Research Article

Approximation Properties of (p, q) -Szász-Mirakjan-Durrmeyer Operators

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In this article, we introduce a new Durrmeyer-type generalization of (p, q) -Szász-Mirakjan operators using the (p, q) -gamma function of the second kind. The moments and central moments are obtained. Then, the Voronovskaja-type asymptotic formula is investigated and point-wise estimates of these operators are studied. Also, some local approximation properties of these operators are investigated by means of modulus of continuity and Peetre $\mathcal{K}$ -functional. Finally, the rate of convergence and weighted approximation of these operators are presented.

1. Introduction

In recent years, (p, q) -analogues of well-known positive operators have been widely constructed and researched since Mursaleen et al. first introduced (p, q) -Bernstein operators [1] and (p, q) -Bernstein-Stancu operators [2]. In [3], Acar first introduced (p, q) -Szász-Mirakjan operators and gave a recurrence relation for the moments of these operators. In [4], Mursaleen et al. proposed a Kantorovich variant of the (p, q) -analogue of Szász-Mirakjan operators under the nondecreasing condition. In [5], Sharma and Gupta introduced (p, q) -Szász-Mirakjan-Kantorovich operators and studied their approximation properties. In [6], Acar et al. constructed King's type (p, q) -Szász-Mirakjan operators preserving x^2 and discussed the order of approximation and weighted approximation properties of these operators. In [7], Aral and Gupta constructed (p, q) -Szász-Mirakjan-Durrmeyer operators using (p, q) -gamma function of the first kind and estimated moments and established some direct results of these operators. In [8], Mursaleen et al. introduced a new modification of Szász-Mirakjan operators based on the (p, q) -calculus and investigated their approximation properties including weighted approximation and Voronovskaya-type

theorem. In [9], Mursaleen et al. proposed two different Kantorovich-type (p, q) -Szász-Mirakjan operators and discussed their error estimated. In [10], Kara and Mahmudov constructed a new (p, q) -Szász-Mirakjan operators as

Definition 1. Let $0 < q < p \leq 1$ and $m \in \mathbb{N}$. For $g : [0, \infty) \rightarrow \mathbb{R}$, (p, q) -Szász-Mirakjan operators can be defined as

$$S_m^{p,q}(g; t) = \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) g\left(\frac{p^{m-k}[k]_{p,q}}{[m]_{p,q}}\right), \quad (1)$$

where $s_{m,k}^{p,q}(t) = (p^{k(k-m)} / q^{(k(k-1))k(k-1)/2}) ([m]_{p,q}^k t^k / [k]_{p,q}!) e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} t)$.

Meantime, quantitative estimates for the convergence in the polynomial weighted spaces and Voronovskaya theorem for new (p, q) -Szász-Mirakjan operators (1) were given. All these achievement motivates us to construct the Durrmeyer analogue of the (p, q) -Szász-Mirakjan operators defined by (1).

Definition 2. Let $0 < q < p \leq 1$ and $m \in \mathbb{N}$. For $g : [0, \infty) \rightarrow \mathbb{R}$, we construct the (p, q) -Szász-Mirakjan-Durrmeyer operators as

$$D_m^{p,q}(g; t) = [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k} p^{k-m} s_{m,k}^{p,q}(u) g(u) d_{p,q} u. \quad (2)$$

Let us recall the basic notations of (p, q) -calculus which can be found in [11]. For any fixed real number $p > 0$ and $q > 0$, the (p, q) -integers $[m]_{p,q}$ are defined as

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \dots + pq^{m-2} + q^{m-1} \\ = \begin{cases} \frac{p^m - q^m}{p - q}, & p \neq q \neq 1, \\ mp^{m-1}, & p = q \neq 1, \\ [m]_q, & p = 1, \\ m, & p = q = 1, \end{cases} \quad (3)$$

where $[m]_q$ denotes the q -integers and $m = 0, 1, 2, \dots$. Also (p, q) -factorial $[m]_{p,q}!$ is defined as follows:

$$[m]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \dots [m]_{p,q}, & m \geq 1, \\ 1, & m = 0. \end{cases} \quad (4)$$

Now, we introduce two types of (p, q) -analogues of exponential function $e_{p,q}(t)$ and $E_{p,q}(t)$ (see [7]):

$$e_{p,q}(t) = \sum_{m=0}^{\infty} \frac{p^{(m(m-1))/2} t^m}{[m]_{p,q}!}, \quad t \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1, \\ E_{p,q}(t) = \sum_{m=0}^{\infty} \frac{q^{(m(m-1))/2} t^m}{[m]_{p,q}!}, \quad t \in \mathbb{R}, |p| < 1 \text{ and } |q| < 1. \quad (5)$$

Let g be an arbitrary function. The improper (p, q) -integral of $g(t)$ on $[0, \infty)$ was defined as (see [12])

$$\int_0^{\infty} g(t) d_{p,q} t = (p - q) \sum_{i=-\infty}^{\infty} \frac{q^i}{p^{i+1}} g\left(\frac{q^i}{p}\right), \quad 0 < \frac{q}{p} < 1. \quad (6)$$

The (p, q) -gamma function of the second kind was defined in [12] as follows:

$$\gamma_{p,q}(z) = \int_0^{\infty} q^{(z(z-1))/2} t^{z-1} e_{p,q}(-pt) d_{p,q} t, \quad \Re(z) > 0. \quad (7)$$

Meantime, the (p, q) -gamma function fulfills the following relation:

$$\gamma_{p,q}(z + 1) = [z]_{p,q} \gamma_{p,q}(z), \quad (8)$$

moreover, for any nonnegative integer $m > 0$, the following relation holds:

$$\gamma_{p,q}(m + 1) = [m]_{p,q}!. \quad (9)$$

2. Auxiliary Results

In order to discuss the approximation properties of the operators $D_m^{p,q}(g; t)$, we need the following lemmas.

Lemma 3 (see ([10], Lemma 4)). For $0 < q < p \leq 1$, $m \in \mathbb{N}$, and $t \in [0, \infty)$, we have

$$S_m^{p,q}(1; t) = 1, S_m^{p,q}(u; t) = t, S_m^{p,q}(u^2; t) = t^2 + \frac{p^{m-1}}{[m]_{p,q}} t, \\ S_m^{p,q}(u^3; t) = t^3 + \frac{(2p + q)p^{m-2}}{[m]_{p,q}} t^2 + \frac{p^{2m-2}}{[m]_{p,q}^2} t, \\ S_m^{p,q}(u^4; t) = t^4 + \frac{(3p^2 + 2qp + q^2)p^{m-3}}{[m]_{p,q}} t^3 \\ + \frac{(3p^2 + 3qp + q^2)p^{2m-4}}{[m]_{p,q}^2} t^2 + \frac{p^{3m-3}}{[m]_{p,q}^3} t. \quad (10)$$

Lemma 4. For $0 < q < p \leq 1$, $r = p/q$, $m \in \mathbb{N}$, and $t \in [0, \infty)$, we have

$$D_m^{p,q}(1; t) = 1, D_m^{p,q}(u; t) = t + \frac{p^m q^{-1}}{[m]_{p,q}}, \\ D_m^{p,q}(u^2; t) = t^2 + \frac{p^m ([2]_{p,q}^2 p^{-1} q^{-2})}{[m]_{p,q}} t + \frac{p^{2m} q^{-3} [2]_{p,q}}{[m]_{p,q}^2}, \\ D_m^{p,q}(u^3; t) = t^3 + \frac{(q^{-1} g_1(r) + 2p^{-1} + p^{-2} q) p^m}{[m]_{p,q}} t^2 \\ + \frac{(p^{-2} + g_2(r) q^{-2} + g_1(r) p^{-1} q^{-1}) p^{2m}}{[m]_{p,q}^2} t \\ + \frac{p^{3m} q^{-3} g_3(r)}{[m]_{p,q}^3}, \\ D_m^{p,q}(u^4; t) = t^4 + \frac{(3p^{-1} + 2qp^{-2} + q^2 p^{-3} + q^{-1} h_1(r)) p^m}{[m]_{p,q}} t^3 \\ + \frac{((3 + (2r + 1)h_1(r))p^{-2} + 3p^{-3}q + p^{-4}q^2 + q^{-2}h_2(r)) p^{2m}}{[m]_{p,q}^2} t^2 \\ + \frac{(p^{-3} + p^{-2}q^{-1}h_1(r) + p^{-1}q^{-2}h_2(r) + q^{-3}h_3(r)) p^{3m}}{[m]_{p,q}^3} t \\ + \frac{q^{-4}h_4(r) p^{4m}}{[m]_{p,q}^4}, \quad (11)$$

where $g_1(r) = \sum_{i=1}^3 [i]_r$, $g_2(r) = \sum_{1 \leq i < j \leq 3} [i]_r [j]_r$, $g_3(r) = [2]_r [3]_r$, $h_1(r) = \sum_{i=1}^4 [i]_r$, $h_2(r) = \sum_{1 \leq i < j \leq 4} [i]_r [j]_r$, $h_3(r) = \sum_{1 \leq i < j < k \leq 4} [i]_r [j]_r [k]_r$, and $h_4(r) = [2]_r [3]_r [4]_r$.

Proof. Using the (p, q) -gamma function of the second kind, we can obtain

$$\begin{aligned}
 D_m^{p,q}(1; t) &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k} p^{k-m} s_{m,k}^{p,q}(u) d_{p,q} u \\
 &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-(k(k+1))/2} p^{(k-m)(k+1)} \\
 &\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} u) d_{p,q} u \\
 &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-(k(k+1))/2} p^{(k-m)(k+1)} \\
 &\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-p([m]_{p,q} p^{k-m} q^{-k} u)) d_{p,q} u \\
 &= \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{(k(k+1))/2} \frac{([m]_{p,q} p^{k-m} q^{-k} u)^k}{[k]_{p,q}!} e_{p,q} \\
 &\quad \cdot (-p([m]_{p,q} p^{k-m} q^{-k} u)) d_{p,q} ([m]_{p,q} p^{k-m} q^{-k} u) \\
 &= \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{\gamma_{p,q}(k+1)}{[k]_{p,q}!} = \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) = 1.
 \end{aligned} \tag{12}$$

and next using $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we have

$$\begin{aligned}
 D_m^{p,q}(u; t) &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k(k+1)/2} p^{(k-m)(k+1)} \\
 &\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} u) u d_{p,q} u \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{m-k} q^{-1}}{[m]_{p,q} [k]_{p,q}!} \int_0^{\infty} q^{-(k+1)(k+2)/2} p^{(k-m)(k+1)} e_{p,q}(-ps) d_{p,q} s \\
 &\quad \cdot ([m]_{p,q} p^{k-m} q^{-k} u)^{k+1} e_{p,q}(-p([m]_{p,q} p^{k-m} q^{-k} u)) \\
 &\quad \times d_{p,q}([m]_{p,q} p^{k-m} q^{-k} u) \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{m-k} q^{-1}}{[m]_{p,q} [k]_{p,q}!} \int_0^{\infty} q^{(k+1)(k+2)/2} p^{k+1} e_{p,q}(-ps) d_{p,q} s \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{m-k} q^{-1}}{[m]_{p,q} [k]_{p,q}!} \gamma_{p,q}(k+2) = \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{m-k} q^{-1} [k+1]_{p,q}}{[m]_{p,q}} \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{m-k} q^{-1} (p^k + q[k]_{p,q})}{[m]_{p,q}} = \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \\
 &\quad + \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{p^m q^{-1}}{[m]_{p,q}} = S_m^{p,q}(u; t) + S_m^{p,q}(1; t) = t + \frac{p^m q^{-1}}{[m]_{p,q}}.
 \end{aligned} \tag{13}$$

Using the equality $[k+1]_{p,q} [k+2]_{p,q} = q^3 [k]_{p,q}^2 + q(2q+p) p^k [k]_{p,q} + [2]_{p,q} p^{2k}$ and Lemma 3, we have

$$\begin{aligned}
 D_m^{p,q}(u^2; t) &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k} p^{k-m} s_{m,k}^{p,q}(u) u^2 d_{p,q} u \\
 &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-(k(k+1))/2} p^{(k-m)(k+1)} \\
 &\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} u) u^2 d_{p,q} u \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{2m-2k}}{[m]_{p,q}^2 [k]_{p,q}!} \int_0^{\infty} q^{-(k(k+1))/2 + k(k+3)} \\
 &\quad \cdot ([m]_{p,q} p^{k-m} q^{-k} u)^{k+2} e_{p,q}(-p([m]_{p,q} p^{k-m} q^{-k} u)) \\
 &\quad \times d_{p,q}([m]_{p,q} p^{k-m} q^{-k} u) \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{2m-2k} q^{-3}}{[m]_{p,q}^2 [k]_{p,q}!} \int_0^{\infty} q^{((k+2)(k+3))/2} s^{k+2} e_{p,q}(-ps) d_{p,q} s \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{2m-2k} q^{-3}}{[m]_{p,q}^2 [k]_{p,q}!} \gamma_{p,q}(k+3) \\
 &= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{2m-2k} q^{-3} [k+1]_{p,q} [k+2]_{p,q}}{[m]_{p,q}^2} \\
 &= \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^2 + \frac{p^m q^{-2} (p+2q)}{[m]_{p,q}} \\
 &\quad \cdot \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} + \frac{p^{2m} q^{-3} [2]_{p,q}}{[m]_{p,q}^2} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \\
 &= S_m^{p,q}(u^2; t) + \frac{p^m q^{-2} (p+2q)}{[m]_{p,q}} S_m^{p,q}(u; t) \\
 &\quad + \frac{p^{2m} q^{-3} [2]_{p,q}}{[m]_{p,q}^2} S_m^{p,q}(1; t) = t^2 + (p^{-1} + 2q^{-1} + pq^{-2}) \\
 &\quad \cdot \frac{p^m}{[m]_{p,q}} t + \frac{p^{2m} q^{-3} [2]_{p,q}}{[m]_{p,q}^2} = t^2 + \frac{p^m ([2]_{p,q}^2 p^{-1} q^{-2})}{[m]_{p,q}} t \\
 &\quad + \frac{p^{2m} q^{-3} [2]_{p,q}}{[m]_{p,q}^2}.
 \end{aligned} \tag{14}$$

Using $[m]_{p,q} = q^{m-1} [m]_r$, we have

$$\begin{aligned}
 [k+1]_{p,q} [k+2]_{p,q} [k+3]_{p,q} &= (q[k]_{p,q} + p^k) (q^2 [k]_{p,q} + [2]_{p,q} p^k) \\
 &\quad \cdot (q^3 [k]_{p,q} + [3]_{p,q} p^k) = q^6 [k]_{p,q}^3 \\
 &\quad + (q^3 [3]_{p,q} + q^4 [2]_{p,q} + q^5) [k]_{p,q}^2 p^k \\
 &\quad + (q[2]_{p,q} [3]_{p,q} + q^2 [3]_{p,q} + q^3 [2]_{p,q}) [k]_{p,q} p^{2k} + [2]_{p,q} [3]_{p,q} p^{3k} \\
 &= q^6 [k]_{p,q}^3 + q^5 ([3]_r + [2]_r + 1) [k]_{p,q}^2 p^k \\
 &\quad + q^4 ([3]_r [2]_r + [3]_r + [2]_r) [k]_{p,q} p^{2k} + q^3 [3]_r [2]_r p^{3k} \\
 &:= q^6 [k]_{p,q}^3 + q^5 g_1(r) [k]_{p,q}^2 p^k + q^4 g_2(r) [k]_{p,q} p^{2k} + q^3 g_3(r) p^{3k}.
 \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned}
D_m^{p,q}(u^3; t) &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k} p^{k-m} s_{m,k}^{p,q}(u) u^3 d_{p,q} u \\
&= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k(k+1)/2} p^{(k-m)(k+1)} \\
&\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} u) u^3 d_{p,q} u \\
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{3m-3k}}{[m]_{p,q}^3 [k]_{p,q}!} \int_0^{\infty} q^{-k(k+1)/2+k(k+4)} \\
&\quad \cdot ([m]_{p,q} p^{k-m} q^{-k} u)^{k+3} e_{p,q}(-p([m]_{p,q} p^{k-m} q^{-k} u)) \\
&\quad \times d_{p,q}([m]_{p,q} p^{k-m} q^{-k} u) = \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{3m-3k} q^{-6}}{[m]_{p,q}^3 [k]_{p,q}!} \\
&\quad \cdot \int_0^{\infty} q^{((k+3)(k+4))/2} s^{k+3} e_{p,q}(-ps) d_{p,q} s \\
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{3m-3k} q^{-6}}{[m]_{p,q}^3 [k]_{p,q}!} \gamma_{p,q}(k+4) \\
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{3m-3k} q^{-6} [k+1]_{p,q} [k+2]_{p,q} [k+3]_{p,q}}{[m]_{p,q}^3} \\
&= \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^3 + \frac{p^m q^{-1} g_1(r)}{[m]_{p,q}} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \\
&\quad \cdot \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^2 + \frac{p^{2m} q^{-2} g_2(r)}{[m]_{p,q}^2} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \\
&\quad + \frac{p^{3m} q^{-3} g_3(r)}{[m]_{p,q}^3} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) = S_m^{p,q}(u^3; t) \\
&\quad + \frac{p^m q^{-1} g_1(r)}{[m]_{p,q}} S_m^{p,q}(u^2; t) + \frac{p^{2m} q^{-2} g_2(r)}{[m]_{p,q}^2} S_m^{p,q}(u; t) \\
&\quad + \frac{p^{3m} q^{-3} g_3(r)}{[m]_{p,q}^3} S_m^{p,q}(1; t) \\
&= t^3 + \frac{(q^{-1} g_1(r) + 2p^{-1} + p^{-2} q) p^m}{[m]_{p,q}} t^2 \\
&\quad + \frac{(p^{-2} + g_2(r) q^{-2} + g_1(r) p^{-1} q^{-1}) p^{2m}}{[m]_{p,q}^2} t + \frac{p^{3m} q^{-3} g_3(r)}{[m]_{p,q}^3}.
\end{aligned} \tag{16}$$

Similarly, using $\prod_{i=1}^4 [k+i]_{p,q} = q^{10} [k]_{p,q}^4 + h_1(r) q^9 [k]_{p,q}^3 p^k + h_2(r) q^8 [k]_{p,q}^2 p^{2k} + h_3(r) q^7 [k]_{p,q} p^{3k} + h_4(r) q^6 p^{4k}$, we can obtain

$$\begin{aligned}
D_m^{p,q}(u^4; t) &= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-k} p^{k-m} s_{m,k}^{p,q}(u) u^4 d_{p,q} u \\
&= [m]_{p,q} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \int_0^{\infty} q^{-(k(k+1))/2} p^{(k-m)(k+1)} \\
&\quad \cdot \frac{([m]_{p,q} u)^k}{[k]_{p,q}!} e_{p,q}(-[m]_{p,q} p^{k-m+1} q^{-k} u) u^4 d_{p,q} u
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{4m-4k}}{[m]_{p,q}^3 [k]_{p,q}!} \int_0^{\infty} q^{-((k(k+1))/2)+(k(k+5))} \\
&\quad \cdot ([m]_{p,q} p^{k-m} q^{-k} u)^{k+4} e_{p,q}(-p([m]_{p,q} p^{k-m} q^{-k} u)) \\
&\quad \times d_{p,q}([m]_{p,q} p^{k-m} q^{-k} u) = \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{4m-4k} q^{-10}}{[m]_{p,q}^4 [k]_{p,q}!} \\
&\quad \cdot \int_0^{\infty} q^{((k+4)(k+5))/2} s^{k+4} e_{p,q}(-ps) d_{p,q} s \\
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{4m-4k} q^{-10}}{[m]_{p,q}^4 [k]_{p,q}!} \gamma_{p,q}(k+5) \\
&= \sum_{k=0}^{\infty} \frac{s_{m,k}^{p,q}(t) p^{4m-4k} q^{-10} [k+1]_{p,q} [k+2]_{p,q} [k+3]_{p,q} [k+4]_{p,q}}{[m]_{p,q}^4} \\
&= \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^4 + \frac{p^m q^{-1} h_1(r)}{[m]_{p,q}} \\
&\quad \cdot \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^3 + \frac{p^{2m} q^{-2} h_2(r)}{[m]_{p,q}^2} \\
&\quad \cdot \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \left(\frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} \right)^2 + \frac{p^{3m} q^{-3} h_3(r)}{[m]_{p,q}^3} \\
&\quad \cdot \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \frac{p^{m-k} [k]_{p,q}}{[m]_{p,q}} + \frac{p^{3m} q^{-4} h_4(r)}{[m]_{p,q}^4} \sum_{k=0}^{\infty} s_{m,k}^{p,q}(t) \\
&= S_m^{p,q}(u^4; t) + \frac{p^m q^{-1} h_1(r)}{[m]_{p,q}} S_m^{p,q}(u^3; t) \\
&\quad + \frac{p^{2m} q^{-2} h_2(r)}{[m]_{p,q}^2} S_m^{p,q}(u^2; t) + \frac{p^{3m} q^{-3} h_3(r)}{[m]_{p,q}^3} S_m^{p,q}(u; t) \\
&\quad + \frac{p^{3m} q^{-4} h_4(r)}{[m]_{p,q}^4} S_m^{p,q}(1; t) \\
&= t^4 + \frac{(3p^{-1} + 2qp^{-2} + q^2 p^{-3} + q^{-1} h_1(r)) p^m}{[m]_{p,q}} t^3 \\
&\quad + \frac{((3 + (2r+1)h_1(r))p^{-2} + 3p^{-3}q + p^{-4}q^2 + q^{-2}h_2(r)) p^{2m}}{[m]_{p,q}^2} t^2 \\
&\quad + \frac{(p^{-3} + p^{-2}q^{-1}h_1(r) + p^{-1}q^{-2}h_2(r) + q^{-3}h_3(r)) p^{3m}}{[m]_{p,q}^3} t \\
&\quad + \frac{q^{-4}h_4(r) p^{4m}}{[m]_{p,q}^4}.
\end{aligned} \tag{17}$$

Lemma 5. Using Lemma 4, we immediately have the following explicit formulas for the central moments:

$$\begin{aligned}
A_m^{p,q}(t) &:= D_m^{p,q}(u-t; t) = \frac{q^{-1} p^m}{[m]_{p,q}}, \\
B_m^{p,q}(t) &:= D_m^{p,q}((u-t)^2; t) = \frac{(p^{-1} + p q^{-2}) p^m}{[m]_{p,q}} t + \frac{q^{-3} [2]_{p,q} p^{2m}}{[m]_{p,q}^2}.
\end{aligned} \tag{18}$$

Lemma 6. The sequences (p_m) , (q_m) satisfy $0 < q_m < p_m \leq 1$ such that $p_m \rightarrow 1$, $q_m \rightarrow 1$ and $p_m^m \rightarrow \beta \in [0, 1]$, $[m]_{p_m, q_m} \rightarrow \infty$ as $m \rightarrow \infty$, $r_m = p_m/q_m$, then

$$\lim_{m \rightarrow \infty} [m]_{p_m, q_m} A_m^{p_m, q_m}(t) = \beta, \quad (19)$$

$$\lim_{m \rightarrow \infty} [m]_{p_m, q_m} B_m^{p_m, q_m}(t) = 2\beta t, \quad (20)$$

$$\lim_{m \rightarrow \infty} [m]_{p_m, q_m} D_m^{p_m, q_m}((u-t)^4; t) = 0. \quad (21)$$

Proof. Applying Lemma 5, we can easily obtain (19) and (20). While $m \rightarrow \infty$, we can rewrite

$$\begin{aligned} D_m^{p_m, q_m}(1; t) &= 1 D_m^{p_m, q_m}(u; t)t + \frac{\beta}{[m]_{p_m, q_m}}; \\ D_m^{p_m, q_m}(u^2; t) &\sim t^2 + \frac{4\beta}{[m]_{p_m, q_m}}t, \\ D_m^{p_m, q_m}(u^3; t) &\sim t^3 + \frac{9\beta}{[m]_{p_m, q_m}}t^2; \\ D_m^{p_m, q_m}(u^4; t) &\sim t^4 + \frac{16\beta}{[m]_{p_m, q_m}}t^3. \end{aligned} \quad (22)$$

Using $D_m^{p_m, q_m}((u-t)^4; t) = D_m^{p_m, q_m}(u^4; t) - 4D_m^{p_m, q_m}(u^3; t)t + 6D_m^{p_m, q_m}(u^2; t)t^2 - 4D_m^{p_m, q_m}(u; t)t^3 + D_m^{p_m, q_m}(1; t)t^4$ and $16 - 9 \times 4 + 4 \times 6 - 4 \times 1 = 0$, we can get (21).

3. Voronovskaja-Type Theorem

Theorem 7. Let $(p_m), (q_m)$ be the sequences defined in Lemma 6 and $g \in C_B[0, \infty)$. Supposing that $g'(t)$ exists at a point $t \in [0, \infty)$, then, we can obtain

$$\lim_{m \rightarrow \infty} [m]_{p_m, q_m} (D_m^{p_m, q_m}(g; t) - g(t)) = \beta(g(t) + t g'(t)), \quad (23)$$

where $C_B[0, \infty)$ denotes the set of all real-valued bounded and continuous functions defined on $[0, \infty)$, endowed with the norm $\|g\| = \sup_{t \in [0, \infty)} |g(t)|$.

Proof. By the Taylor's expansion theorem of function $g \in C_B[0, \infty)$, we can obtain

$$g(u) - g(t) = g'(t)(u-t) + \frac{1}{2} g''(t)(u-t)^2 + \Theta(u, t)(u-t)^2, \quad (24)$$

where $u, t \in [0, \infty)$, $\Theta(u, t)$ is bounded and $\lim_{u \rightarrow t} \Theta(u, t) = 0$. Applying the operator $D_m^{p_m, q_m}$ to the equality above, we can obtain

$$\begin{aligned} D_m^{p_m, q_m}(g; t) - g(t) &= g'(t) A_m^{p_m, q_m}(t) + \frac{1}{2} g''(t) B_m^{p_m, q_m}(t) \\ &\quad + D_m^{p_m, q_m}(\Theta(u, t)(u-t)^2; t) \end{aligned} \quad (25)$$

Since $\lim_{u \rightarrow t} \Theta(u, t) = 0$, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|u-t| < \delta$ and it will imply $|\Theta(u, t)| < \varepsilon$ for all fixed $t \in [0, \infty)$ as m sufficiently large. While if $|u-t| \geq \delta$, then $|\Theta(u, t)| \leq C/\delta^2(u-t)^2$, where $C > 0$ is a constant. Using Lemma 6

$$\begin{aligned} [m]_{p_m, q_m} |D_m^{p_m, q_m}(\Theta(u, t)(u-t)^2; t)| \\ \leq \varepsilon [m]_{p_m, q_m} |D_m^{p_m, q_m}((u-t)^2; t)| \\ + \frac{C}{\delta^2} [m]_{p_m, q_m} D_m^{p_m, q_m}((u-t)^4; t) \rightarrow 0 \quad (m \rightarrow \infty). \end{aligned} \quad (26)$$

The proof is completed.

4. Point-Wise Estimate

In this section, we establish two point-wise estimate of the operators $D_m^{p, q}(g; t)$. First, a function $g \in C[0, \infty)$ is said to satisfy the Lipschitz condition Lip_γ on D (named $g \in \text{Lip}_{C_{g, \gamma}}(\gamma, E)$), $\gamma \in (0, 1]$, $D \subset [0, \infty)$ if

$$|g(u) - g(t)| \leq C_{g, \gamma} |u - t|^\gamma, \quad u \in [0, \infty) \text{ and } t \in D, \quad (27)$$

where $C_{g, \gamma}$ is an absolute positive constant depending only on g and γ .

Theorem 8. Let $0 < q < p \leq 1$, $\gamma \in (0, 1]$ and E be any bounded subset on $[0, \infty)$. If $g \in C_B[0, \infty) \cap \text{Lip}_{C_{g, \gamma}}(\gamma, E)$, then, for all $t \in [0, \infty)$, we have

$$|D_m^{p, q}(g; t) - g(t)| \leq C_{g, \gamma} (B_m^{p, q}(t))^{\gamma/2} + 2d^\gamma(t; E), \quad (28)$$

where $d(t; E)$ denotes the distance between t and E defined by

$$d(t; E) = \inf \{|u - t| : u \in E\}. \quad (29)$$

Proof. Let $\bar{E}$ be the closure of E . Using the properties of infimum, there is at least a point $u_0 \in \bar{E}$ such that $d(t; E) = |t - u_0|$. By the triangle inequality

$$|g(u) - g(t)| \leq |g(u) - g(u_0)| + |g(t) - g(u_0)|, \quad (30)$$

we can obtain

$$\begin{aligned} |D_m^{p, q}(g; t) - g(t)| &\leq D_m^{p, q}(|g(u) - g(u_0)|; t) + D_m^{p, q}(|g(t) - g(u_0)|; t) \\ &\leq C_{g, \gamma} \{D_m^{p, q}(|u - u_0|^\gamma; t) + |t - u_0|^\gamma\} \\ &\leq C_{g, \gamma} \{D_m^{p, q}(|u - t|^\gamma + |t - u_0|^\gamma; t) + |t - u_0|^\gamma\} \\ &= C_{g, \gamma} \{D_m^{p, q}(|u - t|^\gamma; t) + 2|t - u_0|^\gamma\}. \end{aligned} \quad (31)$$

Choosing $p_1 = 2/\gamma$ and $p_2 = 2/(2 - \gamma)$ and using the well-known Hölder inequality, we have

$$\begin{aligned}
|D_m^{p,q}(f; x) - f(x)| &\leq C_{g,\gamma} \left\{ (D_m^{p,q}(|t-x|^{p_1\gamma}; x))^{1/p_1} (D_m^{p,q}(1^{p_2}; x))^{1/p_2} + 2d^\gamma(x; E) \right\} \\
&\leq C_{g,\gamma} \left\{ (D_m^{p,q}((t-x)^2; x))^{\gamma/2} + 2d^\gamma(x; E) \right\} \\
&\leq C_{g,\gamma} \left((B_m^{p,q}(t))^{\gamma/2} + 2d^\gamma(x; E) \right).
\end{aligned} \tag{32}$$

Next, we obtain the local direct estimate of the operators $D_m^{p,q}$, using the Lipschitz-type maximal function of the order γ introduced by Lenze [13] as

$$\tilde{\omega}_\gamma(g; t) = \sup_{t, u \in (0, \infty), t \neq u} \frac{|g(u) - g(t)|}{|u - t|^\gamma}, \gamma \in (0, 1]. \tag{33}$$

Theorem 9. Let $g \in C_B[0, \infty)$ and $\gamma \in (0, 1]$. Then, for all $t \in [0, \infty)$, we have

$$|D_m^{p,q}(g; t) - g(t)| \leq \tilde{\omega}_\gamma(g; t) (B_m^{p,q}(t))^{\gamma/2}. \tag{34}$$

Proof. From equation (33), we have

$$|D_m^{p,q}(g; t) - g(t)| \leq \tilde{\omega}_\gamma(g; t) D_m^{p,q}(|u - t|^\gamma; t). \tag{35}$$

Applying the well-known Hölder inequality, we can get

$$|D_m^{p,q}(g; t) - g(t)| \leq \tilde{\omega}_\gamma(g; t) (D_m^{p,q}((u - t)^2; t))^{\gamma/2} \leq \tilde{\omega}_\gamma(g; t) (B_m^{p,q}(t))^{\gamma/2}. \tag{36}$$

5. Local Approximation

In this section, we establish local approximation theorem for (p, q) -Szász-Mirakjan-Durrmeyer operators. Let us consider the following $\mathcal{K}$ -functional:

$$\mathcal{K}(g; \delta) = \inf_{h \in S^2} \left\{ \|g - h\| + \delta \|h''\| \right\}, \tag{37}$$

where $\delta \in [0, \infty)$ and $S^2 = \{h \in C_B[0, \infty): h', h'' \in C_B[0, \infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of g can be defined by

$$\begin{aligned}
\omega(g; \delta) &= \sup_{0 < |u| \leq \delta} \sup_{t \in [0, \infty)} |g(t + u) - g(t)|, \\
\omega_2(g; \delta) &= \sup_{0 < |u| \leq \delta} \sup_{t \in [0, \infty)} |g(t + 2u) - 2g(t + u) + g(t)|.
\end{aligned} \tag{38}$$

By ([14], p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such

$$\mathcal{K}(g; \delta^2) \leq C\omega_2(g; \delta), \delta > 0. \tag{39}$$

Theorem 10. Let $(p_m), (q_m)$ be the sequences defined in Lemma 6 and $g \in C_B[0, \infty)$. Then, for all $m \in \mathbb{N}$, there exists an absolute positive $C_I = 4C$ such that

$$\begin{aligned}
|D_m^{p_m, q_m}(g; t) - g(t)| &\leq C_I \omega_2 \left(g; \sqrt{(A_m^{p_m, q_m}(t))^2 + B_m^{p_m, q_m}(t)} \right) \\
&\quad + \omega(g; |A_m^{p_m, q_m}(t)|).
\end{aligned} \tag{40}$$

Proof. For a given function $g \in C_B[0, \infty)$, let us define the following new operators:

$$\mathbf{D}_m^{p_m, q_m}(g; t) = D_m^{p_m, q_m}(g; t) - g(A_m^{p_m, q_m}(t) + t) + g(t), g \in [0, \infty). \tag{41}$$

By Lemma 4 and Lemma 5, we obtain

$$\mathbf{D}_m^{p_m, q_m}(1; t) = 1 \text{ and } \mathbf{D}_m^{p_m, q_m}(u; t) = t. \tag{42}$$

Let $t, u \in [0, \infty)$ and $h \in S^2$. By Taylor's expansion formula, we get

$$h(u) = h(t) + h'(t)(u - t) + \int_t^u h''(v)(u - v)dv. \tag{43}$$

Applying $\mathbf{D}_m^{p_m, q_m}$ to the above equality, we can write

$$\begin{aligned}
\mathbf{D}_m^{p_m, q_m}(h; t) - h(t) &= \mathbf{D}_m^{p_m, q_m} \left(\int_t^u h''(v)(u - v)dv; t \right) \\
&\leq D_m^{p_m, q_m} \left(\left| \int_t^u |u - v| h''(v)dv \right|; t \right) \\
&\quad + \left| \int_t^{A_m^{p_m, q_m}(t) + t} h''(u)(A_m^{p_m, q_m}(t) + t - v)dv \right| \\
&\leq D_m^{p_m, q_m}((u - t)^2; t) \|h''\| + (A_m^{p_m, q_m}(t))^2 \|h''\| \\
&= (A_m^{p_m, q_m}(t) + B_m^{p_m, q_m}(t))^2 \|h''\|.
\end{aligned} \tag{44}$$

On the other hand, since $|\mathbf{D}_m^{p_m, q_m}(g; t)| \leq 3\|g\|$. Hence,

$$\begin{aligned}
|D_m^{p_m, q_m}(g; t) - g(t)| &= |\mathbf{D}_m^{p_m, q_m}(g; t) + g(A_m^{p_m, q_m}(t) + t) - 2g(t)| \\
&\leq |\mathbf{D}_m^{p_m, q_m}(g - h; t) - (g - h)(t)| \\
&\quad + |\mathbf{D}_m^{p_m, q_m}(h; x) - h(x)| \\
&\quad + |g(A_m^{p_m, q_m}(x) + x) - g(x)| \\
&\leq 4\|g - h\| + \left((A_m^{p_m, q_m}(t) + B_m^{p_m, q_m}(t))^2 + \right) \|h''\| \\
&\quad + \omega(g; |A_m^{p_m, q_m}(t)|).
\end{aligned} \tag{45}$$

Taking infimum on the right hand side over all $h \in S^2$ and using (39), we obtain the desired assertion.

Corollary 11. Let $(p_m), (q_m)$ be the sequences defined in Lemma 6 and $g \in C_B[0, \infty)$. Then, for any finite interval $I \subset [0, \infty)$, the sequence $\{D_m^{p_m, q_m}(g; t)\}$ converges to g uniformly on I .

6. Rate of Convergence

Let

$$\begin{aligned} B_2[0, \infty) &= \{g : |g(t)| \leq M_g(1+t^2)\}, \\ C_2[0, \infty) &= \{g : g \in B_2[0, \infty) \cap C[0, \infty)\}, \\ C_2^0[0, \infty) &= \left\{g : g \in C_2[0, \infty) \text{ and } \lim_{t \rightarrow \infty} \frac{|g(t)|}{1+t^2} < \infty\right\}, \end{aligned} \quad (46)$$

where M_g is an absolute constant depending only on g . $C_2^0[0, \infty)$ is equipped with the norm $\|g\|_2 = \sup_{t \in [0, \infty)} |g(t)|/(1+t^2)$. As is known, if $f \in C[0, \infty)$ is not uniform, we cannot obtain $\lim_{\delta \rightarrow 0^+} \omega(g; \delta) = 0$. In [15], Ispir defined the following weighted modulus of continuity

$$\Omega(g; \delta) = \sup_{0 < u \leq \delta, t \in (0, \infty)} \frac{|g(u+t) - g(t)|}{(1+u^2)(1+t^2)}, \quad (47)$$

and proved the properties of monotone increasing about $\Omega(g; \delta)$ as $\delta > 0$, $\lim_{\delta \rightarrow 0^+} \Omega(g; \delta) = 0$ and the inequality

$$\Omega(g; \lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(g; \delta), \quad (48)$$

while $\lambda > 0$ and $g \in C_2^0[0, \infty)$. Meantime, we recall the modulus of continuity of g on the interval $[0, \kappa] \subset [0, \infty)$ by

$$\omega_\kappa(g; \delta) = \sup_{u, t \in (0, \kappa], |u-t| \leq \delta} |g(u) - g(t)|, \quad \delta > 0. \quad (49)$$

Theorem 12. Let $g \in C_2[0, \infty)$, $0 < q < p \leq 1$, and $\kappa > 0$, we have

$$\|D_m^{p,q}(g; t) - g\|_{C[0, \kappa]} \leq 4M_g(3+2\kappa^2)B_m^{p,q}(\kappa) + 2\omega_{\kappa+1}\left(f; \sqrt{B_m^{p,q}(\kappa)}\right). \quad (50)$$

Proof. For any $t \in [0, \kappa]$ and $u > \kappa + 1$, we easily have $1 \leq (u - \kappa)^2 \leq (u - t)^2$, thus

$$\begin{aligned} |g(u) - g(t)| &\leq |g(u)| + |g(t)| \leq M_g(2+u^2+t^2) \\ &= M_g(2+t^2+(u-t+t)^2) \\ &\leq M_g(2+2t^2+(u-t)^2) \leq M_g(3+2t^2)(u-t)^2 \\ &\leq M_g(3+2\kappa^2)(u-t)^2, \end{aligned} \quad (51)$$

and for any $t \in [0, \kappa]$, $u \in [0, \kappa + 1]$ and $\delta > 0$, we have

$$|g(u) - g(t)| \leq \omega_{\kappa+1}(|u-t|; t) \leq \left(1 + \frac{|u-t|}{\delta}\right) \omega_{\kappa+1}(g; \delta). \quad (52)$$

For (51) and (52), we can get

$$|g(u) - g(t)| \leq M_g(3+2\kappa^2)(u-t)^2 + \left(1 + \frac{|u-t|}{\delta}\right) \omega_{\kappa+1}(g; \delta). \quad (53)$$

By Schwarz's inequality, for any $t \in [0, \kappa]$, we can get

$$\begin{aligned} |D_m^{p,q}(g; t) - g(t)| &\leq D_m^{p,q}(|g(u) - g(t)|; t) \\ &\leq M_g(3+2\kappa^2)D_m^{p,q}((u-t)^2; t) \\ &\quad + D_m^{p,q}\left(\left(1 + \frac{|u-t|}{\delta}\right); t\right) \omega_{\kappa+1}(g; \delta) \\ &\leq M_g(3+2\kappa^2)D_m^{p,q}((u-t)^2; t) \\ &\quad + \omega_{\kappa+1}(g; \delta) \left(1 + \frac{1}{\delta} \sqrt{D_m^{p,q}((u-t)^2; t)}\right) \\ &\leq M_g(3+2\kappa^2)B_m^{p,q}(t) + \omega_{\kappa+1}(f; \delta) \left(1 + \frac{1}{\delta} \sqrt{B_m^{p,q}(t)}\right) \\ &\leq M_g(3+2\kappa^2)B_m^{p,q}(\kappa) + \omega_{\kappa+1}(g; \delta) \left(1 + \frac{1}{\delta} \sqrt{B_m^{p,q}(\kappa)}\right). \end{aligned} \quad (54)$$

By taking $\delta = \sqrt{B_m^{p,q}(\kappa)}$ and supremum over all $t \in [0, \kappa]$, we accomplish the proof of Theorem 12.

7. Weighted Approximation

In this section, we will discuss the following three theorems about weighted approximation for the operators $D_m^{p,q_m}(g; t)$.

Theorem 13. Let $f \in C_2^0[0, \infty)$ and the sequences (p_m) , (q_m) be the sequences defined in Lemma 6; then, for any $t \in [0, \infty)$, there exists $N \in \mathbb{N}_+$ such that for all $m > N$, the inequality

$$\frac{|D_m^{p_m, q_m}(g; t) - g(t)|}{(1+t^2)^2} \leq 10\Omega\left(f; \frac{1}{\sqrt{[m]_{p_m, q_m}}}\right) \quad (55)$$

holds.

Proof. Using (47) and (48), we can write

$$\begin{aligned} |g(u) - g(t)| &\leq (1+(u-t)^2)(1+t^2)\Omega(f; |u-t|) \\ &\leq 2\left(1 + \frac{|u-t|}{\delta}\right)(1+\delta^2)\Omega(f; \delta)(1+(u-t)^2)(1+t^2) \\ &\leq \begin{cases} 4(1+\delta^2)^2(1+t^2)\Omega(f; \delta), & |u-t| \leq \delta, \\ 4(1+\delta^2)(1+t^2)\Omega(f; \delta) \frac{|u-t|+|u-t|^3}{\delta}, & |u-t| > \delta. \end{cases} \end{aligned} \quad (56)$$

For any $\delta \in (0, 1/2)$ and $t, u \in [0, \infty)$, (56) can be rewritten

$$|g(t) - g(x)| \leq 5(1+t^2)\Omega(f; \delta) \left(\frac{5}{4} + \frac{|u-t|+|u-t|^3}{\delta}\right). \quad (57)$$

Using (20) and (21), there exists $N \in \mathbb{N}_+$ such that for any $m > N$,

$$D_m^{p_m, q_m}((u-t)^2; t) \leq \frac{2}{[m]_{p_m, q_m}} t, D_m^{p_m, q_m}((u-t)^4; t) \leq \frac{t^3}{2}. \quad (58)$$

By Schwarz's inequality, we can obtain

$$D_m^{p_m, q_m}(|u-t|; t) \leq \sqrt{D_m^{p_m, q_m}((u-t)^2; t)} \sqrt{D_m^{p_m, q_m}((u-t)^4; t)} \leq \frac{\sqrt{2t}}{\sqrt{[m]_{p_m, q_m}}}, \quad (59)$$

$$\begin{aligned} D_m^{p_m, q_m}(|u-t|^3; t) &\leq \sqrt{D_m^{p_m, q_m}((u-t)^2; t)} \sqrt{D_m^{p_m, q_m}((u-t)^4; t)} \\ &\leq \frac{t^2}{\sqrt{[m]_{p_m, q_m}}}. \end{aligned} \quad (60)$$

Since $D_m^{p_m, q_m}$ is linear and positive, using (58)–(60), we can obtain

$$\begin{aligned} |D_m^{p_m, q_m}(g; t) - g(t)| &\leq 5(1+t^2)\Omega(g; \delta) \left(\frac{5}{4} + \frac{D_m^{p_m, q_m}(|u-t|+|u-t|^3; t)}{\delta} \right) \\ &\leq 5(1+t^2) \left(\frac{5}{4} + \frac{\sqrt{2t}+t^2}{\delta \sqrt{[m]_{p_m, q_m}}} \right) \Omega(f; \delta). \end{aligned} \quad (61)$$

Choosing $\delta = 1/\sqrt{[m]_{p_m, q_m}}$, we have

$$\begin{aligned} |D_m^{p_m, q_m}(g; t) - g(t)| &\leq 5(1+t^2) \left(\frac{5}{4} + \sqrt{2t}+t^2 \right) \Omega \left(f; \frac{1}{\sqrt{[m]_{p_m, q_m}}} \right) \\ &\leq 10(1+t^2)^2 \Omega \left(f; \frac{1}{\sqrt{[m]_{p_m, q_m}}} \right). \end{aligned} \quad (62)$$

the conclusion holds.

Theorem 14. Let $(p_m), (q_m)$ be the sequences defined in Lemma 6. Then, for any $g \in C_2^0[0, \infty)$, we have

$$\lim_{m \rightarrow \infty} \|D_m^{p_m, q_m}(g; t) - g\|_2 = 0. \quad (63)$$

Proof. By weighted Korovkin theorem in [16], we see that it is sufficient to verify the following three conditions:

$$\lim_{m \rightarrow \infty} \|D_m^{p_m, q_m}(u^k; t) - t^k\|_2 = 0, \quad k = 0, 1, 2. \quad (64)$$

Since $D_m^{p_m, q_m}(1; x) = 1$, then (64) holds true for $k = 0$. By

Lemma 4, we can obtain

$$\begin{aligned} \|D_m^{p_m, q_m}(u; t) - t\|_2 &= \sup_{t \in [0, \infty)} \frac{|D_m^{p_m, q_m}(u; t) - t|}{1+t^2} \\ &\leq \frac{p_m^m q_m^{-1}}{[m]_{p_m, q_m}} \sup_{t \in [0, \infty)} \frac{t}{1+t^2} \\ &= \frac{p_m^m q_m^{-1}}{[m]_{p_m, q_m}} \rightarrow 0, \quad m \rightarrow \infty, \\ \|D_m^{p_m, q_m}(u^2; t) - t^2\|_2 &= \sup_{t \in [0, \infty)} \frac{|D_m^{p_m, q_m}(u^2; t) - t^2|}{1+t^2} \\ &\leq \sup_{t \in [0, \infty)} \frac{t}{1+t^2} \frac{p_m^m ([2]_{p_m, q_m}^2 p_m^{-1} q_m^{-2})}{[m]_{p_m, q_m}} \\ &\quad + \sup_{t \in [0, \infty)} \frac{t}{1+t^2} \frac{p_m^{2m} q_m^{-3} [2]_{p_m, q_m}}{[m]_{p_m, q_m}^2} \\ &= \frac{p_m^m ([2]_{p_m, q_m}^2 p_m^{-1} q_m^{-2})}{[m]_{p_m, q_m}} \\ &\quad + \frac{p_m^{2m} q_m^{-3} [2]_{p_m, q_m}}{[m]_{p_m, q_m}^2} \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (65)$$

Thus, the proof of Theorem 14 is completed.

Theorem 15. Let $(p_m), (q_m)$ be the sequences defined in Lemma 6. Then, for any $g \in C_2^0[0, \infty)$ and $\lambda > 0$, we have

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, \infty)} \frac{|D_m^{p_m, q_m}(g; t) - g(t)|}{(1+t^2)^{1+\lambda}} = 0. \quad (66)$$

Proof. Let $t_0 \in (0, \infty)$ be arbitrary but fixed. Then,

$$\begin{aligned} \sup_{t \in [0, \infty)} \frac{|D_m^{p_m, q_m}(g; t) - g(t)|}{(1+t^2)^{1+\lambda}} &\leq \sup_{t \in [0, t_0]} \frac{|D_m^{p_m, q_m}(g; t) - g(t)|}{(1+t^2)^{1+\lambda}} \\ &\quad + \sup_{t \in (t_0, \infty)} \frac{|D_m^{p_m, q_m}(g; t) - g(t)|}{(1+t^2)^{1+\lambda}} \\ &\leq \|D_m^{p_m, q_m}(g; t) - g\|_{C[0, t_0]} \\ &\quad + \|g\|_2 \sup_{t \in (t_0, \infty)} \frac{|D_m^{p_m, q_m}(1+u^2; t)|}{(1+t^2)^{1+\lambda}} \\ &\quad + \sup_{t \in (t_0, \infty)} \frac{|g(t)|}{(1+t^2)^{1+\lambda}}. \end{aligned} \quad (67)$$

Since $|g(t)| \leq M_g(1+t^2)$, we have $\sup_{t \in (t_0, \infty)} |g(t)|/$

$(1+t^2)^{1+\lambda} \leq M_g \|g\|_2 / (1+t_0^2)^\lambda$. Let $\varepsilon > 0$ be arbitrary, we can choose t_0 to be so large that

$$\frac{M_g \|g\|_2}{(1+t_0^2)^\lambda} < \frac{\varepsilon}{3}. \quad (68)$$

In view of Lemma 4, while $t \in (t_0, \infty)$, we can obtain

$$\|f\|_2 \lim_{t \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{|D_m^{p,q} (1+u^2; x)|}{(1+t^2)^{1+\lambda}} = \|f\|_2 \lim_{t \rightarrow \infty} \frac{1}{(1+t^2)^\lambda} = 0. \quad (69)$$

Hence, we can choose N and t_0 to be so large such that for any $m > N$ the inequality

$$\sup_{t \in [t_0, \infty)} \|g\|_2 \frac{|D_m^{p,q} (1+u^2; t)|}{(1+t^2)^{1+\lambda}} < \frac{\varepsilon}{3}. \quad (70)$$

holds. Also, the first term of the above inequality tends to zero by Theorem 12, that is

$$\|D_m^{p,q} (g; t) - g\|_{C[0, t_0]} < \frac{\varepsilon}{3}. \quad (71)$$

Thus, combining (68)–(71), we obtain the desired result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

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