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Research Article

Exact Solutions of Newell-Whitehead-Segel Equations Using Symmetry Transformations

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In this article, Lie and discrete symmetry transformation groups of linear and nonlinear Newell-Whitehead-Segel (NWS) equations are obtained. By using these symmetry transformation groups, several group invariant solutions of considered NWS equations have been constructed. Furthermore, some more group invariant solutions are generated by using discrete symmetry transformation group. Graphical representations of some obtained solutions are also presented.

1. Introduction

Partial differential equations have numerous physical applications in the fields of fluid mechanics, plasma mechanics, biology, mathematical physics, financial mathematics, applied mathematics, etc., and the Lie group method has a powerful role to discover the solutions of partial differential equations [1, 2]. Lie groups of transformations can be used in several ways to analyze the solutions of partial differential equation (see [3–6]). Lie symmetry generators corresponding to Lie groups are also used to find the group-invariant solutions. These group invariant solutions lead to the exact solutions of partial differential equations. Many researchers have applied this method to find the solutions of different partial differential equations (e.g., [5, 7, 8]). The Lie group method depends upon continuous point transformations. The point transformations other than continuous transformations such as local, global, and discrete point transformations also exist and have many applications. Among other applications [9, 10], discrete symmetries are also helpful to investigate the solutions of partial differential equations [11]. But very rare research work on the discrete symmetries is available up to now. Hydon developed a method to find a complete list for the discrete symmetry transformations of a partial differential equation [12, 13]. This method is based on the Lie algebra of the continuous symmetry generators of differential equation.

The nonequilibrium systems have been usually shown in various extended states, like uniform, pattern states, chaotic, and oscillatory. Several stripe patterns, like stripes of seashells and ripples in sand, occur in a diversity of spatially extended systems which can be explained by a set of equation named amplitude equations. The most important equation among the amplitude equations is the NWS equation that describes the occurrence of the stripe pattern in two-dimensional systems. Furthermore, this equation has been applied to various systems, e.g., nonlinear optics, biological systems, Rayleigh-Benard convection, and chemical reactions. In recent years, several techniques have been developed to solve the NWS equations, e.g., lattice Bolzmann scheme [14], Adomian decomposition method [15], finite difference scheme [16], differential transformation method [17], and reduced differential transformation [18].

But in the current research, several group invariant solutions of the linear and nonlinear NWS equations via the similarity variables, which have been obtained by using continuous and discrete symmetry transformations, are presented.

This paper is arranged into four sections as follows: in Sections 2 and 3, Lie and discrete symmetry group of transformations of linear and nonlinear NWS equations are presented. In Section 4, group invariant solutions of these equations, considering the similarity variables via continuous transformations, are calculated while in Section 5, group invariant solutions of linear NWS equation via discrete transformations are calculated. Finally, in Section 6, the summary of the present work is discussed.

2. Lie Symmetries of Linear and Nonlinear NWS Equations

The main purpose of this paper is to investigate the exact solutions of the following linear and nonlinear NWS equations [18].

$$w_t = w_{xx} - 2w, \tag{1}$$

$$w_t = w_{xx} + 2w - 3w^2, \tag{2}$$

by using their continuous and discrete symmetry transformations.

2.1. Lie Symmetries of a Linear NWS Equation. The method to obtain the Lie symmetries of partial differential equations has been discussed in many books (e.g., see in [1, 3]). Let

$$X = \phi_1(x,t,w)\partial_x + \phi_2(x,t,w)\partial_t + \eta_1(x,t,w)\partial_w \tag{3}$$

be the vector field that generates the symmetry group of equation (1). By applying the second order prolongation of X to equation (1), the following determining equations are obtained:

$$\begin{aligned}(\eta_1)_{t,w} + 2(\phi_2)_t + \frac{1}{4}(\phi_2)_{t,t} = 0, (\eta_1)_{w,w} = 0, (\eta_1)_{w,x} + \frac{1}{2}(\phi_1)_t = 0,\\ (\eta_1)_{x,x} - 2(\phi_2)_t w - (\eta_1)_t - 2\eta_1(x,t,w) + 2(\eta_1)_w w = 0,\\ (\phi_1)_x - \frac{1}{2}(\phi_2)_t = 0, (\phi_1)_{t,t} = 0, (\phi_2)_w = 0,\\ (\phi_2)_x = 0, (\phi_2)t,t,t = 0.\end{aligned} \tag{4}$$

The infinitesimals obtained by solving the above system are

$$\begin{aligned}\eta_1 &= \left(\frac{1}{8}\left(-x^2 - 8t^2 - 2t\right)C_6 - \frac{1}{2}C_4 x - 2C_5 t + C_3\right)w,\\ \phi_1 &= \frac{1}{2}(C_6 x + 2C_4)t + \frac{1}{2}xC_5 + C_2, \phi_2 = \frac{1}{2}C_6 t^2 + C_5 t + C_1.\end{aligned} \tag{5}$$

Thus, linear NWS equation (1) is spanned by the following vector fields:

$$\begin{aligned}X_1 &= \partial_t, X_2 = \partial_x, X_3 = w\partial_w, X_4 = 2t\partial_x - wx\partial_w X_5\\ &= x\partial_x + 2t\partial_t - 4wt\partial_w, X_6\\ &= 4tx\partial_x + 4t^2\partial_t - \left(x^2 + 2t + 8t^2\right)w\partial_w,\end{aligned} \tag{6}$$

which forms a six-dimensional Lie algebra.

2.2. Lie Symmetries of Nonlinear NWS Equation. Similarly by applying the second order prolongation of X to equation (2), the following determining equations are obtained:

$$\begin{aligned}(\phi_1)_t = 0, (\phi_1)_w = 0, (\phi_1)_x = 0, (\phi_2)_t = 0,\\ (\phi_2)_w = 0, (\phi_2)_x = 0, \eta_1(x,t,w) = 0.\end{aligned} \tag{7}$$

The infinitesimals and vector fields for equation (2) are

$$\eta_1 = 0, \phi_1 = C_1, \phi_2 = C_2, X_1 = \partial_x, X_2 = \partial_t. \tag{8}$$

3. Discrete Symmetries of Linear NWS Equation

The commutator relations of the Lie algebra, given in (6), are

$$\begin{aligned}&[X_1, X_4] = 2X_2, [X_1, X_5] = 2X_1 + 4X_3, [X_1, X_6] = 4X_5 - 2X_3,\\ &[X_2, X_4] = -X_3, [X_2, X_5] = X_2, [X_2, X_6] = 2X_4, [X_4, X_5]\\ &\quad = -X_4, [X_5, X_6] = 2X_6.\end{aligned} \tag{9}$$

The structure constants corresponding to above commutator relations are

$$\begin{aligned}c_{12}^5 = -1, c_{16}^1 = \frac{9}{2}, c_{24}^2 = 1, c_{26}^2 = -\frac{3}{2},\\ c_{34}^3 = 1, c_{36}^5 = 1, c_{45}^5 = -1, c_{56}^5 = 3.\end{aligned} \tag{10}$$

Following the method proposed by Hydon in [13], discrete symmetries of equation (1) are obtained and these symmetries are

$$\begin{aligned}&D_1 : (x,t,w) \mapsto (-x,t,-w), D_2 : (x,t,w) \mapsto (-x,t,w),\\ &D_3 : (x,t,w) \mapsto (-x,t,w\exp(-(x+t))).\end{aligned} \tag{11}$$

3.1. Discrete Symmetries of Nonlinear NWS Equation. Again by applying Hydon's method [13] on two-dimensional Lie algebra of equation (2) given in (8), it is found that (2) has only one discrete symmetry transformation

$$D : (x,t,w) \mapsto (-x,t,w). \tag{12}$$

4. Exact Solutions of Linear NWS Equation Obtained by Considering Lie Symmetries

In this section, several group invariant solutions of the linear NWS equation (1) by considering different linear combinations of Lie symmetry generators, which have been obtained in Section 3, are presented.

(1) The similarity variables corresponding to

$$X_1 + X_2 + X_3 = \partial_t + \partial_x + w\partial_w \tag{13}$$

are

$$\nu = t - x, \mu(\nu) = w \exp(-x), \tag{14}$$

which imply that the solution of (1) is in the form

$$w(x, t) = \mu(\nu) \exp(x), \tag{15}$$

where $\mu(\nu)$ satisfies the following ODE

$$\mu_{\nu\nu} - 3\mu_\nu - \mu = 0, \tag{16}$$

with the solution

$$\mu(\nu) = C_1 \exp\left(\frac{1}{2}\left(3 + \sqrt{13}\right)\nu\right) + C_2 \exp\left(-\frac{1}{2}\left(-3 + \sqrt{13}\right)\nu\right). \tag{17}$$

Hence,

$$\begin{aligned} w(x, t) = \exp(x)\bigg(& C_1 \exp\left(\frac{1}{2}\left(3 + \sqrt{13}\right)(t - x)\right) \\ &+ C_2 \exp\left(-\frac{1}{2}\left(-3 + \sqrt{13}\right)(t - x)\right)\bigg) \end{aligned} \tag{18}$$

is the solution of (1)

(2) The similarity variables corresponding to $X_4 = 2t\partial_x - wx\partial_w$ are

$$\nu = t, \mu(\nu) = w \exp\left(\frac{x^2}{4t}\right), \tag{19}$$

which imply that the solution of (1) is in the form

$$w(x, t) = \mu(\nu) \exp\left(-\frac{x^2}{4t}\right), \tag{20}$$

where $\mu(\nu)$ satisfies the ODE

$$2\nu\mu_\nu + 4\nu\mu(\nu) + \mu(\nu) = 0, \tag{21}$$

with the solution

$$\mu(\nu) = C_1 \frac{\exp(-2\nu)}{\sqrt{\nu}}. \tag{22}$$

Hence,

$$w(x, t) = \frac{C_1 \exp\left(\left(-8t^2 - x^2\right)/4t\right)}{\sqrt{t}} \tag{23}$$

is the solution of (1)

(3) The similarity variables corresponding to

$$X_1 + X_2 = \partial_x + \partial_t \tag{24}$$

are

$$\nu = t - x, \mu(\nu) = w, \tag{25}$$

which imply that the solution of (1) is in the form

$$w(x, t) = \mu(\nu), \tag{26}$$

where $\mu(\nu)$ satisfies the ODE

$$\mu_{\nu\nu} - \mu_\nu - 2\mu = 0, \tag{27}$$

with the solution

$$\mu(\nu) = C_1 \exp(2\nu) + C_2 \exp(-\nu). \tag{28}$$

Hence,

$$w(x, t) = C_1 \exp(2t - 2x) + C_2 \exp(x - t) \tag{29}$$

is the solution of (1)

(4) For

$$X_1 + X_4 = \partial_t + 2t\partial_x - wx\partial_w, \tag{30}$$

the similarity variables are

$$\nu = t^2 - x, \mu(\nu) = w \exp\left(-\frac{2}{3}t^3 + xt\right), \tag{31}$$

which imply that the solution of (1) is in the form

$$w(x, t) = \mu(\nu) \exp\left(\frac{2}{3}t^3 - xt\right), \tag{32}$$

where $\mu(\nu)$ satisfies the following ODE

$$\mu_{\nu\nu} - \nu\mu - 2\mu = 0, \tag{33}$$

with the solution

$$\mu(\nu) = C_1 A_i(\nu + 2) + C_2 B_i(\nu + 2). \tag{34}$$

Hence,

$$w(x,t)=\exp\left(\frac{2}{3}t^3-xt\right)\left(C_1A_i\left(t^2-x+2\right)+C_2B_i\left(t^2-x+2\right)\right) \tag{35}$$

is a solution of (1), where A_i and B_i represent the airy functions

(5) For

$$X_2+X_3=\partial_x+w\partial_w, \tag{36}$$

the similarity variables are

$$\nu=t,\ \mu(\nu)=w\exp(-x), \tag{37}$$

which imply that

$$w(x,t)=\exp(x)\mu(\nu), \tag{38}$$

where $\mu(\nu)$ satisfies the ODE

$$\mu_\nu+\mu=0, \tag{39}$$

with the solution

$$\mu(\nu)=C_1\exp(-\nu). \tag{40}$$

Hence,

$$w(x,t)=C_1\exp(x-t) \tag{41}$$

is the solution of (1)

(6) For

$$X_5=x\partial_x+2t\partial_t-4wt\partial_w, \tag{42}$$

the corresponding similarity variables are

$$\nu=\frac{t}{x^2},\ \mu(\nu)=w\exp(2t), \tag{43}$$

which imply that the solution of (1) is in the form

$$w(x,t)=\mu(\nu)\exp(-2t), \tag{44}$$

where $\mu(\nu)$ satisfies the following ODE:

$$4\nu^2\mu_{\nu\nu}+6\nu\mu_\nu-\mu_\nu=0. \tag{45}$$

The solution of above ODE is

$$\mu(\nu)=C_1\operatorname{erf}\left(\frac{1}{2\sqrt{\nu}}\right)+C_2. \tag{46}$$

Hence,

$$w(x,t)=\exp(-2t)\left(C_1\operatorname{erf}\left(\frac{1}{2\sqrt{t/x^2}}\right)+C_2\right) \tag{47}$$

is another solution of (1)

(7) Now for

$$X_6=4tx\partial_x+4t^2\partial_t-\left(x^2+2t+8t^2\right)w\partial_w, \tag{48}$$

the similarity variables are

$$\nu=\frac{t}{x},\ \mu(\nu)=w\sqrt{x}\exp\left(\frac{1}{4}\frac{x^2+8t^2}{t}\right), \tag{49}$$

which imply that the solution of (1) is in the form

$$w(x,t)=\mu(\nu)\frac{\exp\left(-(1/4)\left(\left(x^2+8t^2\right)/4t\right)\right)}{\sqrt{x}}, \tag{50}$$

where $\mu(\nu)$ satisfies the ODE

$$4\nu^2\mu_{\nu\nu}+12\nu\mu_\nu+3\mu=0, \tag{51}$$

with the solution

$$\mu(\nu)=\frac{C_1}{\nu^{3/2}}+\frac{C_2}{\nu^{1/2}}. \tag{52}$$

Hence,

$$w(x,t)=\frac{\left(C_1x+C_2t\right)\exp\left(-(1/4)\left(\left(x^2+8t^2\right)/4t\right)\right)}{t\sqrt{t}} \tag{53}$$

is a solution of (1)

Now, graphical representations for some solutions of (1), obtained in equations (23), (41), and (47) for $C_1=1$ and $C_2=0$, are presented in Figures 1–3, respectively.

4.1. Exact Solutions of Nonlinear NWS Equation Obtained by Considering Lie Symmetries

(1) The similarity variables corresponding to $X_1=\partial_x$, are

$$\nu=t,\ \mu(\nu)=w, \tag{54}$$

which imply that the solution of (2) is in the form

$$w(x,t)=\mu(\nu), \tag{55}$$

where $\mu(\nu)$ satisfies the equation

$$\mu_\nu-2\mu(\nu)+3\mu(\nu)^2=0, \tag{56}$$

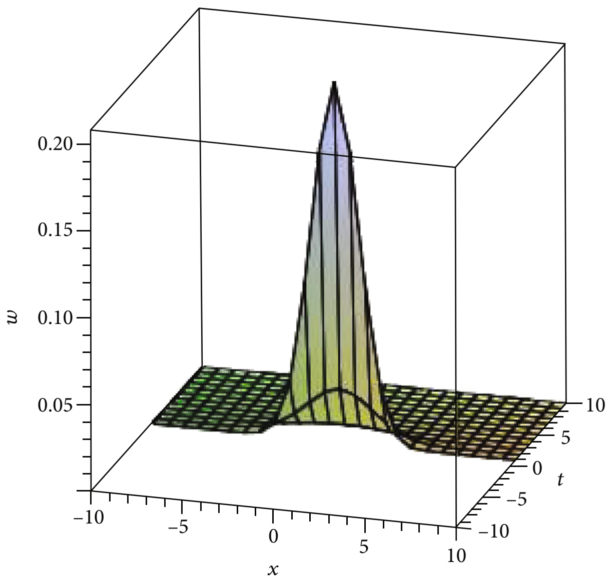

FIGURE 1: Exact sol. of (1) given by (23).

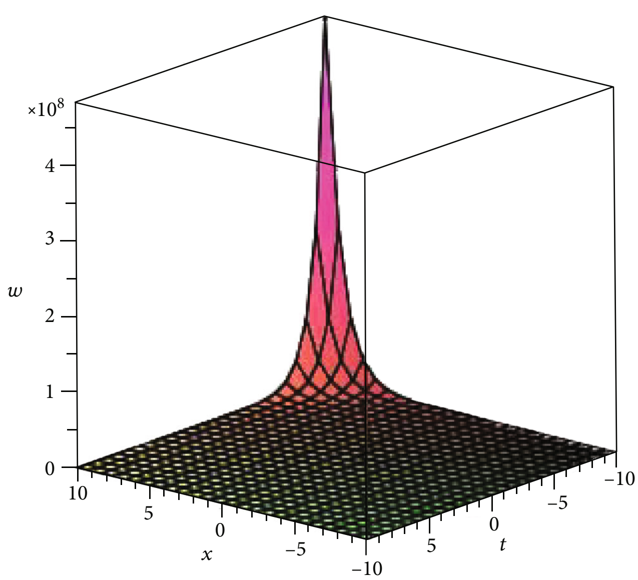

FIGURE 2: Exact sol. of (1) given by (41).

with the solution

$$\mu(\nu) = \frac{2}{3 + 2C_1 \exp(-2\nu)}. \tag{57}$$

Hence,

$$w(x,t) = \frac{2}{3 + 2C_1 \exp(-2t)} \tag{58}$$

is a solution of (2).

The solution of (2) given in (58) for $C_1 = 1$ is presented by its graph in Figure 4

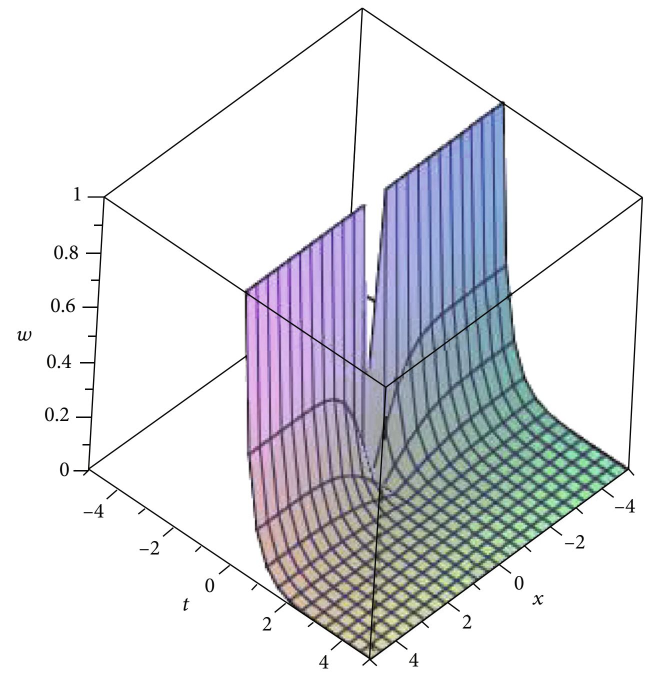

FIGURE 3: Sol. of (1) given by (47).

(2) Now, for $X_2 = \partial_t$, the similarity variables are

$$\nu = x, \mu(\nu) = w, \tag{59}$$

which imply that the solution of (2) is in the form

$$w(x,t) = \mu(\nu), \tag{60}$$

where $\mu(\nu)$ satisfies the equation

$$\mu_{\nu\nu} - 2\mu(\nu) + 3\mu(\nu)^2 = 0. \tag{61}$$

The above ODE has a linear symmetry $\partial/\partial\nu$. By using this linear symmetry and Maple software, ODE (61) is transformed to a first order ODE

$$\frac{du(z)}{dz} = \frac{z(-2+3z)}{u(z)}, \tag{62}$$

which has the solutions

$$u(z) = \pm\sqrt{2z^3 - 2z^2 + C_1} \tag{63}$$

5. Exact Solutions of Linear NWS Equation Obtained by Considering Discrete Symmetry Transformations

In this section, using the discrete symmetry transformation D_3, some further group invariant solutions of (1) have been generated.

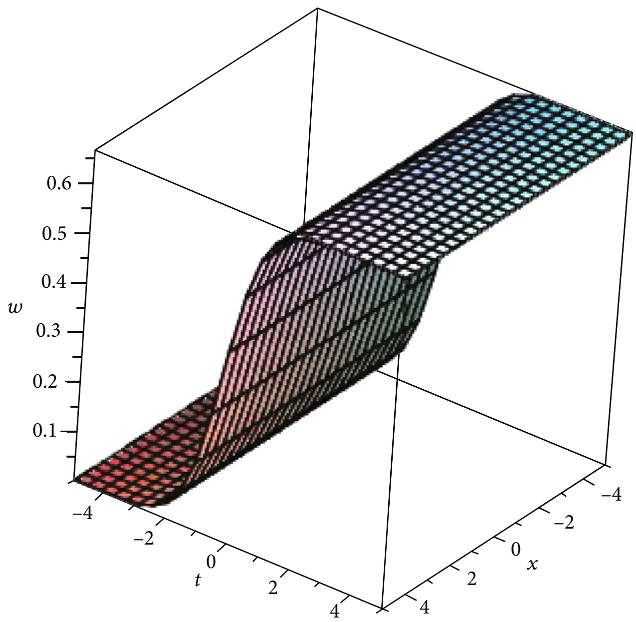

FIGURE 4: Sol. of (2) given in (58).

(1) Since the similarity variables corresponding to $X_1 + X_2 + X_3$ are

$$\nu = t - x, \mu(\nu) = w \exp(-x), \tag{64}$$

by applying the discrete symmetry transformation

$$D_3 : (x, t, w) \mapsto (-x, t, \exp(-(x+t))w), \tag{65}$$

the similarity variables given in (64) will be transformed to

$$\nu = t + x, \mu(\nu) = \exp(-(x+t)) w \exp(x). \tag{66}$$

The above similarity variables imply that the solution of (1) is in the following form:

$$w(x,t) = \exp(t)\mu(\nu), \tag{67}$$

where $\mu(\nu)$ satisfies the following ODE:

$$\mu_{\nu\nu} - \mu_\nu - 3\mu = 0, \tag{68}$$

which has the solution

$$\mu(\nu) = C_1 \exp\left(\frac{1}{2}\left(1+\sqrt{13}\right)\nu\right) + C_2 \exp\left(-\frac{1}{2}\left(-1+\sqrt{13}\right)\nu\right). \tag{69}$$

Hence,

$$\begin{aligned} w(x,t) = \exp(t)\Bigg(& C_1 \exp\left(\frac{1}{2}\left(1+\sqrt{13}\right)(t+x)\right) \\ & + C_2 \exp\left(-\frac{1}{2}\left(-1+\sqrt{13}\right)(t+x)\right)\Bigg) \end{aligned} \tag{70}$$

is a solution of (1)

(2) Since the similarity variables for$X_1 + X_2$are

$$\nu = t - x, \mu(\nu) = w, \tag{71}$$

by applying the discrete symmetry transformation D_3, the similarity variables given in (71) will be transformed to

$$\nu = t + x, \mu(\nu) = w \exp(-(x+t)), \tag{72}$$

which imply that the solution of (1) is in the form

$$w(x,t) = \exp(x+t)\mu(\nu), \tag{73}$$

where $\mu(\nu)$ satisfies the following ODE

$$\mu_{\nu\nu} + \mu_\nu - 2\mu = 0, \tag{74}$$

with the solution

$$\mu(\nu) = C_1 \exp(\nu) + C_2 \exp(-2\nu). \tag{75}$$

Hence,

$$w(x,t) = C_1 \exp(2(x+t)) + C_2 \exp(-(x+t)) \tag{76}$$

is a solution of (1)

(3) Since the similarity variables corresponding to$X_2 + X_3$are

$$\nu = t, \mu(\nu) = w \exp(-x), \tag{77}$$

again by applying the discrete symmetry transformation D_3, the above similarity variables will be transformed to

$$\nu = t, \mu(\nu) = w \exp(-t), \tag{78}$$

which imply that the solution of (1) is in the form

$$w(x,t) = \exp(t)\mu(\nu), \tag{79}$$

where $\mu(\nu)$ satisfies the following ODE

$$\mu_\nu + 3\mu(\nu) = 0. \tag{80}$$

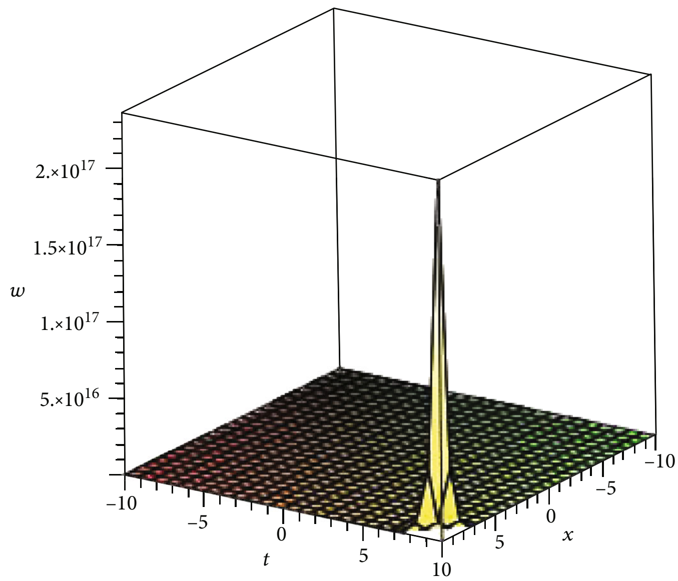

FIGURE 5: Exact sol. of (1) given in (76).

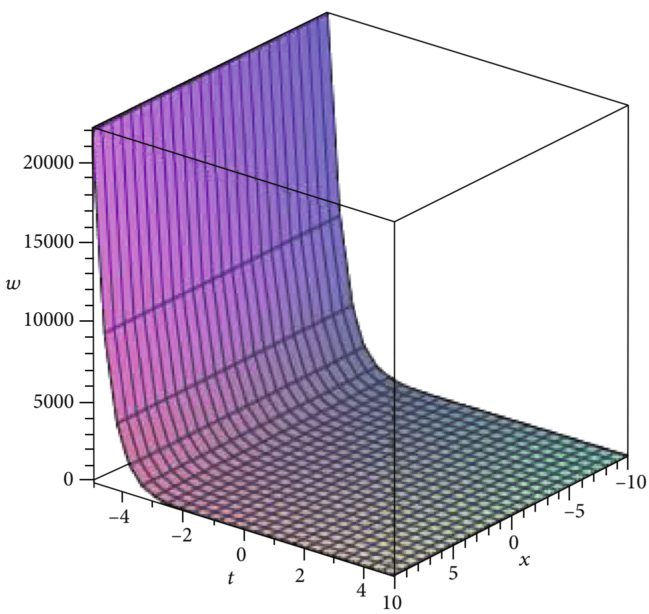

FIGURE 6: Exact sol. of (1) given in (81).

The solution of the above ODE is $\mu(v) = C_1 \exp(-3v)$. Thus,

$$w(x,t) = C_1 \exp(-2t) \tag{81}$$

is a solution of (1)

The solutions of (1) obtained by discrete symmetry transformations, given in (76) and (81) for $C_1 = 1$ and $C_2 = 1$, are presented by their graphs in Figures 5 and 6, respectively.

6. Summary and Conclusions

In this paper, the Lie and discrete symmetry groups of linear and nonlinear NWS equations by using Lie's and Hydon's approach have been presented, and these groups are used to obtain the group invariant solutions of NWS equations. In this research, it is explored that group invariant solutions of partial differential equations by using the discrete symmetry transformations can be also obtained that shows the significance of the discrete symmetry groups. The graphical representations of some solutions are also presented. In Section 2, it was noticed that the linear NWS equation has three discrete symmetry groups of transformations but only D_3 contributes to find the group invariant solutions because the rest two groups are reflection groups, and during the calculations, it was observed that reflection groups do not contribute to find the new group invariant solutions of partial differential equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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