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Research Article

On Convergence Theorems for Generalized Alpha Nonexpansive Mappings in Banach Spaces

Buthinah A. Bin Dehaish and Rawan K. Alharbi

Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Buthinah A. Bin Dehaish; bbindehaish@uj.edu.sa

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The present paper seeks to illustrate approximation theorems to the fixed point for generalized α-nonexpansive mapping with the Mann iteration process. Furthermore, the same results are established with the Ishikawa iteration process in the uniformly convex Banach space setting. The presented results expand and refine many of the recently reported results in the literature.

1. Introduction

Consider a Banach space (BS) X, together with its subset $D(\neq \phi)$. Let us also consider the following notations $\mathrm{Fix}(T)$, $\rightharpoonup$, and $\longrightarrow$ to represent the set of fixed points of T, weak convergence, and strong convergence, correspondingly.

A self-mapping T defined on a subset D is referred to as

(1) nonexpansive provided that $\|T(u) - T(v)\| \leq \|u - v\|$, for all $u, v \in D$

(2) quasi-nonexpansive provided that $\mathrm{Fix}(T) \neq \phi$, and for all $u \in D(T)$ and $v \in \mathrm{Fix}(T)$, the following assertion holds: $\|T(u) - v\| \leq \|u - v\|$

Notably, there is a relationship between a nonexpansive mapping and a quasi-nonexpansive mapping. That is, each nonexpansive mapping satisfying $\mathrm{Fix}(T) \neq \phi$ is quasi-nonexpansive; however, the opposite is not correct generally. Furthermore, the opposite is satisfied as shown in [1] when the linearity condition is added to the quasi-nonexpansive mapping. Thus, a linear quasi-nonexpansive mapping is nonexpansive. Yet, it can be straightforwardly verified that there exist nonlinear quasi-nonexpansive mappings which are continuous and are not nonexpansive; for example,

$$T(u) = \frac{u}{2} \sin \frac{1}{u}, \text{ on } \mathbb{R}^1, \text{ with } T(0) = 0. \tag{1}$$

Nonexpansive mapping and its generalization remain a central topic of interest in the fixed point (FP) theory among different mathematicians and mathematical theorists. Various considerations and a variety of in-depth investigations including generalizations to this mapping have been reported in the literature, in which we notice its development in different branches and under various conditions (see [2–7]). Browder in 1965 [8] and Kirk [9] have shown that self-nonexpansive mappings defined on a convex subset of a uniformly convex Banach space (UCBS) that is closed and bounded have fixed points. In 1974, Senter and Dotson [10] established a strong convergence fixed point theorem with regard to the Mann iteration of a nonexpansive mapping. Furthermore, in 1993, Xu and Tan [11] generalized the results of Reich [12] and Senter and Dotson [10] by using the Ishikawa iterative procedure instead of the Mann process.

Recently, the notion of α-nonexpansive mapping in BS was proposed by Aoyama and Kohsaka [13] in 2011. This notion was further partially extended to a generalized (glz) α-nonexpansive mapping by Pant and Shukla [14] in 2017 as follows: consider a BS X with its subset $D(\neq \phi)$, and the mapping $T : D \longrightarrow D$ is considered to be glz α-nonexpansive provided that $\exists \alpha \in [0, 1)$ such that $\forall u, v \in D$,

$$\begin{aligned} \frac{1}{2}\|u - T(u)\| \leq \|u - v\| \Rightarrow \|T(u) - T(v)\| \\ \leq \alpha\|T(u) - v\| + \alpha\|T(v) - u\| + (1 - 2\alpha)\|u - v\|. \end{aligned} \tag{2}$$

Also, in [14], they have obtained the existence of FP results and convergence theorems by using the iteration process defined by Agarwal et al. [15] that reads

$$\begin{cases} u_1 \in D, \\ u_{n+1} = (1-t_n)T(u_n) + t_n T(v_n), \\ v_n = (1-s_n)u_n + s_n T(u_n), n \in \mathbb{N}, \end{cases} \tag{3}$$

with $\{t_n\}$ and $\{s_n\}$ as sequences belonging to $(0, 1)$. This iteration is known as the S-iteration, and it is independent of both the Ishikawa and Mann iteration processes as demonstrated in [15].

Over the most recent forty years, both the Ishikawa and Mann iteration processes have been effectively utilized by different mathematicians to approximate FP of different types of nonexpansive mappings in BS.

In 1953, Mann [11] devised a methodology that is termed as the Mann iterative process for approximating FP of continuous transformation in BS that reads

$$\begin{cases} u_1 \in D, \\ u_{n+1} = t_n T(u_n) + (1-t_n)u_n, n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{t_n\}$ is a sequence belonging to $[0, 1]$.

Moreover, Ishikawa [16] in 1974 generalized the Mann iterative process from one- to two-step iterations; he also obtained an iterative process to approximate FP of pseducontractive compact mapping in the Hilbert space given below:

$$\begin{cases} u_1 \in D, \\ v_n = (1-s_n)u_n + s_n T(u_n), \\ u_{n+1} = (1-t_n)u_n + t_n T(v_n), n \in \mathbb{N}, \end{cases} \tag{5}$$

with $\{t_n\}$ and $\{s_n\}$ denoting sequences lying in $[0, 1]$ and satisfying some conditions.

Also, observe that the Mann iterative procedure is a particular case of the Ishikawa iteration by the choice of $s_n = 0$, $\forall n \in \mathbb{N}$.

More recently, Piri et al. [17] in 2019 have shown some interesting examples of the glz α-nonexpansive mapping and presented certain comparative convergence behaviors with regard to some powerful iteration procedures including the famous Mann and Ishikawa iterations among others.

As an application, fixed point theory of nonexpansive mapping and its generalization has many applications in different fields such as applications of nonexpansive mapping to solve an integral equation (see [18]) and to solve a variational inequality problem (see [19]). Also, there are applications of some classes of generalized nonexpansive mappings like quasi-nonexpansive mappings under contraction to find the minimum norm fixed point and generalized α-nonexpansive mappings to solve split feasibility problem (see [20, 21]).

However, the present paper is aimed at establishing certain strong and weak convergence theorems of FP for the glz α-nonexpansive mapping via the application of the Mann iteration. Similar results are also set to be established by the application of the Ishikawa iteration process in the sense of UCBS. Remarkably, these results happen to be an extension of the results presented in [1, 11].

2. Preliminaries

Recall that a BS X satisfies the Opial property [22] for every sequence $\{u_n\}$ in X such that $\{u_n\} \rightharpoonup p$; then, $\forall q \in X$ with $p \neq q$,

$$\liminf_{n\longrightarrow\infty} \|u_n - p\| < \liminf_{n\longrightarrow\infty} \|u_n - q\|. \tag{6}$$

For example, all Hilbert spaces, all finite dimensional BS, and $\ell^p(1 < p < \infty)$ have satisfied the Opial property, while $L_p[0, 2\pi](p \neq 2)$ has not satisfied the Opial property [23].

A BS X is uniformly convex provided that for each ε, $(0 < \varepsilon \leq 2)$, $\exists \delta > 0$ such that for any u, $v \in X$ together with $\|u\| = \|v\| = 1$ and $\|u - v\| > \varepsilon$; then, $\|(u + v)/2\| \leq 1 - \delta$ is said to hold.

Let $\{u_n\}$ and $D(\neq \phi)$ be a bounded sequence and subset of a BS X, respectively. Then, $\forall u \in X$, we define

(i) the asymptotic radius of the bounded sequence $\{u_n\}$ at u as

$$r(u, \{u_n\}) = \limsup_{n\longrightarrow\infty} \|u_n - u\|, \tag{7}$$

(ii) the asymptotic radius of the bounded sequence $\{u_n\}$ relative to D as

$$r(D, \{u_n\}) = \inf \{r(u, \{u_n\}) : u \in D\}, \tag{8}$$

(iii) the asymptotic center of the bounded sequence $\{u_n\}$ relative to D as

$$A(D, \{u_n\}) = \{u \in D : r(u, \{u_n\}) = r(D, \{u_n\})\}. \tag{9}$$

We observe that $A(D, \{u_n\}) \neq \phi$. Moreover, if X is UCBS, then $A(D, \{u_n\})$ has exactly one point [23].

Let X^* denote a dual space of BS X. Recall that X possesses the Fréchet differentiable norm provided that for each v in the sphere (unit) S of X, there exists the following limit:

$$\lim_{n\longrightarrow\infty} \frac{\|v + t v_0\| - \|v\|}{t}, \tag{10}$$

which is attained uniformly for $v_0 \in S$.

Thus, as rightly given in [23], $\forall u, w \in X$,

$$\frac{1}{2}\|u\|^2 + \langle w, J(u)\rangle \le \frac{1}{2}\|u+w\|^2 \le \frac{1}{2}\|u\|^2 + \langle w, J(u)\rangle + g(\|w\|), \tag{11}$$

where $J(u) = \partial(1/2)\|u\|^2$ and g is a function (increasing) defined on $\mathbb{R}^+$ of which $\lim_{t\downarrow 0}(g(t)/t) = 0$.

Accordingly, we give an illustrative example for a glz α -nonexpansive mapping in what follows.

Example 1 [14]. Consider $D = [0, 4] \subset \mathbb{R}$ of which a usual norm is endowed on. Let $T : D \longrightarrow D$ be defined by

$$T(u) = \begin{cases} 0, & \text{if } u \ne 4, \\ 2, & \text{if } u = 4. \end{cases} \tag{12}$$

Therefore, T is indeed a glz α-nonexpansive mapping with $\alpha \ge 1/3$.

Definition 2. Mapping which satisfies condition (I) [10]. "Let X be a normed space and let $D \subseteq X$. A map $T : D \longrightarrow D$ satisfies condition (I) provided that there exists a nondecreasing function $h : [0,\infty) \longrightarrow [0,\infty)$ that satisfies $h(0) = 0$ and $h(t) > 0$, for every $t \in (0,\infty)$ such that $\|u - T(u)\| \ge h(d(u, \mathrm{Fix}(T)))$, for each $u \in D$, where $d(u, \mathrm{Fix}(T))$ denotes the distance of u from $\mathrm{Fix}(T)$."

Next, we state some important results that are essentially vital to the present work; these results were introduced in [14, 24] together with their proofs.

Proposition 3. *Consider a BS X together with its subset $D(\ne \phi)$. Let us also consider a glz α-nonexpansive mapping given by $T : D \longrightarrow D$ with a FP $v \in D$. Then, T is quasi-nonexpansive.*

Lemma 4. *Consider a BS X together with its subset $D(\ne \phi)$. Let us also consider a glz α-nonexpansive mapping given by $T : D \longrightarrow D$. Therefore, for every $u, v \in D$,*

$$\|u - T(v)\| \le \|u - v\| + \frac{(3+\alpha)}{(1-\alpha)}\|u - T(u)\|. \tag{13}$$

Proposition 5. Demiclosedness principle [14]. *"Consider a BS X together with the Opial property, and let $D(\ne \phi)$ be a closed subset of X. Let $T : D \longrightarrow D$ be a glz α-nonexpansive mapping. If $\{u_n\} \rightharpoonup z$ and $\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0$, then $T(z) = z$. Meaning, $(I - T)$ is demiclosed at zero, with I denoting the identity mapping on X."*

The lemma below gives the convexity and closedness of the set of FP for the glz α-nonexpansive mapping.

Lemma 6 [14]. *"Consider a glz α-nonexpansive mapping $T : D \longrightarrow D$, where $D(\ne \phi)$ is a subset of a BS X. Then, $\mathrm{Fix}(T)$ is closed. In addition, if D is convex and X is strictly convex, then $\mathrm{Fix}(T)$ is also convex."*

In the sequel, the next lemma will be used to navigate the main results of the paper.

Lemma 7 [24]. *"Consider a UCBS X and $0 < a \le l_n \le b < 1$, $\forall n \in \mathbb{N}$. Moreover, consider the two sequences $\{u_n\}$ and $\{v_n\}$ such that $\limsup_{n\longrightarrow\infty}\|u_n\| \le r$, $\limsup_{n\longrightarrow\infty}\|v_n\| \le r$, and $\lim_{n\longrightarrow\infty}\|l_n u_n + (1 - l_n)v_n\| = r$ hold for some $r \ge 0$. Then, $\lim_{n\longrightarrow\infty}\|u_n - v_n\| = 0$."*

3. Main Results

This section starts off by investigating the weak and strong approximation FP for the glz α-nonexpansive mapping by using the Mann iteration process. Moreover, a similar examination will be looked at by the application of the Ishikawa iteration procedure.

3.1. Main Results for glz α-Nonexpansive with the Mann Iteration

Lemma 8. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\ne \phi)$ of a BS X. Let the sequence $\{u_n\}$ be defined by the Mann iteration (1), and assume ζ to be a FP of T; then, $\lim_{n\longrightarrow\infty}\|u_n - \zeta\|$ exists.*

Proof. By referring to the definition of the Mann iteration (1) and Proposition 3, we get

$$\begin{aligned} \|u_{n+1} - \zeta\| &= \|(1 - t_n)u_n + t_n T(u_n) - \zeta\|, \\ &= \|(1 - t_n)(u_n - \zeta) + t_n(T(u_n) - \zeta)\|, \\ &\le (1 - t_n)\|u_n - \zeta\| + t_n\|T(u_n) - \zeta\|, \\ &\le (1 - t_n)\|u_n - \zeta\| + t_n\|u_n - \zeta\|, = \|u_n - \zeta\|. \end{aligned} \tag{14}$$

Therefore, the sequence $\{\|u_n - \zeta\|\}$ is bounded and nonincreasing. Thus, we conclude that $\lim_{n\longrightarrow\infty}\|u_n - \zeta\|$ exists.

Theorem 9. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\ne \phi)$ of a UCBS X. Let the sequence $\{u_n\}$ with $u_1 \in D$ be defined by the Mann iteration (1). Then, $\mathrm{Fix}(T) \ne \phi$ iff the sequence $\{u_n\}$ is bounded and*

$$\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0. \tag{15}$$

Proof. Consider a bounded sequence $\{u_n\}$, and $\lim_{n\longrightarrow\infty}\|T(u_n) - u_n\| = 0$. As X is UCBS, then $A(D, \{u_n\}) \ne \phi$ and it contains exactly one point.

Let $z \in A(D, \{u_n\})$, and we want to demonstrate that $\mathrm{Fix}(T) \ne \phi$.

Using the asymptotic radius definition as given above, we obtain

$$r(T(z), \{u_n\}) = \limsup_{n\longrightarrow\infty} \|u_n - T(z)\|. \quad (16)$$

Also, using Lemma 4, we get

$$\begin{aligned} r(T(z), \{u_n\}) &= \limsup_{n\longrightarrow\infty} \|u_n - T(z)\|, \\ &\leq \limsup_{n\longrightarrow\infty} \|u_n - z\| + \frac{(3+\alpha)}{(1-\alpha)} \limsup_{n\longrightarrow\infty} \|u_n - T(u_n)\|, \\ &= \limsup_{n\longrightarrow\infty} \|u_n - z\|, = r(z, \{u_n\}). \end{aligned} \quad (17)$$

Hence, $T(z) \in A(D, \{u_n\})$. However, with regard to the uniqueness of the asymptotic center of $\{u_n\}$, we obtain $T(z) = z$. That means $z \in Fix(T)$, and thus, $\text{Fix}(T) \neq \phi$.

Conversely, let $\text{Fix}(T) \neq \phi$ and $w \in \text{Fix}(T)$; then, from Lemma 8, $\lim_{n\longrightarrow\infty} \|u_n - w\|$ exists. Suppose

$$\lim_{n\longrightarrow\infty} \|u_n - w\| = r > . \quad (18)$$

Equation (18) and Proposition 3 yield

$$\limsup_{n\longrightarrow\infty} \|T(u_n) - w\| \leq \limsup_{n\longrightarrow\infty} \|u_n - w\| = r. \quad (19)$$

Hence,

$$\limsup_{n\longrightarrow\infty} \|T(u_n) - w\| \leq r. \quad (20)$$

From equations (18) and (20) and the definition of the Mann iteration (1), we get

$$\begin{aligned} r &= \lim_{n\longrightarrow\infty} \|u_{n+1} - w\| = \lim_{n\longrightarrow\infty} \|(1-t_n)u_n + t_n T(u_n) - w\|, \\ &= \lim_{n\longrightarrow\infty} \|(1-t_n)(u_n - w) + t_n(T(u_n) - w)\|. \end{aligned} \quad (21)$$

In view of equations (18), (20), and (21) and Lemma 7, we deduce that

$$\lim_{n\longrightarrow\infty} \|u_n - w - T(u_n) + w\| = 0. \quad (22)$$

Consequently,

$$\lim_{n\longrightarrow\infty} \|u_n - T(u_n)\| = 0. \quad (23)$$

In order to prove weak convergence of both the Mann and Ishikawa iterative processes to a FP for glz α-nonexpansive mapping, the following lemma is needed.

Lemma 10 [14]. *"Suppose that the conditions of Theorem 9 are fulfilled. Then, $\lim_{n\longrightarrow\infty} \langle u_n, J(p_1 - p_2)\rangle$ exists for any $p_1, p_2 \in Fix(T)$; in particular, $\langle u_0 - v_0, J(p_1 - p_2)\rangle = 0$, $\forall u_0, v_0 \in \eta_w(u_n)$, where $\eta_w(u_n)$ represents the set of all weak limit points of $\{u_n\}$."*

Theorem 11. Weak convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X which satisfies the Opial property or which has a Fréchet differentiable norm such that $(I-T)$ is demiclosed at zero. Let the sequence $\{u_n\}$ be defined by the Mann iteration (1) with $u_1 \in D$ such that a sequence $\{t_n\}$ in $[0,1]$ and $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$. Then, the sequence $\{u_n\}$ converges weakly to a FP of T.*

Proof. Consider $\eta_w(u_n)$ to be the set of all weak limit points of $\{u_n\}$. Then, from the fact that $\text{Fix}(T) \neq \phi$, $\{u_n\}$ is a bounded sequence and

$$\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0, \quad (24)$$

from Theorem 9. Therefore, without loss of generality, let $p \in \eta_w(u_n)$, which means

$$u_n \rightharpoonup p \text{ as } n \longrightarrow \infty. \quad (25)$$

Now, we want to show that $\eta_w(u_n) \subset \text{Fix}(T)$. From (24), (25), and Proposition 5, we have

$$\begin{aligned} u_n &\rightharpoonup p, \\ (I-T)u_n &\longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned} \quad (26)$$

then,

$$(I-T)p = 0 \text{ implies } p = T(p). \quad (27)$$

Thus, $p \in \text{Fix}(T)$, and we deduce that $\eta_w(u_n)$ is a subset of $\text{Fix}(T)$.

Now, to prove that the sequence $\{u_n\}$ converges weakly to a FP of T, it is sufficient to prove that $\eta_w(u_n)$ is a singleton set.

First, we assume X to fulfil the Opial property and suppose p_1 and $p_2 \in \eta_w(u_n)$ such that $p_1 \neq p_2$; then, by the reflexiveness of X, we have

$$\begin{aligned} p_1 &= \text{weak} - \lim_{k\longrightarrow\infty} u_{n_k}, \\ p_2 &= \text{weak} - \lim_{j\longrightarrow\infty} u_{n_j}, \end{aligned} \quad (28)$$

for some $n_k \uparrow \infty, n_j \uparrow \infty$.

By Lemma 8, $\lim_{n\longrightarrow\infty} \|u_n - p_1\|$ exists, since $p_1 \in \eta_w(u_n) \subset \text{Fix}(T)$.

Using the Opial property on X, we get that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|u_n-p_1\|&=\lim_{k\longrightarrow\infty}\left\|u_{n_k}-p_1\right\|<\lim_{k\longrightarrow\infty}\left\|u_{n_k}-p_2\right\|,\\&=\lim_{j\longrightarrow\infty}\left\|u_{n_j}-p_2\right\|<\lim_{j\longrightarrow\infty}\left\|u_{n_j}-p_1\right\|,\\&=\lim_{n\longrightarrow\infty}\|u_n-p_1\|,\end{aligned}\tag{29}$$

arriving at a contradiction.

Consequently, $p_1=p_2$. Hence, $\eta_w(u_n)$ is a singleton. This proves our result for which X satisfies the Opial property.

Secondly, we assume X to have a Fréchert differentiable norm given that $(I-T)$ is demiclosed at zero.

Substituting f_1-f_2 and $t(u_n-f_1)$ for u and w, respectively, in

$$\langle w,J(u)\rangle+\frac{1}{2}\|u\|^2\le\frac{1}{2}\|u+w\|^2\le\langle w,J(u)\rangle+\frac{1}{2}\|u\|^2+g(\|w\|),\tag{30}$$

where $f_1,f_2\in\mathrm{Fix}(T)$ and $0<t<1$, we obtain

$$\begin{aligned}&\frac{1}{2}\|f_1-f_2\|^2+t\langle u_n-f_1,J(f_1-f_2)\rangle,\\&\le\frac{1}{2}\|tu_n+(1-t)(f_1-f_2)\|^2,\\&\le\frac{1}{2}\|f_1-f_2\|^2+t\langle u_n-f_1,J(f_1-f_2)\rangle+g(t\|u_n-f_1\|).\end{aligned}\tag{31}$$

By referring to Lemma 10, the limit $\lim_{n\longrightarrow\infty}\langle u_n-f_1,J(f_1-f_2)\rangle$ exists.

In particular, this implies that

$$\langle p_1-p_2,J(f_1-f_2)\rangle=0,\tag{32}$$

for all $p_1,p_2\in\eta_w(u_n)$ and $f_1,f_2\in\mathrm{Fix}(T)$.

By replacing f_1,f_2 in (32) by p_1,p_2, respectively, we obtain $\|p_1-p_2\|^2=\langle p_1-p_2,J(p_1-p_2)\rangle=0$, since $J(p_1-p_2)=0$.

Thus, $p_1=p_2$. This shows that $\eta_w(u_n)$ must be a singleton.

Theorem 12. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T)\ne\phi$ defined on a closed convex subset $D(\ne\phi)$ of a UCBS X. Then, for arbitrary $u_1\in D$, the sequence $\{u_n\}$ defined by the Mann iteration (1) converges strongly to a member of $Fix(T)$ provided that T satisfies condition (I).*

Proof. Since from Proposition 3, each glz α-nonexpansive mapping that possesses at least one FP is a quasi-nonexpansive mapping, then our conclusion follows from Theorem 2 in [10].

Theorem 13. Strong convergent theorem. *Consider a glz α-nonexpansive self-mapping T with $Fix(T)\ne\phi$ defined on a closed convex subset $D(\ne\phi)$ of a BS X. Let the sequence $\{u_n\}$ be defined by the Mann iteration (1). Then, the sequence $\{u_n\}$ converges strongly to a FP of T provided that*

$$\liminf_{n\longrightarrow\infty}d(u_n,Fix(T))=0.\tag{33}$$

Proof. Assume that the $\liminf_{n\longrightarrow\infty}d(u_n,\mathrm{Fix}(T))=0$, then $\exists\{w_n\}$ a subsequence of $\{u_n\}$ of which

$$\lim_{n\longrightarrow\infty}d(w_n,\mathrm{Fix}(T))=0.\tag{34}$$

By (34), suppose $\{w_{n_j}\}$ again to be a subsequence of $\{w_n\}$ of which

$$\left\|w_{n_j}-z_j\right\|\le\frac{1}{2^j},\quad\forall j\ge1,\tag{35}$$

such that $\{z_j\}$ is a sequence in $\mathrm{Fix}(T)$. Then, by Lemma 8, we have

$$\left\|w_{n_{j+1}}-z_j\right\|\le\left\|w_{n_j}-z_j\right\|\le\frac{1}{2^j}.\tag{36}$$

Now, we want to show that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. By the triangular inequality and (36), we conclude that

$$\|z_{j+1}-z_j\|\le\left\|z_{j+1}-w_{n_{j+1}}\right\|+\left\|w_{n_{j+1}}-z_j\right\|<\frac{1}{2^{j-1}}.\tag{37}$$

A standard argument refers to the fact that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. By Lemma 6, $\mathrm{Fix}(T)$ is a closed subset of the BS X. Thus, $\{z_j\}$ converges to a FP z. Then, we have

$$\left\|w_{n_j}-z\right\|\le\left\|w_{n_j}-z_j\right\|+\|z_j-z\|.\tag{38}$$

Assume $j\longrightarrow\infty$; this means that $\{w_{n_j}\}$ converges strongly to z. Accordingly, $\lim_{n\longrightarrow\infty}\|u_n-z\|$ exists for $z\in Fix(T)$ by Lemma 8. Therefore, the sequence $\{u_n\}$ converges strongly to z.

3.2. Main Results for glz α-Nonexpansive with the Ishikawa Iteration. Now, let us state and prove some lemmas that will be utilized to prove the results as follows.

Lemma 14. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\ne\phi)$ of a BS X. Let the sequence $\{u_n\}$ with $u_1\in D$ be defined by the Ishikawa iteration (2). Suppose that $\zeta\in Fix(T)$; then, the statements given below are true:*

(1) $\max\{\|u_{n+1}-\zeta\|,\|v_n-\zeta\|\}\le\|u_n-\zeta\|,\forall n\in\mathbb{N}$.

(2) $\lim_{n\longrightarrow\infty}\|u_n-\zeta\|$

Proof. (1) By definition of the Ishikawa iteration (2) and Proposition 3, we have

$$\begin{aligned}\|v_n-\zeta\| &= \|(1-s_n)u_n+s_nT(u_n)-\zeta\|,\\ &\le (1-s_n)\|u_n-\zeta\|+s_n\|T(u_n)-\zeta\|,\\ &\le (1-s_n)\|u_n-\zeta\|+s_n\|u_n-\zeta\|, = \|u_n-\zeta\|,\end{aligned} \tag{39}$$

hence

$$\|v_n-\zeta\|\le\|u_n-\zeta\|. \tag{40}$$

Again, using the definition of the Ishikawa iteration (2) and Proposition 3, one gets

$$\begin{aligned}\|u_{n+1}-\zeta\| &= \|(1-t_n)u_n+t_nT(v_n)-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|T(v_n)-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|v_n-\zeta\|,\\ &\le (1-t_n)\|u_n-\zeta\|+t_n\|u_n-\zeta\|, = \|u_n-\zeta\|,\end{aligned} \tag{41}$$

thus,

$$\|u_{n+1}-\zeta\|\le\|u_n-\zeta\|. \tag{42}$$

Now, from (40) and (42), we get that

$$\max\{\|u_{n+1}-\zeta\|, \|v_n-\zeta\|\}\le\|u_n-\zeta\|, \quad \forall n\in\mathbb{N}. \tag{43}$$

(2) Using (42), the sequence $\{\|u_n-\zeta\|\}$ is deduced to be nonincreasing and bounded. Thus, $\lim_{n\longrightarrow\infty}\|u_n-\zeta\|$ exists.

Theorem 15. *Consider a glz α-nonexpansive self-mapping T defined on a closed convex subset $D(\neq\phi)$ of a UCBS X. Let the sequence $\{u_n\}$ with $u_1\in D$ be defined by the Ishikawa iteration (2). Then, $Fix(T)\neq\phi$ iff the sequence $\{u_n\}$ is bounded and also*

$$\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\|=0. \tag{44}$$

Proof. Consider a bounded sequence $\{u_n\}$ and $\lim_{n\longrightarrow\infty}\|T(u_n)-u_n\|=0$. Therefore, we get $\mathrm{Fix}(T)\neq\phi$ by following the same steps of the analogous part in the proof of Theorem 9.

Conversely, assume $\mathrm{Fix}(T)\neq\phi$ and $u_0\in\mathrm{Fix}(T)$, so from Lemma 14, $\lim_{n\longrightarrow\infty}\|u_n-u_0\|$ exists. Suppose

$$\lim_{n\longrightarrow\infty}\|u_n-u_0\|=r. \tag{45}$$

From equation (45) and Lemma 14, we have

$$\limsup_{n\longrightarrow\infty}\|v_n-u_0\|\le\limsup_{n\longrightarrow\infty}\|u_n-u_0\|=r. \tag{46}$$

Hence,

$$\limsup_{n\longrightarrow\infty}\|v_n-u_0\|\le r. \tag{47}$$

From equation (47) and Proposition 3, we get

$$\limsup_{n\longrightarrow\infty}\|T(v_n)-u_0\|\le\limsup_{n\longrightarrow\infty}\|v_n-u_0\|\le r. \tag{48}$$

Thus,

$$\limsup_{n\longrightarrow\infty}\|T(v_n)-u_0\|\le r. \tag{49}$$

Now, by the definition of the Ishikawa iteration (2), one gets

$$r=\lim_{n\longrightarrow\infty}\|u_{n+1}-u_0\|=\lim_{n\longrightarrow\infty}\|(1-t_n)u_n+t_nT(v_n)-u_0\| \tag{50}$$

$$=\lim_{n\longrightarrow\infty}\|(1-t_n)(u_n-u_0)+t_n(T(v_n)-u_0)\|. \tag{51}$$

In view of equations (45), (49), and (50) and Lemma 7, one obtains

$$\lim_{n\longrightarrow\infty}\|T(v_n)-u_n\|=0. \tag{52}$$

Again, by the definition of the Ishikawa iteration (2), we have

$$\|u_{n+1}-u_n\|=\|(1-t_n)u_n+t_nT(v_n)-u_n\|=t_n\|T(v_n)-u_n\|. \tag{53}$$

Now, letting $n\longrightarrow\infty$, we get

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\|=t_n\lim_{n\longrightarrow\infty}\|T(v_n)-u_n\|=0. \tag{54}$$

Hence,

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-u_n\|=0. \tag{55}$$

Then, we conclude

$$\begin{aligned}\|u_{n+1}-T(v_n)\| &= \|u_{n+1}-u_n+u_n-T(v_n)\|\\ &\le\|u_{n+1}-u_n\|+\|u_n-T(v_n)\|.\end{aligned} \tag{56}$$

By equations (52) and (55), we deduce

$$\lim_{n\longrightarrow\infty}\|u_{n+1}-T(v_n)\|=0. \tag{57}$$

We observe

$$\begin{aligned}\|u_{n+1}-u_0\| &= \|u_{n+1}-T(v_n)+T(v_n)-u_0\|\\ &\le\|u_{n+1}-T(v_n)\|+\|T(v_n)-u_0\|.\end{aligned} \tag{58}$$

Now, taking the liminf of the last inequality, we get that

$$r=\liminf_{n\longrightarrow\infty}\|u_{n+1}-u_0\|\le\liminf_{n\longrightarrow\infty}\|u_{n+1}-T(v_n)\|+\liminf_{n\longrightarrow\infty}\|v_n-u_0\|. \tag{59}$$

Thus,

$$r \leq \liminf_{n\longrightarrow\infty} \|v_n - u_0\|. \quad (60)$$

From equations (47) and (60), we get

$$r = \lim_{n\longrightarrow\infty} \|v_n - u_0\| = \lim_{n\longrightarrow\infty} \|(1-s_n)u_n + s_n T(u_n) - u_0\|,$$
$$= \lim_{n\longrightarrow\infty} \|(1-s_n)(u_n - u_0) + s_n(T(u_n) - u_0)\|. \quad (61)$$

Finally, from equations (45), (47), and (60) and via Lemma 7, one concludes that

$$\lim_{n\longrightarrow\infty} \|u_n - u_0 - T(u_n) + u_0\| = 0. \quad (62)$$

Hence,

$$\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0. \quad (63)$$

Theorem 16. Weak convergent theorem. *Consider a glz α -nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X which satisfies the Opial property or which has a Fréchet differentiable norm such that $(I-T)$ is demiclosed at zero. So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) together with restricting $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=1}^{\infty} t_n (1-t_n)s_n < \infty$ converges weakly to a FP of T.*

Proof. The methodology of the proof is identical to that of Theorem 11.

Theorem 17. Strong convergent theorem. *Consider a glz α -nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X, and suppose in addition that T satisfies condition (I). So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration given in (2) converges strongly to a FP p of T.*

Proof. Since $\mathrm{Fix}(T) \neq \phi$, Theorem 15 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\longrightarrow\infty} \|u_n - T(u_n)\| = 0$.

Also, from condition (I), we guarantee that

$$\|u_n - T(u_n)\| \geq h(d(u_n, \mathrm{Fix}(T))), \quad \forall n \geq 1. \quad (64)$$

Thus, $d(u_n, \mathrm{Fix}(T)) \longrightarrow 0$ as $n \longrightarrow \infty$ follows from equation (64). A standard argument happens when there exists $p \in \mathrm{Fix}(T)$ of which $u_n \longrightarrow p \in \mathrm{Fix}(T)$ as $n \longrightarrow \infty$.

Theorem 18. Strong convergent theorem. *Consider a glz α -nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a BS X. So for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) converges strongly to a FP of T provided that*

$$\liminf_{n\longrightarrow\infty} d(u_n, Fix(T)) = 0. \quad (65)$$

Proof. Suppose that $\liminf_{n\longrightarrow\infty} d(u_n, \mathrm{Fix}(T)) = 0$, so a subsequence $\{w_n\}$ of the sequence $\{u_n\}$ exists of which $\lim_{n\longrightarrow\infty} d(w_n, \mathrm{Fix}(T)) = 0$. In view of the previous step, consider a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $\|w_{n_j} - z_j\| \leq 1/2^j, \forall j \geq 1$, where $\{z_j\}$ is a sequence in $\mathrm{Fix}(T)$.

So, with the help of Lemma 14, we guarantee that

$$\left\|w_{n_{j+1}} - z_j\right\| \leq \left\|w_{n_j} - z_j\right\| \leq \frac{1}{2^j}. \quad (66)$$

Now, we want to show that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$. From (66), we conclude that

$$\|z_{j+1} - z_j\| \leq \left\|z_{j+1} - w_{n_{j+1}}\right\| + \left\|w_{n_{j+1}} - z_j\right\| < \frac{1}{2^{j-1}}. \quad (67)$$

A standard argument proves that $\{z_j\}$ is a Cauchy sequence in $\mathrm{Fix}(T)$.

By referring to Lemma 6, we get that $\mathrm{Fix}(T)$ is a closed subset of BS X; then, $\{z_j\}$ converges to a FP z.

Now, we have

$$\left\|w_{n_j} - z\right\| \leq \left\|w_{n_j} - z_j\right\| + \|z_j - z\|. \quad (68)$$

Assume $j \longrightarrow \infty$; this means that $\{w_{n_j}\}$ converges strongly to z.

By Lemma 14, the limit $\lim_{n\longrightarrow\infty} \|u_n - z\|$ exists for $z \in \mathrm{Fix}(T)$.

Thus, $\{u_n\}$ converges strongly to z, where $z \in \mathrm{Fix}(T)$.

Theorem 19. Strong convergent theorem. *Consider a glz α -nonexpansive self-mapping T with $Fix(T) \neq \phi$ defined on a closed convex subset $D(\neq \phi)$ of a UCBS X. Suppose also that the range of D under T is included in a subset of X that is compact (i.e., $T(D) \subseteq C \subseteq X$, where C is compact). Then, for any initial value $u_1 \in D$, the sequence $\{u_n\}$ defined by the Ishikawa iteration (2) converges strongly to a FP of T.*

Proof. Given that $\mathrm{Fix}(T) \neq \phi$, Theorem 15 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\longrightarrow\infty} \|T(u_n) - u_n\| = 0$.

Now, from precompactness of $T(D)$, we conclude that $T(D) \subseteq C$, where C is compact.

Hence, $T\{u_n\} \subseteq C$ affirming $T(u_n) \longrightarrow u_n$ as $n \longrightarrow \infty$.

Then, $\{u_n\}$ has a convergent subsequence $\{u_{n_k}\}$, so $\{u_{n_k}\} \longrightarrow z$ as $k \longrightarrow \infty$.

Again, from Theorem 15, one gets

$$\lim_{k\longrightarrow\infty} \left\|u_{n_k} - Tu_{n_k}\right\| = 0, \quad (69)$$

which implies $\|z - T(z)\| = 0$ which further implies $z \in \mathrm{Fix}(T)$. Since by Lemma 14, for $z \in \mathrm{Fix}(T)$, $\lim_{n\longrightarrow\infty} \|u_n - z\|$ exists, this means that the sequence $\{u_n\}$ converges strongly to $z \in \mathrm{Fix}(T)$.

4. Conclusion

In conclusion, the class of generalized α-nonexpansive mapping has been extensively examined in a uniformly convex Banach space setting. Results of the existence of the fixed point have been established and proven in Theorems 9 and 15 via the applications of the Mann and Ishikawa iterations, respectively. The established results corresponded to the results of Theorem 5.6 in [14].

Moreover, to approximate the fixed point of a generalized α-nonexpansive mapping, we made use of the Mann and Ishikawa iterations and proved strong convergence results. For instance, the established Theorems 12 and 13 via the Mann iteration came as a special state of Theorem 2 in [10] and corresponded to Theorem 5.9 in [14], correspondingly; while through the Ishikawa iteration, Theorems 17, 18, and 19 generalized Theorem 2.4 in [1] and corresponded to Theorem 5.9 in [14] and Theorem 2 in [11], respectively. Furthermore, with regard to weak convergence results, Theorems 11 and 16 for the Mann and Ishikawa iterations, respectively, generalized Theorem 2 in [12] and Theorem 2.3 in [1], respectively, by considering a generalized α-nonexpansive mapping instead of a nonexpansive mapping.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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