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Research Article

Characterizations of Double Commutant Property on $\mathscr{B}(\mathscr{H})$

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Let $\mathscr{H}$ be a complex Hilbert space. Denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. In this paper, we investigate the non-self-adjoint subalgebras of $\mathscr{B}(\mathscr{H})$ of the form $\mathscr{T}+\mathscr{B}$, where $\mathscr{B}$ is a block-closed bimodule over a masa and $\mathscr{T}$ is a subalgebra of the masa. We establish a sufficient and necessary condition such that the subalgebras of the form $\mathscr{T}+\mathscr{B}$ has the double commutant property in some particular cases.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space. We denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Given a nonempty subset $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$,the commutant of $\mathscr{A}$ is the set $\mathscr{A}' := \{T \in \mathscr{B}(\mathscr{H}): TA = AT \text{ for all } A \in \mathscr{A}\}$. The double commutant of $\mathscr{A}$ is $\mathscr{A}'' := (\mathscr{A}')'$. Clearly, $\mathscr{A} \subseteq \mathscr{A}''$. von Neumann's double commutant theorem states that if $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ is a self-adjoint algebra of operators whose kernel $\ker \mathscr{A} := \bigcap_{A\in\mathscr{A}} \ker A = 0$, then the closure of $\mathscr{A}$ in any of the weak operator, strong operator, and weak* topologies is the double commutant $\mathscr{A}''$. In fact, if $\mathscr{A}$ is a WOT-closed, unital $\mathscr{C}^*$-subalgebra of $\mathscr{B}(\mathscr{H})$, then $\mathscr{A} = \mathscr{A}''$. In this paper, we analyze the settings of non-self-adjoint subalgebras of $\mathscr{B}(\mathscr{H})$ whose double commutant coincides with themselves. We say that such algebras satisfy the double commutant property.

In the past several decades, a great deal of effort has been devoted to the study of the subalgebras of $\mathscr{B}(\mathscr{H})$ with the double commutant property. For a few references, see [1–8]. In recent years, there has been renewed interest in the study of double commutant property [9–15]. For singly generated algebras, Ruston [16] showed that every algebraic operator in $\mathscr{B}(\mathscr{H})$ has the double commutant property. Turner [8] proved that a normal operator satisfies the double commutant property if and only if it is reductive. For nonsingly generated algebras, Davidson and Pitts [2] researched the noncommutative analytic Toeplitz algebra with the double commutant property. Marcoux and Mastnak [12] analyzed the non-self-adjoint subalgebras of $\mathscr{B}(\mathscr{H})$ whose double commutant agrees with themselves; specifically, they considered the class of algebras of the form $\mathscr{D}+\mathscr{R}$ in finite dimensional space, where $\mathscr{R}$ is a bimodule over a masa and $\mathscr{D}$ is a unital subalgebras of the masa.

In this note, we will investigate the subalgebras of $\mathscr{B}(\mathscr{H})$ with the double commutant property, which extends the result in [13] extensively.

2. Preliminaries

Let $\mathscr{H}$ be a Hilbert space, if $0 \neq x, y \in \mathscr{H}$, we denote by $x \otimes y$ the rank-one operator on $\mathscr{H}$ given by $x \otimes y(z) := \langle z, y\rangle x$. For a subalgebra $\mathscr{W}$ of $\mathscr{B}(\mathscr{H})$, we define the annihilator of $\mathscr{W}$ as $\mathscr{W}^{\perp} = \{T \in \mathscr{B}(\mathscr{H}): TW = 0 = WT \text{ for all } W \in \mathscr{W}\}$. Given a collection $\{P_\alpha\}_\alpha$ of orthogonal projections in $\mathscr{B}(\mathscr{H})$, we denote by $\vee_\alpha P_\alpha$ the orthogonal projection onto $\vee\{\operatorname{Ran} P_\alpha\}_\alpha$. Note that all projections considered on the manuscript are orthogonal projections. Given projections P and Q in $\mathscr{B}(\mathscr{H})$, we define the P, Q-block of $\mathscr{B}(\mathscr{H})$ as follows:

$$\mathscr{B}_{P,Q} := Q\mathscr{B}(\mathscr{H})P = \{QTP : T \in \mathscr{B}(\mathscr{H})\}. \quad (1)$$

Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a maximal abelian self-adjoint subalgebra (that is, $\mathscr{M}$ is a masa), $\{P_\gamma, Q_\gamma\}_{\gamma\in\Gamma}$ be a collection of projections in $\mathscr{M}$, then we say that a subspace $\mathscr{B}$ of $\mathscr{B}(\mathscr{H})$ is block-generated over $\mathscr{M}$ if

$$\mathscr{B} = \vee\left\{\mathscr{B}_{P_\gamma,Q_\gamma} : \gamma\in\Gamma\right\}. \quad (2)$$

With loss of generality, we assume that each $P_\gamma \neq 0 \neq Q_\gamma$.

Definition 1. Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a masa and let $\mathscr{B} = \vee\{\mathscr{B}_{P\gamma}, {}_{Q\gamma} : \gamma\in\Gamma\}$ be a block-generated bimodule for some family of projections $\{P_\gamma, Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathscr{M}$. We say that $\mathscr{B}$ is disconnected if there exist $\varnothing \neq \Gamma_1, \Gamma_2 \subseteq \Gamma$, and projections $E_1, F_1, E_2, F_2 \in \mathscr{M}$ so that

(1) $\Gamma = \Gamma_1 \cup \Gamma_2$

(2) $\{0\} \neq \vee\{\mathscr{B}_{P_\gamma,Q_\gamma} : \gamma\in\Gamma_k\} \subseteq \mathscr{B}_{E_k,F_k}, k = 1, 2$

(3) $E_1 \vee F_1$ is orthogonal to $E_2 \vee F_2$

Otherwise, we say that $\mathscr{B}$ is connected.

Marcoux and Mastnak proved the following proposition in [12]. Now, we give another simpler proof.

Proposition 2 (see [12]). *Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a masa and let $\mathscr{B} = \vee\{\mathscr{B}_{P\gamma}, Q_\gamma : \gamma\in\Gamma\}$ be a block-generated bimodule over $\mathscr{M}$ with $P_\gamma \neq 0 \neq Q_\gamma$ for all γ. Then,*

(1) $\mathscr{B}^\perp = \mathscr{B}_{Q_0^\perp,P_0^\perp}$, where $P_0 = \vee_\gamma P_\gamma$ and $Q_0 = \vee_\gamma Q_\gamma$

(2) $\mathscr{B}$ is connected if and only if $\mathscr{B}' = \mathscr{B}^\perp + \mathbb{C}I$

Proof.

(1) It is clear that $\mathscr{B}_{Q_0^\perp,P_0^\perp} \subseteq \mathscr{B}^\perp$. We only need to show that $\mathscr{B}^\perp \subseteq \mathscr{B}_{Q_0^\perp,P_0^\perp}$.

$\forall T \in \mathscr{B}^\perp$, we have $TQ_\gamma\mathscr{B}(\mathscr{H})P_\gamma = Q_\gamma\mathscr{B}(\mathscr{H})P_\gamma T = 0$, $\forall\gamma \in \Gamma$. Since $\mathscr{B}(\mathscr{H})$ is prime, we obtain $TQ_\gamma = 0 = P_\gamma T, \forall_\gamma \in \Gamma$. So, $TQ_0 = 0 = P_0T$. This implies that $T = TQ_0^\perp$ and $T = P_0^\perp T$; therefore, $T = P_0^\perp TQ_0^\perp$, so we have $T \in \mathscr{B}_{Q_0^\perp,P_0^\perp}$.

(2) If $\mathscr{B}$ is connected, it is easy to vertify that $\mathscr{B}^\perp + \mathbb{C}I \subseteq \mathscr{B}'$; we will prove that $\mathscr{B}' \subseteq \mathscr{B}^\perp + \mathbb{C}I$.

$\forall T \in \mathscr{B}'$, let $A = x\otimes y \in \mathscr{B}(\mathscr{H})$, then $TQ_\gamma x\otimes yP_\gamma = Q_\gamma x\otimes yP_\gamma T$. This implies that for each $\gamma\in\Gamma$, there exists $\lambda_\gamma \in \mathbb{C}$ so that $TQ_\gamma = \lambda_\gamma Q_\gamma$ and $P_\gamma T = \lambda_\gamma P_\gamma$. We claim that $\lambda_{\gamma1} = \lambda_{\gamma2}, \forall_{\gamma1,\gamma2} \in \Gamma$.

Let $\Gamma_1 = \{\gamma\in\Gamma : \lambda_\gamma = \lambda_{\gamma0}\}$, then $\gamma_0 \in \Gamma_1$, so $\Gamma_1 \neq \varnothing$. Let $\Gamma_2 = \{\gamma\in\Gamma : \lambda_\gamma \neq \lambda_{\gamma0}\}$. Suppose that $\Gamma_2 \neq \varnothing$. For $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, we have

$$\lambda_{\gamma_1} Q_{\gamma_1} Q_{\gamma_2} = TQ_{\gamma_1} Q_{\gamma_2} = TQ_{\gamma_2} Q_{\gamma_1} = \lambda_{\gamma_2} Q_{\gamma_2} Q_{\gamma_1} = \lambda_{\gamma_2} Q_{\gamma_1} Q_{\gamma_2}. \quad (3)$$

Since $\lambda_{\gamma1} \neq \lambda_{\gamma2}$, we get $Q_{\gamma1}Q_{\gamma2} = 0$. Similarly, we have $P_{\gamma2}Q_{\gamma1} = 0$, $P_{\gamma1}Q_{\gamma2} = 0$, and $P_{\gamma1}P_{\gamma2} = 0$. Let

$$E_1 = \vee_{\gamma\in\Gamma_1}\left\{P_\gamma \vee Q_\gamma\right\},$$
$$E_2 = \vee_{\gamma\in\Gamma_2}\left\{P_\gamma \vee Q_\gamma\right\}, \quad (4)$$

then $E_1 \perp E_2$, and $\vee\{\mathscr{B}_{P\gamma}, {}_{Q\gamma} : \gamma\in\Gamma_k\} \subseteq \mathscr{B}_{Ek,Ek}, {}_k = 1, 2$. This contradicts the connection of $\mathscr{B}$. Therefore, for all $\gamma\in\Gamma$, there exists $\lambda\in\mathbb{C}$ so that $TQ_\gamma = \lambda Q_\gamma$ and $P_\gamma T = \lambda P_\gamma$. Thus, $(T-\lambda I)Q_\gamma = 0 = P_\gamma(T-\lambda I)$ for all $\gamma\in\Gamma$. It ensures that $(T-\lambda I)B = B(T-\lambda I) = 0$ for all $B\in\mathscr{B}$. So, we have $T-\lambda I \in \mathscr{B}^\perp$, i.e., $T \in \mathscr{B}^\perp + \mathbb{C}I$.

On the other hand, suppose that $\mathscr{B}$ is disconnected, then there exists E_1, E_2, F_1, F_2 as Definition 1. Let $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ and $T = \lambda_1(E_1 \vee F_1) + \lambda_2(E_2 \vee F_2)$. $\forall B\in\mathscr{B}$, write $B = B_1 + B_2$, where $B_i \in \vee\{B_{P\gamma}, Q_\gamma, \gamma\in\Gamma_i\}$. The fact that $TB = \lambda_1 B_1 + \lambda_2 B_2 = BT$ implies that $T\in\mathscr{B}'$. However, if $B_1, B_2 \neq 0$, $\forall\delta\in\mathbb{C}$, we have

$$(T-\delta I)B = TB - \delta B = (\lambda_1 - \delta)B_1 + (\lambda_2 - \delta)B_2 \neq 0. \quad (5)$$

Thus, $T - \delta I \notin \mathscr{B}^\perp$, $T \notin \mathscr{B}^\perp + \mathbb{C}I$. This is a contradiction. □

Proposition 3 (see [13]). *Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a masa and let $\mathscr{B} = \vee\{\mathscr{B}_{P\gamma}, Q_\gamma : \gamma\in\Gamma\}$ be a block-generated $\mathscr{M}$-bimodule for some family of projections $\{P_\gamma, Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathscr{M}$. Then, there is a partition $\{\Gamma_i : i\in\Omega\}$ of Γ so that the subspaces $\mathscr{B}_i = \vee_{\lambda\in\Gamma i}\mathscr{B}_{P\lambda}, {}_{Q\lambda}$ are connected for each $i\in\Omega$, and $i \neq j \in \Omega$ implies that $\mathscr{B}_i \vee \mathscr{B}_j$ is disconnected.*

By the proposition above, we can decompose $\mathscr{B}$ into a direct sum $\mathscr{B} = \oplus_i \mathscr{B}_i$, where each $\mathscr{B}_i$ is a connected subspace of $\mathscr{B}(\mathscr{H})$.

Definition 4. Let $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ be a masa and let $\mathscr{B}$ be a block-generated bimodule over $\mathscr{M}$. Let $\mathscr{B} = \oplus_i \mathscr{B}_i$ be the decomposition of $\mathscr{B}$ as in Proposition 3 for each $i\in\Omega$; let

$$E_i := \vee\left\{P_\gamma : \gamma\in\Gamma_i\right\},$$
$$F_i := \vee\left\{Q_\gamma : \gamma\in\Gamma_i\right\}. \quad (6)$$

We define $\mathscr{B}_{E_i,F_i}$ as the block closure of $\mathscr{B}_i$ and $\mathscr{B}_c = \oplus_i \mathscr{B}_{E_i,F_i}$ as the block closure of $\mathscr{B}$.

By the connection of $\mathscr{B}_i$ and Proposition 2, we have $\mathscr{B}_i' = \mathbb{C}I + \mathscr{B}_{F_i^\perp,E_i^\perp} = (\mathscr{B}_{E_i,F_i})'$. Now,

$$\mathscr{B}' = (\oplus_i \mathscr{B}_i)' = \cap_i \mathscr{B}_i' = \cap_i \left(\mathscr{B}_{E_i,F_i}\right)' = \mathscr{B}_c'. \quad (7)$$

Let $\mathcal{T} \subseteq \mathcal{M}$ and suppose that $\mathcal{A} = \mathcal{T} + \mathcal{B}$ satisfies $\mathcal{A} = \mathcal{A}''$. Then

$$\mathcal{A}' = (\mathcal{T} + \mathcal{B})' = \mathcal{T}' \cap \mathcal{B}' = \mathcal{T}' \cap \mathcal{B}_c' \subseteq \mathcal{B}_c', \quad (8)$$

it follows that $\mathcal{B}_c \subseteq \mathcal{B}_c' \subseteq \mathcal{A}'' = \mathcal{A}$. It is obvious that $\mathcal{B} \subseteq \mathcal{B}_c$; we obtain $\mathcal{A} = \mathcal{T} + \mathcal{B} = \mathcal{T} + \mathcal{B}_c$. So, we can consider the form $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,F_i}$ if $\mathcal{A} = \mathcal{T} + \mathcal{B}$ satisfies $\mathcal{A} = \mathcal{A}''$ in the following.

3. The Double Commutant Theorem on $\mathcal{B}(\mathcal{H})$

In this section, we discuss non-self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$ which have the double commutant property. We concentrate on the case $\mathcal{A} = \mathcal{T} + \mathcal{B}$, where $\mathcal{T}$ is a subalgebra of $\mathcal{M}$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is a block-generated $\mathcal{M}$-bimodule. First, we consider the case $\mathcal{B} = \mathcal{B}_{E,E^\perp}$.

Theorem 5. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa. Suppose that $\mathcal{A} = \mathcal{T} + \mathcal{B}_{E,E^\perp}$, where E is a projection in $\mathcal{M}$ and $\mathcal{T}$ is a subspace of $\mathcal{M}$. If $\mathcal{A} = \mathcal{A}''$, then*

(1) $\mathcal{T} = \mathcal{T}''$

(2) for any nonzero projection P in $\mathcal{T}$, we have $P \nleq E, P \nleq E^\perp$

(3) Span$\{Ran(E^\perp AE)$: $E^\perp AE \in \mathcal{T}', A \in \mathcal{B}(\mathcal{H})\}$ is dense in $E^\perp$, where Span denotes the linear expansion of set

Proof. Since $\mathcal{A}' = \mathcal{T}' \cap \mathcal{B}_{E,E^\perp}'$, by Proposition 2, $\mathcal{B}_{E,E^\perp}' = \mathbb{C}I + \mathcal{B}_{E,E^\perp}$, we have

$$\mathcal{A}' = \mathbb{C}I + \left\{E^\perp TE : T \in \mathcal{B}(\mathcal{H}), E^\perp TE \in \mathcal{T}'\right\}. \quad (9)$$

(1) It is clear that $\mathcal{A}' \subseteq \mathcal{T}'$. So, we have $\mathcal{T}'' \subseteq \mathcal{A}'' = \mathcal{A}$. Let $S \in \mathcal{T}''$, then there exists $U \in \mathcal{T}$ and $T \in \mathcal{B}(\mathcal{H})$ such that $S = U + E^\perp TE$. Since $E \in \mathcal{M}, \mathcal{T} \subseteq \mathcal{M}$, we get $E \in \mathcal{T}'$. It follows that $ES = SE, UE = EU$. Thus, we obtain

$$E^\perp TE = E^\perp (S - U)E = E^\perp SE - E^\perp UE = E^\perp ES - E^\perp EU = 0. \quad (10)$$

Therefore, $S = U \in \mathcal{T}$. This implies that $\mathcal{T}'' \subseteq \mathcal{T}$. Hence, $\mathcal{T}'' = \mathcal{T}$.

(2) We assume that there exists nonzero projection $P \in \mathcal{T}$ so that $P \leq E^\perp$.

Then, $EP = PE = 0$. Let $T \in \mathcal{B}(\mathcal{H})$ such that $E^\perp TE \in \mathcal{T}'$, then $PE^\perp TE = E^\perp TEP = 0$. It follows from (9) that

$$\mathcal{A}' = \mathbb{C}I + \left\{\left(E^\perp - P\right) TEP^\perp : E^\perp TE \in T', T \in \mathcal{B}(\mathcal{H})\right\}. \quad (11)$$

So, $(E^\perp - P)\mathcal{B}(\mathcal{H})P \subseteq \mathcal{A}'' = \mathcal{T} + E^\perp \mathcal{B}(\mathcal{H})E$. Hence, $\forall B \in \mathcal{B}(\mathcal{H})$, there exists $U \in \mathcal{T}$ and $A \in \mathcal{B}(\mathcal{H})$ such that $(E^\perp - P)BP = U + E^\perp AE$. Therefore,

$$\begin{aligned}\left(E^\perp - P\right)BP &= \left(E^\perp - P\right)\left(U + E^\perp AE\right)P \\ &= \left(E^\perp - P\right)UP + \left(E^\perp - P\right)E^\perp AEP \\ &= 0.\end{aligned} \quad (12)$$

It implies that $(E^\perp - P)\mathcal{B}(\mathcal{H})P = \{0\}$. Since $\mathcal{B}(\mathcal{H})$ is prime, we get $P = E^\perp$. Together with (11), we obtain $\mathcal{A}' = \mathbb{C}I$. Thus, $\mathcal{A} = \mathcal{A}'' = \mathcal{B}(\mathcal{H})$. This is a contradiction.

Similarly, we get $P \nleq E$.

(3) Let P be a projection onto $\overline{\text{Span}}\{\text{Ran}(E^\perp AE)$: $E^\perp AE \in T', A \in \mathcal{B}(\mathcal{H})\}$.

Then, $P \leq E^\perp$. For any $E^\perp AE \in \mathcal{T}'$, we have $(E^\perp - P)E^\perp AE = 0$ and $E^\perp AE(E^\perp - P) = 0$. Hence, $(E^\perp - P) \in \mathcal{A}'' = \mathcal{A}$. Therefore, there exists $T \in \mathcal{T}$ and $B \in \mathcal{B}(\mathcal{H})$ such that $E^\perp - P = T + E^\perp BE$. It follows that $E^\perp BE = 0$. So, $(E^\perp - P) \in \mathcal{T}$. From (2), we get $E^\perp - P = 0$, i.e., $P = E^\perp$. ☐

Then, we concern the general case $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,F_i}, i \in \Omega$.

Theorem 6. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa, and $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,F_i}$ be a block-closed bimodule over $\mathcal{M}$. If $\mathcal{A} = \mathcal{T} + \mathcal{B}$ satifies $\mathcal{A} = \mathcal{A}''$, then the following are equivalent:*

(1) $\sum_i E_i = I$ (convergence being in the SOT)

(2) $\sum_i F_i = I$ (convergence being in the SOT)

Proof. (1)⇒(2). Let $T \in \mathcal{A}'$, then we have $T \in \mathcal{B}'$. Thus, for all $S \in \mathcal{B}(\mathcal{H})$, $TF_iSE_i = F_iSE_iT, i \in \Omega$. Then, for $0 \neq x, y \in \mathcal{H}$, $TF_ix \otimes yE_i = F_ix \otimes yE_iT, i \in \Omega$, it implies that there exists $\eta_i \in \mathbb{C}$ so that $E_iT = \eta_i E_i, i \in \Omega$. Therefore, $T = \sum_i E_iT = \oplus_i \eta_i E_i$. It follows that

$$\mathcal{A}' \subseteq \oplus_i \mathbb{C}E_i. \quad (13)$$

For any $i \in \Omega$, since $\sum_j E_j = I$, we have $F_i = (\sum_j E_j)F_i = E_iF_i$. It implies that $F_i \leq E_i, i \in \Omega$. From (13), we get $E_i B(H) E_i \subseteq \mathcal{A}'' = \mathcal{A} = \mathcal{T} + \mathcal{B}$ for any $i \in \Omega$. So,

$$\begin{aligned}(E_i - F_i)\mathcal{B}(\mathcal{H})F_i &= (E_i - F_i)(E_i\mathcal{B}(\mathcal{H})E_i)F_i \subseteq (E_i - F_i)(\mathcal{T} + \mathcal{B})F_i \\ &= (E_i - F_i)TF_i + (E_i - F_i)F_i\mathcal{B}(\mathcal{H})E_iF_i \\ &= \{0\}.\end{aligned} \quad (14)$$

Thus, $(E_i - F_i)\mathcal{B}(\mathcal{H})F_i = \{0\}$. Therefore, $E_i - F_i = 0$, $E_i = F_i, i \in \Omega$. Hence, $\sum_i F_i = \sum_i E_i = I$.

(2)⇒(1). The argument is similar to (1)⇒(2). ☐

Particularly, if $\mathcal{T} = \mathbb{C}I$, then we obtain the necessary and sufficient conditions for $\mathcal{A}$ to satisfy the double commutant property.

Theorem 7. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,F_i}$ be a block-closed bimodule over $\mathcal{M}$. Suppose that $\mathcal{A} = \mathbb{C}I + \mathcal{B}$. Then, $\mathcal{A} = \mathcal{A}''$ if and only if one of the following is true:*

(1) $\sum_i E_i = I$ and $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,E_i}$

(2) $\sum_i E_i \neq I \neq \sum_i F_i$

Proof. First, we prove the necessary part. Let $\mathcal{A} = \mathcal{A}''$. If $\sum_i E_i = I$, then from the proof of Theorem 6, we have $F_i = E_i, i \in \Omega$. So, $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,E_i}$; if $\sum_i E_i \neq I$, then $\sum_i E_i \neq I \neq \sum_i F_i$. Otherwise, $\sum_i F_i = I$. Thus, we get $\sum_i E_i = I$ from Theorem 6. This is a contradiction.

Now, we consider the sufficient part. If $\sum_i E_i = I$ and $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,E_i}$, then $\mathcal{A} = \mathbb{C}I + \mathcal{B} = \mathcal{B}$ is the von Neumann algebra with identity element. Hence, from von Neumann's double commutant theorem, we have $\mathcal{A} = \mathcal{A}''$; if $\sum_i E_i \neq I \neq \sum_i F_i$, then $E_0^\perp \neq 0 \neq F_0^\perp$, where $E_0 = \vee_i E_i$, $F_0 = \vee_i F_i$. By Proposition 2,

$$\begin{aligned}\mathcal{A}' &= \mathcal{B}' = \cap_i \mathcal{B}'_{E_i,F_i} = \cap_i \left(\mathbb{C}I + \mathcal{B}^\perp_{E_i,F_i}\right) \\ &= \cap_i \left(\mathbb{C}I + \mathcal{B}_{F_i^\perp,E_i^\perp}\right) \supseteq \cap_i \mathcal{B}_{F_i^\perp,E_i^\perp} \\ &= \mathcal{B}_{F_0^\perp,E_0^\perp}.\end{aligned} \tag{15}$$

So, $\mathcal{A}'' \subseteq (\mathcal{B}_{F_0^\perp,E_0^\perp})' = \mathbb{C}I + (\mathcal{B}_{F_0^\perp,E_0^\perp})^\perp = \mathbb{C}I + \mathcal{B}_{E_0,F_0}$.

Let $X \in \mathcal{A}''$. Then, there exists $T \in \mathcal{B}(\mathcal{H})$ and $\eta \in \mathbb{C}$ so that $X = F_0 T E_0 + \eta I$. For any $i \in \Omega$, let P_i be a projection onto $\mathrm{Ran}(E_i \vee F_i)$ and let $P_0 = I - \sum_i P_i$. It is clear that $P_i \in \mathcal{A}'$, $i \in \Omega \cup \{0\}$. Thus, $XP_i = P_i X, i \in \Omega \cup \{0\}$. Therefore, we can write $X = \sum_{i\in\Omega\cup\{0\}} X_i$, where $X_i = P_i X P_i$. $\forall_i \in \Omega$,

$$X_i = P_i X P_i = P_i(F_0 T E_0 + \eta I)P_i = F_i T E_i + \eta P_i. \tag{16}$$

Since $X_0 = P_0 X P_0 = P_0(F_0 T E_0 + \eta I)P_0 = \eta P_0$, we have

$$X = \eta P_0 + \sum_{i\in\Omega}(F_i T E_i + \eta P_i) = \eta I + \sum_{i\in\Omega} F_i T E_i \in \mathcal{A}. \tag{17}$$

So, $\mathcal{A}'' \subseteq \mathcal{A}$. Hence, $\mathcal{A} = \mathcal{A}''$. ☐

Now, we concern the class of algebras of the form $\mathcal{A} = \mathcal{T} + \mathcal{B}$ which satisfies $\mathcal{A} = \mathcal{A}''$, where $\mathcal{T} \subseteq \mathcal{M}$ spans by projections, $\mathcal{B} \in \mathcal{B}(\mathcal{H})$ is a block-closed $\mathcal{M}$ bimodule, and $\mathcal{T} \cap \mathcal{B} = \{0\}$.

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and $P \in \mathcal{M}$ be an orthogonal projection. For $\mathcal{A} \subseteq \mathcal{B}(\mathrm{Ran}P)$, we denote the relative commutant of $\mathcal{A}$ with respect to $\mathcal{B}(\mathrm{Ran}P)$ as $\mathcal{A}^\dagger = \{T \in \mathcal{B}(\mathrm{Ran}P)$: $AT = TA$ for all $A \in \mathcal{A}\}$ and the relative annihilator of $\mathcal{A}$ with respect to $\mathcal{B}(\mathrm{Ran}P)$ as $\mathcal{A}^0 = \{T \in \mathcal{B}(\mathrm{Ran}P)$: $AT = 0 = TA$ for all $A \in \mathcal{A}\}$.

Theorem 8. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa, $\mathcal{B} = \oplus_i B_{E_i,F_i}$ be a block-closed $\mathcal{M}$-bimodule, and P_0 be the maximal projection spanned by the projections over $\mathcal{B}$, i.e., $P_0 = \oplus_i F_i E_i$. Suppose that $\mathcal{T} = Span\{P_1, P_2, \cdots, P_s\}$ is a subalgebra of $\mathcal{M}$, where P_0, $P_1, P_2, \cdots, P_s$ are mutually orthogonal projections which ranks are no less than 2 and $\sum_{i=0}^s P_i = I$. Suppose that $\mathcal{A} = \mathcal{T} + \mathcal{B}$, $\mathcal{T} \cap \mathcal{B} = \{0\}$. Then the following are equivalent:*

(1) $\mathcal{A} = \mathcal{A}''$

(2) For each $1 \leq j \leq s$, $P_j \nleq \sum_i E_i$, and $P_j \nleq \sum_i F_i$

To prove the theorem, we need several lemmas.

Lemma 9. *Let $D, E, F \in \mathcal{M}$ be nonzero projections. Let $T \in \mathcal{B}(\mathcal{H})$ and $TX = XT$ for all $X \in \mathcal{B}_{D^\perp E, DF}$, then $T(DF) = \lambda(DF)$ for some $\lambda \in \mathbb{C}$.*

Proof. Since $TX = XT$ for all $X \in \mathcal{B}_{D^\perp E,DF}$, we have $TDF\mathcal{B}(\mathcal{H})D^\perp E = DF\mathcal{B}(\mathcal{H})D^\perp ET$. For $x_0, y_0 \in \mathcal{H}$, $TDFx_0 \otimes y_0 D^\perp E = DFx_0 \otimes y_0 D^\perp ET$. Thus, $TDFx_0 = \lambda DFx_0$ and $(D^\perp ET)^* y_0 = \bar{\lambda}(D^\perp E)^* y_0$ for some $\lambda \in \mathbb{C}$. We also have $TDF_x \otimes y_0 D^\perp E = DFx \otimes y_0 D^\perp ET$, $\forall x \in \mathcal{H}$. So, we get $TDFx = \lambda DFx$ for all $x \in H$. Thus, $T(DF) = \lambda(DF)$. ☐

Lemma 10. *Let $\mathcal{B} = \oplus_i \mathcal{B}_{E_i,F_i}$, then $\mathcal{B}' = \mathcal{B}_{F_0^\perp,E_0^\perp} + \overline{Span}\{(E_i \vee F_i): i \in \Omega\}$, where $E_0 = \vee_i E_i$ and $F_0 = \vee_i F_i$.*

Proof. It is clear that $\mathcal{B}' \supseteq \mathcal{B}_{F_0^\perp,E_0^\perp} + \overline{\mathrm{Span}}\{(E_i \vee F_i): i \in \Omega\}$.

On the other hand, $\forall T \in \mathcal{B}'$, since $\mathcal{B}' = \cap_i \mathcal{B}'_{E_i,F_i} = \cap_i(\mathcal{B}^\perp_{E_i,F_i} + \mathbb{C}I)$, there exists $T_i \in \mathcal{B}^\perp_{E_i,F_i}$ and $\lambda_i \in \mathbb{C}$ for all $i \in \Omega$ so that $T = T_i + \lambda_i I$. Since $T_i \in \mathcal{B}^\perp_{E_i,F_i} = \mathcal{B}_{F_i^\perp,E_i^\perp}$, we have $T_i F_i = 0, i \in \Omega$. It follows that $\lambda_i F_i = (T - T_i)F_i = TF_i, i \in \Omega$. Thus, $|\lambda_i| \leq \|T\|$ for each $i \in \Omega$. Let $X = \sum_i \lambda_i (E_i \vee F_i)$, then $X \in \overline{\mathrm{Span}}\{(E_i \vee F_i): i \in \Omega\}$ and

$$\begin{aligned}T - X &= T_i + \lambda_i[I - (E_i \vee F_i)] - \sum_{k\neq i} \lambda_k (E_k \vee F_k) \\ &= T_i + \lambda_i(I - E_i - F_i + E_i F_i) - \sum_{k\neq i} \lambda_k (E_k \vee F_k) \\ &= T_i + \lambda_i(I - E_i)(I - F_i) - \sum_{k\neq i} \lambda_k (E_k \vee F_k).\end{aligned} \tag{18}$$

Because $(I - E_i)(I - F_i) \in \mathcal{B}^\perp_{E_i,F_i}$ and $E_k \vee F_k \in \mathcal{B}^\perp_{E_i,F_i}$, $k \neq i$, we have $(T - X) \in \mathcal{B}^\perp_{E_i,F_i}$ for all $i \in \Omega$. Furthermore, $(T - X) \in \cap_i \mathcal{B}^\perp_{E_i,F_i} = \mathcal{B}_{F_0^\perp,E_0^\perp}$. So, $T \in \mathcal{B}_{F_0^\perp,E_0^\perp} + \overline{\mathrm{Span}}\{(E_i \vee F_i): i \in \Omega\}$. ☐

Proof of Theorem 8. (1)⇒(2). With loss of generality, we only need to show that $P_1 \leq \sum_i F_i$ and $P_1 \nleq \sum_i F_i$. We argue by contradiction. Assume $P_1 \leq \sum_i F_i$.

$\forall i \in \Omega$, $P_1 E_i \leq (\sum_i F_i)E_i = F_i E_i \leq P_0$. So, $P_1 E_i = P_0 P_1 E_i = 0$. Thus, $P_1^\perp E_i = E_i, i \in \Omega$. In fact, $\forall T \in \{P_1\}'$, $TP_1 = P_1 T$. We can write $T = P_1 T P_1 + P_1 T P_1^\perp + P_1^\perp T P_1 + P_1^\perp T P_1^\perp$. But $P_1 T P_1^\perp + T P_1 P_1^\perp = 0 = P_1^\perp P_1$. So, $T = P_1 T P_1 + P_1^\perp T P_1^\perp$. Hence,

$\{P_1\}' \subseteq P_1\mathscr{B}(\mathscr{H})P_1 + P_1^{\perp}\mathscr{B}(\mathscr{H})P_1^{\perp}$. It is clear that $P_1\mathscr{B}(\mathscr{H}) P_1 + P_1^{\perp}\mathscr{B}(\mathscr{H})P_1^{\perp} \subseteq \{P_1\}'$. Therefore, $\{P_1\}' = P_1\mathscr{B}(\mathscr{H})P_1 + P_1^{\perp}\mathscr{B}(\mathscr{H})P_1^{\perp}$.

Since $\mathscr{A}' \subseteq \{P_1\}' = P_1\mathscr{B}(\mathscr{H})P_1 + P_1^{\perp}\mathscr{B}(\mathscr{H})P_1^{\perp}$, we have $\forall T \in \mathscr{A}'$, $TP_1 = P_1T$.

From above, we can write $T = T_1 + T_4$, where $T_1 = P_1T P_1$, $T_4 = P_1^{\perp}TP_1^{\perp}$. Since $\mathscr{B}_{P_1^{\perp}E_j,P_1F_j} = P_1\mathscr{B}_{E_j,F_j}P_1^{\perp} \in \mathscr{A}$, for all $S \in \mathscr{B}_{P_1^{\perp}E_j,P_1F_j}$, we have $TS = ST$. Then, $T_1S = ST_4$. For each $i \in \Omega$, let $Q_i = P_1F_i$, then $\sum_i Q_i = P_1\sum_i F_i = P_1$. Thus, there exists $i \in \Omega$ such that $Q_i \neq 0$. By reindexing if necessary, we assume that $Q_1 \neq 0$.

By Lemma 9, for each $Q_1 \neq 0$, there exists $\alpha_i \in C$ so that $TQ_i = \alpha_i Q_i$. So, with the decomposition of $H = \operatorname{Ran}Q_1 \oplus \operatorname{Ran}Q_2 \oplus \cdots \oplus \operatorname{Ran}P_1^{\perp}$, we can write

$$T = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T_4 \end{pmatrix}. \tag{19}$$

Because of the arbitrary of T,

$$\mathscr{A}' \subseteq \mathbb{C}Q_1 \oplus \cdots \oplus P_1^{\perp}\mathscr{B}(\mathscr{H})P_1^{\perp}. \tag{20}$$

Thus, $Q_1\mathscr{B}(\mathscr{H})Q_1 \subseteq \mathscr{A}'' = \mathscr{A}$. For all $0 \neq A \in Q_1\mathscr{B}(\mathscr{H}) Q_1$, $A = Q_1AQ_1$. Since $A \in \mathscr{A}$, we can write $A = \sum_j^s 1^{\lambda_j P_j} + \oplus_i F_iA_iE_i$ for some $\lambda_j \in \mathbb{C}$, $1 \le j \le s$, and $A_i \in \mathscr{B}(\mathscr{H})$, $i \in \Omega$. Then, $A = AQ_1 = \lambda_1 Q_1$, it implies that $Q_1\mathscr{B}(\mathscr{H})\,Q_1 \subseteq \mathbb{C}Q_1$ and $Q_1 = P_1$. P_1 is a rank-1 projection. It is a contradiction.

(2)⇒(1). It is clear that $A \subseteq \mathscr{A}''$. We need to prove that $\mathscr{A}'' \subseteq \mathscr{A}$. Let $T \in \mathscr{A}''$. With the decomposition of $H = \operatorname{Ran} P_0 \oplus \operatorname{Ran}P_1 \oplus \cdots \oplus \operatorname{Ran}P_s$, we can write $T = [T_{i,j}]0 \le i, j \le s$, where $T_{i,j} = P_iTP_j$. Given $0 \le j, k \le s$, let $A_{j,k} = \{PjAPk \mid \operatorname{Ran} Pk : A \in A\} \subseteq B(\operatorname{Ran}Pk, \operatorname{Ran}Pj)$ and $(\mathscr{A}'')j,k = \{P_jAP_k \mid \operatorname{Ran}P_k : A \in \mathscr{A}''\}$.

Claim 11. $\mathscr{A}_{k,k}^{\dagger\dagger} = \mathscr{A}_{k,k}$, $0 \le k \le s$.

Fix $0 \le k \le s$, for each $i \in \Omega$, we define $E_i[k] = E_iP_k$, $F_i[k] = F_iP_k$ and $E_0[k] = \sum_i E_i[k]$, $F_0[k] = \sum_i F_i[k]$. By the hypothesis, $\sum_i E_i[k] \neq P_k \neq \sum_i F_i[k]$, $1 \le k \le s$. It is clear that P_k is the identify element of $\mathscr{B}(\operatorname{Ran}P_k)$. So, by Theorem 7, we have $\mathscr{A}_{k,k}^{\dagger\dagger} = \mathscr{A}_{k,k}$, $1 \le k \le s$.

For $k = 0$, since $P_0 = \oplus_i F_iE_i$, we have $F_iP_0 = F_iE_i = P_0E_i = E_iP_0$. It implies that $E_i[0] = F_i[0]$, and $\sum_i E_i[0] = \sum_i F_iE_i = P_0$, by Theorem 7, $\mathscr{A}_{0,0}^{\dagger\dagger} = \mathscr{A}_{0,0}$.

Claim 12. $(\mathscr{A}'')_{k,k} \subseteq \mathscr{A}_{k,k}$, $0 \le k \le s$.

Since for $0 \le k \le s$, $P_k \in \mathscr{A} \subseteq \mathscr{A}''$, we have $T_{k,k} = P_kTP_k \in \mathscr{A}''$ and $T_{k,k} \in \mathscr{B}(\operatorname{Ran}P_k)$. Let $W \in \mathscr{A}'$, then $WP_k = P_kW$, $0 \le k \le s$. So $W = W_0 \oplus W_1 \oplus \cdots \oplus W_s$, where $W_k = P_kWP_k$. Thus, $WT = TW$ implies that

$$W_kT_{k,r} = T_{k,r}W_r, 0 \le k, r \le s. \tag{21}$$

In particular, $W_kT_{k,k} = T_{k,k}W_k$, $0 \le k \le s$. Therefore, we have

$$T_{k,k} \in \left(P_k\mathscr{A}'P_k\right)', 0 \le k \le s. \tag{22}$$

For $1 \le k \le s$, define $E_0[k]^0 = P_k - \sum_i E_i[k]$, $F_0[k]_0 = P_k - \sum_i F_i[k]$. Let $A_k \in \mathscr{A}_{k,k}^{\dagger} \subseteq \mathscr{B}(\operatorname{Ran}P_k)$, $1 \le k \le s$. By Lemma 10,

$$A_k \in \mathscr{B}_{F_0[k]^0,E_0[k]^0} + \overline{\operatorname{Span}}\{(E_i[k] \vee F_i[k]) : i \in \Omega\}. \tag{23}$$

Hence, there exists $B_k \in \mathscr{B}_{F_0[k]^0,E_0[k]^0}$ and $\alpha_i \in \mathbb{C}, i \in \Omega$, so that $A_k = B_k + \sum_i \alpha_i(E_i[k] \vee F_i[k])$. Let $Y = B_k + \sum_i \alpha_i(E_i \vee F_i)$; now, we prove $Y \in \mathscr{A}'$. Since $E_i \vee F_i \in \mathscr{A}'$ for each $i \in \Omega$, we need to show that $B_k \in \mathscr{A}'$. In fact,

$$B_k = E_0[k]^0B_kF_0[k]^0 = P_kE_0[k]^0B_kF_0[k]^0P_k. \tag{24}$$

If $1 \le j \neq k \le s$, then $B_kP_j = P_jB_k = 0$; if $1 \le j = k \le s$, then $P_kB_k = B_k = B_kP_k$. Thus, $B_k \in \mathscr{T}'$. Furthermore, for all $X \in \mathscr{B}$, $X = \sum_i F_iXE_i$, then

$$\begin{aligned} XB_k &= \sum_i XE_iP_kE_0[k]^0B_k = \sum_i XE_i[k]E_0[k]^0B_k = 0, \\ B_kX &= B_kF_0[k]^0P_k\sum_i F_iX = B_kF_0[k]^0F_0[k]X = 0. \end{aligned} \tag{25}$$

Hence, $B_k \in \mathscr{B}'$. Therefore, $B_k \in \mathscr{A}'$. Now, $A_k = P_kYP_k \in P_k\mathscr{A}'P_k$; this implies that $\mathscr{A}_{k,k}^{\dagger} \subseteq P_kA_0P_k$. So, from (22), we have $T_{k,k} \in (P_k\mathscr{A}'^{P}_{k})^{\dagger} \subseteq \mathscr{A}_{k,k}^{\dagger\dagger} = \mathscr{A}_{k,k}$.

Thus, $(\mathscr{A})'_{k,k} \subseteq \mathscr{A}_{k,k}$, $1 \le k \le s$.

Now, we consider $\mathscr{A}_{0,0} = \sum_i E_i[0]\mathscr{B}(\mathscr{H})E_i[0]$. It is easy to verify that $\mathscr{A}_{0,0}^{\dagger} = \oplus_{Ei[0]\neq 0}\mathbb{C}E_i[0]$. Given $Z \in \mathscr{A}_{0,0}^{\dagger}$, there exists $\alpha_i \in \mathbb{C}$ so that $Z = \sum_{E_i[0]\neq 0}\alpha_iE_i[0]$. Let $Y = \sum_{E_i[0]\neq 0}\alpha_i(E_i \vee F_i)$, then $Y \in \mathscr{A}'$ and $Z = P_0YP_0 \in P_0\mathscr{A}'P_0$. It implies that $\mathscr{A}_{0,0}^{\dagger} \subseteq P_0\mathscr{A}'P_0$. So, $T_{0,0} \in (P_0\mathscr{A}'P_0)^{\dagger} \subseteq \mathscr{A}_{0,0}^{\dagger\dagger} = \mathscr{A}_{0,0}$ by (22). Hence, $(\mathscr{A})'_{0,0} \subseteq \mathscr{A}_{0,0}$.

Claim 13. $(\mathscr{A})'_{0,k} \subseteq \mathscr{A}_{0,k}$, $(\mathscr{A})'_{k,0} \subseteq \mathscr{A}_{k,0}$, $1 \le k \le s$.

Let $B \in \mathscr{B}_{F_0[k]^0,E_0[k]^0} \subseteq \mathscr{B}(\operatorname{Ran}P_k)$, $1 \le k \le s$, then from Claim 12, we have $B = P_kBP_k$ and $B \in \mathscr{A}'$. Therefore, $B_0 = P_0BP_0 = 0$, $B_k = P_kBP_k = B$. So, from (21), we get

$$0 = B_0T_{0,k} = T_{0,k}B_k = T_{0,k}B. \tag{26}$$

Let $y_0 \in H$ satisfying $F_0[k]^0y_0 \neq 0$. From the above, $\forall x \in \mathscr{H}$, $T_{0,k}E_0[k]^0x \otimes y_0F_0[k]^0 = 0$. Thus, $T_{0,k}E_0[k]^0 = 0$. It implies that

$$T_{0,k} = T_{0,k}E_0[k]. \tag{27}$$

For each $i \in \Omega$, since $(E_i \vee F_i) \in \mathscr{A}'$ and $T_{0,k} = P_0TP_k \in \mathscr{A}''$, we get $(E_i \vee F_i)T_{0,k} = T_{0,k}(E_i \vee F_i)$. Thus,

$$E_i[0]T_{0,k} = P_0(E_i \vee F_i)P_0 T_{0,k} = T_{0,k}P_k(E_i \vee F_i)P_k = T_{0,k}(E_i[k] \vee F_i[k]). \quad (28)$$

Together with (27) and (28), we have

$$F_i[0]T_{0,k} = E_i[0]T_{0,k} = T_{0,k}E_0[k](E_i[k] \vee F_i[k]) = T_{0,k}E_i[k]. \quad (29)$$

So, $T_{0,k} \in \oplus_i B_{E_i[k],F_i[0]} \in \mathscr{A}_{0,k}$.

A similar argument shows that $T_{k,0} \in \mathscr{A}_{k,0}$.

Claim 14. $(\mathscr{A}'')_{m,n} \subseteq A_{m,n}$, $1 \le m \ne n \le s$.

Let $Y \in \mathscr{B}_{F_0[m]^0, E_0[m]^0}$, then $Y = P_m Y P_m$ and $Y \in \mathscr{A}'$ by Claim 12. So, we have $Y_n = P_n Y P_n = 0$, $Y_m = P_m Y P_m = Y$. Together with (21), we have

$$0 = T_{m,n}Y_n = Y_m T_{m,n} = Y T_{m,n}. \quad (30)$$

Thus, $T_{m,n} = F_0[m]T_{m,n}$. Similarly, $T_{m,n} = T_{m,n}E_0[n]$. Therefore,

$$T_{m,n} = F_0[m]T_{m,n}E_0[n]. \quad (31)$$

Since $(E_i \vee F_i) \in \mathscr{A}'$, $i \in \Omega$, $T_{m,n} \in \mathscr{A}''$, we have $(E_i \vee F_i)P_m T_{m,n} = T_{m,n}P_n(E_i \vee F_i)$. So, $(E_i[m] \vee F_i[m])T_{m,n} = T_{m,n}(E_i[n] \vee F_i[n])$. By (31), $F_i[m]T_{m,n} = (E_i[m] \vee F_i[m])F_0[m]T_{m,n} = T_{m,n} E_0[n](E_i[n] \vee F_i[n]) = T_{m,n}E_i[n]$. Thus, $T_{m,n} \in \oplus_i B_{E_i[n],F_i[m]} \in \mathscr{A}_{m,n}$.

Now from the above, for any $0 \le i, j \le s$, we have $(\mathscr{A}'')_{i,j} \subseteq \mathscr{A}_{i,j}$. So, $\mathscr{A}'' \subseteq \mathscr{A}$.

We complete the proof.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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