Research Article

Caputo Fractional Derivative Hadamard Inequalities for Strongly m -Convex Functions

Xue Feng¹, Baolin Feng², Ghulam Farid³, Sidra Bibi⁴, Qi Xiaoyan¹ and Ze Wu¹

¹Aviation University of Air Force, Fundamental Departmental, Changchun, Jilin 130000, China

²Mudanjiang Normal University, School of Mathematical Sciences Mudanjiang, Heilongjiang 157011, China

³Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan

⁴Govt. Girls Primary School, Kamra Khurd, Attock 43570, Pakistan

Correspondence should be addressed to Xue Feng; fx20071981@163.com and Ghulam Farid; faridphdsms@hotmail.com

Received 12 December 2020; Revised 22 January 2021; Accepted 1 April 2021; Published 22 April 2021

Academic Editor: Maria Alessandra Ragusa

Copyright © 2021 Xue Feng et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, two versions of the Hadamard inequality are obtained by using Caputo fractional derivatives and strongly m -convex functions. The established results will provide refinements of well-known Caputo fractional derivative Hadamard inequalities for m -convex and convex functions. Also, error estimations of Caputo fractional derivative Hadamard inequalities are proved and show that these are better than error estimations already existing in literature.

1. Introduction

Strongly convex function was introduced by Polyak in [1]. Strong convexity is a strengthening of the notion of convexity; some properties of strongly convex functions are just “stronger versions” of known properties of convex functions. Strongly convex functions have been used for proving the convergence of a gradient-type algorithm for minimizing a function. They play an important role in the optimization theory and mathematical economics.

Definition 1. Let D be a convex subset of $\mathbb{X}$, $(\mathbb{X}, \|\cdot\|)$ be a normed space. A function $\psi : D \subset \mathbb{X} \rightarrow \mathbb{R}$ is called strongly convex function with modulus C if it satisfies

$$\psi(az + (1-z)b) \leq z\psi(a) + (1-z)\psi(b) - Cz(1-z)\|a-b\|^2, \quad (1)$$

$\forall a, b \in D, z \in [0, 1]$ and $C \geq 0$.

Many authors have been inventing the properties and applications of strongly convex function, for detailed information, see [2–6].

The concepts of m -convex functions and strongly m -convex functions were introduced in [7, 8], respectively. Toader [7] gave the idea of m -convex functions as follows.

Definition 2. A function $\psi : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $z \in [0, 1]$, we have

$$\psi(zx + m(1-z)y) \leq z\psi(x) + m(1-z)\psi(y). \quad (2)$$

Lara et al. introduced strongly m -convex functions as follows:

Definition 3 (see [8]). A function $\psi : I \rightarrow \mathbb{R}$ is called strongly m -convex function with modulus $C \geq 0$ if

$$\psi(za + m(1-z)b) \leq z\psi(a) + m(1-z)\psi(b) - C mz(1-z)|a-b|^2, \quad (3)$$

for $a, b \in I$ and $z \in [0, 1]$.

Nowadays, fractional integral inequalities are in the study of several researchers (see, [9–17] and references therein),

where they have used different kinds of well-known functions and fractional integral operators. Here, in this paper, we are interested to produce fractional integral inequalities for Caputo fractional derivatives of strongly m -convex functions. The Caputo fractional derivative operators are defined as follows.

Definition 4 (see [18]). Let $\psi \in AC^n[a, b]$. The Caputo fractional derivatives of order $\beta > 0$ of ψ are defined as follows:

$$\begin{aligned} {}^C D_{a+}^\beta \psi(x) &= \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\psi^{(n)}(z)}{(x-z)^{\beta-n+1}} dz, x > a, \\ {}^C D_{b-}^\beta \psi(x) &= \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \frac{\psi^{(n)}(z)}{(z-x)^{\beta-n+1}} dz, x < b, \end{aligned} \quad (4)$$

where $n = [\beta] + 1$. If $\beta = n \in \{1, 2, 3, \dots\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^C D_{a+}^\beta \psi)(x)$ coincides with $\psi^{(n)}(x)$, whereas $({}^C D_{a-}^\beta \psi)(x)$ coincides with $\psi^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular, we have

$$({}^C D_{a+}^0 \psi)(x) = ({}^C D_{b-}^0 \psi)(x) = \psi(x), \quad (5)$$

where $n = 1$ and $\beta = 0$.

The Hadamard inequality is another interpretation of convex function. It is stated as follows.

Definition 5 (see [19]). Let $\psi : I \rightarrow \mathbb{R}$ be a convex function on interval $I \subset \mathbb{R}$ and $a, b \in I$ where $a < b$. Then, the following inequality holds:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \psi(x) dx \leq \frac{\psi(a) + \psi(b)}{2}. \quad (6)$$

If order in (6) is reversed, then it holds for concave function.

Farid et al. [20] have proved the following Hadamard inequality for Caputo fractional derivatives of convex functions:

Theorem 6. Let $\psi : [a, b] \rightarrow \mathbb{R}$ be the function with $\psi \in AC^n[a, b]$ and $0 \leq a < b$. Also, let $\psi^{(n)}$ be positive and convex function on $[a, b]$. Then, the following inequality holds for Caputo fractional derivatives:

$$\begin{aligned} \psi^{(n)}\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \left[({}^C D_{a+}^\beta \psi)(b) + (-1)^n ({}^C D_{b-}^\beta \psi)(a) \right] \\ &\leq \frac{[\psi^{(n)}(a) + \psi^{(n)}(b)]}{2}. \end{aligned} \quad (7)$$

They also established the following identity.

Lemma 7 (see [20]). Let $\psi : [a, b] \rightarrow \mathbb{R}, 0 \leq a < b$, be the function such that $\psi \in AC^n[a, b]$. Then, the following equality for Caputo fractional derivatives holds:

$$\begin{aligned} &\frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \\ &\quad \cdot \left[({}^C D_{a+}^\beta \psi)(b) + (-1)^n ({}^C D_{b-}^\beta \psi)(a) \right] \\ &= \frac{b-a}{2} \int_0^1 \left[(1-z)^{n-\beta} - z^{n-\beta} \psi^{(n+1)}(za + (1-z)b) \right] dz. \end{aligned} \quad (8)$$

Farid et al. [20] also proved the following inequality for Caputo fractional derivatives.

Theorem 8. Let $\psi : [a, b] \rightarrow \mathbb{R}, 0 \leq a < b$ be the function with $\psi \in AC^{n+1}[a, b]$ and also let $|\psi^{(n+1)}|$ be convex on $[a, b]$. Then, the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} &\left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \right. \\ &\quad \cdot \left. \left[({}^C D_{a+}^\beta \psi)(b) + (-1)^n ({}^C D_{b-}^\beta \psi)(a) \right] \right| \\ &\leq \frac{(b-a)}{2(n-\beta+1)} \left(1 - \frac{1}{2^{n-\beta}} \right) \left[|\psi^{(n+1)}(a)| + |\psi^{(n+1)}(b)| \right]. \end{aligned} \quad (9)$$

Kang et al. [21] proved the following version of the Hadamard inequality for Caputo fractional derivatives.

Theorem 9. Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a positive function with $\psi \in AC^n[a, b]$ and $0 \leq a < b$. If $\psi^{(n)}$ is convex function on $[a, b]$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} \psi^{(n)}\left(\frac{a+b}{2}\right) &\leq \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[({}^C D_{((a+b)/2)+}^\beta \psi)(b) \right. \\ &\quad \left. + (-1)^n ({}^C D_{((a+b)/2)-}^\beta \psi)(a) \right] \\ &\leq \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2}. \end{aligned} \quad (10)$$

Farid et al. [22] established the following identity.

Lemma 10. Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < mb, m \in (0, 1)$. If $\psi \in AC^{n+1}[a, b]$, then

the following equality for Caputo fractional derivatives holds:

$$\begin{aligned} & \frac{2^{n-(\beta/k)-1} k \Gamma_k(n - (\beta/k) + k)}{(mb - a)^{n-(\beta/k)}} \left[\left({}^C D_{((a+bm)/2)+}^{\beta,k} \psi \right)(mb) \right. \\ & \quad \left. + m^{n-(\beta/k)+1} (-1)^n \left({}^C D_{((a+bm)/2m)-}^{\beta,k} \psi \right) \left(\frac{a}{m} \right) \right] \\ & \quad - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \\ & = \frac{mb-a}{4} \left[\int_0^1 z^{n-(\beta/k)} \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) \right. \\ & \quad \left. - \int_0^1 z^{n-(\beta/k)} \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) dz \right], \beta > 0. \end{aligned} \quad (11)$$

Kang et al. [21] also proved the following inequalities for Caputo fractional derivatives.

Theorem 11. Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $\psi \in AC^{n+1}[a, b]$ and $a < b$. If $|\psi^{(n+1)}|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[\left({}^C D_{((a+b)/2)+}^{\beta} \psi \right)(b) \right. \right. \\ & \quad \left. \left. + (-1)^n \left({}^C D_{((a+b)/2)-}^{\beta} \psi \right)(a) \right] - \psi^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4(n-\beta+1)^{1/p}} \left(\frac{1}{2(n-\beta+1)(n-\beta+2)} \right)^{1/q} \\ & \quad \cdot \left[\left((n-\beta+1) |\psi^{(n+1)}(a)|^q + (n-\beta+3) |\psi^{(n+1)}(b)|^q \right)^{1/q} \right. \\ & \quad \left. + \left((n-\beta+3) |\psi^{(n+1)}(a)|^q + (\beta+1) |\psi^{(n+1)}(b)|^q \right)^{1/q} \right], \beta > 0. \end{aligned} \quad (12)$$

Theorem 12 (see [21]). Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a function with $\psi \in AC^{n+1}[a, b]$ and $a < b$. If $|\psi^{(n+1)}|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[\left({}^C D_{((a+b)/2)+}^{\beta} \psi \right)(b) \right. \right. \\ & \quad \left. \left. + (-1)^n \left({}^C D_{((a+b)/2)-}^{\beta} \psi \right)(a) \right] - \psi^{(n)} \left(\frac{a+b}{2} \right) \right| \\ & \leq \frac{b-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \left[\left(\frac{|\psi^{(n+1)}(a)|^q + 3|\psi^{(n+1)}(b)|^q}{4} \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{3|\psi^{(n+1)}(a)|^q + |\psi^{(n+1)}(b)|^q}{4} \right)^{1/q} \right] \\ & \leq \frac{b-a}{4} \left(\frac{4}{3(np-\beta p+1)} \right)^{1/p} \\ & \quad \cdot \left[|\psi^{(n+1)}(a)| + |\psi^{(n+1)}(b)| \right], \beta > 0, \frac{1}{p} + \frac{1}{q} = 1. \end{aligned} \quad (13)$$

We will study all of the above fractional inequalities for strongly m -convex functions and at the same time will obtain their generalizations and refinements. In Section 2, we will give refinements of two versions of the Hadamard inequality for Caputo fractional derivatives. We will connect their particular cases with some well-known results. In Section 3, by applying known identities, we will give refinements of error estimations of the Hadamard inequalities.

2. Main Results

The following result is the generalization of Theorem 6 which in a particular case also provides its refinement.

Theorem 13. Let $\psi \in AC^n[a, b]$, $0 \leq a < mb$ be a positive function. If $\psi^{(n)}$ is a strongly m -convex function with modulus $C \geq 0$, $m \in (0, 1]$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \psi^{(n)} \left(\frac{bm+a}{2} \right) + \frac{mC(n-\beta)}{4(n-\beta+2)} \\ & \quad \cdot \left\{ (b-a)^2 + \frac{2(b-a)((a/m)-mb)}{(n-\beta+1)} + \frac{2((a/m)-mb)^2}{(n-\beta)(n-\beta+1)} \right\} \\ & \leq \frac{\Gamma(n-\beta+1)}{2(bm-a)^{n-\beta}} \left[m^{n-\beta+1} (-1)^n \left({}^C D_b^{\beta} \psi \right) \left(\frac{a}{m} \right) + \left({}^C D_a^{\beta} \psi \right)(mb) \right] \\ & \leq \frac{(n-\beta)}{2(n-\beta+1)} \left\{ \frac{[m^2 \psi^{(n)}(a/m^2) + m \psi^{(n)}(b)]}{(n-\beta)} \right. \\ & \quad \left. + [m \psi^{(n)}(b) + \psi^{(n)}(a)] - \frac{Cm((b-a)^2 + (b-(a/m^2))^2)}{(n-\beta+2)} \right\}, \end{aligned} \quad (14)$$

with $\beta > 0$.

Proof. Since $\psi^{(n)}$ is strongly m -convex function with modulus C , for $x, y \in [a, b]$, we have

$$\psi^{(n)} \left(\frac{mx+y}{2} \right) \leq \frac{m\psi^{(n)}(x) + \psi^{(n)}(y)}{2} - \frac{mC}{4} |x-y|^2. \quad (15)$$

Let $x = (1-z)(a/m) + zb \leq b$ and $y = m(1-z)b + za \geq a$, $z \in [0, 1]$. Then, we have

$$\begin{aligned} 2\psi^{(n)} \left(\frac{bm+a}{2} \right) & \leq m\psi^{(n)} \left((1-z) \frac{a}{m} + zb \right) \\ & \quad + \psi^{(n)}(m(1-z)b + za) \\ & \quad - \frac{mC}{2} \left((1-z) \frac{a}{m} + zb \right. \\ & \quad \left. - (m(1-z)b + za) \right)^2. \end{aligned} \quad (16)$$

By multiplying (16) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} & 2\psi^{(n)}\left(\frac{bm+a}{2}\right)\int_0^1 z^{n-\beta-1} dz \\ & \leq \int_0^1 m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right)z^{n-\beta-1} dz \\ & \quad + \int_0^1 \psi^{(n)}(m(1-z)b + za)z^{n-\beta-1} dz \\ & \quad - \frac{mC}{2}\int_0^1 \left(z^{n-\beta-1} \times (1-z)\frac{a}{m} + zb - (m(1-z)b + za)^2\right) dz. \end{aligned} \quad (17)$$

By using change of variables and computing the last integral, from (17), we get

$$\begin{aligned} & \frac{2}{n-\beta}\psi^{(n)}\left(\frac{bm+a}{2}\right) \\ & \leq \frac{1}{bm-a}\left(\int_{\frac{a}{m}}^b m^2\psi^{(n)}(x)\left(\frac{mx-a}{bm-a}\right)^{n-\beta-1} dx \right. \\ & \quad \left. + \int_a^{mb} \psi^{(n)}(y)\left(\frac{bm-y}{bm-a}\right)^{n-\beta-1} dy\right) \\ & \quad - \frac{mC}{2}\left\{\frac{(b-a)^2}{n-\beta+2} + \frac{2(b-a)((a/m)-mb)}{(n-\beta+1)(n-\beta+2)} \right. \\ & \quad \left. + \frac{2((a/m)-mb)^2}{(n-\beta)(n-\beta+1)(n-\beta+2)}\right\}. \end{aligned} \quad (18)$$

Further, it takes the following form

$$\begin{aligned} \psi^{(n)}\left(\frac{bm+a}{2}\right) & \leq \frac{\Gamma(n-\beta+1)}{2(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^CD_b^\beta\psi\right)\left(\frac{a}{m}\right) \right. \\ & \quad \left. + \left({}^CD_a^\beta\psi\right)(mb)\right] - \frac{mC(n-\beta)}{4(n-\beta+2)} \\ & \quad \cdot \left\{(b-a)^2 + \frac{2(b-a)((a/m)-mb)}{(n-\beta+1)} \right. \\ & \quad \left. + \frac{2((a/m)-mb)^2}{(n-\beta)(n-\beta+1)}\right\}. \end{aligned} \quad (19)$$

Since $\psi^{(n)}$ is strongly m -convex function with modulus C , for $z \in [0, 1]$, then one has

$$\begin{aligned} & m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right) + \psi^{(n)}(m(1-z)b + za) \\ & \leq m\psi^{(n)}(b) + m^2(1-z)\psi^{(n)}\left(\frac{a}{m^2}\right) + z\psi^{(n)}(a) \\ & \quad + m(1-z)\psi^{(n)}(b) - Cmz(1-z)(b-a)^2 \\ & \quad - Cm^2z(1-z)\left(b - \frac{a}{m^2}\right)^2. \end{aligned} \quad (20)$$

By multiplying (20) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 m\psi^{(n)}\left((1-z)\frac{a}{m} + zb\right)z^{n-\beta-1} dz \\ & \quad + \int_0^1 \psi^{(n)}(m(1-z)b + za)z^{n-\beta-1} dz \\ & \leq \int_0^1 m\psi^{(n)}(b)z^{n-\beta} dz + \int_0^1 m^2\psi^{(n)}\left(\frac{a}{m^2}\right)(1-z)z^{n-\beta-1} dz \\ & \quad + \int_0^1 \psi^{(n)}(a)z^{n-\beta} dz + \int_0^1 m\psi^{(n)}(b)(1-z)z^{n-\beta-1} dz \\ & \quad - Cm\left((b-a)^2 + \left(b - \frac{a}{m^2}\right)\right)\int_0^1 (1-z)z^{n-\beta} dz. \end{aligned} \quad (21)$$

By using change of variables and computing the last integral, from (21), we get

$$\begin{aligned} & \frac{1}{bm-a}\int_{a/m}^b m^2\psi^{(n)}(x)\left(\frac{mx-a}{bm-a}\right)^{n-\beta-1} dx \\ & \quad + \frac{1}{bm-a}\int_a^{mb} \psi^{(n)}(y)\left(\frac{bm-y}{bm-a}\right)^{n-\beta-1} dy \\ & \leq \frac{[m^2\psi^{(n)}(a/m^2) + m\psi^{(n)}(b)]}{(n-\beta)(n-\beta+1)} + \frac{[m\psi^{(n)}(b) + \psi^{(n)}(a)]}{n-\beta+1} \\ & \quad - \frac{Cm\left((b-a)^2 + \left(b - (a/m^2)\right)^2\right)}{(n-\beta+1)(n-\beta+2)}. \end{aligned} \quad (22)$$

Further, it takes the following form

$$\begin{aligned} & \frac{\Gamma(n-\beta+1)}{2(bm-a)^{n-\beta}}\left[m^{n-\beta+1}(-1)^n\left({}^CD_b^\beta\psi\right)\left(\frac{a}{m}\right) + \left({}^CD_a^\beta\psi\right)(mb)\right] \\ & \leq \frac{(n-\beta)}{2(n-\beta+1)}\left\{\frac{[m^2\psi^{(n)}(a/m^2) + m\psi^{(n)}(b)]}{(n-\beta)} \right. \\ & \quad \left. + [m\psi^{(n)}(b) + \psi^{(n)}(a)] \right. \\ & \quad \left. - \frac{Cm\left((b-a)^2 + \left(b - (a/m^2)\right)^2\right)}{(n-\beta+2)}\right\}. \end{aligned} \quad (23)$$

Inequalities (19) and (11) constituted the required inequality.

The consequences of Theorem 13 are stated in the following corollary and remark:

Corollary 14. By setting $m = 1$ in inequality (14), we will get ([23], Theorem 6)

$$\begin{aligned} & \psi^{(n)}\left(\frac{b+a}{2}\right) + \frac{C(b-a)^2[(\beta-n+2)+(n-\beta)^2]}{4(n-\beta+1)(n-\beta+2)} \\ & \leq \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \left[(-1)^n \left({}^C D_b^\beta \psi \right)(a) + \left({}^C D_a^\beta \psi \right)(b) \right] \\ & \leq \frac{[\psi^{(n)}(a) + \psi^{(n)}(b)]}{2} - \frac{C(n-\beta)(b-a)^2}{(n-\beta+1)(n-\beta+2)}. \end{aligned} \quad (24)$$

Remark 15. If $C = 0$ and $m = 1$ in (14), then we will get the fractional Hadamard inequality stated in Theorem 6.

The upcoming result is the refinement of another version of the Hadamard inequality for Caputo fractional derivatives stated in a theorem in [19].

Theorem 16. Under the assumptions of Theorem 13, the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned} & \psi^{(n)}\left(\frac{bm+a}{2}\right) + \frac{mC(n-\beta)}{8(n-\beta+2)} \\ & \cdot \left\{ \frac{(b-a)^2}{2} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{(n-\beta+1)} \right. \\ & \left. + \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)} \right\} \\ & \leq \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2)}^\beta \psi \right)\left(\frac{a}{m}\right) \right. \\ & \left. + \left({}^C D_{((a+bm)/2)}^\beta \psi \right)(mb) \right] \\ & \leq \frac{(n-\beta)}{4(n-\beta+1)} \left\{ \frac{(n-\beta+2)[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)]}{(n-\beta)} \right. \\ & \left. + [m\psi^{(n)}(b) + \psi^{(n)}(a)] \right. \\ & \left. - \frac{mC(n-\beta+3)[(b-a)^2 + m(b-(a/m^2))^2]}{2(n-\beta+2)} \right\}, \end{aligned} \quad (25)$$

with $\beta > 0$.

Proof. Let $x = (a/m)((2-z)/2) + b(z/2)$ and $y = a(z/2) + m((2-z)/2)b$, $z \in [0, 1]$ in (15), then we have

$$\begin{aligned} & 2\psi^{(n)}\left(\frac{bm+a}{2}\right) \\ & \leq m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right) + \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) \\ & - \frac{mC}{2}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2} - a\frac{z}{2} - m\left(\frac{2-z}{2}\right)b\right)^2. \end{aligned} \quad (26)$$

By multiplying (26) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned} & 2\psi^{(n)}\left(\frac{bm+a}{2}\right) \int_0^1 z^{n-\beta-1} dz \\ & \leq \int_0^1 m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right) z^{n-\beta-1} dz \\ & + \int_0^1 \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) z^{n-\beta-1} dz \\ & - \frac{mC}{2} \int_0^1 \left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2} - a\frac{z}{2} - m\left(\frac{2-z}{2}\right)b\right)^2 z^{n-\beta-1} dz. \end{aligned} \quad (27)$$

By using change of variables and computing the last integral, from (27), we get

$$\begin{aligned} & \frac{2}{n-\beta} \psi^{(n)}\left(\frac{bm+a}{2}\right) \\ & \leq m \int_{a/m}^{((a+bm)/2m)} \left(\frac{2m(y-(a/m))}{bm-a}\right)^{n-\beta-1} \psi^{(n)}(y) \frac{2mdy}{bm-a} \\ & + \int_{((a+bm)/2)}^{mb} \left(\frac{2(mb-x)}{bm-a}\right)^{n-\beta-1} \psi^{(n)}(x) \frac{2dx}{bm-a} \\ & - \frac{mC}{2} \left\{ \frac{(b-a)^2}{4(n-\beta+2)} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{2(n-\beta+1)(n-\beta+2)} \right. \\ & \left. + \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{4(n-\beta)(n-\beta+1)(n-\beta+2)} \right\}. \end{aligned} \quad (28)$$

Further, it takes the following form:

$$\begin{aligned} & \psi^{(n)}\left(\frac{bm+a}{2}\right) \\ & \leq \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(bm-a)^{n-\beta}} \left[m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)}^\beta \psi \right)\left(\frac{a}{m}\right) \right. \\ & \left. + \left({}^C D_{((a+bm)/2)}^\beta \psi \right)(mb) \right] \\ & - \frac{mC(n-\beta)}{8(n-\beta+2)} \left\{ \frac{(b-a)^2}{2} + \frac{(b-a)((a/m)-mb)(n-\beta+3)}{(n-\beta+1)} \right. \\ & \left. + \frac{((a/m)-mb)^2[(n-\beta)^2+5n-5\beta+8]}{2(n-\beta)(n-\beta+1)} \right\}. \end{aligned} \quad (29)$$

Since $\psi^{(n)}$ is strongly m -convex function and $z \in [0, 1]$, we have the following inequality:

$$\begin{aligned} & m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right) + \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right) \\ & \leq m^2\left(\frac{2-z}{2}\right)\psi^{(n)}\left(\frac{a}{m^2}\right) + m\frac{z}{2}\psi^{(n)}(b) + \frac{z}{2}\psi^{(n)}(a) \\ & + m\left(\frac{2-z}{2}\right)\psi^{(n)}(b) - m^2C\left(\frac{z(2-z)}{4}\right)\left(b - \frac{a}{m^2}\right)^2 \\ & - mC\left(\frac{z(2-z)}{4}\right)(b-a)^2. \end{aligned} \quad (30)$$

By multiplying (30) with $z^{n-\beta-1}$ on both sides and making integration over $[0, 1]$, we get

$$\begin{aligned}
& \int_0^1 m\psi^{(n)}\left(\frac{a}{m}\left(\frac{2-z}{2}\right) + b\frac{z}{2}\right)z^{n-\beta-1}dz \\
& + \int_0^1 \psi^{(n)}\left(a\frac{z}{2} + m\left(\frac{2-z}{2}\right)b\right)z^{n-\beta-1}dz \\
& \leq \int_0^1 m^2\left(\frac{2-z}{2}\right)\psi^{(n)}\left(\frac{a}{m^2}\right)z^{n-\beta-1}dz \\
& + \int_0^1 m\frac{z}{2}\psi^{(n)}(b)z^{n-\beta-1}dz + \int_0^1 \frac{z}{2}\psi^{(n)}(a)z^{n-\beta-1}dz \\
& + \int_0^1 m\left(\frac{2-z}{2}\right)\psi^{(n)}(b)z^{n-\beta-1}dz \\
& - \int_0^1 mC\left(\frac{z(2-z)}{4}\right)\left[m\left(b - \frac{a}{m^2}\right)^2 + (b-a)^2\right]z^{n-\beta-1}dz.
\end{aligned} \tag{31}$$

By using change of variables and computing the last integral, from (31), we get

$$\begin{aligned}
& m \int_{a/m}^{((a+bm)/2m)} \left(\frac{2m(y - (a/m))}{bm - a}\right)^{n-\beta-1} \psi^{(n)}(y) \frac{2mdy}{bm - a} \\
& + \int_{((a+bm)/2)}^{mb} \left(\frac{2(mb - x)}{bm - a}\right)^{n-\beta-1} \psi^{(n)}(x) \frac{2dx}{bm - a} \\
& \leq \frac{(n - \beta + 2)[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)]}{2(n - \beta)(n - \beta + 1)} \\
& + \frac{m\psi^{(n)}(b) + \psi^{(n)}(a)}{2(n - \beta + 1)} \\
& - \frac{mC(n - \beta + 3)[(b - a)^2 + m(b - (a/m^2))^2]}{4(n - \beta + 1)(n - \beta + 2)}.
\end{aligned} \tag{32}$$

Further, it takes the following form

$$\begin{aligned}
& \frac{2^{n-\beta-1}\Gamma(n - \beta + 1)}{(bm - a)^{n-\beta}} \left[m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)^-}^\beta \psi\right)\left(\frac{a}{m}\right) \right. \\
& \quad \left. + \left({}^C D_{((a+bm)/2)^+}^\beta \psi\right)(mb) \right] \\
& \leq \frac{(n - \beta)}{4} \left\{ \frac{(n - \beta + 2)[m\psi^{(n)}(b) + m^2\psi^{(n)}(a/m^2)]}{(n - \beta)(n - \beta + 1)} \right. \\
& \quad + \frac{m\psi^{(n)}(b) + \psi^{(n)}(a)}{(n - \beta + 1)} \\
& \quad \left. - \frac{mC(n - \beta + 3)[(b - a)^2 + m(b - (a/m^2))^2]}{2(n - \beta + 1)(n - \beta + 2)} \right\}.
\end{aligned} \tag{33}$$

From (29) and (33), (25) can be obtained.

Corollary 17. By setting $m = 1$ in inequality (25), we will get ([23], Theorem 7)

$$\begin{aligned}
& \psi^{(n)}\left(\frac{b+a}{2}\right) + \frac{C(b-a)^2}{2(n-\beta+1)(n-\beta+2)} \\
& \leq \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[(-1)^n \left({}^C D_{((a+b)/2)^-}^\beta \psi\right)(a) \right. \\
& \quad \left. + \left({}^C D_{((a+b)/2)^+}^\beta \psi\right)(b) \right] \\
& \leq \frac{\psi^{(n)}(b) + \psi^{(n)}(a)}{2} - \frac{C(b-a)^2(n-\beta)(n-\beta+3)}{4(n-\beta+1)(n-\beta+2)}.
\end{aligned} \tag{34}$$

Remark 18. If $C = 0$ and $m = 1$ in (25), then we will get the fractional Hadamard inequality stated in Theorem 9.

3. Error Bounds of Fractional Hadamard Inequalities

In this section, we give refinements of the error bounds of fractional Hadamard inequalities for Caputo fractional derivatives.

Theorem 19. Let $\psi \in AC^{n+1}[a, b]$, $a < b$ be a differentiable mapping on (a, b) . If $|\psi^{(n+1)}|$ is a strongly m -convex function on $[a, mb]$, $m \in (0, 1]$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned}
& \left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n - \beta + 1)}{2(b - a)^{n-\beta}} \right. \\
& \quad \cdot \left[\left({}^C D_{a^+}^\beta \psi\right)(b) + (-1)^n \left({}^C D_{b^-}^\beta \psi\right)(a) \right] \left. \right| \\
& \leq \frac{(b - a)}{2(n - \beta + 1)} \left(1 - \frac{1}{2^{n-\beta}} \right) \\
& \quad \cdot \left[\left| \psi^{(n+1)}(a) \right| + m \left| \psi^{(n+1)}\left(\frac{b}{m}\right) \right| \right] \\
& \quad - \frac{Cm(b - a)((b/m) - a)^2}{(n - \beta + 2)(n - \beta + 3)} \left(1 - \frac{n - \beta + 4}{2^{n-\beta+2}} \right),
\end{aligned} \tag{35}$$

with $\beta > 0$.

Proof. Since $|\psi'|$ is strongly m -convex function on $[a, b]$, for $z \in [0, 1]$, we have

$$\begin{aligned}
& |f'(za + (1 - z)b)| = \left| f' \left(za + m(1 - z)\frac{b}{m} \right) \right| \\
& \leq z|f'(a)| + m(1 - z)|f'(b/m)| \\
& \quad - Cmz(1 - z) \left(\frac{b}{m} - a \right)^2.
\end{aligned} \tag{36}$$

By applying Lemma 7 and the strongly m -convexity of $|\psi^{(n+1)}|$, we find

$$\begin{aligned}
 & \left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \right. \\
 & \quad \cdot \left[\left({}^C D_{a^+}^\beta \psi \right)(b) + (-1)^n \left({}^C D_{b^-}^\beta \psi \right)(a) \right] \Big| \\
 & \leq \frac{b-a}{2} \int_0^1 (1-z)^{n-\beta} - z^{n-\beta} \left| \psi^{(n+1)} \left(za + m(1-z)\frac{b}{m} \right) \right| dz \\
 & \leq \frac{b-a}{2} \int_0^1 (1-z)^{n-\beta} - z^{n-\beta} \left(z \left| \psi^{(n+1)}(a) \right| \right. \\
 & \quad \left. + m(1-z) \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| - Cmz(1-z) \left(\frac{b}{m} - a \right)^2 \right) dz \\
 & \leq \frac{b-a}{2} \left[\int_0^{1/2} \left((1-z)^{n-\beta} - z^{n-\beta} \right) \left(z \left| \psi^{(n+1)}(a) \right| \right. \right. \\
 & \quad \left. \left. + m(1-z) \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| - Cmz(1-z) \left(\frac{b}{m} - a \right)^2 \right) dz \right. \\
 & \quad \left. + \int_{1/2}^1 \left(z^{n-\beta} - (1-z)^{n-\beta} \right) \left(z \left| \psi^{(n+1)}(a) \right| \right. \right. \\
 & \quad \left. \left. + m(1-z) \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| - Cmz(1-z) \left(\frac{b}{m} - a \right)^2 \right) dz \right]. \quad (37)
 \end{aligned}$$

In the following, we compute integrals appearing on the right side of inequality (37):

$$\begin{aligned}
 & \int_0^{1/2} \left((1-z)^{n-\beta} - z^{n-\beta} \right) \\
 & \quad \cdot \left(z \left| \psi^{(n+1)}(a) \right| + m(1-z) \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \right. \\
 & \quad \left. - Cmz(1-z) \left(\frac{b}{m} - a \right)^2 \right) dz \\
 & = \left| \psi^{(n+1)}(a) \right| \int_0^{1/2} \left(z(1-z)^{n-\beta} - z^{n-\beta+1} \right) dz \\
 & \quad + m \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \int_0^{1/2} \left((1-z)^{n-\beta+1} - (1-z)z^{n-\beta} \right) dz \\
 & \quad - Cm \left(\frac{b}{m} - a \right)^2 \left(\int_0^{1/2} z(1-z)^{n-\beta+1} dz \right. \\
 & \quad \left. - \int_0^{1/2} z^{n-\beta+1}(1-z) dz \right) \\
 & = \left| \psi^{(n+1)}(a) \right| \left(\frac{1}{(n-\beta+1)(n-\beta+2)} - \frac{(1/2)^{n-\beta+1}}{n-\beta+1} \right) \\
 & \quad + m \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \left(\frac{1}{(n-\beta+2)} - \frac{(1/2)^{n-\beta+1}}{n-\beta+1} \right) \\
 & \quad - \frac{Cm((b/m)-a)^2}{(n-\beta+2)(n-\beta+3)} \left[1 - \frac{(n-\beta+4)}{2^{n-\beta+2}} \right], \quad (38)
 \end{aligned}$$

$$\begin{aligned}
 & \int_{1/2}^1 \left(z^{n-\beta} - (1-z)^{n-\beta} \right) \\
 & \quad \cdot \left(z \left| \psi^{(n+1)}(a) \right| + m(1-z) \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \right. \\
 & \quad \left. - Cmz(1-z) \left(\frac{b}{m} - a \right)^2 \right) dz \\
 & = \left| \psi^{(n+1)}(a) \right| \int_{1/2}^1 \left(z^{n-\beta+1} - z(1-z)^{n-\beta} \right) dz \\
 & \quad + m \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \int_{1/2}^1 \left((1-z)z^{n-\beta} - (1-z)^{n-\beta+1} \right) dz \\
 & \quad - Cm \left(\frac{b}{m} - a \right)^2 \left(\int_{1/2}^1 (1-z)z^{n-\beta+1} dz \right. \\
 & \quad \left. - \int_{1/2}^1 z(1-z)^{n-\beta+1} dz \right) \\
 & = \left| \psi^{(n+1)}(a) \right| \left(\frac{1}{n-\beta+2} - \frac{(1/2)^{n-\beta+1}}{n-\beta+1} \right) \\
 & \quad + m \left| \psi^{(n+1)} \left(\frac{b}{m} \right) \right| \\
 & \quad \cdot \left(\frac{1}{(n-\beta+1)(n-\beta+2)} - \frac{(1/2)^{n-\beta+1}}{n-\beta+1} \right) \\
 & \quad - \frac{Cm((b/m)-a)^2}{(n-\beta+2)(n-\beta+3)} \left[1 - \frac{n-\beta+4}{2^{n-\beta+2}} \right]. \quad (39)
 \end{aligned}$$

By putting the values of (38) and (39) in (37), we get (35).

Corollary 20. By setting $m = 1$ in inequality (35), we will get ([23], Theorem 8)

$$\begin{aligned}
 & \left| \frac{\psi^{(n)}(a) + \psi^{(n)}(b)}{2} - \frac{\Gamma(n-\beta+1)}{2(b-a)^{n-\beta}} \right. \\
 & \quad \cdot \left[\left({}^C D_{a^+}^\beta \psi \right)(b) + (-1)^n \left({}^C D_{b^-}^\beta \psi \right)(a) \right] \Big| \\
 & \leq \frac{(b-a)}{2(n-\beta+1)} \left(1 - \frac{1}{2^{n-\beta}} \right) \left[\left| \psi^{(n+1)}(a) \right| + \left| \psi^{(n+1)}(b) \right| \right] \\
 & \quad - \frac{C(b-a)^3}{(n-\beta+2)(n-\beta+3)} \left(1 - \frac{n-\beta+4}{2^{n-\beta+2}} \right). \quad (40)
 \end{aligned}$$

Remark 21. If $C = 0$ and $m = 1$ in (35), we get the fractional Hadamard inequality which is stated in Theorem 8.

By using Lemma (11), we give the following Caputo fractional derivative inequality.

Theorem 22. Let $\psi \in AC^{n+1}[a, b]$, $a < b$, be a differentiable mapping on (a, b) . If $|\psi^{(n+1)}|^q$ is strongly m -convex function

on $[a, mb]$, $m \in (0, 1]$, for $q \geq 1$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \psi \right)(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1} (-1)^n \left({}^C D_{((a+mb)/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\
 & \leq \frac{mb-a}{4(n-\beta+1)^{1/p}} \left(\frac{1}{2(n-\beta+1)(n-\beta+2)} \right)^{1/q} \\
 & \quad \cdot \left[\left((n-\beta+1) \left| \psi^{(n+1)}(a) \right|^q \right. \right. \\
 & \quad \left. \left. + (n-\beta+3)m \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-a)^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right. \\
 & \quad \left. + \left((n-\beta+3)m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q \right. \right. \\
 & \quad \left. \left. + (n-\beta+1) \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-(a/m^2))^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right]. \quad (41)
 \end{aligned}$$

Proof. By taking $k = 1$ in Lemma 10 and using the strongly m -convexity of $|\psi^{(n+1)}|$, we have

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \psi \right)(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1} (-1)^n \left({}^C D_{((a+mb)/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left[\int_0^1 z^{n-\beta} \left(\left| \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) \right| \right. \right. \\
 & \quad \left. \left. + \left| \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) \right| \right) dz \right] \\
 & \leq \frac{mb-a}{4} \left[\left(\left| \psi^{(n+1)}(a) \right| + \left| \psi^{(n+1)}(b) \right| \right) \int_0^1 \frac{z^{n-\beta+1}}{2} dz \right. \\
 & \quad \left. + \left(\left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right| + \left| \psi^{(n+1)}(b) \right| \right) \int_0^1 \frac{(2-z)z^{n-\beta}}{2} dz \right. \\
 & \quad \left. - \frac{Cm}{4} \left[(b-a)^2 - \left(b - \frac{a}{m^2} \right) \right] \int_0^1 (2-z)z^{n-\beta+1} dz \right] \\
 & \leq \frac{mb-a}{4(n-\beta+2)} \left[\left(\left| \psi^{(n+1)}(a) \right| + \left| \psi^{(n+1)}(b) \right| \right) \right. \\
 & \quad \left. + \frac{m \left[\left| \psi^{(n+1)}(a/m^2) \right| + \left| \psi^{(n+1)}(b) \right| \right]}{2(n-\beta+1)} \right. \\
 & \quad \left. - \frac{Cm}{4} \left[(b-a)^2 - \left(b - \frac{a}{m^2} \right) \right] \left(\frac{n-\beta+4}{(n-\beta+3)} \right) \right]. \quad (42)
 \end{aligned}$$

Now, applying Lemma 10, using power mean inequality, we have

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \psi \right)(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1} (-1)^n \left({}^C D_{((a+mb)/2m)^-}^\beta \psi \right) \left(\frac{a}{m} \right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)} \left(\frac{a+mb}{2} \right) + m \psi^{(n)} \left(\frac{a+mb}{2m} \right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left[\int_0^1 z^{n-\beta} \left(\left| \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) \right| \right. \right. \\
 & \quad \left. \left. + \left| \psi^{(n+1)} \left(\frac{2-z}{2m} a + \frac{z}{2} b \right) \right| \right) dz \right] \\
 & \leq \frac{mb-a}{4} \left(\int_0^1 z^{n-\beta} dz \right)^{1-(1/q)} \\
 & \quad \cdot \left[\left(\int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(\frac{z}{2} a + m \left(\frac{2-z}{2} \right) b \right) \right|^q dz \right)^{1/q} \right. \\
 & \quad \left. + \left(\int_0^1 z^{n-\beta} \left| \psi^{(n+1)} \left(a \left(\frac{2-z}{2m} \right) + b \frac{z}{2} \right) \right|^q dz \right)^{1/q} \right]. \quad (43)
 \end{aligned}$$

By using strongly m -convexity of $|\psi^{(n+1)}|^q$, we have

$$\begin{aligned}
 & \leq \frac{mb-a}{4(n-\beta+1)^{1/p}} \left[\left(\left| \psi^{(n+1)}(a) \right|^q \int_0^1 \frac{z^{n-\beta+1}}{2} dz \right. \right. \\
 & \quad \left. \left. + m \left| \psi^{(n+1)}(b) \right|^q \int_0^1 \frac{2z^{n-\beta} - z^{n-\beta+1}}{2} dz \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-a)^2}{4} \int_0^1 (2z^{n-\beta+1} - z^{n-\beta+2}) dz \right)^{1/q} \right. \\
 & \quad \left. + \left(m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q \int_0^1 \frac{2z^{n-\beta} - z^{n-\beta+1}}{2} dz \right. \right. \\
 & \quad \left. \left. + \left| \psi^{(n+1)}(b) \right|^q \int_0^1 \frac{z^{n-\beta+1}}{2} dz \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-(a/m^2))^2}{4} \int_0^1 (2z^{n-\beta+1} - z^{n-\beta+2}) dz \right)^{1/q} \right] \\
 & = \frac{mb-a}{4(n-\beta+1)^{1/p}} \left(\frac{1}{2(n-\beta+1)(n-\beta+2)} \right)^{1/q} \\
 & \quad \cdot \left[\left((n-\beta+1) \left| \psi^{(n+1)}(a) \right|^q + (n-\beta+3)m \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-a)^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right. \\
 & \quad \left. + \left((n-\beta+3)m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q \right. \right. \\
 & \quad \left. \left. + (n-\beta+1) \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-(a/m^2))^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right], \quad (44)
 \end{aligned}$$

inequality (41) is obtained.

Corollary 23. By setting $m = 1$ in inequality (41), we will get ([23], Theorem 9)

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[\left({}^C D_{((a+b)/2)^+}^\beta \right) \psi(b) \right. \right. \\
 & \quad \left. \left. + (-1)^n \left({}^C D_{((a+b)/2)^-}^\beta \right) \psi(a) \right] - \psi^{(n)}\left(\frac{a+b}{2}\right) \right| \\
 & \leq \frac{b-a}{4(n-\beta+1)^{1/p}} \left(\frac{1}{2(n-\beta+1)(n-\beta+2)} \right)^{1/q} \\
 & \quad \cdot \left[\left((n-\beta+1) \left| \psi^{(n+1)}(a) \right|^q + (n-\beta+3) \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{C(b-a)^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right. \\
 & \quad \left. + \left((n-\beta+3) \left| \psi^{(n+1)}(a) \right|^q + (n-\beta+1) \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
 & \quad \left. \left. - \frac{C(b-a)^2(n-\beta+1)(n-\beta+4)}{2(n-\beta+3)} \right)^{1/q} \right]. \quad (45)
 \end{aligned}$$

Remark 24. (i) If $C = 0$ and $m = 1$ in (41), then we will get inequality stated in Theorem 11.

Theorem 25. Let $\psi \in AC^{n+1}[a, b]$, $a < b$, be a differentiable mapping on (a, b) . If $|\psi^{(n+1)}|^q$ is a strongly m -convex function on $[a, mb]$, $m \in (0, 1]$, for $q > 1$, then the following inequality for Caputo fractional derivatives holds:

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \right) \psi(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)^-}^\beta \right) \psi\left(\frac{a}{m}\right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right) \right] \right| \\
 & \leq \frac{bm-a}{16} \left(\frac{4}{np-\beta p+1} \right)^{1/p} \\
 & \quad \cdot \left[\left(\left(\left| \psi^{(n+1)}(a) \right| + (3m)^{1/q} \left| \psi^{(n+1)}(b) \right| \right)^q \right. \right. \\
 & \quad \left. \left. - \frac{2Cm(b-a)^2}{3} \right)^{1/q} \right. \\
 & \quad \left. + \left(\left((3m)^{1/q} \left| \psi^{(n+1)}\left(\frac{a}{m^2}\right) \right| + \left| \psi^{(n+1)}(b) \right| \right)^q \right. \right. \\
 & \quad \left. \left. - \frac{2Cm(b-(a/m^2))^2}{3} \right)^{1/q} \right]. \quad (46)
 \end{aligned}$$

Proof. By taking $k = 1$ in Lemma 10 and with the help of modulus property, we get

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \right) \psi(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)^-}^\beta \right) \psi\left(\frac{a}{m}\right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left[\int_0^1 z^{n-\beta} \left| \psi^{(n+1)}\left(\frac{z}{2}a + m\frac{(2-z)}{2}b\right) \right| dz \right. \\
 & \quad \left. + \int_0^1 z^{n-\beta} \left| \psi^{(n+1)}\left(\frac{2-z}{2m}a + \frac{z}{2}b\right) \right| dz \right]. \quad (47)
 \end{aligned}$$

Now applying Holder's inequality, we get

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \right) \psi(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)^-}^\beta \right) \psi\left(\frac{a}{m}\right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \\
 & \quad \cdot \left[\left(\int_0^1 \left| \psi^{(n+1)}\left(\frac{z}{2}a + m\frac{(2-z)}{2}b\right) \right| dz \right)^{1/q} \right. \\
 & \quad \left. + \left(\int_0^1 z^{n-\alpha} \left| \psi^{(n+1)}\left(\frac{2-z}{2m}a + \frac{z}{2}b\right) \right| dz \right)^{1/q} \right]. \quad (48)
 \end{aligned}$$

Using strongly m -convexity of $|\psi^{(n+1)}|^q$, we get

$$\begin{aligned}
 & \left| \frac{2^{n-\beta-1}\Gamma(n-\beta+1)}{(mb-a)^{n-\beta}} \left[\left({}^C D_{((a+bm)/2)^+}^\beta \right) \psi(mb) \right. \right. \\
 & \quad \left. \left. + m^{n-\beta+1}(-1)^n \left({}^C D_{((a+bm)/2m)^-}^\beta \right) \psi\left(\frac{a}{m}\right) \right] \right. \\
 & \quad \left. - \frac{1}{2} \left[\psi^{(n)}\left(\frac{a+mb}{2}\right) + m\psi^{(n)}\left(\frac{a+mb}{2m}\right) \right] \right| \\
 & \leq \frac{mb-a}{4} \left(\frac{1}{np-\beta p+1} \right)^{1/p} \left[\left(\left| \psi^{(n+1)}(a) \right|^q \int_0^1 \frac{z}{2} dz \right. \right. \\
 & \quad \left. \left. + m \left| \psi^{(n+1)}(b) \right|^q \int_0^1 \left(\frac{2-z}{2} \right) dz \right. \right. \\
 & \quad \left. \left. - \frac{Cm(b-a)^2}{4} \int_0^1 (2z-z^2) dz \right)^{1/q} \right. \\
 & \quad \left. + \left(m \left| \psi^{(n+1)}\left(\frac{a}{m^2}\right) \right|^q \int_0^1 \left(\frac{2-z}{2} \right) dz + \left| \psi^{(n+1)}(b) \right|^q \int_0^1 \frac{z}{2} dz \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& - \frac{Cm(b - (a/m^2))^2}{4} \int_0^1 (2z - z^2) dz \Big)^{1/q} \Big] \\
& = \frac{mb - a}{16} \left(\frac{4}{np - \beta p + 1} \right)^{1/p} \left[\left(\left| \psi^{(n+1)}(a) \right|^q \right. \right. \\
& \quad \left. \left. + 3m \left| \psi^{(n+1)}(b) \right|^q - \frac{2Cm(b-a)^2}{3} \right)^{1/q} \right. \\
& \quad \left. + \left(3m \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right|^q + \left| \psi^{(n+1)}(b) \right|^q \right. \right. \\
& \quad \left. \left. - \frac{2Cm(b - (a/m^2))^2}{3} \right)^{1/q} \right] \\
& \leq \frac{bm - a}{16} \left(\frac{4}{np - \beta p + 1} \right)^{1/p} \left[\left(\left| \psi^{(n+1)}(a) \right| \right. \right. \\
& \quad \left. \left. + (3m)^{1/q} \left| \psi^{(n+1)}(b) \right| \right)^q - \frac{2Cm(b-a)^2}{3} \right)^{1/q} \\
& \quad + \left(\left((3m)^{1/q} \left| \psi^{(n+1)} \left(\frac{a}{m^2} \right) \right| + \left| \psi^{(n+1)}(b) \right| \right)^q \right. \\
& \quad \left. \left. - \frac{2Cm(b - (a/m^2))^2}{3} \right)^{1/q} \right]. \tag{49}
\end{aligned}$$

Here, we have used the fact that $(a_1 + b_1)^q \geq (a_1)^q + (b_1)^q$, where $q > 1$, $a_1, b_1 \geq 0$. This completes the proof.

Corollary 26. By setting $m = 1$ in inequality (46), we will get ([23], Theorem 10)

$$\begin{aligned}
& \left| \frac{2^{n-\beta-1} \Gamma(n-\beta+1)}{(b-a)^{n-\beta}} \left[\left({}^C D_{((a+b)/2)^+}^\beta \right) (b) \right. \right. \\
& \quad \left. \left. + (-1)^n \left({}^C D_{((a+b)/2)^-}^\beta \right) (a) \right] - \psi^{(n)} \left(\frac{a+b}{2} \right) \right| \\
& \leq \frac{b-a}{16} \left(\frac{4}{np - \beta p + 1} \right)^{1/p} \left[\left(\left| \psi^{(n+1)}(a) \right| \right. \right. \\
& \quad \left. \left. + (3)^{1/q} \left| \psi^{(n+1)}(b) \right| \right)^q - \frac{2C(b-a)^2}{3} \right)^{1/q} \\
& \quad + \left(\left((3)^{1/q} \left| \psi^{(n+1)}(a) \right| + \left| \psi^{(n+1)}(b) \right| \right)^q \right. \\
& \quad \left. \left. - \frac{2C(b-a)^2}{3} \right)^{1/q} \right]. \tag{50}
\end{aligned}$$

Remark 27. If $C = 0$ and $m = 1$ in (46), then we will get inequality stated in Theorem 12.

Data Availability

There is no data required for this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics Doklady*, vol. 7, pp. 72–75, 1966.
- [2] J. Makò and A. Hézy, "On strongly convex functions," *Carpathian Journal of Mathematics*, vol. 32, no. 1, pp. 87–95, 2016.
- [3] N. Merentes and K. Nikodem, "Remarks on strongly convex functions," *Aequationes Math*, vol. 80, no. 1-2, pp. 193–199, 2010.
- [4] K. Nikodem, "On strongly convex functions and related classes of functions," in *Handbook of Functional Equations. Springer Optimization and Its Applications*, vol. 95, T. Rassias, Ed., Springer, New York, NY, USA, 2014.
- [5] K. Nikodem and Z. Páles, "Characterizations of inner product spaces by strongly convex functions," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 83–87, 2011.
- [6] J. P. Vial, "Strong convexity of sets and functions," *Journal of Mathematical Economics*, vol. 9, no. 1-2, pp. 187–205, 1982.
- [7] G. H. Toader, "Some generalisations of the convexity," *Proc. Colloq. Approx. Optim*, 1984, pp. 329–338, Cluj-Napoca (Romania), 1984.
- [8] T. Lara, N. Merentes, R. Quintero, and E. Rosales, "On strongly m-convex functions," *Mathematica Aeterna*, vol. 5, no. 3, pp. 521–535, 2015.
- [9] P. Agarwal, M. Jleli, and M. Tomar, "Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals," *Journal of Inequalities and Applications*, vol. 2017, no. 1, 2017.
- [10] F. Chen, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity," *Chinese Journal of Mathematics*, vol. 2014, Article ID 173293, 7 pages, 2014.
- [11] Y. Dong, M. Zeb, G. Farid, and S. Bibi, "Hadamard inequalities for strongly α , m-convex functions via Caputo fractional derivatives," *Journal of Mathematics*, vol. 2021, Article ID 6691151, 16 pages, 2021.
- [12] G. Farid, A. Javed, and A. U. Rehman, "On Hadamard inequalities for n – times differentiable functions which are relative convex via Caputo k-fractional derivatives," *Nonlinear Analysis Forum*, vol. 22, no. 2, pp. 17–28, 2017.
- [13] G. Farid, A. Javed, A. U. Rehman, and M. I. Qureshi, "On Hadamard-type inequalities for differentiable functions via Caputo k-fractional derivatives," *Cogent Mathematics*, vol. 4, no. 1, article 1355429, 2017.
- [14] G. Farid, A. U. Rehman, and M. Zahra, "On Hadamard inequalities for k-fractional integrals," *Nonlinear Functional Analysis and Applications*, vol. 21, no. 3, pp. 463–478, 2016.
- [15] M. E. Özdemir, A. O. Akdemri, and E. Set, "On $(h - m)$ – convexity and Hadamard-type inequalities," *Transylvanian Journal of Mathematics and Mechanics*, vol. 8, no. 1, pp. 51–58, 2016.
- [16] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013.

- [17] A. Waheed, A. U. Rehman, M. I. Qureshi, F. A. Shah, K. A. Khan, and G. Farid, "On Caputo k -fractional derivatives and associated inequalities," *IEEE Access*, vol. 7, pp. 32137–32145, 2019.
- [18] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [19] S. S. Dragomir and C. E. M. Pearce, "Selected topics on Hermite-Hadamard inequalities and applications," *Mathematics Preprint Archive*, vol. 1, pp. 463–817, 2003.
- [20] G. Farid, A. Javed, and S. Naqvi, "Hadamard and Fejer Hadamard inequalities and related results via Caputo fractional derivatives," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 3, pp. 16–30, 2017.
- [21] S. M. Kang, G. Farid, W. Nazeer, and S. Naqvi, "A version of the Hadamard inequality for Caputo fractional derivatives and related results," *Journal of Computational Analysis and Applications*, vol. 27, no. 6, pp. 962–972, 2019.
- [22] G. Farid, A. Javed, and A. U. Rehman, "Fractional integral inequalities of Hadamard type for m -convex functions via Caputo k -fractional derivatives," *Journal of Fractional Calculus and Applications*, vol. 10, no. 1, pp. 120–134, 2019.
- [23] G. Farid, A. U. Rehman, S. Bibi, and Y.-M. Chu, "Refinements of two fractional versions of Hadamard inequalities for Caputo fractional derivatives and related results," *Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 1–10, 2021.