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Research Article

Isometries between Spaces of Vector-Valued Differentiable Functions

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This article characterizes the isometries between spaces of all differentiable functions from a compact interval of the real line into a strictly convex Banach space.

1. Introduction

The main purpose of this article is to characterize the isometries of the space of all (continuously) differentiable functions from a compact interval of the real line into a strictly convex Banach space.

Cambern ([1], Theorem 6.5.5) and Pathak [2] investigated the surjective linear isometries over spaces of complex-valued differentiable functions on the interval $[0, 1]$ and gave a representation for such operators, and Jarosz and Pathak [3] studied those operators over differentiable function spaces defined on te compact subsets of the real line (without isolated points). Also, Wang [4] studied the isometries between spaces of scalar-valued (real case and complex case) differentiable functions (and vanish at infinity) on the locally compact subsets of the Euclidean spaces (without isolated points) and gave a representation for such operators.

On the other hand, Botelho and Jamison [5] extended these results to the surjective linear isometries on spaces of 1-time differentiable functions on $[0, 1]$ with values in a finite-dimensional Hilbert space. In [5], the main result is valid whenever the (real) dimension of the Hilbert space is bigger than one. Compare with the assumptions in ([6], Theorem 2.13). In [6], the authors investigated the isometries between spaces of p-times differentiable functions (and vanish at infinity) on an open subset of the real line with values in a strictly convex Banach space. Also, Li and Wang [7] studied the isometries between spaces of p-times differentiable functions (and vanish at infinity) on an open subset of the Euclidean space with values in a reflexive and strictly convex Banach space. In [7], there is a gap in the proof of Theorem 3.5 (page 553); in the proof of Claim 1, why $\{\partial^\gamma Tg(\tau(x)) : \gamma \in \Gamma\}$ must have at most one nonzero term? Compare with the proofs in ([4], Lemma 2.5) and ([2], Lemma 2.3).

Suppose that $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ are Banach spaces (on the real line). Denote the space of all C^1-functions $f : [0, 1] \to V$ by $C^1([0, 1], V)$, and on this space, we consider the following norm:

$$\|f\|_1 \coloneqq \max_{t\in[0,1]}\left\{\|f(t)\| + \left\|f'(t)\right\|\right\} = \max_{t\in[0,1]}\left\{\|f(t)\|_1\right\} < \infty. \tag{1}$$

Let $T : C^1([0, 1], V) \to C^1([0, 1], V)$ be a surjective linear isometry. In [5], it is shown when V is a finite dimensional Hilbert space, there exists a linear isometry $J : V \to V$ such that either $T(f)(t) = J(f(t))$ or $T(f)(t) = J(f(1 - t))$, for all $f \in C^1([0, 1], V)$ and $t \in [0, 1]$. In this article, we extend the above result to a surjective isometry $T : C^1([0, 1], V) \to C^1([0, 1], W)$, whenever Banach spaces V and W are strictly convex.

In [6], the authors described the surjective isometries between spaces of p-times differentiable functions (and vanish at infinity) on an open subset of the real line with values in a strictly convex Banach space (with dimension greater than one). Such a representation is as $T(f)(t) = J_t(f \circ \tau(t))$, where $J_t : V \to W$ is a surjective linear isomorphism (for all $t \in [0,1]$) and τ is a diffeomorphism (over $[0,1]$), and its proof is based on the main result of [8]. By using the proof of ([6], Lemma 2.12), we recover their representation in our setting. Therefore, our description is better and its proof is somewhat shorter and more elementary than the one in [6]. Also, in the appendix, we provide a proof for Theorem 13 for special case $= W = \mathbf{R}$. In a forthcoming article, we extend our result to the class of p-times differentiable functions.

2. Characterization of T-Set in $C^1([0,1], V)$

Suppose that $(V, \|\cdot\|)$ is a Banach space on the real line. For $v_1, v_2 \in V \setminus \{0\}$ and $x \in [0,1]$, we define

$$S(x, v_1, v_2) := \Big\{ f \in C^1([0,1]0,1], V) : \|f\|_1 = \|f(x)\|_1 \& f(x) \in \langle v_1 \rangle_+ \& f'(x) \in \langle v_2 \rangle_+ \Big\}, \tag{2}$$

where $\langle v \rangle_+ := \{u \in V : u = \lambda v, \text{for some } \lambda \geq 0\}$.

A T-set in a Banach space U is a subset S of U with the property that for any finite collection $x_1, \cdots, x_n \in S$, $\|\sum_i^n x_i\| = \sum_i^n \|x_i\|$, and such that S is maximal with respect to this property.

In this section, we show that every T-set of $C^1([0,1], V)$ has a unique representation by some $S(x, v_1, v_2)$, up to $\langle v_1 \rangle_+$ and $\langle v_2 \rangle_+$, whenever V is strictly convex (compare with ([9], Lemma 7.2.2)). We start this section with two elementary Lemmas.

Lemma 1. *For any $x \in]0,1[$ and $\eta > 0$ such that $0 < x - \eta < x + \eta < 1$, there exists $\phi \in C^1([0,1], \mathbf{R})$ in such a way that $\phi(x) = 0$,* supp $(\phi) \subset]x - \eta, x + \eta[$ *and*

$$\|\phi\|_1 = \|\phi(x)\|_1 > \|\phi(t)\|_1, \tag{3}$$

for all $t \in [0,1] \setminus \{x\}$. Moreover, when $x = 0$ or $x = 1$, we can state and prove a similar result.

Proof. For $0 < \delta < \varepsilon < 1/4$, define $f_{\varepsilon,\delta} : [-\varepsilon, \varepsilon] \to \mathbf{R}$ as follows:

$$f_{\varepsilon,\delta}(t) := \begin{cases} \sqrt{t+\delta} - \sqrt{\delta} & \text{if } t \geq 0, \\ -\sqrt{-t+\delta} + \sqrt{\delta} & \text{if } t \leq 0. \end{cases} \tag{4}$$

Then, it is easy to verify that $f_{\varepsilon,\delta} \in C^1([-\varepsilon, \varepsilon], \mathbf{R})$ and

$$\|f_{\varepsilon,\delta}\|_1 = \|f_{\varepsilon,\delta}(0)\|_1 = 0 + \frac{1}{2\sqrt{\delta}} > \|f_{\varepsilon,\delta}(t)\|_1, \tag{5}$$

for all $t \in [-\varepsilon, \varepsilon] \setminus \{0\}$. It is clear that $f_{\varepsilon,\delta}$ can be extended as a C^1-function on $[-1/2, 1/2]$ with support in $[-\varepsilon_+, \varepsilon_+]$, for any $\varepsilon_+ \in]\varepsilon, 1/2[$. Denote this extension of $f_{\varepsilon,\delta}$ by $\tilde{f}_{\varepsilon,\delta}$. Now, by choosing $\delta > 0$ small enough (fixing ε and ε_+), we have

$$\left\|\tilde{f}_{\varepsilon,\delta}\right\|_1 = \left\|\tilde{f}_{\varepsilon,\delta}(0)\right\|_1 > \left\|\tilde{f}_{\varepsilon,\delta}(t)\right\|_1, \tag{6}$$

for all $t \in [-1/2, 1/2] \setminus \{0\}$. Now, one can easily construct the desired function ϕ as in Lemma 1.

For $v \in V$, the function $\widehat{v} \in C^1([0,1], V)$ denotes the constant function with value v.

Lemma 2. *Let V be a Banach space, $v_1 \in V$ and $v_2 \in V \setminus \{0\}$. For any $x \in [0,1]$, there exists $\phi \in C^1([0,1], V)$ in such a way that $\phi(x) = v_1$, $\phi'(x) = v_2$ and*

$$\|\phi\|_1 = \|\phi(x)\|_1 > \|\phi(t)\|_1, \tag{7}$$

for all $t \in [0,1] \setminus \{x\}$.

Proof. Let ϕ_0 be a function with the properties as in Lemma 1. Then, $\phi := \widehat{v}_1 + (\phi_0/\phi_0'(x))\,\widehat{v}_2$ is the desired function.

Theorem 3. *Let V be a strictly convex Banach space. Then, every T-set of $C^1([0,1], V)$ is as $S(x, v_1, v_2)$, for some $x \in [0,1]$ and $v_1, v_2 \in V \setminus \{0\}$, and vice versa $S(x, v_1, v_2)$ is a T -set of $C^1([0,1], V)$.*

Proof. First, we show that $S(x, v_1, v_2)$ is a T-set of $C^1([0,1], V)$, for any $x \in [0,1]$ and $v_1, v_2 \in V \setminus \{0\}$. To do this, it is enough to show the maximality property for $S(x, v_1, v_2)$. Suppose that $h \in C^1([0,1], V)$ such that $\|h + f\|_1 = \|h\|_1 + \|f\|_1$, for any $f \in S(x, v_1, v_2)$, we show that $h \in S(x, v_1, v_2)$. Suppose that $\|h\|_1 > \|h(x)\|_1$. So, there exists an open neighborhood of x in $[0,1]$, say U_x, such that $\|h\|_1 > \|h(t)\|_1$, for all $t \in U_x$. By Lemma 1, there exists $f_1 \in S(x, v_1, v_2)$ such that $f_1(x) = 0$, $f_1'(x) = v_2$ and supp $(f_1) \subset U_x$, and also

$$\|f_1\|_1 = \|f_1(x)\|_1 > \|f_1(t)\|_1, \tag{8}$$

for all $t \in [0,1] \setminus \{x\}$. Then, we have

$$\|h(t) + f_1(t)\|_1 \leq \|h(t)\|_1 + \|f_1(t)\|_1 < \|h\|_1 + \|f_1\|_1, \tag{9}$$

for all $t \in [0,1]$. This is a contradiction. Therefore, we get $\|h\|_1 = \|h(x)\|_1$.

Next, we prove that $h(x) \in \langle v_1 \rangle_+$ and $h'(x) \in \langle v_2 \rangle_+$. We know that $f_2 := \widehat{v}_1 + f_1 \in S(x, v_1, v_2)$ (see the proof of Lemma 2) and

$$\|h(t) + f_2(t)\|_1 \leq \|h(t)\|_1 + \|f_2(t)\|_1 < \|h\|_1 + \|f_2\|_1, \tag{10}$$

for all $t \in [0,1] \setminus \{x\}$. Also, we have

$$\begin{aligned} \|h + f_2\|_1 &= \|h(x) + f_2(x)\|_1 \\ &= \|h(x) + f_2(x)\| + \|h'(x) + f_2'(x)\| \\ &\le \|h(x)\| + \|f_2(x)\| + \|h'(x)\| + \|f_2'(x)\| \\ &= \|h(x)\| + \|h'(x)\| + \|f_2(x)\| + \|f_2'(x)\| \\ &\le \|h\|_1 + \|f_2\|_1. \end{aligned} \tag{11}$$

By Equation (10) and the assumptions, we see that all inequalities in Equation (11) are equalities. This implies that

$$\begin{aligned} \|h(x) + f_2(x)\| &= \|h(x)\| + \|f_2(x)\|, \\ \|h'(x) + f_2'(x)\| &= \|h'(x)\| + \|f_2'(x)\|. \end{aligned} \tag{12}$$

Since V is strictly convex, this implies that $h(x) \in \langle f_2(x)\rangle_+$ and $h'(x) \in \langle f_2'(x)\rangle_+$ and then $h(x) \in \langle v_1\rangle_+$ and $h'(x) \in \langle v_2\rangle_+$.

To prove the converse, suppose that S is a T-set in $C^1([0,1], V)$. For $f \in S$, define

$$K_f \coloneqq \left\{ t \in [0,1] \colon \|f\|_1 = \|f(t)\|_1 \right\}. \tag{13}$$

It is clear that K_f is a compact nonempty subset of $[0,1]$. Also, for any finite collection of elements $f_1, f_2, \cdots, f_n \in S$, we have $\bigcap_{i=1}^n K_{f_i} \neq \varnothing$. By contradiction, suppose that $\bigcap_{i=1}^n K_{f_i} = \varnothing$. This implies that for any $t \in [0,1]$, there exists some f_{j_t}, for $1 \le j_t \le n$, such that $\|f_{j_t}(t)\|_1 < \|f_{j_t}\|_1$. Then, we obtain

$$\left\| \sum_{i=1}^n f_i(t) \right\|_1 \le \sum_{i=1}^n \|f_i(t)\|_1 < \sum_{i=1}^n \|f_i\|_1, \tag{14}$$

for all $t \in [0,1]$. So, we get

$$\left\| \sum_{i=1}^n f_i(t) \right\|_1 < \sum_{i=1}^n \|f_i\|_1, \tag{15}$$

for all $t \in [0,1]$. This is a contradiction. So, $\bigcap_{i=1}^n K_{f_i} \neq \varnothing$. This implies that $\bigcap_{f \in S} K_f \neq \varnothing$. Let $x \in \bigcap_{f \in S} K_f$. Finally, by the maximality property of S, since V is strictly convex, we see that $S = S(x, v_1, v_2)$, for some $v_1, v_2 \in V \setminus \{0\}$.

Next, we state a few simple facts about T-sets in $C^1([0,1], V)$, whenever V is a strictly convex Banach space.

Proposition 4. *Let V be a strictly convex Banach space. Consider the space $C^1([0,1], V)$, then we have*

(i) $S(x, v_1, v_2)$ is not trivial, i.e., $S(x, v_1, v_2) \neq \{0\}$

(ii) $S(x, v_1, v_2) \cap S(\bar{x}, \bar{v}_1, \bar{v}_2)$ is not trivial, whenever $x \neq \bar{x}$

(iii) If $x \neq \bar{x}$, or $v_1 \notin \langle \bar{v}_1\rangle_+$, or $v_2 \notin \langle \bar{v}_2\rangle_+$, then $S(x, v_1, v_2) \neq S(\bar{x}, \bar{v}_1, \bar{v}_2)$

(iv) If $v_1 \notin \langle \bar{v}_1\rangle_+$, then

$$S(x, v_1, v_2) \bigcap S(x, \bar{v}_1, v_2) = \left\{ f \in C^1([0,1], V) \colon f(x) = 0 \right\} \bigcap S(x, v_1, v_2). \tag{16}$$

(v) If $v_2 \notin \langle \bar{v}_2\rangle_+$, then

$$S(x, v_1, v_2) \bigcap S(x, v_1, \bar{v}_2) = \left\{ f \in C^1([0,1], V) \colon f'(x) = 0 \right\} \bigcap S(x, v_1, v_2). \tag{17}$$

(vi) If $v_1 \notin \langle \bar{v}_1\rangle_+$ and $v_2 \notin \langle \bar{v}_2\rangle_+$, then

$$S(x, v_1, v_2) \bigcap S(x, \bar{v}_1, \bar{v}_2) = \{0\}. \tag{18}$$

Proof. It is straightforward (using Lemma 1 and Lemma 2).

3. Main Results

In this section, by using results of the previous section about T-sets in $C^1([0,1], V)$, we obtain a few important properties of a given isometry $T : C^1([0,1], V) \to C^1([0,1], W)$ in order to characterize such isometry.

Proposition 5. *Let $T : C^1([0,1], V) \to C^1([0,1], W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Then,*

(i) T maps a T-set in $C^1([0,1], V)$ to a T-set in $C^1([0,1], W)$. In particular, for any $x \in [0,1]$ and $v_1, v_2 \in V \setminus \{0\}$, there exist $y \in [0,1]$ and $w_1, w_2 \in W \setminus \{0\}$ such that $T(S(x, v_1, v_2)) = S(y, w_1, w_2)$

(ii) If $T(S(x, v_1, v_2)) = S(y, w_1, w_2)$ and $T(S(x, \bar{v}_1, \bar{v}_2)) = S(\bar{y}, \bar{w}_1, \bar{w}_2)$, then $y = \bar{y}$

Proof. (i) Since T is an isometry, by definition, it is easy to see that T maps a T-set to a T-set. Now, by Theorem 3, the proof is complete. (ii) By contradiction, suppose that $y \neq \bar{y}$.

First, we assume that $\dim(V) > 1$. By Proposition 4 (vi), there exist $u_1, u_2 \in V \setminus \{0\}$ such that

$$\begin{aligned} S(x, u_1, u_2) \bigcap S(x, v_1, v_2) &= \{0\}, \\ S(x, u_1, u_2) \bigcap S(x, \bar{v}_1, \bar{v}_2) &= \{0\}. \end{aligned} \tag{19}$$

Therefore, we obtain

$$\begin{aligned} T(S(x, u_1, u_2)) \bigcap S(y, w_1, w_2) &= \{0\}, \\ T(S(x, u_1, u_2)) \bigcap S(\bar{y}, \bar{w}_1, \bar{w}_2) &= \{0\}. \end{aligned} \tag{20}$$

On the other hand, by part (i), we know that $T(S(x, u_1, u_2)) = S(z, \zeta_1, \zeta_2)$, for some $z \in [0,1]$ and $\zeta_1, \zeta_2 \in W \setminus \{0\}$. Also, either $z \neq y$ or $z \neq \bar{y}$. This is a contradiction, by Proposition 4 (ii).

Next, we assume that $V = \mathbf{R}$ and $\dim(W) > 1$. Then, there exist $\zeta_1, \zeta_2 \in W \setminus \{0\}$ and $\bar{x} \in [0,1] \setminus \{x\}$ such that $T(S(\bar{x}, \widehat{v}_1, \widehat{v}_2)) = S(y, \zeta_1, \zeta_2)$, for some $\widehat{v}_1, \widehat{v}_2 \in V \setminus \{0\}$. Now, by considering T^{-1} and since $T^{-1}(S(y, w_1, w_2)) = S(x, v_1, v_2)$ and $T^{-1}(S(y, \zeta_1, \zeta_2)) = S(\bar{x}, \widehat{v}_1, \widehat{v}_2)$, we obtain a contradiction as before (note that $x \neq \bar{x}$).

Finally, we assume that $V = W = \mathbf{R}$ (see [3], page 202). Also, we present a proof for this in the appendix.

Corollary 6. *Let $T : C^1([0,1], V) \to C^1([0,1], W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Then, there exists a bijection $\Phi_T : [0,1] \to [0,1]$ such that $T(S(x, v_1, v_2)) = S(\Phi_T(x), w_1, w_2)$, $\Phi_T(x)$ does not depend on $v_1, v_2 \in V$ (and $w_1, w_2 \in W$).*

Proof. It is an immediate consequence of Proposition 5.

Theorem 7. *Let $T : C^1([0,1], V) \to C^1([0,1], W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Then, T maps constant functions to constant functions. In particular, T induces a linear isometry from V onto W.*

Proof. Suppose that $v \in V \setminus \{0\}$ and the function Φ_T is as in Corollary 6. It is clear that $\widehat{v} \in S(x, v, v_2)$, for all $x \in [0,1]$ and $v_2 \in V$. Therefore, by Proposition 4 (iv) and (v) and Proposition 5 (ii), for any $x \in [0,1]$, we see that either $T(\widehat{v})(\Phi_T(x)) = 0$ or $(T(v\wedge))'(\Phi_T(x)) = 0$. Now, the theorem is an immediate consequence of the following simple fact from real analysis:

(i) Let $h : \mathbf{R} \to V$ be a differentiable function such that for any $x \in \mathbf{R}$, either $h(x) = 0$ or $h'(x) = 0$. Then, h is a constant function

Proposition 8. *Let $T : C^1([0,1], V) \to C^1([0,1], W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Consider $x \in [0,1]$ and $v_1, v_2 \in V \setminus \{0\}$ and suppose that $T(S(x, v_1, v_2)) = S(y, w_1, w_2)$, for some $y \in [0,1]$ and $w_1, w_2 \in W \setminus \{0\}$. Then,*

(i) If $\bar{v}_2 \in V \setminus \{0\}$, then,

$$T(S(x, v_1, \bar{v}_2)) = S(y, w_1, \bar{w}_2), \tag{21}$$

for some $\bar{w}_2 \in W \setminus \{0\}$

(ii) If $\bar{v}_1 \in V \setminus \{0\}$, then,

$$T(S(x, \bar{v}_1, v_2)) = S(y, \bar{w}_1, w_2), \tag{22}$$

for some $\bar{w}_1 \in W \setminus \{0\}$

Proof. (i) By Proposition 5 (ii), we know that $T(S(x, v_1, \bar{v}_2)) = S(y, \bar{w}_1, \bar{w}_2)$, for some $\bar{w}_1, \bar{w}_2 \in W \setminus \{0\}$. On the other hand, the constant function $\widehat{v}_1 \in C^1([0,1], V)$ belongs to $S(x, v_1, v)$, for all $v \in V$. Therefore, by Theorem 7, $T(\widehat{v}_1) \in C^1([0,1], W)$ is a constant function with value in $\langle w_1 \rangle_+ \cap \langle \bar{w}_1 \rangle_+$. This implies that $\bar{w}_1 \in \langle w_1 \rangle_+$ (note that $T(\widehat{v}_1) \neq 0$). So, this completes the proof. (ii) The proof is an immediate consequence of part (i) applied to T^{-1} (by contradiction and using Proposition 4 (vi)).

Lemma 9. *Suppose that $f \in C^2([0,1], V)$, i.e., f is continuously twice differentiable, such that $f(x) = 0$ and $f'(x) \neq 0$, for some $x \in]0,1[$. For any $\eta > 0$ such that $0 < x - \eta < x + \eta < 1$, there exists $\phi \in C^1([0,1], V)$ such that $\phi(x) = 0$, $\phi'(x) \in \langle f'(x) \rangle_+$, $\operatorname{supp}(\phi) \subset]x - \eta, x + \eta[$ and*

$$\begin{aligned} \|\phi\|_1 &= \|\phi(x)\|_1 > \|\phi(t)\|_1, \\ \|\phi + f\|_1 &= \|\phi(x) + f(x)\|_1 > \|\phi(t) + f(t)\|_1, \end{aligned} \tag{23}$$

for all $t \in [0,1] \setminus \{x\}$. Moreover, when $x = 0$ or $x = 1$, we can state and prove a similar result.

Proof. Since $f \in C^2([0,1], V)$, there is a positive constant M such that

$$\left\| f'(s) - f'(t) \right\| \leq M|s - t|, \tag{24}$$

for all $s, t \in [0,1]$. By using this fact, the rest of proof is similar to the proof of Lemma 1, with a slight modification.

Remark 10. Lemma 9 is meaningful when $f'(x) = 0$. In fact, for any $v \in V \setminus \{0\}$, there exists $\phi \in C^1([0,1], V)$ satisfying the conditions in Lemma 9, except, replacing the condition of $\phi'(x) \in \langle f'(x) \rangle_+$ with $\phi'(x) \in \langle v \rangle_+ \setminus \{0\}$.

Theorem 11. *Let $T : C^1([0,1], V) \to C^1([0,1], W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Suppose that Φ_T is as in Corollary 6 and consider $x \in [0,1]$. Suppose that $f \in C^2([0,1], V)$ satisfies the condition $f(x) = 0$, then $T(f)(\Phi_T(x)) = 0$.*

Proof. By Lemma 9 and Remark 10, there exists a function $\phi \in C^1([0,1], V)$ in such a way that $\phi(x) = 0$:

$$\begin{aligned} \|\phi\|_1 &= \|\phi(x)\|_1 > \|\phi(t)\|_1, \\ \|\phi + f\|_1 &= \|\phi(x) + f(x)\|_1 > \|\phi(t) + f(t)\|_1, \end{aligned} \tag{25}$$

for all $t \in [0,1] \setminus \{x\}$. This implies that $\phi + f \in S(x, v, v_2)$, for all $v \in V \setminus \{0\}$ and some $v_2 \in V \setminus \{0\}$. Now, by Proposition 8 (ii), we see that $T(\phi + f) \in S(\Phi_T(x), w, w_2)$, for all $w \in W \setminus \{0\}$ and some $w_2 \in W \setminus \{0\}$. So, by Proposition 4 (iv), we have $T(\phi + f)(\Phi_T(x)) = 0$. Similarly, we can show that $T(\phi)(\Phi_T(x)) = 0$. This completes the proof of theorem.

Corollary 12. *In Theorem 11, we can replace the condition $f \in C^2([0,1], V)$ with the weaker assumption $f \in C^1([0,1], V)$.*

Proof. It is an immediate consequence of the density of $C^2([0,1],V)$ in $C^1([0,1],V)$ (with $\|\cdot\|_1$ norm).

Theorem 13. *Let $T : C^1([0,1],V) \to C^1([0,1],W)$ be a surjective linear isometry, where V and W are two strictly convex Banach spaces. Then, there exists a surjective linear isometry $J : V \to W$, such that either $T(f)(t) = J(f(t))$ or $T(f)(t) = J(f(1-t))$, for all $f \in C^1([0,1],V)$ and $t \in [0,1]$.*

Proof. By Theorem 7, we know that T maps the constant function $\widehat{v} \in C^1([0,1],V)$ with value $v \in V$ to the constant function $\widehat{w} \in C^1([0,1],W)$, for some $w \in W$. So, T induces a surjective linear isometry $J : V \to W$. Now, let f be an arbitrary element of $C^1([0,1],V)$ and let x be an arbitrary element of $[0,1]$. Define $g := f - \widehat{f(x)} \in C^1([0,1],V)$. It is clear that $g(x) = 0$, so by Corollary 12, we obtain $T(g)(\Phi_T(x)) = 0$. This implies that $T(f)(\Phi_T(x)) = J(f(x))$, for all $f \in C^1([0,1],V)$ and $x \in [0,1]$. Finally, by a standard argument, we can show that Φ_T is differentiable on $[0,1]$ and the absolute value of its derivative is a constant function with value 1. This completes the proof of theorem.

Remark 14. Theorem 13 remains meaningful and valid for a surjective linear isometry $T : C^1([a,b],V) \to C^1([c,d],W)$ as well, and as a consequence, we see that the intervals $[a,b]$ and $[c,d]$ should have equal length.

Appendix

In this appendix, we provide a proof for Proposition 5 (ii) whenever $V = W = \mathbf{R}$. To do this, we need to show that $y = \bar{y}$.

Without loss of generality, we may assume that $v_1 = v_2 = \bar{v}_1 = 1$ and $\bar{v}_2 = -1$ and also $w_i = \pm 1$ and $\bar{w}_i = \pm 1$, for $i = 1, 2$.

On the other hand, by Proposition 1.3 (or Equation (1.2)) in [5] (see also ([1], 6.5)), for any $w \in [0,1]$, $\theta_1 = \pm 1$ and $\theta_2 = \pm 1$, there exist $z = z(w,\theta_1,\theta_2) \in [0,1]$, $\Theta_1 = \Theta_1(w,\theta_1,\theta_2) = \pm 1$, and $\Theta_2 = \Theta_2(w,\theta_1,\theta_2) = \pm 1$, such that

$$\theta_1 f(w) + \theta_2 f'(w) = \Theta_1 g(z) + \Theta_2 g'(z), \tag{A.1}$$

for all $f \in C^1([0,1],V)$, where $g := Tf$. In particular, there exist $z_1, z_2 \in [0,1]$, $\alpha_i = \pm 1$, and $\beta_i = \pm 1$, for $i = 1, 2$, such that

$$\begin{aligned} f(x) + f'(x) &= \alpha_1 g(z_1) + \alpha_2 g'(z_1), \\ f(x) - f'(x) &= \beta_1 g(z_2) + \beta_2 g'(z_2), \end{aligned} \tag{A.2}$$

for all $f \in C^1([0,1],V)$, where $g := Tf$.

Now, we show that $y = z_1$, $\bar{y} = z_2$, $\alpha_2 = w_2$, $\beta_2 = \bar{w}_2$ and $\alpha_1 = \beta_1 = w_1 = \bar{w}_1$. To do this, by Lemma 1, there exists a function $\phi \in S(y, w_1, w_2)$ in such a way that $\phi(y) = 0$, $\phi'(y) = w_2$, and also

$$1 = \|\phi\|_1 = \|\phi(y)\|_1 > \|\phi(s)\|_1, \tag{A.3}$$

for all $s \in [0,1] \setminus \{y\}$. Since $\Phi := T^{-1}(\phi) \in S(x, v_1, v_2)$ and

$$1 = \Phi(x) + \Phi'(x) = \alpha_1 \phi(z_1) + \alpha_2 \phi'(z_1) = \alpha_2 w_2. \tag{A.4}$$

This implies that $y = z_1$ and $\alpha_2 = w_2$. Similarly, we can show that $\bar{y} = z_2$ and $\beta_2 = \bar{w}_2$.

Next, we show that T maps a constant function to a constant function. Suppose that the image of $f_0 \in C^1([0,1],V)$ under T is the constant function $g_0 \in C^1([0,1],W)$ with value 1. Then, by Equation (A.1), we have

$$\begin{aligned} f_0(w) + f_0'(w) &= \gamma_1, \\ f_0(w) - f_0'(w) &= \eta_1, \end{aligned} \tag{A.5}$$

for all $w \in [0,1]$, where $\gamma_1 = \pm 1$ and $\eta_1 = \pm 1$. Then, we obtain

$$2 f_0(w) = \gamma_1 + \eta_1, \tag{A.6}$$

for all $w \in [0,1]$. Since f_0 is continuous and $[0,1]$ is connected, we see that f_0 should be constant with value $\alpha_1 = \beta_1 = w_1 = \bar{w}_1$.

By a simple argument similar to the proof of ([2], Lemma 1.4), we can show that the map $w \mapsto z$ is a well-defined homeomorphism from $[0,1]$ onto itself (for fixed values θ_1 and θ_2 in Equation (A.1)); we denote this homeomorphism by $h^T_{\theta_1,\theta_2} = h_{\theta_1,\theta_2}$.

Let $A_{1,1}$ denote the set of all $w \in [0,1]$ such that

$$f(w) + f'(w) = \alpha_1 g(z) + \alpha_2 g'(z), \tag{A.7}$$

for all $f \in C^1([0,1],V)$, where $g := Tf$ and $z = h_{1,1}(w)$. One can easily show that $A_{1,1}$ is a closed and open set in $[0,1]$, and since $x \in A_{1,1}$, we obtain $A_{1,1} = [0,1]$. Similarly, we can define $A_{1,-1}$ and show that $A_{1,-1} = [0,1]$. Therefore, we have

$$f(w) + f'(w) = \alpha_1 g(h_{1,1}(w)) + \alpha_2 g'(h_{1,1}(w)), \tag{A.8}$$

$$f(w) - f'(w) = \beta_1 g(h_{1,-1}(w)) + \beta_2 g'(h_{1,-1}(w)), \tag{A.9}$$

for all $f \in C^1([0,1],V)$ and all $w \in [0,1]$, where $g := Tf$ (note that $\alpha_2 = w_2$, $\beta_2 = \bar{w}_2$ and $\alpha_1 = \beta_1 = w_1 = \bar{w}_1$). If $\alpha_2 = \beta_2$, by considering T^{-1} and the corresponding homeomorphism $h^{T^{-1}}_{\alpha_1,\alpha_2}$, we obtain a contradiction.

So, we have $\alpha_2 = -\beta_2$. Now, by choosing $f(t) := \exp(t)$ in Equation (A.8), we see that $g(s) := Tf(s)$ satisfies

$$\begin{aligned} 2\exp(t) &= w_1 g(h_{1,1}(t)) + w_2 g'(h_{1,1}(t)), \\ 0 &= w_1 g(h_{1,-1}(t)) - w_2 g'(h_{1,-1}(t)), \end{aligned} \tag{A.10}$$

for all $t \in [0,1]$. From the second equation, we see that $g(s) = \lambda \exp(w_1 w_2 s)$, for some constant $\lambda \in \mathbf{R}$. By the first equation, we have

$$\exp(t) = \lambda w_1 \exp(w_1 w_2 h_{1,1}(t)), \tag{A.11}$$

for all $t \in [0, 1]$. In particular, for $t = 0$, we get

$$1 = \lambda w_1 \exp(w_1 w_2 h_{1,1}(0)), \quad (A.12)$$

and $\lambda w_1 > 0$. On the other hand, since $h_{1,1}$ is a homeomorphism on $[0, 1]$, $h_{1,1}(0)$ is equal to 0 or 1.

If $h_{1,1}(0) = 0$, we obtain $\lambda w_1 = 1$ and then we get

$$\exp(t) = \exp(w_1 w_2 h_{1,1}(t)), \quad (A.13)$$

for all $t \in [0, 1]$. This implies that $w_1 w_2 = 1$ and $h_{1,1}(t) = t$, for all $t \in [0, 1]$. Then, since $y = h_{1,1}(x))$, we obtain $y = x$.

If $h_{1,1}(0) = 1$, we obtain $\lambda w_1 = \exp(-w_1 w_2)$ and then we get

$$\exp(t) = \exp(w_1 w_2 (-1 + h_{1,1}(t))), \quad (A.14)$$

for all $t \in [0, 1]$. This implies that $w_1 w_2 = -1$ and $h_{1,1}(t) = 1 - t$, for all $t \in [0, 1]$. Then, since $y = h_{1,1}(x))$, we obtain $y = 1 - x$.

Therefore, either $y = x$, whenever $w_1 w_2 > 0$ or $y = 1 - x$, whenever $w_1 w_2 < 0$. Similarly, by using $f(t) := \exp(-t)$ and $h_{1,-1}$, we can show that either $\bar{y} = x$, whenever $w_1 w_2 > 0$ or $\bar{y} = 1 - x$, whenever $w_1 w_2 < 0$. Finally, we obtain $y = \bar{y}$, as desired.

Remark 15. By the above constructions, one can provide a direct proof for Theorem 13, whenever $V = W = \mathbf{R}$. Also, in ([1], 6.5), one can find a proof for Theorem 13, whenever $V = W$ is the complex plane (note that its dimension over the real line is 2).

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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