mathscr{P}}_{mn,q}f\right)(u,v)\right| \le \left(\frac{1}{\delta_1}\sqrt{\frac{u(h-u-v)}{[m]_q}} + \frac{1}{\delta_2}\sqrt{\frac{v(h-u-v)}{[n]_q}} + 1\right) w(f; \delta_1, \delta_2). \tag{54}$$



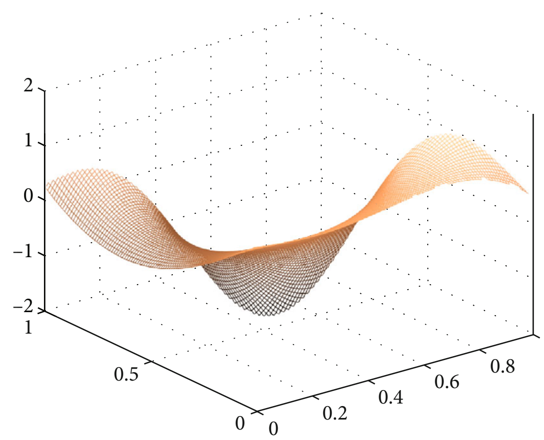
(a) $f(u, v) = \sin(10u) + \cos(5v)$



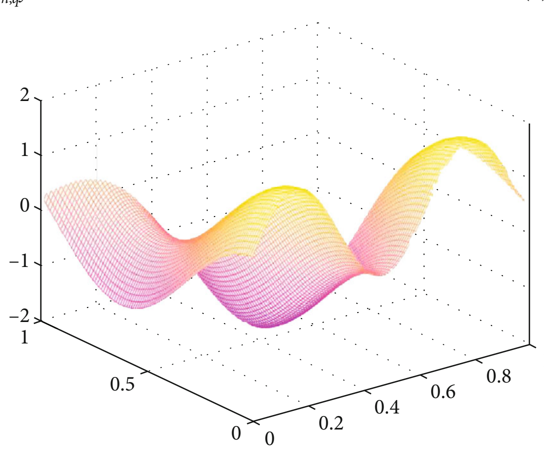
(b) Operator $B^{u}_{m,q}f$



(c) $B^{v}_{n,q}f$

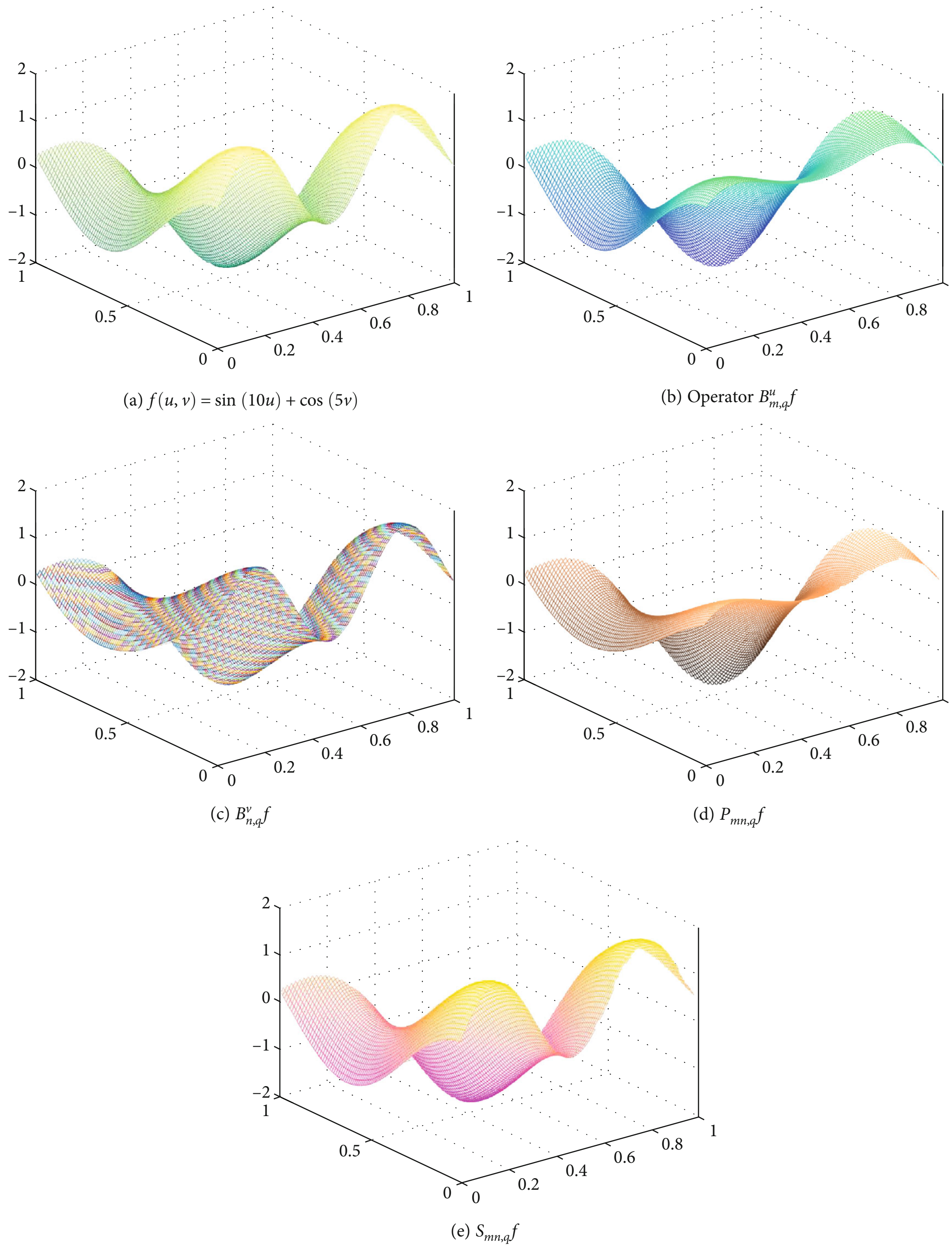


(d) $P_{mn,q}f$



(e) $S_{mn,q}f$

Figure 1: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 6$, $n = 6$, and $q = 0.70$.



(a) $f(u, v) = \sin(10u) + \cos(5v)$

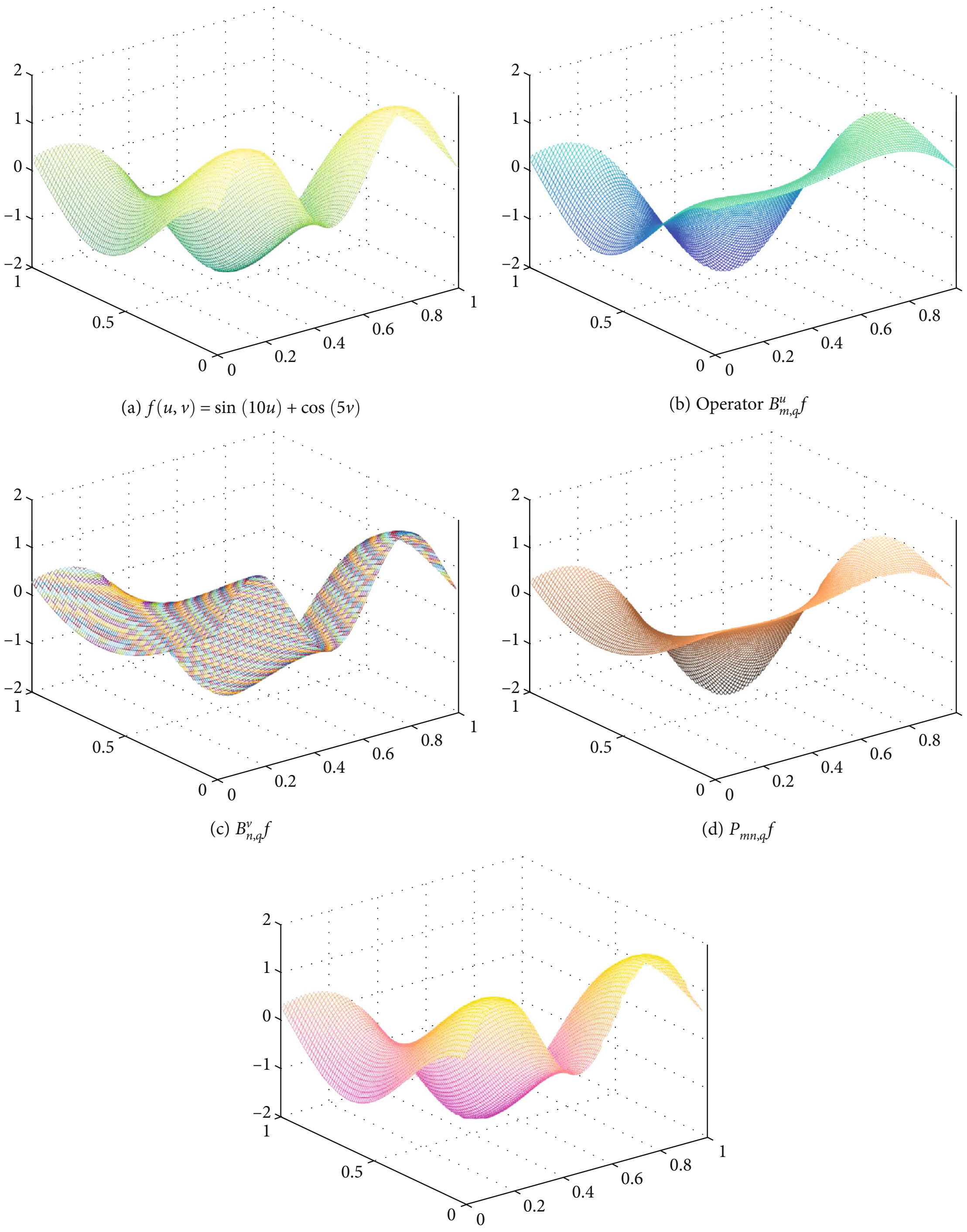
(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

Figure 2: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 6$, $n = 6$, and $q = 0.99$.



(a) $f(u,v) = \sin(10u) + \cos(5v)$

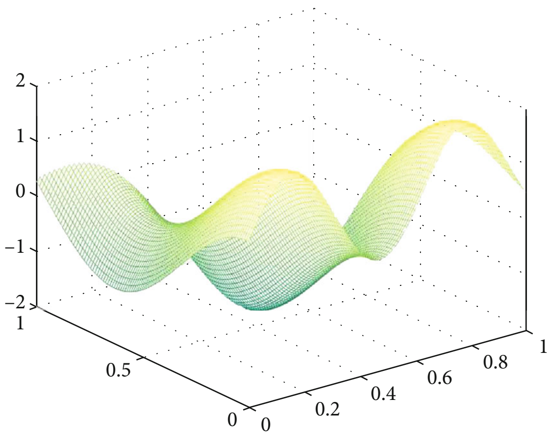
(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

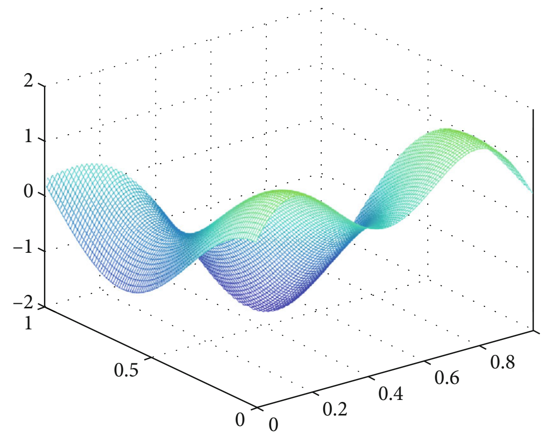
(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

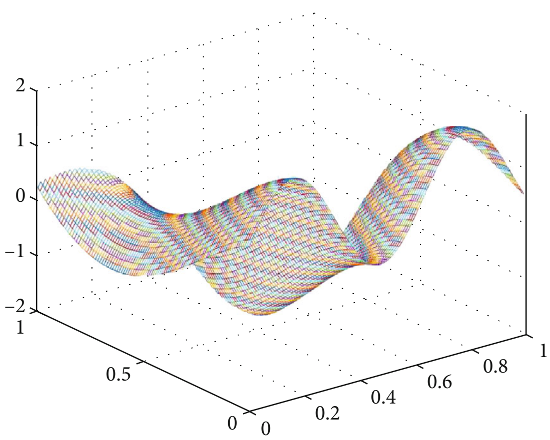
Figure 3: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h=1$, $m=15$, $n=15$, and $q=0.70$.



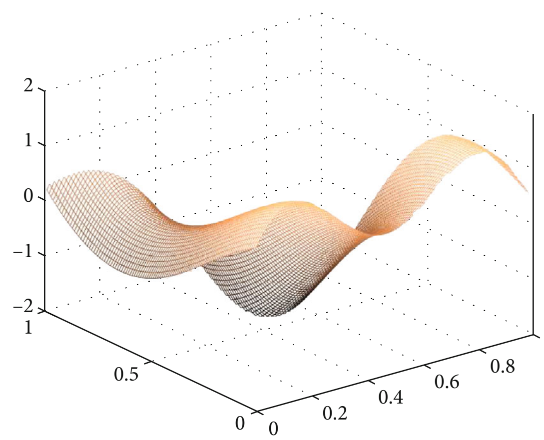
(a) $f(u,v) = \sin(10u) + \cos(5v)$



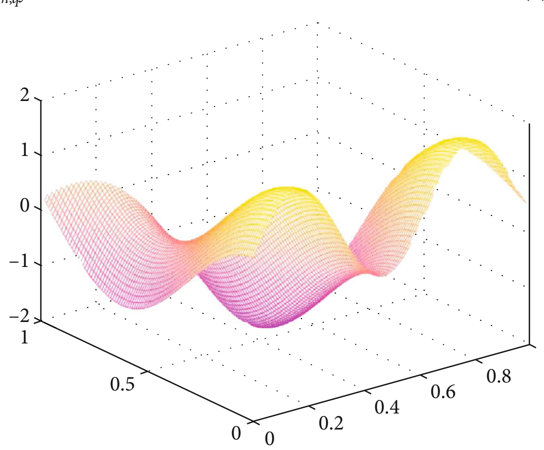
(b) Operator $B^{u}_{m,q}f$



(c) $B^{v}_{n,q}f$

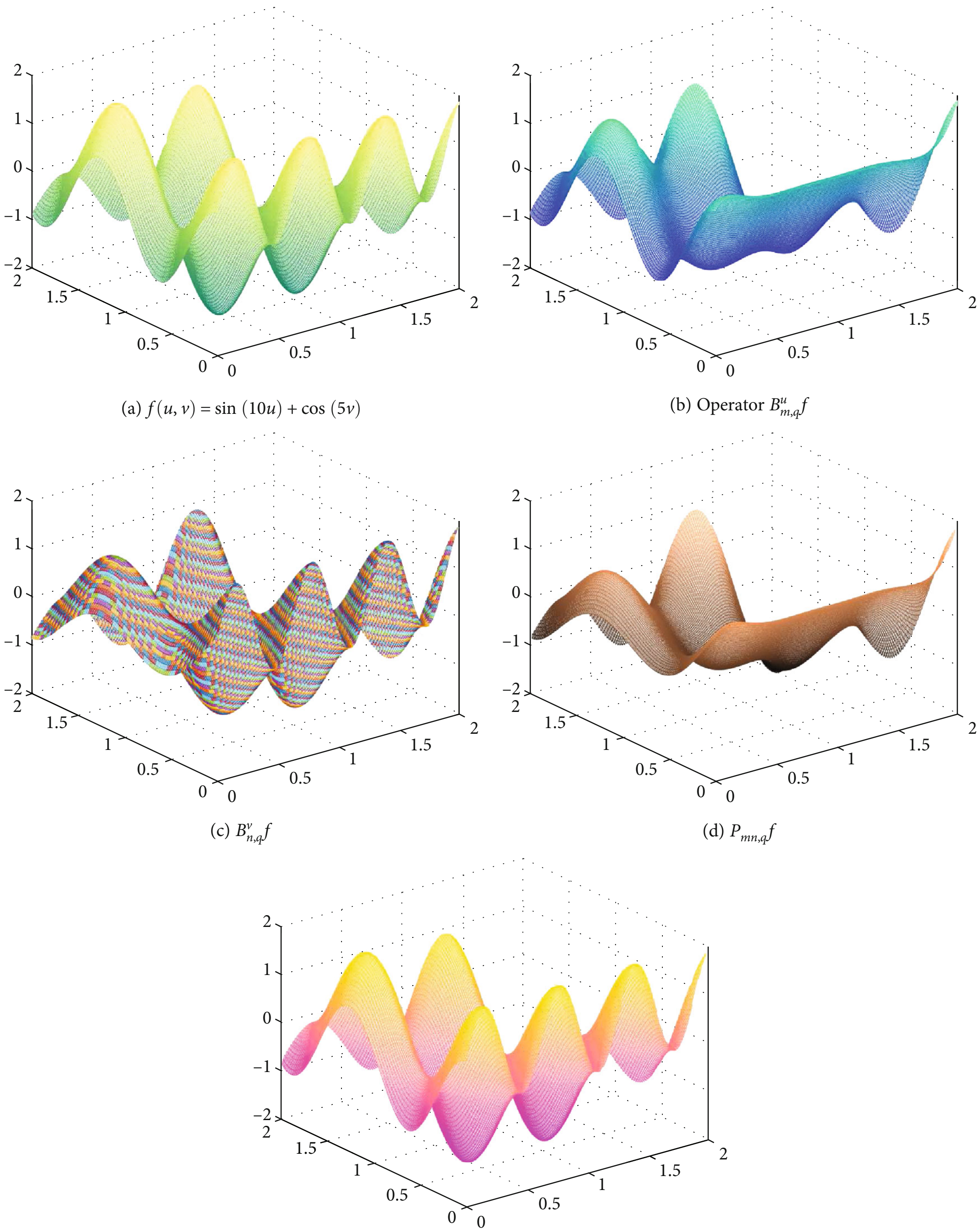


(d) $P_{mn,q}f$



(e) $S_{mn,q}f$

FIGURE 4: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h = 1$, $m = 15$, $n = 10$, and $q = 0.99$.



(a) $f(u,v)=\sin(10u)+\cos(5v)$

(b) Operator $B^{u}_{m,q}f$

(c) $B^{v}_{n,q}f$

(d) $P_{mn,q}f$

(e) $S_{mn,q}f$

FIGURE 5: Operators $\mathscr{B}^{u}_{m,q}f$, $\mathscr{B}^{v}_{n,q}f$, $\mathscr{P}_{mn,q}f$, and $\mathscr{S}_{mn,q}f$ approximating function on triangular domain for $h=2$, $m=10$, $n=10$, and $q=0.99$.

Since

$$\frac{u(h-u-v)}{[m]_q} \le \frac{h^2}{4[m]_q},$$
$$\frac{v(h-u-v)}{[n]_q} \le \frac{h^2}{4[n]_q}, \quad \text{for all } (u,v) \in \mathcal{T}_h. \tag{55}$$

We have

$$|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)| \le \left(\frac{h}{2\delta_1\sqrt{[m]_q}} + \frac{h}{2\delta_2\sqrt{[n]_q}} + 1\right) w(f;\delta_1,\delta_2)|$$
$$\cdot \left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)| \le (1+h)w\left(f;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right). \tag{56}$$

4. Boolean Sum Operators

Let

$$\mathcal{S}_{mn,q} := \mathcal{B}^u_{m,q} \bigoplus \mathcal{B}^v_{n,q} = \mathcal{B}^u_{m,q} + \mathcal{B}^v_{n,q} - \mathcal{B}^u_{m,q}\mathcal{B}^v_{n,q},$$
$$\mathcal{T}_{nm,q} := \mathcal{B}^v_{n,q} \bigoplus \mathcal{B}^u_{m,q} = \mathcal{B}^v_{n,q} + \mathcal{B}^u_{m,q} - \mathcal{B}^v_{n,q}\mathcal{B}^u_{m,q}, \tag{57}$$

be the Boolean sums of the Phillips-type Bernstein operators $\mathcal{B}^u_{m,q}$ and $\mathcal{B}^v_{n,q}$.

Theorem 11. *For the real-valued function f defined on $\mathcal{T}_h$, we have*

$$\mathcal{S}_{mn,q}f\big|_{\partial\mathcal{T}_h} = f\big|_{\partial\mathcal{T}_h}. \tag{58}$$

Proof. We have

$$\mathcal{S}_{mn,q}f = \left(\mathcal{B}^u_{m,q} + \mathcal{B}^v_{n,q} - \mathcal{B}^u_{m,q}\mathcal{B}^v_{n,q}\right)f. \tag{59}$$

The interpolation properties of $\mathcal{B}^u_{m,q}$,$\mathcal{B}^v_{n,q}$ together with properties (i)–(iii) of the operator $\mathcal{P}_{mn,q}$ imply that

$$\left(\mathcal{S}_{mn,q}f\right)(u,0) = \left(\mathcal{B}^u_{m,q}f\right)(u,0) + f(u,0) - \left(\mathcal{B}^u_{m,q}f\right)(u,0) = f(u,0),$$
$$\left(\mathcal{S}_{mn,q}f\right)(0,v) = f(0,v) - \left(\mathcal{B}^v_{n,q}f\right)(0,v) + \left(\mathcal{B}^v_{n,q}f\right)(0,v) = f(0,v),$$
$$\left(\mathcal{S}_{mn,q}f\right)(u,h-u) = f(u,h-u) + f(u,h-u) - f(u,h-u) = f(u,h-u), \tag{60}$$

for all $u, v \in [0,h]$.

Let $\mathcal{R}^{\mathcal{S}}_{mn,q}f$ be the remainder of the Boolean sum approximation formula

$$f = \mathcal{S}_{mn,q}f + \mathcal{R}^{\mathcal{S}}_{mn,q}f. \tag{61}$$

Theorem 12. *If $f \in C(\mathcal{T}_h)$, then*

$$\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}f\right)(u,v)\right| \le \left(1+\frac{h}{2}\right)w\left(f(.,v);\frac{1}{\sqrt{[m]_q}}\right) + \left(1+\frac{h}{2}\right)w\left(f(u,.);\frac{1}{\sqrt{[n]_q}}\right) + (1+h)w \cdot \left(f;\frac{1}{\sqrt{[m]_q}},\frac{1}{\sqrt{[n]_q}}\right), \tag{62}$$

for all $(u,v) \in \mathcal{T}_h$.

Proof. From the equality

$$f - \mathcal{S}_{mn,q}f = f - \mathcal{B}^u_{m,q}f + f - \mathcal{B}^v_{n,q}f - \left(f - \mathcal{P}_{mn,q}f\right), \tag{63}$$

we get

$$\left|\left(\mathcal{R}^{\mathcal{S}}_{mn,q}f\right)(u,v)\right| \le \left|\left(\mathcal{R}^u_{m,q}f\right)(u,v)\right| + \left|\left(\mathcal{R}^v_{n,q}f\right)(u,v)\right| + \left|\left(\mathcal{R}^{\mathcal{P}}_{mn,q}f\right)(u,v)\right|. \tag{64}$$

Now, from (25), (44), and (50), we follow the proof (62).

Remark 13. Analogous relations can be obtained for the remainders of the product approximation formula

$$f = \mathcal{Q}_{nm,q}f + \mathcal{R}^{\mathcal{Q}}_{nm,q}f = \mathcal{B}^v_{n,q}\mathcal{B}^u_{m,q}f + \mathcal{R}^{\mathcal{Q}}_{nm,q}f, \tag{65}$$

and for the Boolean sum formula

$$f = \mathcal{T}_{nm,q}f + \mathcal{R}^{\mathcal{T}}_{nm,q}f = \left(\mathcal{B}^v_{n,q} \oplus \mathcal{B}^u_{m,q}\right)f + \mathcal{R}^{\mathcal{T}}_{nm,q}f. \tag{66}$$

5. Graphical Analysis

Let us consider a function for graphical analysis. In Figure 1(a), we have presented the graph of function $f(u,v) = \sin(10u) + \cos(5v)$ on triangular domain. The graph of Phillips Bernstein operator $\mathcal{B}^u_{m,q}f$ based on quantum analogue on triangular domain is shown in Figure 1(b). Similarly, other operators $\mathcal{B}^v_{n,q}f$, $\mathcal{P}_{mn,q}f$, and $\mathcal{S}_{mn,q}f$ approximating function are shown in Figures 1(c)–1(e) for various values of q, m, n, and h. One can observe from Figures 1–5 that operators are approximating function better as q approaches to 1 for fixed value of m and n.

Also from these figures, one can observe that operator is approximating function better with increasing values of m and n and by fixing q on triangular domain.

Thus, we have constructed Phillips-type q-Bernstein operators over triangular domain which hold the end point interpolation property on some edges and vertices of triangle.

Hence, it can be concluded that after introducing one extra parameter q in Lupas Bernstein operators, we have more modeling flexibility for approximation on triangular domain.

Data Availability

No data are available.

Conflicts of Interest

The authors declare that they have no competing interests.

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