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Research Article

New Generalizations of Set Valued Interpolative Hardy-Rogers Type Contractions in b-Metric Spaces

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Debnath and De La Sen introduced the notion of set valued interpolative Hardy-Rogers type contraction mappings on b-metric spaces and proved that on a complete b-metric space, whose all closed and bounded subsets are compact, the set valued interpolative Hardy-Rogers type contraction mapping has a fixed point. This article presents generalizations of above results by omitting the assumption that all closed and bounded subsets are compact.

1. Introduction

There are numerous studies on interpolation inequalities in literature. In 1999, Chua [1] gave some weighted Sobolev interpolation inequalities on product spaces. Badr and Russ [2] proved some Littlewood-Paley inequalities and interpolation results for Sobolev spaces. Interpolation is considered as one of the central concepts in pure logic. Various interpolation properties find their applications in computer science and have many deep purely logical consequences (see [3, 4]). Gogatishvili and Koskela [5] presented variant interpolation properties of Besov spaces defined on metric spaces. Going in the same direction in the setting of metric spaces via contraction mappings, Karapinar [6] presented the concept of an interpolative Kannan contraction mapping and proved that this mapping admits a fixed point on complete metric spaces. Later on, this notation has been extended into several directions (see [7–18]).

In [6], Karapinar presented the interpolative Kannan contraction as follows: a mapping $K : (W, d_W) \to (W, d_W)$ is an interpolative Kannan contraction if

$$d_W\left(Kw^a, Kw^b\right)\right] \le \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{1-\iota_1} \tag{1}$$

for all $w^a, w^b \in W$ with $w^a \neq Kw^a$, where $\delta \in [0, 1)$ and $\iota_1 \in (0, 1)$. This inequality was further refined by Karapinar et al. [7] by

$$d_W\left(Kw^a, Kw^b\right)\right] \le \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{1-\iota_1} \tag{2}$$

for all $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$, $\iota_1 \in (0,1)$, and $\text{fix}(K) = \{w^a \in W : Kw^a = w^a\}$.

Gaba and Karapinar [9] further modified the interpolative Kannan contraction concept in the following way: a mapping $K : (W, d_W) \to (W, d_W)$ is a $(\delta, \iota_1, \iota_2)$-interpolative Kannan contraction, if

$$d_W\left(Kw^a, Kw^b\right)\Big] \leq \delta[d_W(w^a, Kw^a)]^{\iota_1}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_2} \quad (3)$$

for all $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$, $\iota_1, \iota_2 \in (0,1)$ with $\iota_1 + \iota_2 < 1$. Karapinar et al. [10] gave the interpolative Hardy-Rogers type contraction as follows: a mapping $K : (W, d_W) \to (W, d_W)$ is called an interpolative Hardy-Rogers type contraction if

$$\begin{aligned} &d_W\left(Kw^a, Kw^b\right) \\ &\quad \leq \delta\left[\left[d_W\left(w^a, w^b\right)\right]^{\iota_1}[d_W(w^a, Kw^a)]^{\iota_2}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_3}\right. \\ &\quad\quad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a, Kw^b\right) + d_W\left(Kw^a, w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right] \end{aligned} \quad (4)$$

for each $w^a, w^b \in W \setminus \text{fix}(K)$, where $\delta \in [0,1)$ and $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

Later on, Debnath and De La Sen [12] extended the above definition to set valued interpolative Hardy-Rogers type contraction mappings on b-metric spaces and proved that on complete b-metric spaces, whose all closed and bounded subsets are compact, the set valued interpolative Hardy-Rogers type contraction mapping has a fixed point.

On the other hand, Bakhtin [19] and Czerwik [20] introduced the notion of b-metric spaces.

Definition 1 (see [19, 20]). Let W be a nonempty set and $d_W : W \times W \to [0,\infty)$ be a function so that for all $i, j, \ell \in X$ and some $\rho \geq 1$,

$$\begin{aligned} &d_W(i,j) = 0 \Leftrightarrow i = j, \\ &d_W(i,j) = d_W(j,i), \\ &d_W(i,j) \leq \rho[d_W(i,\ell) + d_W(\ell,j)]. \end{aligned} \quad (5)$$

Then, d_W is a b-metric on W, and (W, d_W, ρ) is called a b-metric space with a coefficient $\rho \geq 1$.

For related works in this setting, see [21–23]. From now on, (W, d_W, ρ) is a b-metric space with a coefficient $\rho \geq 1$. In the whole paper, $\rho \geq 1$ is the coefficient of the b-metric space.

Definition 2 (see [20]). We have the following:

(a) A sequence $\{\eta_n\}$ in W is said to be Cauchy if $\lim_{n,m\to\infty} d_W(\eta_n, \eta_m) = 0$

(b) A sequence $\{\eta_n\}$ in W is said to be convergent to η if $\lim_{n,m\to\infty} d_W(\eta_n, \eta) = 0$

(c) (W, d_W, ρ) is said to be complete if every Cauchy sequence $\{\eta_n\}$ in W is convergent

Denote by $CB(W)$ the set of nonempty closed bounded subsets of W. For $A, B \in CB(X)$, consider

$$\varDelta_W(A,B) = \sup\{d_W(\omega, B)\,;\,\omega \in A\}, \quad (6)$$

where $d_W(\omega, B) = \inf\{d_W(\omega,\mu), \mu \in B\}$. The functional $H_W : CB(W) \times CB(W) \to [0,\infty)$ defined by

$$H_W(A,B) = \max\{\varDelta_W(A,B), \varDelta_W(B,A)\} \quad (7)$$

is known as the Pompieu-Hausdorff b-metric on $CB(W)$. We state the following known lemma.

Lemma 3 (see [24]). *Let (W, d_W, ρ) be a b-metric space ($\rho \geq 1$). Let $A, B \in CB(W)$ and $a \in A$. We have the two following statements:*

(i) For each $\varepsilon > 0$, there is $b \in B$ so that

$$d_W(a,b) \leq H_W(A,B) + \varepsilon \quad (8)$$

(ii) For each $h > 1$, there is $\nu \in B$ so that

$$d_W(a,\nu) \leq hH_W(A,B) \quad (9)$$

This article presents two new generalizations of set valued interpolative Hardy-Rogers type contraction mappings. Namely, we ensure the existence of fixed points of such maps on a complete b-metric space without considering the assumption that all closed and bounded subsets must be compact. Two examples are also presented.

2. Main Results

First, we define the notion of ξ-interpolative Hardy-Rogers type contractions.

Definition 4. Consider a b-metric space (W, d_W, ρ). Also, consider maps $K : W \to CB(W)$ and $\xi : W \times W \to \mathbb{R} \setminus \{0\}$. Such a map K is called an ξ-interpolative Hardy-Rogers type contraction if

$$\begin{aligned} &\left[H_W\left(Kw^a, Kw^b\right)\right]^{\xi(w^a, w^b)} \\ &\quad \leq \delta\left[\left[d_W\left(w^a, w^b\right)\right]^{\iota_1}[d_W(w^a, Kw^a)]^{\iota_2}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_3}\right. \\ &\quad\quad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a, Kw^b\right) + d_W\left(Kw^a, w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right] \end{aligned} \quad (10)$$

for each $w^a, w^b \in W$ with

$$\min\left\{d_W\left(w^a, w^b\right), d_W(w^a, Kw^a), d_W\left(w^b, Kw^b\right)\right\} > 0, \tag{11}$$

where $\delta \in [0, 1/\rho^2)$ and $\iota_1, \iota_2, \iota_3 \in (0, 1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

The following result ensures the existence of a fixed point of ξ-interpolative Hardy-Rogers type contractions.

Theorem 5. *Consider a complete b-metric space (W, d_W, ρ) and consider an ξ-interpolative Hardy-Rogers type contraction map K. Also, consider the given assertions.*

(I) There must exist $w_0^a \in W$ and $w_1^a \in Kw_0^a$ such that $\xi(w_0^a, w_1^a) = 1$

(II) For each $w^a, w^b \in W$ with $\xi(w^a, w^b) = 1$, we have $\xi(w^c, w^d) = 1 \forall w^c \in Kw^a, w^d \in Kw^b$

(III) For each $\{w_m^a\}$ in W with $w_m^a \to w$ and $\xi(w_m^a, w_{m+1}^a) = 1 \forall m \in \mathbb{N}$, we have $\xi(w_m^a, w) = 1 \forall m \in \mathbb{N}$

Then, K must have a fixed point in W.

Proof. By assertion (I) there are $w_0^a \in W$ and $w_1^a \in Kw_0^a$ with $\xi(w_0^a, w_1^a) = 1$. If

$$\min\{d_W(w_0^a, w_1^a), d_W(w_0^a, Kw_0^a), d_W(w_1^a, Kw_1^a)\} = 0, \tag{12}$$

then K has a fixed point. Suppose that

$$\min\{d_W(w_0^a, w_1^a), d_W(w_0^a, Kw_0^a), d_W(w_1^a, Kw_1^a)\} > 0. \tag{13}$$

By (10), we obtain

$$\begin{aligned} &H_W(Kw_0^a, Kw_1^a) \\ &\quad = [H_W(Kw_0^a, Kw_1^a)]^{\xi(w_0^a, w_1^a)} \\ &\quad \le \delta\Bigg[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3} \\ &\qquad \times \left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]. \end{aligned} \tag{14}$$

This leads to

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a) \\ &\quad \le \frac{1}{\sqrt{\delta}} H_W(Kw_0^a, Kw_1^a) \\ &\quad \le \sqrt{\delta}\Bigg[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3} \\ &\qquad \times \left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a) + d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]. \end{aligned} \tag{15}$$

Since $1/\sqrt{\delta} > 1$, there is $w_2^a \in Kw_1^a$ such that

$$d_W(w_1^a, w_2^a) \le \frac{1}{\sqrt{\delta}} d_W(w_1^a, Kw_1^a). \tag{16}$$

Thus, by (15),

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}\Bigg[&[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3} \\ &\times \left[\frac{1}{2\rho}(d_W(w_0^a, w_2^a) + d_W(w_1^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]. \end{aligned} \tag{17}$$

Note that $d_W(w_0^a, w_2^a) \le \rho[d_W(w_0^a, w_1^a) + d_W(w_1^a, w_2^a)] \le 2\rho \max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}$. Hence, by (17), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}\Big[&[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3} \\ &\times [\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}]^{1-\iota_1-\iota_2-\iota_3}\Big]. \end{aligned} \tag{18}$$

Now, we consider $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_0^a, w_1^a)$. Then, by (18), we get

$$\begin{aligned} d_W(w_1^a, w_2^a) \le \sqrt{\delta}\Big[&[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_0^a, w_1^a)]^{\iota_3} \\ &\times [d_W(w_0^a, w_1^a)]^{1-\iota_1-\iota_2-\iota_3}\Big]. \end{aligned} \tag{19}$$

This implies

$$d_W(w_1^a, w_2^a) \le \sqrt{\delta} d_W(w_0^a, w_1^a). \tag{20}$$

Note that when we take $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_1^a, w_2^a)$ in (18), then we get $d_W(w_1^a, w_2^a) = 0$, that is, $w_1^a \in Kw_1^a$; hence, this choice is not possible. As $\xi(w_0^a, w_1^a) = 1$ and $w_1^a \in Kw_0^a$ and $w_2^a \in Kw_1^a$, then by assertion (II), we get $\xi(w_1^a, w_2^a) = 1$. Again, we consider

$$\min\{d_W(w_1^a, w_2^a), d_W(w_1^a, Kw_1^a), d_W(w_2^a, Kw_2^a)\} > 0, \tag{21}$$

then by (10), we get

$$\begin{aligned} &\frac{1}{\sqrt{\delta}} d_W(w_2^a, Kw_2^a) \\ &\quad \le \frac{1}{\sqrt{\delta}} H_W(Kw_1^a, Kw_2^a) \\ &\quad = \frac{1}{\sqrt{\delta}} [H_W(Kw_1^a, Kw_2^a)]^{\xi(w_1^a, w_2^a)} \\ &\quad \le \sqrt{\delta}\Bigg[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, Kw_1^a)]^{\iota_2}[d_W(w_2^a, Kw_2^a)]^{\iota_3} \\ &\qquad \times \left[\frac{1}{2\rho}(d_W(w_1^a, Kw_2^a) + d_W(Kw_1^a, w_2^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]. \end{aligned} \tag{22}$$

Since $1/\sqrt{\delta} > 1$, there is $w_3^a \in Kw_2^a$ such that

$$d_W(w_2^a, w_3^a) \le \frac{1}{\sqrt{\delta}} d_W(w_2^a, Kw_2^a). \tag{23}$$

Thus, by (22), we conclude

$$d_W(w_2^a, w_3^a) \le \sqrt{\delta}\Bigg[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_2^a, w_3^a)]^{\iota_3} \times \left[\frac{1}{2\rho}(d_W(w_1^a, w_3^a) + d_W(w_2^a, w_2^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]. \tag{24}$$

Note that $d_W(w_1^a, w_3^a) \le \rho[d_W(w_1^a, w_2^a) + d_W(w_2^a, w_3^a)] \le 2\rho \max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\}$. Hence, by (24), we get

$$d_W(w_2^a, w_3^a) \le \sqrt{\delta}\Big[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_2^a, w_3^a)]^{\iota_3} \times [\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\}]^{1-\iota_1-\iota_2-\iota_3}\Big]. \tag{25}$$

Now, we consider $\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\} = d_W(w_1^a, w_2^a)$. Then, by (18), we get

$$d_W(w_2^a, w_3^a) \le \sqrt{\delta}\Big[[d_W(w_1^a, w_2^a)]^{\iota_1}[d_W(w_1^a, w_2^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3} \times [d_W(w_1^a, w_2^a)]^{1-\iota_1-\iota_2-\iota_3}\Big]. \tag{26}$$

This yields that

$$d_W(w_2^a, w_3^a) \le \sqrt{\delta} d_W(w_1^a, w_2^a). \tag{27}$$

Note that if we take $\max\{d_W(w_1^a, w_2^a), d_W(w_2^a, w_3^a)\} = d_W(w_2^a, w_3^a)$ in (25), then $d_W(w_2^a, w_3^a) = 0$, that is, $w_2^a \in Kw_2^a$, which is not possible. From (27) and (20), we get

$$d_W(w_2^a, w_3^a) \le \left(\sqrt{\delta}\right)^2 d_W(w_0^a, w_1^a). \tag{28}$$

Proceeding in this way, we can obtain a sequence $\{w_m^a\}$ in W with $w_{m+1}^a \in Kw_m^a$, $\xi(w_m^a, w_{m+1}^a) = 1$ for all $m \in \mathbb{W}$ and

$$d_W(w_m^a, w_{m+1}^a) \le \left(\sqrt{\delta}\right)^m d_W(w_0^a, w_1^a) \forall m \in \mathbb{N}. \tag{29}$$

Also, by the construction of $\{w_m^a\}$, we get

$$\min\{d_W(w_m^a, w_{m+1}^a), d_W(w_m^a, Kw_m^a), d_W(w_{m+1}^a, Kw_{m+1}^a)\} > 0 \forall m \in \mathbb{N}. \tag{30}$$

By a triangular inequality, we have for $n > m$,

$$d_W(w_n^a, w_m^a) \le \sum_{j=m}^{n-1} \rho^j d_W\left(w_j^a, w_{j+1}^a\right) \le \sum_{j=m}^{n-1} \rho^j \left(\sqrt{\delta}\right)^j d_W(w_0^a, w_1^a). \tag{31}$$

Since the above series is convergent, $\{w_m^a\}$ is a Cauchy sequence in W. Completeness of W gives w_*^a in W such that $w_m^a \to w_*^a$. By considering assertion (III), we get $\xi(w_m^a, w_*^a) = 1 \forall m \in \mathbb{N}$. Here, we claim $w_*^a \in Kw_*^a$. If the claim is wrong, then $\min\{d_W(w_m^a, w_*^a), d_W(w_m^a, Kw_m^a), d_W(w_*^a, Kw_*^a)\} > 0$ for all $m > m_0$, for some $m_0 \in \mathbb{N}$. From (10), we get

$$\begin{aligned}
&d_W(w_{m+1}^a, Kw_*^a)\\
&\quad\le H_W(Kw_m^a, Kw_*^a)\\
&\quad= [H_W(Kw_m^a, Kw_*^a)]^{\xi(w_m^a, w_*^a)}\\
&\quad\le \delta\Bigg[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3}\\
&\qquad\times\left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(Kw_m^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg]\\
&\quad\le \delta\Bigg[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3}\\
&\qquad\times\left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a) + d_W(w_{m+1}^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\Bigg] \forall m > m_0.
\end{aligned} \tag{32}$$

From the above, we get $\lim_{m\to\infty} d_W(w_{m+1}^a, Kw_*^a) = 0$. By the triangular inequality, we have

$$d_W(w_*^a, Kw_*^a) \le \rho[d_W(w_*^a, w_{m+1}^a) + d_W(w_{m+1}^a, Kw_*^a)] \forall m \in \mathbb{N}. \tag{33}$$

By taking the limit $m \to \infty$, we get $d_W(w_*^a, Kw_*^a) = 0$, that is, $w_*^a \in Kw_*^a$. Therefore, our claim is valid.

Example 1. Consider $W = \mathbb{Z}$ with $d_W(w_n, w_m) = (w_n - w_m)^2$ for all $w_n, w_m \in W$. Define $K : W \to CB(W)$ by

$$K(w_n) = \begin{cases} \{0\}, & w_n \in \{0, 1, 2, 3, \cdots\} \\ \left\{-(w_n - 2)^2\right\}, & w_n \in \{-1, -2, -3, \cdots\} \end{cases} \tag{34}$$

and $\xi : W \times W \to \mathbb{R} \setminus \{0\}$ by

$$\xi(w_n, w_m) = \begin{cases} 1, & w_n, w_m \in \{0, 1, 2, 3, \cdots\} \\ -[|w_n| + |w_m| + 4], & \text{otherwise}. \end{cases} \tag{35}$$

Note that

Case 1. If $w_n, w_m \ge 0$ with $w_n \ne w_m$, we get $H_W(Kw_n, Kw_m)^{\xi(w_n, w_m)} = 0$.

Case 2. If $w_n, w_m < 0$ with $w_n \neq w_m$, we get

$$H_W(Kw_n, Kw_m)^{\xi(w_n,w_m)} = \frac{1}{\left[\left(-(w_n-2)^2 + (w_m-2)^2\right)^2\right]^{|w_n|+|w_m|+4}}. \quad (36)$$

Case 3. If $w_n < 0$ and $w_m \geq 0$, we get $H_W(Kw_n, Kw_m)^{\xi(w_n,w_m)} = 1/[(-(w_n-2)^2)^2]^{|w_n|+|w_m|+4}$.

After calculating the values, it is easy to see that

For Case1: if $w_n, w_m > 0$ with $w_n \neq w_m$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= 0 < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (37)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

For Case2: if $w_n, w_m < 0$ with $w_n \neq w_m$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= \frac{1}{\left[\left(-(w_n-2)^2 + (w_m-2)^2\right)^2\right]^{|w_n|+|w_m|+4}} \\ &\leq \frac{1}{(49)^7} < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (38)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

For Case3: if $w_n < 0$ and $w_m > 0$, we get

$$\begin{aligned}[H_W(Kw_n, Kw_m)]^{\xi(w_n,w_m)} &= \frac{1}{\left[\left(-(w_n-2)^2\right)^2\right]^{|w_n|+|w_m|+4}} \leq \frac{1}{(81)^6} < \frac{1}{5}\left[1\cdot 1\cdot 1\cdot \frac{1}{4}\right] \\ &\leq \frac{1}{5}\left[[d_W(w_n,w_m)]^{\iota_1}[d_W(w_n,Kw_n)]^{\iota_2}[d_W(w_m,Kw_m)]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}(d_W(w_n,Kw_m)+d_W(Kw_n,w_m))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (39)$$

for each $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$. By keeping these calculations in mind, one can check that all the hypotheses of Theorem 5 are valid. Hence, K must have a fixed point.

The following definition presents a multiplicative ξ-interpolative Hardy-Rogers type contraction.

Definition 6. Consider a b-metric space (W, d_W, ρ). Also, consider the maps $K : W \to CB(W)$ and $\xi : W \times W \to [0, \infty)$. Such K is called a multiplicative ξ-interpolative Hardy-Rogers type contraction if

$$\begin{aligned}\xi\left(w^a, w^b\right)H_W\left(Kw^a, Kw^b\right) &\leq \delta\left[\left[d_W\left(w^a, w^b\right)\right]^{\iota_1}[d_W(w^a, Kw^a)]^{\iota_2}\left[d_W\left(w^b, Kw^b\right)\right]^{\iota_3}\right. \\ &\quad \left.\times\left[\frac{1}{2\rho}\left(d_W\left(w^a, Kw^b\right)+d_W\left(Kw^a, w^b\right)\right)\right]^{1-\iota_1-\iota_2-\iota_3}\right]\end{aligned} \quad (40)$$

for each $w^a, w^b \in W$ with

$$\min\left\{d_W\left(w^a, w^b\right), d_W(w^a, Kw^a), d_W\left(w^b, Kw^b\right)\right\} > 0, \quad (41)$$

where $\delta \in (0, 1/\rho^2)$ and $\iota_1, \iota_2, \iota_3 \in (0,1)$ with $\iota_1 + \iota_2 + \iota_3 < 1$.

The following result concerns the existence of fixed points for the above-defined mapping.

Theorem 7. *Consider a complete b-metric space (W, d_W, ρ) and consider a multiplicative ξ-interpolative Hardy-Rogers type contraction map K. Also, consider the given assertions:*

(i) There must exist $w_0^a \in W$ and $w_1^a \in Kw_0^a$ such that $\xi(w_0^a, w_1^a) \geq 1$

(ii) For each $w^a, w^b \in W$ with $\xi(w^a, w^b) \geq 1$, we have $\xi(w^c, w^d) \geq 1 \forall w^c \in Kw^a, w^d \in Kw^b$

(iii) For each $\{w_m^a\}$ in W with $w_m^a \to w$ and $\xi(w_m^a, w_{m+1}^a) \geq 1 \forall m \in \mathbb{N}$, we have $\xi(w_m^a, w) \geq 1 \forall m \in \mathbb{N}$

Then K possesses a fixed point in W.

Proof. Assertion (i) implies the existence of $w_0^a \in W$ and $w_1^a \in Kw_0^a$ with $\xi(w_0^a, w_1^a) \geq 1$. We consider

$$\min\{d_W(w_0^a, w_1^a), d_W(w_0^a, Kw_0^a), d_W(w_1^a, Kw_1^a)\} > 0. \quad (42)$$

Otherwise, K has a fixed point. Then, by (40), we obtain

$$\begin{aligned}H_W(Kw_0^a, Kw_1^a)\\ &\le \xi(w_0^a, w_1^a)H_W(Kw_0^a, Kw_1^a)\\ &\le \delta\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3}\right.\\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a)+d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right].\end{aligned} \tag{43}$$

This yields that

$$\begin{aligned}\frac{1}{\sqrt{\delta}}d_W(w_1^a, Kw_1^a)\\ &\le \frac{1}{\sqrt{\delta}}H_W(Kw_0^a, Kw_1^a)\\ &\le \sqrt{\delta}\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, Kw_0^a)]^{\iota_2}[d_W(w_1^a, Kw_1^a)]^{\iota_3}\right.\\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, Kw_1^a)+d_W(Kw_0^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right].\end{aligned} \tag{44}$$

Since $1/\sqrt{\delta} > 1$, there is $w_2^a \in Kw_1^a$ satisfying

$$d_W(w_1^a, w_2^a) \le \frac{1}{\sqrt{\delta}}d_W(w_1^a, Kw_1^a). \tag{45}$$

Thus, by (44), we get

$$\begin{aligned}d_W(w_1^a, w_2^a) &\le \sqrt{\delta}\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3}\right.\\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_0^a, w_2^a)+d_W(w_1^a, w_1^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right].\end{aligned} \tag{46}$$

Since $d_W(w_0^a, w_2^a) \le \rho[d_W(w_0^a, w_1^a) + d_W(w_1^a, w_2^a)] \le 2\rho \max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}$, we get using (46),

$$\begin{aligned}d_W(w_1^a, w_2^a) &\le \sqrt{\delta}\left[[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_1^a, w_2^a)]^{\iota_3}\right.\\ &\left.\times[\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\}]^{1-\iota_1-\iota_2-\iota_3}\right].\end{aligned} \tag{47}$$

Consider $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_0^a, w_1^a)$. Then, by (47), we get

$$\begin{aligned}d_W(w_1^a, w_2^a) &\le \sqrt{\delta}[d_W(w_0^a, w_1^a)]^{\iota_1}[d_W(w_0^a, w_1^a)]^{\iota_2}[d_W(w_0^a, w_1^a)]^{\iota_3}\\ &\times[d_W(w_0^a, w_1^a)]^{1-\iota_1-\iota_2-\iota_3}].\end{aligned} \tag{48}$$

This implies that

$$d_W(w_1^a, w_2^a) \le \sqrt{\delta}d_W(w_0^a, w_1^a). \tag{49}$$

If we take $\max\{d_W(w_0^a, w_1^a), d_W(w_1^a, w_2^a)\} = d_W(w_1^a, w_2^a)$ in (47), then we get $d_W(w_1^a, w_2^a) = 0$, that is, $w_1^a \in Kw_1^a$, which is not possible. Since $\xi(w_0^a, w_1^a) \ge 1$, $w_1^a \in Kw_0^a$, and $w_2^a \in Kw_1^a$, by assertion (ii), we get $\xi(w_1^a, w_2^a) \ge 1$. Applying (40) and again assertion (ii), we can obtain a sequence $\{w_m^a\}$ in W with $w_{m+1}^a \in Kw_m^a$, $\xi(w_m^a, w_{m+1}^a) \ge 1$ for all $m \in \mathbb{W}$ and

$$d_W(w_m^a, w_{m+1}^a) \le \left(\sqrt{\delta}\right)^m d_W(w_0^a, w_1^a) \forall m \in \mathbb{N}. \tag{50}$$

Also, by construction of $\{w_m^a\}$, we know that

$$\begin{aligned}&\min\{d_W(w_m^a, w_{m+1}^a), d_W(w_m^a, Kw_m^a), d_W(w_{m+1}^a, Kw_{m+1}^a)\}\\ &> 0 \forall m \in \mathbb{N}.\end{aligned} \tag{51}$$

By a triangular inequality, we have for $n > m$,

$$d_W(w_n^a, w_m^a) \le \sum_{j=m}^{n-1}\rho^j d_W\left(w_j^a, w_{j+1}^a\right) \le \sum_{j=m}^{n-1}\rho^j\left(\sqrt{\delta}\right)^j d_W(w_0^a, w_1^a). \tag{52}$$

This implies that $\{w_m^a\}$ is a Cauchy sequence in W. Since W is complete, $w_m^a \to w_*^a \in W$. By assertion (iii), we get $\xi(w_m^a, w_*^a) \ge 1$ for all $m \in \mathbb{N}$. Now, we claim that $w_*^a \in Kw_*^a$. Assume the claim is wrong, then $\min\{d_W(w_m^a, w_*^a), d_W(w_m^a, Kw_m^a), d_W(w_*^a, Kw_*^a)\} > 0$ for all $m > m_0$ for some $m_0 \in \mathbb{N}$. Then by (40), we get

$$\begin{aligned}d_W(w_{m+1}^a, Kw_*^a)\\ &\le H_W(Kw_m^a, Kw_*^a)\\ &= \xi(w_m^a, w_*^a)H_W(Kw_m^a, Kw_*^a)\\ &\le \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3}\right.\\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a)+d_W(Kw_m^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\\ &\le \delta\left[[d_W(w_m^a, w_*^a)]^{\iota_1}[d_W(w_m^a, Kw_m^a)]^{\iota_2}[d_W(w_*^a, Kw_*^a)]^{\iota_3}\right.\\ &\left.\times\left[\frac{1}{2\rho}(d_W(w_m^a, Kw_*^a)+d_W(w_{m+1}^a, w_*^a))\right]^{1-\iota_1-\iota_2-\iota_3}\right]\forall m > m_0.\end{aligned} \tag{53}$$

From the above inequality, we get $\lim_{m\to\infty}d_W(w_{m+1}^a, Kw_*^a) = 0$. By a triangular inequality, we have

$$d_W(w_*^a, Kw_*^a) \le \rho[d_W(w_*^a, w_{m+1}^a) + d_W(w_{m+1}^a, Kw_*^a)] \forall m \in \mathbb{N}. \tag{54}$$

Hence, by taking the limit $m \longrightarrow \infty$, we get $d_W(w_*^a, K w_*^a) = 0$, that is, $w_*^a \in K w_*^a$.

Example 2. Consider $W = \mathbb{Z}$ with $d_W(w_n, w_m) = |w_n - w_m|$ for all $w_n, w_m \in W$. Define $K : W \longrightarrow CB(W)$ by

$$K(w_n) = \begin{cases} \{0\}, & w_n \in \{0, 1, 2, 3, \cdots\} \\ \{0, 2w_n\}, & w_n \in \{-1, -2, -3, \cdots\} \end{cases} \quad (55)$$

and $\xi : W \times W \to \mathbb{R} - \{0\}$ by

$$\xi(w_n, w_m) = \begin{cases} 1, & w_n, w_m \in \{0, 1, 2, 3, \cdots\} \\ 0, & \text{otherwise.} \end{cases} \quad (56)$$

One can see that all the hypotheses of Theorem 7 are valid. Hence, K must have a fixed point.

Remark 8. Note that ([12], Theorem 2) is not applicable in Example 2. It suffices to take $x = -1$ and $y = -2$, then $Kx = \{0, -2\}$ and $Ky = \{0, -4\}$. Thus, we have $H(Kx, Ky) = 2$, $d(x, y) = 1$, $d(x, Kx) = 1$, $d(y, Ky) = 2$, $d(y, Kx) = 0$, and $d(x, Ky) = 1$. One then writes

$$H(Kx, Ky) = 2 > \delta \left[(1^{\iota_1})(1^{\iota_2})(2^{\iota_3}) \left(\left(\frac{1}{2}\right)^{1-\iota_1-\iota_2-\iota_3} \right) \right] \quad (57)$$

for all $\delta, \iota_1, \iota_2, \iota_3 \in (0, 1)$. Thus, our main results generalize and improve the result given in [12]. Moreover, when considering the single valued case in Theorem 5 and Theorem 7, that is, for a self-mapping $K : W \longrightarrow W$, we get generalizations of the main results in [9].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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