Research Article

Existence of Three Solutions for Nonlinear Operator Equations and Applications to Second-Order Differential Equations

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The existence of three solutions for nonlinear operator equations is established via index theory for linear self-adjoint operator equations, critical point reduction method, and three critical points theorems obtained by Brezis-Nirenberg, Ricceri, and Averna-Bonanno. Applying the results to second-order Hamiltonian systems satisfying generalized periodic boundary conditions or Sturm-Liouville boundary conditions and elliptic partial differential equations satisfying Dirichlet boundary value conditions, we obtain some new theorems concerning the existence of three solutions.

1. Introduction and Main Results

In their excellent paper [1] in 1991, Brezis-Nirenberg proved the existence of two nonzero critical points under suitable assumptions by a negative gradient flow and linking. The obtained result was applied to second-order Hamiltonian systems and yielded the existence of three periodic solutions for these systems. After that, in [2], Tang obtained one existence of at least three periodic solutions for second-order Hamiltonian systems by using two nonzero critical points theorem in [1]. This result generalizes the corresponding result in [1]. In addition, in [3], one new existence of at least three periodic solutions for second-order Hamiltonian systems was obtained by using the critical point reduction method and two nonzero critical points theorem in [1, 4].

Moreover, after the three critical points theorem (see [5, 6]) of Ricceri appearing, it has proved to be one of the most often widely used to solve differential equations, such as elliptic partial differential equations satisfying Dirichlet boundary value conditions (see [7–12]) and second-order Hamiltonian systems satisfying periodic boundary value conditions (see [3, 13]). Recently, a general three critical points theorem is given by Averna-Bonanno [14]. After that, in [15], the existence of at least three periodic solutions was established for

a class of p -Hamiltonian systems by two general three critical points theorems in [6, 14].

It is well known that the above-mentioned problems can be displayed as a self-adjoint operator equation. For the multiplicity of self-adjoint operator equation, Chang proved the existence of three distinct solutions under suitable assumptions via Morse theory in his excellent paper [16] in 1981. The obtained results were applied to second-order Hamiltonian systems and wave equations and yielded the existence of three periodic solutions for these systems. Similar results can be found in [17–21].

Inspired by the ideas of [16, 18, 19], in this paper, we reconsider in the framework of the operator equations some theorems proved in [1–3, 9, 10, 15, 22] and generalize these theorems to nonlinear self-adjoint operator equations. Our technical approach is based on some three critical points theorems obtained by Brezis-Nirenberg [1], Averna-Bonanno [14], and Ricceri [6] or the critical point reduction method in [4]. As applications, we consider the existence of three solutions for second-order Hamiltonian systems satisfying generalized periodic boundary conditions or Sturm-Liouville boundary conditions and elliptic partial differential equations satisfying Dirichlet boundary value conditions. Meanwhile, we give some examples (see Examples 1–4 in

Section 4.1) to illustrate the effectiveness of our results and point out that our results improve and generalize the corresponding results in [1–3, 9, 10, 15, 22] via several remarks (see Remarks 18, 20, 22, 24, 29, and 33 in Section 4).

Before expressing our main results, for the convenience of the readers, we give a brief introduction to the index theory and the setting of the space; for some details, we refer to [23, 24] (or see Section 2).

Let X be a real infinite-dimensional separable Hilbert space with inner product $(\cdot, \cdot)_X$ and the corresponding norm $\|\cdot\|_X$. Let $A : D(A) \subset X \rightarrow X$ be an unbounded linear self-adjoint operator with $\sigma(A) = \sigma_d(A)$ bounded from below. This implies that there is an orthonormal basis $\{e_j\}_{j=1}^\infty$ of X and $\lambda_1 \leq \lambda_2 \leq \dots$ such that $Ae_j = \lambda_j e_j$, $D(A) = \{\sum_{j=1}^\infty c_j e_j \mid \sum_{j=1}^\infty \lambda_j^2 c_j^2 < \infty\}$. Meanwhile, let $Z \equiv D(|A|^{1/2}) = \{\sum_{j=1}^\infty c_j e_j \mid \sum_{j=1}^\infty |\lambda_j| c_j^2 < \infty\}$ with the norm $\|x\|_Z^2 = \sum_{j=1}^\infty (1 + |\lambda_j|) c_j^2$. For any $x = \sum_{j=1}^\infty c_j e_j \in Z$, $y = \sum_{j=1}^\infty d_j e_j \in Z$, we define a bilinear form

$$a(x, y) = \sum_{j=1}^\infty \lambda_j c_j d_j. \quad (1)$$

By $(Ax, y)_X = a(x, y)$ if $x \in D(A)$, $y \in Z$, we can see that $a(x, y)$ is the extension of $(Ax, y)_X$ on Z . Moreover, let $B \in L_s(X)$ be the usual space consisting of bounded symmetric operators in X . For any $B \in L_s(X)$, we define

$$\begin{aligned} \nu_A(B) &= \dim \ker(A - B), \\ i_A(B) &= \sum_{\lambda < 0} \nu_A(B + \lambda Id), \end{aligned} \quad (2)$$

which are introduced by Dong (see Definition 3.1.1 and Proposition 3.1.4 in [23] or Definition 7.1.1 in [24]).

Let us consider the following two nonlinear operator equations:

$$Ax - B_0 x = \nabla \Phi(x), \quad (3)$$

$$Ax - B_0 x + B_1 x = \eta \nabla \Phi(x) + \mu \nabla \Psi(x), \quad (4)$$

where $\eta, \mu \in [0, +\infty)$, $B_0, B_1 \in L_s(X)$ with $i_A(B_0) = 0$, $\nu_A(B_0) \neq 0$, and B_1 is positive definite for all $x \in X$, that is, there exists a positive constant $\underline{\lambda}$ such that $(B_1 x, x)_X \geq \underline{\lambda} \|x\|_X^2$ for all $x \in X$. Meanwhile, Φ and Ψ satisfy, respectively.

$(\Phi_0) \Phi \in C^1(Z, R)$ is weakly continuous with compact derivative, that is, $x_n \rightharpoonup x_0$ in Z implies that $\Phi(x_n) \rightarrow \Phi(x_0)$ and $\Phi'(x_n) \rightarrow \Phi'(x_0)$. Moreover, for every $x \in Z$, there exists $\nabla \Phi(x) \in X$ such that $\Phi'(x)y = (\nabla \Phi(x), y)_X$ for all $y \in Z$.

$(\Psi_0) \Psi \in C^1(Z, R)$ is weakly continuous with compact derivative. Moreover, for every $x \in Z$, there exists $\nabla \Psi(x) \in X$ such that $\Psi'(x)y = (\nabla \Psi(x), y)_X$ for all $y \in Z$.

Noticing that $i_A(B_0) = 0$, $\nu_A(B_0) \neq 0$, we have $\min \sigma(A - B_0) = 0$. Then, we can define another inner product

$$(x, y) = a(x, y) - (B_0 x, y)_X + (x, y)_X, \quad \forall x, y \in Z, \quad (5)$$

with the corresponding norm

$$\|x\| = (a(x, x) - (B_0 x, x)_X + \|x\|_X^2)^{1/2}, \quad \forall x \in Z. \quad (6)$$

Clearly, $\|\cdot\|$ is equivalent to $\|\cdot\|_Z$.
Let

$$\|x\|_{B_1} = (a(x, x) - (B_0 x, x)_X + \|x\|_X^2)^{1/2}, \quad \forall x \in Z. \quad (7)$$

Since B_1 is positive definite, there exist $\bar{\lambda} \geq \underline{\lambda} > 0$ such that $\bar{\lambda} \|x\|_X^2 \geq (B_1 x, x)_X \geq \underline{\lambda} \|x\|_X^2$ for all $x \in X$. So we have

$$\min \{1, \underline{\lambda}\} \|x\|^2 \leq \|x\|_{B_1}^2 \leq \max \{1, \bar{\lambda}\} \|x\|^2. \quad (8)$$

Observe that the embedding $Z \hookrightarrow X$ is compact. Further, assume that there exists a Banach space X_0 satisfying $Z \subset X_0 \subset X$ and the embeddings $Z \hookrightarrow X_0$ and $X_0 \hookrightarrow X$ are compact and continuous, respectively. Thus, there exists $k_1 > 0$ such that $\|x\|_{X_0} \leq k_1 \|x\|$ for all $x \in Z$. Let

$$k_2 = \sup_{x \in Z \setminus \{0\}} \frac{\|x\|_{X_0}}{\|x\|_{B_1}}. \quad (9)$$

From (8), we have

$$0 < k_2 \leq \tilde{k} = k_1 (\min \{1, \underline{\lambda}\})^{-1/2}. \quad (10)$$

Let X_1 be a nontrivial subspace of X . For $B_1, B_2 \in L_s(X)$, we write $B_1 \leq B_2$ with respect to X_1 if and only if $(B_1 x, x)_X \leq (B_2 x, x)_X$ for all $x \in X_1$; we write $B_1 < B_2$ w.r.t. X_1 if and only if $(B_1 x, x)_X < (B_2 x, x)_X$ for all $x \in X_1 \setminus \{0\}$. If $X_1 = X$, then we just write $B_1 \leq B_2$ or $B_1 < B_2$.

Next, we use the index $(i_A(B), \nu_A(B)) \in N \times N$ defined in [23, 24] (see Section 2) for all $B \in L_s(X)$ to reach our main results.

Theorem 1. Assume that Φ satisfies (Φ_0) and

(Φ_1) there exist $c_0 > 0$ and $B_2 \in \mathcal{L}_s(X)$ with $B_2 \geq 0$, $\nu_A(B_0 + B_2) = 0$ and $i_A(B_0 + B_2) = \nu_A(B_0)$, such that

$$|(\nabla \Phi(x), x_X)| \leq (B_2 x, x)_X + c_0 \|x\|, \quad (11)$$

for all $x \in X$

(Φ_2) there exists a positive definite $B_3 \in \mathcal{L}_s(X)$ such that $\Phi(x) - (1/2)(B_3 x, x)_X$ is convex

Then, problem (3) has at least one solution with saddle character in Z (i.e., the solution is a saddle point).

Assume in addition that

(Φ_3) there exist $\delta > 0$, $B_{01}, B_{02} \in \mathcal{L}_s(X)$ with $B_{02} > B_{01} > B_0$, $\nu_A(B_{0i}) \neq 0$ ($i = 1, 2$) and $i_A(B_{02}) = i_A(B_{01}) + \nu_A(B_{01})$, such that

$$\frac{1}{2}((B_{01} - B_0)x, x)_X \leq \Phi(x) \leq \frac{1}{2}((B_{02} - B_0)x, x)_X, \quad (12)$$

for all $\|x\| \leq \delta$. Then, problem (3) has at least three distinct solutions in Z .

Theorem 2. Assume that Φ satisfies (Φ_0) and (Φ_4) there exist $c_1, c_2 > 0$ and $\gamma_0 \in (0, 1)$ such that

$$|(\nabla\Phi(x), y)_X| \leq c_1 \|x\|_{X_0}^{\gamma_0} \cdot \|y\|_{X_0} + c_2 \|y\|_{X_0}, \quad (13)$$

for all $x, y \in X$

(Φ_5) there exists $c_3 > (c_1^2 k_1^2 / 2\varepsilon_2^2) > 0$ such that

$$\limsup_{\|x\| \rightarrow +\infty} \frac{\Phi(x)}{\|x\|^{2\gamma_0}} < -c_3, \quad (14)$$

for all $x \in \ker(A - B_0)$, where the constants k_1 and ε_2 are given by (9) and (49), respectively.

Then, problem (3) has at least one solution in Z . Further, assume that Φ satisfies (Φ_3) . Then, problem (3) has at least three distinct solutions in Z .

Theorem 3. Assume that Φ satisfies (Φ_0) and

$$(\Phi_6)\Phi(\theta) = 0$$

(Φ_7) there exist $c_3, c_4 \in \mathbb{R}^+$ and $\gamma \in (0, 2)$ such that $\Phi(x)$

$$\leq c_3 \|x\|_{X_0}^\gamma + c_4 \text{ for all } x \in Z$$

(Φ_8) there exist a positive constant $\tilde{c}$ and a $x_0 \in \ker(A - B_0)$ with $\tilde{c} < \tilde{k} \|x_0\|_X \sqrt{\lambda}$, such that

$$\sup_{x \in E} \Phi(x) < \frac{\tilde{c}^2}{\tilde{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0), \quad (15)$$

where $E = \{x \in Z \mid \|x\|_{X_0} \leq \tilde{c}\}$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive number r_0 with the following property: for each $\eta \in \Lambda$ and for every $\Psi : X \rightarrow \mathbb{R}$ satisfies (Ψ_0) , there exists $\rho > 0$ such that, for each $\mu \in [0, \rho]$, problem (4) has at least three distinct solutions whose norms in Z are less than r_0 .

Theorem 4. Assume that $x_0, c_3, c_4, \tilde{c}, \gamma$, and E are the same as in Theorem 3, Φ satisfies (Φ_0) , (Φ_6) , (Φ_7) , and (Φ_9)

$$\sup_{x \in E} \Phi(x) < \frac{\tilde{c}^2}{\tilde{c}^2 + \tilde{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0). \quad (16)$$

Then, for each

$$\eta \in \left(\frac{\tilde{\lambda} \|x_0\|_X^2}{2 \left(\Phi(x_0) - \sup_{x \in E} \Phi(x) \right)}, \frac{\tilde{c}^2}{2 \tilde{k}^2 \sup_{x \in E} \Phi(x)} \right), \quad (17)$$

problem (4) admits at least three distinct solutions provided $\mu = 0$.

Remark 5. For some examples of the scalar equations corresponding to Theorems 1–4, we will give in Section 4, respectively.

The paper will be organized as follows. In Section 2, we recall first a saddle point reduction theorem under rather

general assumptions in [4], a general two nonzero critical points theorem in [1], and two general three critical points theorems in [6, 14] as well as one proposition in [25]. Then, we recall some useful conclusions of index theory for linear self-adjoint operator equations from [23, 24]. Finally, we quote a lemma in [26], which verifies that (3) and (4) possess a variational construction in Z . In Section 3, we prove Theorems 1–4. In Section 4, we will investigate its applications to second-order Hamiltonian systems satisfying generalized periodic boundary conditions or Sturm-Liouville boundary conditions and second-order elliptic partial differential equations satisfying Dirichlet boundary value conditions. Meanwhile, we give some examples and some remarks to illustrate that the corresponding results of [1–3, 9, 10, 15, 22] are special cases of these results in a sense.

2. Preliminaries

In order to prove our main results, we recall first four lemmas and one proposition due to [1, 4, 6, 14, 25], respectively.

Lemma 6 (see [4], Theorem 2.3). Let Z_1, Z_2 be Hilbert spaces, and let $\psi \in C^1(Z_1 \times Z_2, \mathbb{R})$. Suppose that ψ satisfies $-D_1\psi(\cdot, x_2) : Z_1 \rightarrow Z_1$ is α -monotone for all $x_2 \in Z_2$, that is, there exists $\alpha > 0$ such that

$$\begin{aligned} & \langle -D_1\psi(x_{1,1}, x_2) + D_1\psi(x_{1,2}, x_2), x_{1,1} - x_{1,2} \rangle \\ & \geq \alpha \|x_{1,1} - x_{1,2}\|^2 \forall x_{1,1}, x_{1,2} \in Z_1. \end{aligned} \quad (18)$$

Then, there exists a map $\phi \in C(Z_2, Z_1)$ such that $\phi(x_2) = x_1(x_2)$ is the unique saddle point of $\psi(\cdot, x_2)$ for every $x_2 \in Z_2$. Moreover, the map $\phi : Z_2 \rightarrow Z_1$ defined by

$$\phi(x_2) = \psi(x_1(x_2), x_2) = \sup_{x_1 \in Z_1} \psi(x_1, x_2) \quad (19)$$

is continuously differentiable, and its derivative is given by

$$\phi'(x_2) = D_2\psi(x_1(x_2), x_2) \text{ for every } x_2 \in Z_2. \quad (20)$$

Lemma 7 (see [1], Theorem 4). Let Z be a reflexive real Banach space with a direct sum decomposition $Z = Z_1 \oplus Z_2$ and $\dim Z_1 < +\infty$. Let $I \in C^1(Z, \mathbb{R})$ with $I(\theta) = 0$, satisfying (PS) condition and assume that I is bounded below and $\inf_Z I < 0$. Assume also that there is $\delta > 0$ such that

$$\begin{cases} I(x) \leq 0, & \text{for } x \in Z_1, \|x\| \leq \delta, \\ I(x) \geq 0, & \text{for } x \in Z_2, \|x\| \leq \delta. \end{cases} \quad (21)$$

Then, I has at least two nonzero critical points.

Lemma 8 (see [6], Theorem 1). Let Z be a reflexive real Banach space, $\bar{I} \subseteq \mathbb{R}$ an interval, $J : Z \rightarrow \mathbb{R}$ a sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on Z^* , and J bounded on each bounded subset of Z , $\Phi : Z \rightarrow \mathbb{R}$ a C^1 functional with

compact derivative. Assume that

$$\lim_{\|x\| \rightarrow +\infty} (J(x) - \eta\Phi(x)) = +\infty, \quad (22)$$

for all $\eta \in \tilde{I}$ and that there exists $\beta \in R$ such that

$$\sup_{\eta \in \tilde{I}} \inf_{x \in Z} (J(x) + \eta(\beta - \Phi(x))) < \inf_{x \in Z} \sup_{\eta \in \tilde{I}} (J(x) + \eta(\beta - \Phi(x))). \quad (23)$$

Then, there exist a nonempty open set $\Lambda \in \tilde{I}$ and a positive number r_0 with the following property: for every $\eta \in \Lambda$ and every C^1 functional $\Psi : Z \rightarrow R$ with compact derivative, there exists $\rho > 0$ such that, for each $\mu \in [0, \rho]$, the equation

$$J'(x) - \eta\Phi'(x) - \mu\Psi'(x) = 0 \quad (24)$$

has at least three solutions whose norms in Z are less than r_0 .

Proposition 9 (see [25], Proposition 3.1). Let Z be a nonempty set and J, Φ two real functions on Z . Assume that there are $\tilde{r} > 0$ and $x_0, x_1 \in Z$ such that

$$\begin{aligned} J(x_1) &= \Phi(x_1) = 0, J(x_0) > \tilde{r}, \\ \sup_{x \in J^{-1}((-\infty, \tilde{r}])} \Phi(x) &< \tilde{r} \frac{\Phi(x_0)}{J(x_0)}. \end{aligned} \quad (25)$$

Then, for each $\beta \in R$ satisfying

$$\sup_{x \in J^{-1}((-\infty, \tilde{r}])} \Phi(x) < \beta < \tilde{r} \frac{\Phi(x_0)}{J(x_0)}, \quad (26)$$

one has

$$\sup_{\eta \geq 0} \inf_{x \in Z} (J(x) + \eta(\beta - \Phi(x))) < \inf_{x \in Z} \sup_{\eta \geq 0} (J(x) + \eta(\beta - \Phi(x))). \quad (27)$$

Lemma 10 (see [14], Theorem B). Let Z be a reflexive Banach space, $J : Z \rightarrow R$ a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on Z^* , $\Phi : Z \rightarrow R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put, for each $\tilde{r} > \inf_Z J$,

$$\begin{aligned} \varphi_1(\tilde{r}) &= \inf_{x \in J^{-1}\left(\left(-\infty, r^-\right)\right)} \frac{\sup_{J^{-1}((-\infty, \tilde{r}))} \omega \Phi - \Phi(x)}{\tilde{r} - J(x)}, \varphi_2(\tilde{r}) \\ &= \inf_{x \in J^{-1}((-\infty, \tilde{r}))} \sup_{y \in J^{-1}([\tilde{r}, +\infty))} \frac{\Phi(y) - \Phi(x)}{J(y) - J(x)}, \end{aligned} \quad (28)$$

where $J^1((\infty, \tilde{r}))^\omega$ is the closure of $J^{-1}((-\infty, \tilde{r}))$ in the weak topology, and assume that

(i) there is $\tilde{r} \in R$ such that $\tilde{r} > \inf_Z J$ and $\varphi_1(\tilde{r}) < \varphi_2(\tilde{r})$

(ii) $\lim_{\|x\| \rightarrow +\infty} (J(x) - \eta\Phi(x)) = +\infty$ for all $\eta \in (1/\varphi_2(\tilde{r}), 1/\varphi_1(\tilde{r}))$

Then, for each $\eta \in (1/\varphi_2(\tilde{r}), 1/\varphi_1(\tilde{r}))$, the equation $J'(x) - \eta\Phi'(x) = 0$ has at least three solutions in Z .

Next, we recall some definitions and propositions about index theory in [23, 24].

Definition 11 (see [24], page 108). For any $B \in L_s(X)$, we define

$$\psi_{a,B}(x, y) = a(x, y) - (Bx, y)_X, \forall x, y \in Z. \quad (29)$$

For any $x, y \in Z$ if $\psi_{a,B}(x, y) = 0$, we say that x and y are $\psi_{a,B}$ -orthogonal. For any two subspaces Z_1 and Z_2 of Z if $\psi_{a,B}(x, y) = 0$ for any $x \in Z_1, y \in Z_2$, we say that Z_1 and Z_2 are $\psi_{a,B}$ -orthogonal.

Proposition 12 (see [23], Proposition 2.1.1; [24], Proposition 7.2.1). For any $B \in L_s(X)$, the space Z has a $\psi_{a,B}$ -orthogonal decomposition

$$Z = Z_a^+(B) \oplus Z_a^0(B) \oplus Z_a^-(B), \quad (30)$$

such that $\psi_{a,B}$ is positive definite, null, and negative definite on $Z_a^+(B)$, $Z_a^0(B)$, and $Z_a^-(B)$, respectively. Moreover, $Z_a^0(B)$ and $Z_a^-(B)$ are finitely dimensional.

Definition 13 (see [23], Definition 2.1.2; [24], Definition 7.2.1). For any $B \in L_s(X)$, we define $\nu_a(B) = \dim Z_a^0(B)$, $i_a(B) = \dim Z_a^-(B)$.

Proposition 14 (1) (see [24], Proposition 7.2.2 (i)). For any $B \in L_s(X)$, we have

$$\nu_A(B) = \nu_a(B), i_A(B) = i_a(B), \ker(A - B) = Z_a^0(B). \quad (31)$$

(2) ([27], Proposition 2.6 (5)). For any $B_1, B_2 \in L_s(X)$, if $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$, and $(-\psi_{a,B_1}(x_1, x_1))^{1/2} + (\psi_{a,B_2}(x_2, x_2))^{1/2}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_2)$. In particular, for any $B_1 \in L_s(X)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^+(B_1)$ and $(-\psi_{a,B_1}(x_1, x_1))^{1/2} + (\psi_{a,B_1}(x_2, x_2))^{1/2}$ is also an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_1)$.

Finally, let us consider two functionals I and I_1 defined by

$$I(x) = \frac{1}{2}a(x, x) - \frac{1}{2}(B_0x, x)_X - \Phi(x), \quad (32)$$

$$I_1(x) = \frac{1}{2}a(x, x) - \frac{1}{2}(B_0x, x)_X + \frac{1}{2}(B_1x, x)_X - \eta\Phi(x) - \mu\Psi(x), \quad (33)$$

for every $x \in Z$ and $\eta, \mu \in [0, +\infty)$, respectively. Under assumption (Φ_0) and (Ψ_0) , from Theorem 2 in [28], we can verify

that $I, I_1 \in C^1(Z, R)$ are weakly lower semicontinuous on Z and I', I'_1 are weakly continuous with

$$I'(x)y = a(x, y) - (B_0x, y)_X - \Phi'(x)y, \quad (34)$$

$$I'_1(x)y = a(x, y) - (B_0x, y)_X + (B_1x, y)_X - \eta\Phi'(x)y - \mu\Psi'(x)y, \quad (35)$$

for every $x, y \in Z$ and $\eta, \mu \in [0, +\infty)$.

The following important lemma is an immediate conclusion of Lemma 6 in [26].

Lemma 15. Assume that (Φ_0) and (Ψ_0) hold. Then, a critical point of $I(x)$ (or $I_1(x)$) is a solution for problem (3) (or (4)).

3. Proofs of the Theorems

In this section, we give the proofs of Theorems 1–4, respectively. First, with the aid of the critical point reduction method in [4] and the three critical points theorem in [1], we prove Theorem 1. For Theorem 2, we use Theorem 1 in [28] and the three critical points theorem in [1] to give its proof. In the Proof of Theorem 3, our technical approach is based on the three critical points theorem of Ricceri [6]. Finally, we use Proposition 3.1 in [25] and Theorem B in [14] to prove Theorem 4.

Proof of Theorem 1. By assumption $i_A(B_0) = 0$, $\nu_A(B_0) \neq 0$, Definition 13, and Proposition 12, we know that $Z = Z_a^0(B_0) \oplus Z_a^+(B_0)$. Let $Z_1 = Z_a^0(B_0)$, $Z_2 = Z_a^+(B_0)$. The functional φ is defined by

$$\varphi(x_2) = \sup_{x_1 \in Z_1} \psi(x_1 + x_2) = \sup_{x_1 \in Z_1} I(x_1 + x_2), \quad x_2 \in Z_2. \quad (36)$$

From the convexity of $\Phi(x) - (1/2)(B_3x, x)_X$ and assumption (Φ_0) , we can obtain

$$(\nabla\Phi(x) - \nabla\Phi(y), x - y)_X \geq (B_3(x - y), x - y)_X, \quad x, y \in Z. \quad (37)$$

Since $B_3 \in \mathcal{L}_s(X)$ is positive definite, there exists $\varepsilon > 0$ such that $(B_3x, x)_X \geq \varepsilon\|x\|_X^2$ for all $x \in X$, which shows that

$$(\nabla\Phi(x) - \nabla\Phi(y), x - y)_X \geq \varepsilon\|x - y\|_X^2, \quad x, y \in Z. \quad (38)$$

Thus, for each fixed $x_2 \in Z_2$ and any $x_{1,1}, x_{1,2} \in Z_1$, we obtain

$$(\nabla\Phi(x_{1,1} + x_2) - \nabla\Phi(x_{1,2} + x_2), x_{1,1} - x_{1,2})_X \geq \varepsilon\|x_{1,1} - x_{1,2}\|_X^2. \quad (39)$$

Since $Z = Z_1 \oplus Z_2 = Z_a^0(B_0) \oplus Z_a^+(B_0)$, from (39), Defini-

tion 11, and Proposition 12, we have

$$\begin{aligned} & \left\langle -\psi'(x_{1,1} + x_2) - \left(-\psi'(x_{1,2} + x_2)\right), x_{1,1} - x_{1,2} \right\rangle \\ &= \left\langle -I'(x_{1,1} + x_2) - \left(-I'(x_{1,2} + x_2)\right), x_{1,1} - x_{1,2} \right\rangle \\ &= -a(x_{1,1} - x_{1,2}, x_{1,1} - x_{1,2}) + (B_0(x_{1,1} - x_{1,2}), x_{1,1} - x_{1,2})_X \\ &\quad + (\nabla\Phi(x_{1,1} + x_2) - \nabla\Phi(x_{1,2} + x_2), x_{1,1} - x_{1,2})_X \\ &= -\psi_{a,B_0}(x_{1,1} - x_{1,2}, x_{1,1} - x_{1,2}) \\ &\quad + (\nabla\Phi(x_{1,1} + x_2) - \nabla\Phi(x_{1,2} + x_2), x_{1,1} - x_{1,2})_X \\ &\geq \varepsilon\|x_{1,1} - x_{1,2}\|_X^2, \end{aligned} \quad (40)$$

via $\psi_{a,B_0}(x, y) = 0$ for all $x, y \in Z_a^0(B_0)$. Noticing that $Z_1 = Z_a^0(B_0)$ is finite-dimensional, we know that there exists $\varepsilon_0 > 0$ such that

$$\left\langle -\psi'(x_{1,1} + x_2) - \left(-\psi'(x_{1,2} + x_2)\right), x_{1,1} - x_{1,2} \right\rangle \geq \varepsilon\varepsilon_0\|x_{1,1} - x_{1,2}\|_X^2, \quad (41)$$

which implies that $\psi(x) = I(x)$ satisfies the $\varepsilon\varepsilon_0$ -monotonicity of Lemma 6. Thus, by Lemma 6, there exists a continuous mapping $\phi : Z_2 \rightarrow Z_1$ such that $\varphi(x_2) = \psi(\phi(x_2) + x_2)$ for all $x_2 \in Z_2$, where $\varphi : Z_2 \rightarrow R$ defined by (19) is continuous differentiable and $\varphi'(x_2) = \psi'(\phi(x_2) + x_2)|_{Z_2}$ for all $x_2 \in Z_2$. This means that if $x_2 \in Z_2$ is a critical point of φ , $\phi(x_2) + x_2$ is a critical point of ψ and I .

Again by assumption (Φ_1) , for every $x_2 \in Z_2$, we have

$$|\Phi(x_2) - \Phi(\theta)| = \left| \int_0^1 (\nabla\Phi(sx_2), x_2)_X ds \right| \leq \frac{1}{2}(B_2x_2, x_2)_X + c_0\|x_2\|. \quad (42)$$

Thus, from (19), we have

$$\begin{aligned} \varphi(x_2) &\geq \psi(x_2) = I(x_2) \geq \frac{1}{2}[a(x_2, x_2) - ((B_0 + B_2)x_2, x_2)_X] \\ &\quad - c_0\|x_2\| - \Phi(\theta) \\ &= \frac{1}{2}\psi_{a,B_0+B_2}(x_2, x_2) - c_0\|x_2\| - \Phi(\theta), \end{aligned} \quad (43)$$

as $x_1 = \theta$. Since $\nu_A(B_0 + B_2) = 0$ and $i_A(B_0 + B_2) = \nu_A(B_0)$, by (2) of Proposition 14, we can see that $(\psi_{a,B_0+B_2}(x_2, x_2))^{1/2}$ is an equivalent norm on Z for $x_2 \in Z_2 = Z_a^+(B_0) = Z_a^+(B_0 + B_2)$. Hence, there exists $\varepsilon_1 > 0$ such that

$$\varphi(x_2) \geq \frac{\varepsilon_1}{2}\|x_2\|^2 - c_0\|x_2\| - \Phi(\theta) \geq \frac{\varepsilon_1}{2}\|x_2\|^2 - c_0\|x_2\| - \Phi(\theta) \rightarrow +\infty, \quad (44)$$

as $\|x_2\| \rightarrow +\infty$. Noticing that I is weakly lower semicontinuous on Z , then there exists $x_{2,0} \in Z_2$ such that $\varphi(x_{2,0}) =$

$\min_{x_2 \in Z_2} \varphi(x_2)$, and hence, $x_{2,0} + \phi(x_{2,0})$ is a solution with saddle point character of problem (3) in Z .

Moreover, if condition (Φ_3) holds, then $Z = Z_a^-(B_{01}) \oplus Z_a^0(B_{01}) \oplus Z_a^0(B_{02}) \oplus Z_a^+(B_{02})$ via (2) of Proposition 14. Since $B_{01} > B_0$, from Definition 11, we have $\psi_{a,B_0}(x, x) > \psi_{a,B_{01}}(x, x)$ for all $x \in Z \setminus \{\theta\}$ and $\psi_{a,B_0}(x, x) > 0$ for all $x \in Z_a^0(B_{01}) \setminus \{\theta\}$ with $Z_a^0(B_{01}) \subset Z_a^+(B_0)$. Let $Z_{2,1} = (Z_a^-(B_{01}) \oplus Z_a^0(B_{01})) \cap Z_a^+(B_0)$ and $Z_{2,2}$ be the orthogonal complement of $Z_{2,1}$ in Z_2 . We claim that $\phi(\theta) = \theta$. Indeed, (Φ_3) means that $\Phi(\theta) = 0$ and $\nabla\Phi(\theta) = \theta$. By (Φ_2) , (39), and $\phi(\theta) \in Z_1 = Z_a^0(B_0)$, we have

$$\begin{aligned} 0 &= \langle \psi'(\phi(\theta)), -\phi(\theta) \rangle = \langle I'(\phi(\theta)), -\phi(\theta) \rangle = \langle -\nabla\Phi(\phi(\theta)), -\phi(\theta) \rangle_X \\ &= \langle \nabla\Phi(\theta) - \nabla\Phi(\phi(\theta)), -\phi(\theta) \rangle_X \geq \varepsilon \|\phi(\theta)\|_X^2 \geq 0, \end{aligned} \quad (45)$$

which implies that $\phi(\theta) = \theta$. By the continuity of ϕ , we know that there exists $0 < r < \delta/2$ such that $\|\phi(x_2)\| < \delta/2$ as $\|x_2\| \leq r$. Thus, from (12) of (Φ_3) and Definition 11, we obtain

$$\begin{aligned} \varphi(x_{2,1}) &= \psi(x_{2,1} + \phi(x_{2,1})) = I(x_{2,1} + \phi(x_{2,1})) \\ &\leq \frac{1}{2} [a(x_{2,1} + \phi(x_{2,1}), x_{2,1} + \phi(x_{2,1})) \\ &\quad - (B_{01}(x_{2,1} + \phi(x_{2,1})), x_{2,1} + \phi(x_{2,1}))_X] \\ &= \frac{1}{2} \psi_{a,B_{01}}(x_{2,1} + \phi(x_{2,1}), x_{2,1} + \phi(x_{2,1})) \leq 0, \end{aligned} \quad (46)$$

for all $x_{2,1} \in Z_{2,1}$ with $\|x_{2,1}\| \leq r$ via $B_{01} > B_0$ and $x_{2,1} + \phi(x_{2,1}) \in (Z_a^-(B_{01}) \oplus Z_a^0(B_{01})) \cup Z^0(B_0)$, and

$$\begin{aligned} \varphi(x_{2,2}) &\geq \psi(x_{2,2}) = I(x_{2,2}) \\ &\geq \frac{1}{2} [a(x_{2,2}, x_{2,2}) - (B_{02}x_{2,2}, x_{2,2})_X] = \frac{1}{2} \psi_{a,B_{02}}(x_{2,2}, x_{2,2}) \geq 0, \end{aligned} \quad (47)$$

for all $x_{2,2} \in Z_{2,2}$ with $\|x_{2,2}\| \leq r$ via $Z_{2,2} = Z_a^+(B_{01}) \cap Z_a^+(B_0)$ and $Z_a^+(B_{01}) = Z_a^+(B_{02}) \oplus Z_a^0(B_{02})$.

Since I is weakly lower semicontinuous on Z , so is φ on Z_2 . From the weakly lower semicontinuity and coerciveness of φ , we can see that φ satisfies (PS) condition and is bounded below on Z_2 .

If $\inf \{\varphi(x_2) : x_2 \in Z_2\} = 0$, then all $x_{2,1} \in Z_{2,1}$ with $\|x_{2,1}\| \leq r$ are minima of φ , which shows that φ has infinite critical points. If $\inf \{\varphi(x_2) : x_2 \in Z_2\} < 0$, then φ has at least two non-zero critical points via Lemma 7. Consequently, problem (3) has at least two nontrivial solutions in Z . Further, since $\Phi(\theta) = 0$, we know that problem (3) has trivial solution θ . Hence, problem (3) has at least three distinct solutions in Z . The proof is complete.

Proof of Theorem 2. By assumption $i_A(B_0) = 0$, $v_A(B_0) \neq 0$, Definition 13, and Proposition 12, we have $Z = Z_a^0(B_0) \oplus Z_a^+(B_0)$. Let $Z_1 = Z_a^0(B_0)$, $Z_2 = Z_a^+(B_0)$. For $x \in Z$, we have $x =$

$x_1 + x_2$ with $x_1 \in Z_1$ and $x_2 \in Z_2$. It follows from (Φ_4) that

$$\begin{aligned} |\Phi(x) - \Phi(x_1)| &= \left| \int_0^1 (\nabla\Phi(x_1 + sx_2), x_2)_X ds \right| \\ &\leq c_1 \|x_2\|_{X_0} \int_0^1 \|x_1 + sx_2\|_{X_0}^{\gamma_0} ds + c_2 \|x_2\|_{X_0} \\ &\leq c_1 \|x_2\|_{X_0} \|x_1\|_{X_0}^{\gamma_0} + \frac{c_1}{\gamma_0 + 1} \|x_2\|_{X_0}^{\gamma_0+1} + c_2 \|x_2\|_{X_0} \\ &\leq \frac{c_1 k_1 \varepsilon}{2} \|x_2\|^2 + \frac{c_1 k_1}{2\varepsilon} \|x_1\|^{2\gamma_0} + \frac{c_1 k_1}{\gamma_0 + 1} \|x_2\|^{\gamma_0+1} \\ &\quad + c_2 k_1 \|x_2\|, \end{aligned} \quad (48)$$

where ε is any positive real number. Since $x_2 \in Z_2 = Z_a^+(B_0)$, by (2) of Proposition 14, we know that $(\psi_{a,B_0}(x_2, x_2))^{1/2}$ is an equivalent norm on Z for $x_2 \in Z_2$. Hence, there exists $\varepsilon_2 > 0$ such that

$$(\psi_{a,B_0}(x_2, x_2))^{1/2} \geq \varepsilon_2 \|x_2\|, \quad (49)$$

for all $x_2 \in Z_2$, and

$$\begin{aligned} I(x) &= \frac{1}{2} \psi_{a,B_0}(x_2, x_2) - [\Phi(x) - \Phi(x_1)] - \Phi(x_1) \\ &\geq \frac{\varepsilon_2^2}{2} \|x_2\|^2 - |\Phi(x) - \Phi(x_1)| - \Phi(x_1) \\ &\geq \frac{\varepsilon_2^2 - c_1 k_1 \varepsilon}{2} \|x_2\|^2 - \|x_1\|^{2\gamma_0} \left[\frac{c_1 k_1}{2\varepsilon} + \frac{\Phi(x_1)}{\|x_1\|^{2\gamma_0}} \right] \\ &\quad - \frac{c_1 k_1}{\gamma_0 + 1} \|x_2\|^{\gamma_0+1} - c_2 k_1 \|x_2\|. \end{aligned} \quad (50)$$

By $c_3 > c_1^2 k_1^2 / \varepsilon_2^2 > 0$, we can choose an $\varepsilon_0 \in (0, 1)$ such that $c_3 > (1 + \varepsilon_0)(c_1^2 k_1^2 / \varepsilon_2^2)$. Taking $\varepsilon = \varepsilon_2^2 / (c_1 k_1 (1 + \varepsilon_0))$, by (Φ_5) , we have $\varepsilon_2^2 - c_1 k_1 \varepsilon = (1 - (1/(1 + \varepsilon_0)))\varepsilon_2^2 > 0$ and

$$\frac{c_1 k_1}{2\varepsilon} + \frac{\Phi(x_1)}{\|x_1\|^{2\gamma_0}} < \frac{c_1 k_1}{2\varepsilon} - c_3 < -\frac{1 + \varepsilon_0}{2} \cdot \frac{c_1^2 k_1^2}{\varepsilon_2^2} < 0, \quad (51)$$

for $x_1 \in Z_1 = \ker(A - B_0)$. As $\|x\| \rightarrow +\infty$ if and only if $\|x_1\| + \|x_2\| \rightarrow +\infty$, from the above equality and (Φ_5) , we obtain that

$$I(x) \rightarrow +\infty, \quad (52)$$

as $\|x\| \rightarrow +\infty$. Noticing that I is weakly lower semicontinuous on Z , by Theorem 1 in [28], we know that problem (3) has one solution in Z .

Moreover, if condition (Φ_3) holds, then $Z = Z_a^-(B_{01}) \oplus Z_a^0(B_{01}) \oplus Z_a^0(B_{02}) \oplus Z_a^+(B_{02})$ via (2) of Proposition 14. Let $\bar{Z}_1 = Z_a^-(B_{01}) \oplus Z_a^0(B_{01})$ and $\bar{Z}_2 = Z_a^0(B_{02}) \oplus Z_a^+(B_{02})$. Then, we have $Z = \bar{Z}_1 \oplus \bar{Z}_2$ and $\dim \bar{Z}_1 < +\infty$. From (12) of (Φ_3)

and Definition 11, we obtain

$$I(x_1) \leq \frac{1}{2} [a(x_1, x_1) - (B_{01}x_1, x_1)_X] = \frac{1}{2} \Psi_{a, B_{01}}(x_1, x_1) \leq 0, \quad (53)$$

for all $x_1 \in \bar{Z}_1$ with $\|x_1\| \leq \delta$, and

$$I(x_2) \geq \frac{1}{2} [a(x_2, x_2) - (B_{02}x_2, x_2)_X] = \frac{1}{2} \Psi_{a, B_{02}}(x_2, x_2) \geq 0, \quad (54)$$

for all $x_2 \in \bar{Z}_2$ with $\|x_2\| \leq \delta$.

By the weakly lower semicontinuity and coerciveness of I , we can see that I satisfies (PS) condition and is bounded below on Z . Noticing that $I(x) = -\Phi(x)$ for all $x \in \ker(A - B_0)$, from (12) of (Φ_3) , we have $I(x) < 0$ for all $x \in \ker(A - B_0)$ with $0 < \|x\| < \delta$ and $I(\theta) = 0$, which implies that $\inf_Z I < 0$. Thus, I has at least two nonzero critical points via Lemma 7. Thanks to θ is also a trivial solution of problem (3), we know that problem (3) has at least three distinct solutions in Z . The proof is complete.

Proof of Theorem 3. Let

$$J(x) = \frac{1}{2} a(x, x) - \frac{1}{2} (B_0 x, x)_X + \frac{1}{2} (B_1 x, x)_X = \frac{\|x\|_{B_1}^2}{2}, \quad (55)$$

for each $x \in Z$. From (33) and (35), we can see that a solution of $J'(x) - \eta \Phi'(x) - \mu \Psi'(x) = 0$ must be a critical point of $I_1(x)$, which means that problem (4) also has a solution via Lemma 15.

Obviously, J is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. From the conditions (Φ_0) and (Ψ_0) , we know that Φ and Ψ are continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, J is bounded on each bounded subset of Z via (8).

Next, we prove that J admits a continuous inverse on Z . In fact, from (8) and (35), we have

$$\langle J'(x) - J'(y), x - y \rangle = \|x - y\|_{B_1}^2 \geq \min \{1, \underline{\lambda}\} \|x - y\|^2, \quad (56)$$

for all $x, y \in Z$, which shows that J' is uniformly monotone on Z . Moreover, using standard arguments ensure that J' also turns out to be coercive and hemicontinuous in Z . Applying Theorem 26.A of [29], we know that J admits a continuous inverse on Z .

Thanks to (Φ_7) and (8), for each $\eta > 0$, we have

$$\lim_{\|x\| \rightarrow +\infty} (J(x) - \eta \Phi(x)) \geq \lim_{\|x\| \rightarrow +\infty} \left(\frac{\|x\|_{B_1}^2}{2} - \eta (c_3 \|x\|_{X_0}^\gamma + c_4) \right) = +\infty, \quad (57)$$

via $\gamma \in (0, 2)$ and $\|x\|_{X_0} \leq k_1 \|x\|$. This implies that the assumption (22) of Lemma 8 holds.

For $x_0 \in \ker(A - B_0)$ in (Φ_8) , by the definition of $\|\cdot\|_{B_1}$, we have

$$\sqrt{\underline{\lambda}} \|x_0\|_X \geq \|x_0\|_{B_1} = ((B_1 x_0, x_0)_X)^{1/2} \geq \sqrt{\underline{\lambda}} \|x_0\|_X. \quad (58)$$

From $\tilde{c} < \tilde{k} \|x_0\|_X \sqrt{\underline{\lambda}}$, we see that

$$\|x_0\|_{B_1} > \frac{\tilde{c}}{\tilde{k}}. \quad (59)$$

Hence, thanks to (15) of (Φ_8) and (58), we have

$$\sup_{x \in E} \Phi(x) < \frac{\tilde{c}^2}{\tilde{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0) \leq \frac{\tilde{c}^2}{\tilde{k}^2 \|x_0\|_{B_1}^2} \Phi(x_0). \quad (60)$$

Now, taking $\tilde{r} = (1/2)(\tilde{c}/\tilde{k})^2$. By (59), we have

$$J(x_0) = \frac{\|x_0\|_{B_1}^2}{2} > \tilde{r} > 0 = J(\theta). \quad (61)$$

From (9) and (10), we obtain $\|x\|_{X_0} \leq \tilde{k} \|x\|_{B_1}$ for all $x \in Z$. Hence, for each $x \in Z$ such that

$$J(x) = \frac{\|x\|_{B_1}^2}{2} \leq \tilde{r}, \quad (62)$$

we have $(1/2)(\|x\|_{X_0}/k)^2 \leq \tilde{r} = (1/2)(\tilde{c}/k)^2$, which implies that

$$\|x\|_{X_0} \leq \tilde{c}. \quad (63)$$

It follows from (60) and (63) that

$$\sup_{x \in J^{-1}((-\infty, \tilde{r}))} \Phi(x) \leq \sup_{x \in E} \Phi(x) < \frac{\tilde{c}^2}{\tilde{k}^2 \|x_0\|_{B_1}^2} \Phi(x_0) = \tilde{r} \frac{\Phi(x_0)}{J(x_0)}. \quad (64)$$

Put β satisfying

$$\sup_{x \in J^{-1}((-\infty, \tilde{r}))} \Phi(x) < \beta < \tilde{r} \frac{\Phi(x_0)}{J(x_0)}, \quad (65)$$

and $x_1 = \theta$, by (61), (64), and Proposition 9, we have

$$\sup_{\eta \geq 0} \inf_{x \in Z} (J(x) + \eta(\beta - \Phi(x))) < \inf_{x \in Z} \sup_{\eta \geq 0} (J(x) + \eta(\beta - \Phi(x))), \quad (66)$$

which implies that the assumption (23) of Lemma 8 holds as $\tilde{I} = [0, +\infty)$.

Finally, set $\tilde{I} = [0, +\infty)$, we can see that all the assumptions of Lemma 8 are satisfied. Hence, our conclusion follows from Lemma 8.

Proof of Theorem 4. In order to use Lemma 10 to prove Theorem 4, we consider the functional I_1 defined by (33) as $\mu = 0$. From the Proof of Theorem 3, it is not difficult to see that we only need to verify the validity of (i) in Lemma 10.

From the Proof of Theorem 3, we know that $J(\theta) = \Phi(\theta) = 0$, $\tilde{r} = (1/2)(\tilde{c}/k)^2$, and

$$J(x_0) = \frac{\|x_0\|_{B_1}^2}{2} > \tilde{r} > 0 = \inf_Z J. \quad (67)$$

This shows that $\theta \in J^{-1}((-\infty, \tilde{r}))$ and $x_0 \in J^{-1}([\tilde{r}, +\infty))$. From (63) and (Φ_g) , we have

$$\sup_{x \in J^{-1}((-\infty, \tilde{r}])} \Phi(x) \leq \sup_{x \in E} \Phi(x) < \frac{\tilde{c}^2}{\tilde{c}^2 + \bar{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0) < \Phi(x_0), \quad (68)$$

which implies that

$$\Phi(x_0) - \Phi(x) > 0, \quad (69)$$

for each $x \in J^{-1}((-\infty, \tilde{r}]) \subset Z$. Consequently, it follows from (Φ_g) , (58) and (69) that

$$\begin{aligned} \varphi_1(\tilde{r}) &= \inf_{x \in J^{-1}((-\infty, \tilde{r}))} \frac{\sup_{J^{-1}((-\infty, \tilde{r}])} \Phi - \Phi(x)}{\tilde{r} - J(x)} \leq \frac{\sup_{J^{-1}((-\infty, \tilde{r}])} \Phi - \Phi(\theta)}{\tilde{r} - J(\theta)} \\ &= \frac{\sup_{J^{-1}((-\infty, \tilde{r}])} \Phi}{\tilde{r}} \leq \frac{\sup_{x \in E} \Phi(x)}{\tilde{r}} = \frac{2\tilde{k}^2 \sup_{x \in E} \Phi(x)}{\tilde{c}^2} \\ &< \frac{2\tilde{k}^2}{\tilde{c}^2 + \bar{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0), \end{aligned} \quad (70)$$

$$\begin{aligned} \varphi_2(\tilde{r}) &\geq \inf_{x \in J^{-1}((-\infty, \tilde{r}))} \frac{\Phi(x_0) - \Phi(x)}{J(x_0) - J(x)} \geq \frac{\inf_{x \in J^{-1}((-\infty, \tilde{r}))} \Phi(x_0) - \Phi(x)}{J(x_0)} \\ &\geq \frac{2}{\|x_0\|_{B_1}^2} \left(\Phi(x_0) - \sup_{x \in J^{-1}((-\infty, \tilde{r}))} \Phi(x) \right) \\ &\geq \frac{2}{\bar{\lambda} \|x_0\|_X^2} \left(\Phi(x_0) - \sup_{x \in E} \Phi(x) \right) \\ &> \frac{2}{\bar{\lambda} \|x_0\|_X^2} \left(\Phi(x_0) - \frac{\tilde{c}^2}{\tilde{c}^2 + \bar{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0) \right) \\ &\geq \frac{2\tilde{k}^2}{\tilde{c}^2 + \bar{\lambda} \tilde{k}^2 \|x_0\|_X^2} \Phi(x_0). \end{aligned} \quad (71)$$

From (70) and (71), we obtain

$$\varphi_2(\tilde{r}) > \varphi_1(\tilde{r}),$$

$$\eta \in \left(\frac{\bar{\lambda} \|x_0\|_X^2}{2 \left(\Phi(x_0) - \sup_{x \in E} \Phi(x) \right)}, \frac{\tilde{c}^2}{2\tilde{k}^2 \sup_{x \in E} \Phi(x)} \right) \subset \left(\frac{1}{\varphi_2(\tilde{r})}, \frac{1}{\varphi_1(\tilde{r})} \right). \quad (72)$$

Therefore, (i) of Lemma 10 is satisfied. The proof is complete.

4. Applications to Second-Order Hamiltonian Systems and Elliptic Partial Differential Equations

In this section, as three examples of applications of the main results, we investigate second-order Hamiltonian systems satisfying generalized periodic boundary value conditions, second-order Hamiltonian systems satisfying Sturm-Liouville boundary value conditions, and elliptic partial differential equations satisfying Dirichlet boundary value conditions. For systematic researches of Hamiltonian systems, we refer to the excellent books [28, 30, 31]. Meanwhile, for researches of elliptic partial differential equations, we refer to [32–34].

4.1. Second-Order Hamiltonian Systems Satisfying Generalized Periodic Boundary Value Conditions. As a first example, we discuss the solvability of the following two generalized periodic boundary value problems

$$\begin{cases} -\ddot{x} - \bar{B}_0(t)x = \nabla_x V(t, x), & \text{a.e. } t \in [0, 1], \\ x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \end{cases} \quad (73)$$

$$\begin{cases} -\ddot{x} - \bar{B}_0(t)x + \bar{B}_1(t)x = \eta \nabla_x V(t, x) + \mu \nabla_x F(t, x), & \text{a.e. } t \in [0, 1], \\ x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \end{cases} \quad (74)$$

where $\eta, \mu \in [0, +\infty)$, $\bar{B}_0(t), \bar{B}_1(t) \in L^\infty([0, 1], L_s(R^n)) = \{B(t) = (b_{jk})_{n \times n} | b_{jk}(t) = b_{kj}(t), t \in [0, 1], b_{jk}(t) \in L^\infty([0, 1])\}$, $M, N \in GL(n) = \{A = (a_{jk})_{n \times n} | a_{jk} \in \text{Rand} \det(A) \neq 0\}$, and $MN^T = I_n$, where I_n is the unit matrix of order n , and $\nabla_x V(t, x)$ and $\nabla_x F(t, x)$ denote the gradient of $V(t, x)$ and $F(t, x)$ w.r.t. $x \in R^n$, respectively.

Let $X = L^2([0, 1], R^n)$. We define $A_1 : D(A_1) \rightarrow X$ by $(A_1 x)(t) = -\ddot{x}(t)$ where $D(A_1) = \{x \in H^2([0, 1], R^n) | x \text{ satisfies } x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0)\}$. Set $(B_i x)(t) = \bar{B}_i(t)x(t)$ ($i = 0, 1$). Then, $D(B_i) = X$. From Corollary 1.21 in [26], we see that A_1 is self-adjoint and $\sigma(A_1) = \sigma_d(A_1) \subset [0, +\infty)$. Define $i_M(\bar{B}_0) = i_{A_1}(B_0)$, $v_M(\bar{B}_0) = v_{A_1}(B_0)$, that is, $v_M(\bar{B}_0)$ is the dimension of the solution subspace of (73) as $V(t, x) \equiv 0$ and $i_M(\bar{B}_0) = \sum_{\lambda < 0} v_M(\bar{B}_0 + \lambda I_n)$.

Remark 16 (see [23], Example 2.4.3; [24], Remark 7.1.3). Let $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ be the eigenvalues of a constant $n \times n$ symmetric matrix B . For $\zeta \in R \setminus \{0\}$ with $\xi_0 = \arccos 2/(\zeta^{-1} + \zeta)$, we have

$$\begin{aligned} i_{\zeta I_n}(B) &= \sum_{k=1}^n \# \left\{ j \in N : (2j\pi + \xi_0)^2 < \alpha_k \right\} \\ &\quad + \sum_{k=1}^n \# \left\{ j \in N : (2\pi - \xi_0 + 2j\pi)^2 < \alpha_k \right\}, \end{aligned} \quad (75)$$

$$\begin{aligned} \nu_{\zeta_{I_n}}(B) &= \sum_{k=1}^n \# \left\{ j \in N : (2j\pi + \xi_0)^2 = \alpha_k \right\} \\ &+ \sum_{k=1}^n \# \left\{ j \in N : (2\pi - \xi_0 + 2j\pi)^2 = \alpha_k \right\}, \end{aligned} \quad (76)$$

where $\#S$ denotes the number of elements in a set S . In particular, formulae (75) and (76) when $\zeta = 1$ were given first by Mawhin and Willem in [28].

Now, we assume that $\bar{B}_1(t)$ is positive definite for all $x \in R^n$, that is, there exists a positive constant $\underline{\lambda}$ such that $(\bar{B}_1(t)x, x) \geq \underline{\lambda}|x|^2$ for all $x \in R^n$, and $V, F : [0, 1] \times R^n \rightarrow R$ satisfy the following condition:

$(H_0)V(t, x)$ and $F(t, x)$ are measurable in t for every $x \in R^n$ and continuously differentiable in x for a.e. $t \in [0, 1]$.

Further, assume that $i_M(\bar{B}_0) = 0$, $\nu_M(\bar{B}_0) \neq 0$. Meanwhile, set $Z_1 = \{x \in H^1([0, 1], R^n) \mid x(1) = Mx(0)\}$. By Corollary 1.21 in [26] again, we know that $Z_1 = D(|A_1|^{1/2})$ and the norm

$$\|x\|_{Z_1} = \left(\int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2] dt \right)^{1/2}, \quad (77)$$

for each $x \in Z_1$. Noticing that $i_M(\bar{B}_0) = 0$, $\nu_M(\bar{B}_0) \neq 0$, we can define another equivalent norm

$$\|x\| = \left(\int_0^1 [|\dot{x}(t)|^2 - (\bar{B}_0(t)x(t), x(t)) + |x(t)|^2] dt \right)^{1/2}, \quad (78)$$

for each $x \in Z_1$.

Let

$$\begin{aligned} \|x\|_{\bar{B}_1} &= \left(\int_0^1 [|\dot{x}(t)|^2 - (\bar{B}_0(t)x(t), x(t)) \right. \\ &\quad \left. + (\bar{B}_1(t)x(t), x(t))] dt \right)^{1/2}, \quad \forall x \in Z_1. \end{aligned} \quad (79)$$

Since $\bar{B}_1(t) \in L^\infty([0, 1], L_s(R^n))$ is positive definite, there exist $\bar{\lambda} > \underline{\lambda} > 0$ such that $\bar{\lambda}\|x\|_{L^2}^2 \geq \int_0^1 (\bar{B}_1(t)x(t), x(t)) dt \geq \underline{\lambda}\|x\|_{L^2}^2$ for all $x \in X$. So, we have

$$\min \{1, \underline{\lambda}\} \|x\|^2 \leq \|x\|_{\bar{B}_1}^2 \leq \max \{1, \bar{\lambda}\} \|x\|^2. \quad (80)$$

Observe that the embedding $Z_1 \hookrightarrow L^2 = X$ is compact. Let $\|\cdot\|_\infty$ be the norm of $L^\infty([0, 1], R^n)$. Then, there is a constant $\delta_0 > 0$ such that

$$|x| \leq \|x\|_\infty \leq \delta_0 \|x\|, \quad (81)$$

for any $x \in Z_1$. Let

$$k_2 = \sup_{x \in Z_1 \setminus \{0\}} \frac{\|x\|_\infty}{\|x\|_{\bar{B}_1}}. \quad (82)$$

From (80) and (81), we have

$$0 < k_2 \leq \tilde{k} = \delta_0 (\min \{1, \underline{\lambda}\})^{-1/2}. \quad (83)$$

For any $\bar{B}_1(t), \bar{B}_2(t) \in L^\infty([0, 1], L_s(R^n))$, we write $\bar{B}_1 \leq \bar{B}_2$ if $\bar{B}_1(t) \leq \bar{B}_2(t)$ for a.e. $t \in [0, 1]$ and define $\bar{B}_1 < \bar{B}_2$ if $\bar{B}_1(t) < \bar{B}_2(t)$ a.e. $t \in [0, 1]$. From Theorems 1–4, we have the following four results hold.

Theorem 17. Assume that $V(t, x)$ satisfies (H_0) and

(H_1) there exist $g \in L^1([0, 1], R^+)$ and $\bar{B}_2 \in L^\infty([0, 1], L_s(R^n))$ with $\bar{B}_2 \geq 0$, $\nu_M(\bar{B}_0 + \bar{B}_2) = 0$ and $i_M(\bar{B}_0 + \bar{B}_2) = \nu_M(\bar{B}_0)$ such that

$$|\nabla_x V(t, x), x| \leq (\bar{B}_2(t)x, x) + g(t)|x|, \quad (84)$$

for all $x \in R^n$ and a.e. $t \in [0, 1]$

(H_2) there exists a positive definite $\bar{B}_3 \in L^\infty([0, 1], L_s(R^n))$ such that $V(t, x) - 1/2(\bar{B}_3(t)x, x)$ is convex in x for a.e. $t \in [0, 1]$

Then, problem (73) has at least one solution with saddle character in Z_1 .

Assume in addition that

(H_3) there exist $\delta_1 > 0$, $\bar{B}_{01}, \bar{B}_{02} \in L^\infty([0, 1], L_s(R^n))$ such that $\bar{B}_{02} > \bar{B}_{01} > \bar{B}_0$ and $\nu_M(\bar{B}_{0i}) \neq 0 (i = 1, 2)$, $i_M(\bar{B}_{02}) = i_M(\bar{B}_{01}) + \nu_M(\bar{B}_{01})$, and for all $|x| \leq \delta_1$

$$\frac{1}{2}((\bar{B}_{01}(t) - \bar{B}_0(t))x, x) \leq V(t, x) \leq \frac{1}{2}((\bar{B}_{02}(t) - \bar{B}_0(t))x, x), \quad (85)$$

for a.e. $t \in [0, 1]$. Then, problem (73) has at least three distinct solutions in Z_1 .

Proof. Set

$$\Phi(x) = \int_0^1 V(t, x) dt, \forall x \in Z_1. \quad (86)$$

Then, $\Phi \in C^1(Z_1, R)$ is weakly continuous with compact derivative, and for every $x \in Z_1$,

$$\Phi'(x)y = \int_0^1 (\nabla_x V(t, x), y) dt, \forall y \in Z_1, \quad (87)$$

because of (H_0) and (84). Hence, (Φ_0) holds.

Thanks to (81) and (84) of (H_1) , we have

$$\begin{aligned} |(\nabla \Phi(x), x)_{L^2}| &\leq \int_0^1 (\bar{B}_2(t)x(t), x(t)) dt + \int_0^1 g(t)|x(t)| dt \\ &\leq (B_2x, x)_{L^2} + \|x\|_\infty \int_0^1 g(t) dt \\ &\leq (B_2x, x)_{L^2} + c_0 \|x\|, \end{aligned} \quad (88)$$

for all $x \in L^2$, where $c_0 = \delta_0 \int_0^1 g(t) dt$. This shows that (Φ_1) holds.

Since $\bar{B}_3(t)$ is positive definite, that is, there exists a positive constant ε_0 such that $(\bar{B}_3(t)x, x) \geq \varepsilon_0 \|x\|^2$ for all $x \in R^n$, which implies that $(B_3x, x)_{L^2} \geq \varepsilon_0 \|x\|_{L^2}^2$ for all $x \in L^2$. Thus, (Φ_2) follows from (H_2) .

Clearly, $\bar{B}_{02} > \bar{B}_{01} > \bar{B}_0 \implies B_{02} > B_{01} > B_0$. Noticing that $|x| \leq \|x\|_\infty \leq \delta_0 \|x\|$ for any $x \in Z_1$, we have $|x| \leq \delta_1$ as $\|x\| \leq \delta_1 / \delta_0 = \delta$. So (12) follows from (85), which shows that (Φ_3) holds. Now, we can apply Theorem 1 to obtain Theorem 17.

Remark 18. In particular, set $\bar{B}_0(t) \equiv 0$, $\bar{B}_2(t) = f(t)I_n$, $\bar{B}_3(t) = h(t)I_n$, $\bar{B}_{01}(t) = m^2(2\pi)^2 I_n$, $\bar{B}_{02}(t) = (m+1)^2(2\pi)^2 I_n$, $m \in \{1, 2, \dots\}$ and $M = I_n$, where $f(t), h(t) \in L^1([0, 1], R^+)$ with $0 < f(t) < 4\pi^2$ and $\inf_{t \in [0, 1]} h(t) > 0$. Then, $Z_1 = \{x \in H^1([0, 1], R^n) \mid x(1) = x(0)\}$, $\sigma(A_1) = \{(2m\pi)^2 \mid m \in N\}$, and $\ker(A_1 - \bar{B}_0) = R^n$. Hence, the following problem

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0 \quad (89)$$

has a similar conclusion of Theorem 17. In addition, for the interval $[0, T]$ considered in second-order Hamiltonian systems satisfying periodic boundary value conditions. As $T = 1$, the conclusion of Theorem 2.2 in [3] holds if we replace $0 < f(t) < 4\pi^2$ and $\inf_{t \in [0, 1]} h(t) > 0$ with $0 < \int_0^1 f(t) dt < 12$

and $\int_0^1 h(t) dt > 0$, respectively. Clearly, $\inf_{t \in [0, 1]} h(t) > 0 \implies \int_0^1 h(t) dt > 0$. But for $f \in L^1([0, 1], R^+)$, we can see that $\int_0^1 f(t) dt < 12 \implies 0 < f(t) < 4\pi^2$ and $0 < f(t) < 4\pi^2 \implies \int_0^1 f(t) dt < 12$. Indeed, if

$$f(t) = \begin{cases} 8\pi^2, & x \in \left[0, \frac{1}{4\pi^2}\right], \\ 1 - \frac{1}{4\pi^2}, & x \in \left(\frac{1}{4\pi^2}, 1\right], \end{cases} \quad (90)$$

then $\int_0^1 f(t) dt = 3$ and $f(t) > 4\pi^2$ for $x \in [0, 1/4\pi^2]$; if $12 < f(t) < 4\pi^2$, then $\int_0^1 f(t) dt > 12$. So Theorem 17 is a new result and in a sense a development of Theorem 2.2 in [3].

Next, we give some examples about the problem (89).

Example 1. Let $f(t) = 2\pi^2(1 + \kappa_0 \sin^2 t)$, $g(t) = 4\pi^2$, $h(t) = 2\pi^2$, $\bar{B}_{01}(t) = (2\pi)^2 I_n$, $\bar{B}_{02}(t) = 4(2\pi)^2 I_n$. For each $\kappa_0 \in (0, 1/\sin^2 1)$, by the convexity of

$$Y(s) = \begin{cases} 2\pi^2 ks - \pi^2, & s \geq 1, \\ \pi^2 ks^2, & 0 \leq s \leq 1, \end{cases} \quad (91)$$

where $k > 0$, we can verify that

$$V(t, x) = \begin{cases} \pi^2(1 + \kappa_0 \sin^2 t)|x|^2 + 2\pi^2(1 + \kappa_0 \sin^2 t)|x| - \pi^2(1 + \kappa_0 \sin^2 t), & |x| \geq 1, \\ 2\pi^2(1 + \kappa_0 \sin^2 t)|x|^2, & |x| \leq 1 \end{cases} \quad (92)$$

satisfies the conditions of Theorem 17. So problem (89) has at least three distinct solutions. Noticing that $\int_0^1 f(t) dt > 2\pi^2 > 12$, we know that Example 1 does not satisfy the conditions of Theorem 2.2 in [3].

Theorem 19. Assume that $V(t, x)$ satisfies (H_0) and (H_4) there exist $f, g \in L^1([0, 1], R^+)$ and $\gamma_0 \in (0, 1)$ such that

$$|\nabla_x V(t, x)| \leq f(t)|x|^{\gamma_0} + g(t), \quad (93)$$

for all $x \in R^n$ and a.e. $t \in [0, 1]$

(H_5) there exists $c_3 > (\delta_0^2/2\varepsilon_2^2)(\int_0^1 f(t) dt)^2 > 0$ such that

$$\limsup_{\|x\| \rightarrow +\infty} \|x\|^{-2\gamma_0} \int_0^1 V(t, x) dt < -c_3, \quad (94)$$

for all $x \in \ker(A_1 - \bar{B}_0)$, where the constants δ_0 and ε_2 are given by (81) and (49), respectively.

Then, problem (73) has at least one solution in Z_1 . Further, assume that $V(t, x)$ satisfies (H_3) . Then, problem (73) has at least three distinct solutions in Z_1 .

Proof. Similar to the proof of Theorem 17, we need only to show that (Φ_4) follows from (H_4) . By (H_4) , we have

$$\begin{aligned} |(\nabla \Phi(x), y)_{L^2}| &\leq \int_0^1 f(t) |x(t)|^{\gamma_0} \cdot |y(t)| dt + \int_0^1 g(t) |y(t)| dt \\ &\leq c_1 \|x\|_\infty^{\gamma_0} \cdot \|y\|_\infty + c_2 \|x\|_\infty, \end{aligned} \quad (95)$$

for all $x, y \in L^2$, where $c_1 = \int_0^1 f(t) dt$, $c_2 = \int_0^1 g(t) dt$. This shows that (Φ_4) holds. Hence, we can apply Theorem 2 to obtain Theorem 19.

Remark 20. In particular, set $\bar{B}_0(t) \equiv 0$, $\bar{B}_{01}(t) = m^2(2\pi)^2 I_n$, $\bar{B}_{02}(t) = (m+1)^2(2\pi)^2 I_n$, $m \in \{1, 2, \dots\}$ and $M = I_n$. Then, the following problem (89) has a similar conclusion of Theorem 19. For the interval $[0, T]$ considered in second-order Hamiltonian systems satisfying periodic boundary value

conditions, as $T = 1$, the conclusion of Theorem 4 in [2] holds if we replace (H_5) with

$$\lim_{|x| \rightarrow +\infty} |x|^{-2\gamma_0} \int_0^1 V(t, x) dt = -\infty. \quad (96)$$

Clearly, our result generalizes Theorem 4 in [2].

In addition, as $T = 1$, in Theorem 7 of [1], assume that the smooth periodic function $V(t, x)$ satisfies (H_3) ,

$$(H'_5) V(t, x) \longrightarrow -\infty \text{ as } |x| \longrightarrow +\infty \text{ uniformly in } t, \quad (97)$$

and for some $x_0 \in R^n$,

$$\int_0^1 V(t, x_0) dt > 0. \quad (98)$$

Then, problem (89) has at least three distinct solutions. Clearly, condition (H_4) is stronger than condition of Theorem 7 in [1], and condition (H_5) is weaker than condition (H'_5) . Moreover, we can see that the below Example 2 satisfies the conditions of Theorem 19 but does not satisfy Theorem 7 in [1]. So Theorem 19 is also a generalization of Theorem 7 in [1] in a sense. Next, we give an example about problem (89).

Example 2. Let

$$V(t, x) = \begin{cases} (\sin 2\pi t)|x|^{\frac{7}{2}} + \left(\frac{1}{5} - t\right)|x|^{\frac{3}{2}} + \left[4\pi^2 \sin^2 \pi t - \frac{7}{4} \sin 2\pi t + 8\pi^2, \right. \\ \left. + \frac{3}{2} \left(t - \frac{1}{5}\right)\right]|x| - 2\pi^2 \sin^2 \pi t + \frac{3}{4} \sin 2\pi t - 4\pi^2 - \frac{1}{2} \left(t - \frac{1}{5}\right), & |x| \geq 1, \\ 2\pi^2 (2 + \sin^2 \pi t) |x|^2, & |x| \leq 1, \end{cases} \quad (99)$$

for all $x \in R^n$ and $t \in [0, 1]$. It is easy to see that

$$\nabla_x V(t, x) = \begin{cases} \frac{7}{4} (\sin 2\pi t) \frac{x}{|x|^{1/4}} + \frac{3}{2} \left(\frac{1}{5} - t\right) \frac{x}{|x|^{1/2}} + \left[4\pi^2 \sin^2 \pi t - \frac{7}{4} \sin 2\pi t, \right. \\ \left. + 8\pi^2 + \frac{3}{2} \left(t - \frac{1}{5}\right)\right] \frac{x}{|x|}, & |x| \geq 1, \\ 4\pi^2 (2 + \sin^2 \pi t) x, & |x| \leq 1, \end{cases} \quad (100)$$

$$|\nabla_x V(t, x)| \leq \frac{7}{4} |\sin 2\pi t| \cdot \left|x\right|^{\frac{3}{4}} + \frac{6}{5} \left|x\right|^{\frac{1}{2}} + 12\pi^2 + \frac{6}{5},$$

for all $x \in R^n$ and a.e. $t \in [0, 1]$. Note that for $\varepsilon > 0$, we have

$$\frac{6}{5} |x|^{1/2} \leq \frac{7}{4} \varepsilon |x|^{3/4}, \quad (101)$$

as $x \in R^n$ and $|x| \geq 1/\varepsilon^4$, which implies that $(6/5)|x|^{1/2} \leq (7/4)\varepsilon|x|^{3/4} + (6/5\varepsilon^2)$ and

$$|\nabla_x V(t, x)| \leq \frac{7}{4} (|\sin 2\pi t| + \varepsilon) |x|^{3/4} + \frac{6}{5\varepsilon^2} + 13\pi^2, \quad (102)$$

for all $x \in R^n$ and a.e. $t \in [0, 1]$. The above shows (H_4) holds with $\gamma_0 = 3/4$, $f(t) = (7/4)(|\sin 2\pi t| + \varepsilon)$, $g(t) = (6/5\varepsilon^2) + 13$

π^2 . By Wirtinger's inequality and Sobolev inequality of Proposition 1.3 in [28], we have $\ker(A_1 - \bar{B}_0) = R^n$, $|x| = \|x\|_\infty = \|x\|$ for all $x \in R^n$, and $\delta_0^2 = 1/12$, $\varepsilon_2^2 = 4\pi^2/(1 + 4\pi^2)$, $(\int_0^1 f(t) dt)^2 = ((7/2\pi) + (7\varepsilon/4))^2$ via a simple calculation. Noticing that

$$\begin{aligned} \limsup_{|x| \rightarrow +\infty} |x|^{-2\gamma_0} \int_0^1 V(t, x) dt &= -\frac{3}{10} < -\frac{1 + 4\pi^2}{96\pi^2} \left(\frac{7}{2\pi} + \frac{7\varepsilon}{4}\right)^2 \\ &= -\frac{\delta_0^2}{2\varepsilon_2^2} \left(\int_0^1 f(t) dt\right)^2 \end{aligned} \quad (103)$$

as $\varepsilon > 0$ sufficiently small, we know that (H_5) holds. Set $\bar{B}_{01}(t) = (2\pi)^2 I_n$, $\bar{B}_{02}(t) = 4(2\pi)^2 I_n$. By $\bar{B}_0(t) \equiv 0$ and $V(t, x) = 2\pi^2(2 + \sin^2 \pi t)|x|^2$ as $|x| \leq 1$, we can verify that (H_3) holds. By Theorem 19, problem (89) has at least three distinct solutions. Obviously, Example 19 does not satisfy the conditions of Theorem 4 in [2] and Theorem 7 in [1].

Theorem 21. Assume that $V(t, x)$ satisfies (H_0) and $(H_6) V(t, 0) = 0$ for all $t \in [0, 1]$
 (H_7) there exist $q(t) \in L^1([0, 1], \mathbb{R}^+)$ and $\gamma \in (0, 2)$ such that $V(t, x) \leq q(t)(1 + |x|^\gamma)$ for all $x \in \mathbb{R}^n$
 (H_8) there exist a positive constant $\tilde{c}$ and a $x_0 \in \ker(A_1 - \bar{B}_0)$ with $\tilde{c} < \tilde{k}\|x_0\|_{L^2} \sqrt{\lambda}$, such that

$$\sup_{(t,x) \in [0,1] \times E_1} V(t, x) < \frac{\tilde{c}^2}{\tilde{\lambda} \tilde{k}^2 \|x_0\|_{L^2}^2} \int_0^1 V(t, x_0(t)) dt, \quad (104)$$

where $E_1 = \{x \in \mathbb{R}^n \mid |x| \leq \tilde{c}\}$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive number r_0 with the following property: for each $\eta \in \Lambda$ and for every $F(t, x): [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (H_0) , there exists $\rho > 0$ such that, for each $\mu \in [0, \rho]$, problem (74) has at least three distinct solutions whose norms in Z_1 are less than r_0 .

Proof. Clearly, (H_6) implies that $\Phi(\theta) = 0$. Similar to the proof of Theorem 17, set

$$\Psi(x) = \int_0^1 F(t, x) dt, \forall x \in Z_1. \quad (105)$$

Then, (Ψ_0) holds. Again set $X_0 = L^\infty([0, 1], \mathbb{R}^n)$, we have $Z_1 \subset X_0 \subset X = L^2$ and the embeddings $Z_1 \hookrightarrow X_0$ and $X_0 \hookrightarrow X = L^2$ are compact and continuous, respectively. By (H_7) , we obtain $\Phi(x) \leq \int_0^1 q(t) dt + \|x\|_\infty^\gamma \int_0^1 q(t) dt = c_2 + c_1 \|x\|_{X_0}^\gamma$ for all $x \in Z_1$, which shows that (Φ_7) holds. Thanks to (81), we know that $E = \{x \in Z \mid \|x\|_\infty \leq \tilde{c}\} \subset E_1 = \{x \in \mathbb{R}^n \mid |x| \leq \tilde{c}\}$. So (15) follows from (104), which shows that (Φ_8) holds. Now, we can apply Theorem 3 to obtain Theorem 21.

Remark 22. In particular, set $\bar{B}_0(t) \equiv 0$ and $M = I_n$. Then, the following problem

$$\begin{cases} -\ddot{x} + \bar{B}_1(t)x = \eta \nabla_x V(t, x) + \mu \nabla_x F(t, x), & \text{a.e. } t \in [0, 1], \\ x(1) - x(0) = \dot{x}(1) - \dot{x}(0) = 0, \end{cases} \quad (106)$$

has a similar conclusion of Theorem 21, where $\tilde{k} = \sqrt{2}$. In addition, our result slightly generalizes Theorem 1 in [15] as $T = 1, p = 2$ in two aspects. At first, Theorem 21 requires $V(t, x)$ being measurable in t for every $x \in \mathbb{R}^n$ instead of continuous in t for every $x \in \mathbb{R}^n$; secondly Theorem 21 requires $\bar{B}_1(t) \in L^\infty([0, 1], L_s(\mathbb{R}^n))$ instead of $\bar{B}_1(t) \in C([0, 1], L_s(\mathbb{R}^n))$.

An example of problem (106) corresponding to Theorem 21 is given below.

Example 3. Let $n = 2$,

$$\bar{B}_1(t) = \begin{pmatrix} f_1(t) & 0 \\ 0 & f_1(t) \end{pmatrix}, \quad (107)$$

and $f_1(t) \in L^\infty([0, 1], \mathbb{R})$ with $\sup f_1(t) = 2, \inf f_1(t) = 1$. Then, $\bar{\lambda} = 2, \underline{\lambda} = 1$. Let

$$V(x_1, x_2) = (x_1^2 + x_2^2)^{1/2} e^{1-(x_1^2+x_2^2)} + (x_1^2 + x_2^2)^{3/4}, \forall (x_1, x_2) \in \mathbb{R}^2. \quad (108)$$

Then, for every $F(t, x): [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (H_0) , all the hypotheses of Theorem 21 are satisfied with $\tilde{k} = \sqrt{2}, \tilde{c} = 1, x_0 = (\sqrt{3}, \sqrt{6}), \gamma = 3/2$ and $q(t)$ is a large enough constant. So, problem (106) has at least three distinct solutions. Noting that $f_1(t) \in C([0, 1], \mathbb{R})$ is not necessarily true, we can see that the above example does not satisfy the conditions of Theorem 1 in [15].

Theorem 23. Assume that $x_0, \tilde{c}, \gamma, q(t)$, and E_1 are the same as in Theorem 21, $V(t, x)$ satisfies $(H_0), (H_6), (H_7)$, and (H_9)

$$\sup_{(t,x) \in [0,1] \times E_1} V(t, x) < \frac{\tilde{c}^2}{\tilde{c}^2 + \tilde{\lambda} \tilde{k}^2 \|x_0\|_{L^2}^2} \int_0^1 V(t, x_0(t)) dt. \quad (109)$$

Then, for each

$$\eta \in \left(\frac{\tilde{\lambda} \|x_0\|_{L^2}^2}{2 \left(\int_0^1 V(t, x_0(t)) dt - \sup_{(t,x) \in [0,1] \times E_1} V(t, x) \right)}, \frac{\tilde{c}^2}{2 \tilde{k}^2 \sup_{(t,x) \in [0,1] \times E_1} V(t, x)} \right) = E_2, \quad (110)$$

problem (74) admits at least three distinct solutions provided $\mu = 0$.

Proof. Similar to the proof of Theorem 21, we need only to show that (Φ_9) follows from (H_9) . Noticing that $E \subset E_1$, we have

$$\sup_{x \in E} \Phi(x) = \sup_{x \in E} \int_0^1 V(t, x) dt \leq \sup_{(t,x) \in [0,1] \times E_1} V(t, x), \quad (111)$$

which means that (Φ_9) holds and

$$\eta \in E_2 \subset \left(\frac{\tilde{\lambda} \|x_0\|_{L^2}^2}{2 \left(\Phi(x_0) - \sup_{x \in E} \Phi(x) \right)}, \frac{\tilde{c}^2}{2 \tilde{k}^2 \sup_{x \in E} \Phi(x)} \right). \quad (112)$$

Therefore, we can apply Theorem 4 to obtain Theorem 23.

Remark 24. In particular, set $\bar{B}_0(t) \equiv 0$ and $M = I_n$. Then, the following problem (106) has a similar conclusion of Theorem

23. By the above Remark 5, we can see that our result slightly generalizes Theorem 2 of [15] as $T = 1, p = 2$ in two aspects.

Finally, we give an example of problem (106) corresponding to Theorem 23.

Example 4. Let $n = 2$,

$$\bar{B}_1(t) = \begin{pmatrix} f_1(t) & 0 \\ 0 & f_1(t) \end{pmatrix}, \quad (113)$$

and $f_1(t) \in L^\infty([0, 1], \mathbb{R})$ with $\sup f_1(t) = 2, \inf f_1(t) = 1$. Then, $\bar{\lambda} = 2, \underline{\lambda} = 1$. Let

$$V(x_1, x_2) = \frac{1}{9} (x_1^2 + x_2^2)^{12} e^{1-(x_1^2+x_2^2)}, \forall (x_1, x_2) \in \mathbb{R}^2. \quad (114)$$

Then, all the hypotheses of Theorem 23 are satisfied with $\tilde{k} = \sqrt{2}, \tilde{c} = 1, x_0 = (\sqrt{3}, \sqrt{6}), \gamma = 3/2$ and $q(t)$ is a large enough constant. So, for each $\eta \in (81e^8/(9^{12} - e^8), 9/4)$, problem (106) has at least three distinct solutions provided $\mu = 0$. Noting that $f_1(t) \in C([0, 1], \mathbb{R})$ is not necessarily true, we can see that the above example does not satisfy the conditions of Theorem 2 in [15].

4.2. Second-Order Hamiltonian Systems Satisfying Sturm-Liouville Boundary Value Conditions. As a second example, we discuss the solvability of the following two Sturm-Liouville boundary value problems

$$\begin{cases} -\ddot{x} - \bar{B}_0(t)x = \nabla_x V(t, x), \text{ a.e. } t \in [0, 1], \\ x(0) \cos \alpha - \dot{x}(0) \sin \alpha = 0, x(1) \cos \beta - \dot{x}(1) \sin \beta = 0, \end{cases} \quad (115)$$

$$\begin{cases} -\ddot{x} - \bar{B}_0(t)x + \tilde{B}_1(t)x = \eta \nabla_x V(t, x) + \mu \nabla_x F(t, x), \text{ a.e. } t \in [0, 1], \\ x(0) \cos \alpha - \dot{x}(0) \sin \alpha = 0, x(1) \cos \beta - \dot{x}(1) \sin \beta = 0, \end{cases} \quad (116)$$

where $\eta, \mu \in [0, +\infty), \bar{B}_0(t), \tilde{B}_1(t) \in L^\infty([0, 1], L_s(\mathbb{R}^n)), \alpha \in [0, \pi), \beta \in (0, \pi]$, and $\nabla_x V(t, x)$ and $\nabla_x F(t, x)$ denote the gradient of $V(t, x)$ and $F(t, x)$ for $x \in \mathbb{R}^n$, respectively.

Let $X = L^2([0, 1], \mathbb{R}^n)$. We define $A_2 : D(A_2) \rightarrow X$ by $(A_2 x)(t) = -\ddot{x}(t)$ where $D(A_2) = \{x \in H^2([0, 1], \mathbb{R}^n) \mid x \text{ satisfies } x(0) \cos \alpha - \dot{x}(0) \sin \alpha = 0, x(1) \cos \beta - \dot{x}(1) \sin \beta = 0\}$. Set $(B_i x)(t) = \tilde{B}_i(t)x(t) (i = 0, 1)$. Then, $D(B_i) = X$. A_2 is self-adjoint and $\sigma(A_2) = \sigma_d(A_2)$ is bounded from below via Proposition 1.17 in [26]. Define $i_{\alpha, \beta}(\bar{B}_0) = i_{A_2}(B_0), \nu_{\alpha, \beta}(\bar{B}_1) = \nu_{A_2}(B_1)$, that is, $\nu_{\alpha, \beta}(\bar{B}_0)$ is the dimension of the solution subspace of (115) with $V(t, x) \equiv 0$.

Now, we assume that $i_{\alpha, \beta}(\bar{B}_0) = 0, \nu_{\alpha, \beta}(\bar{B}_0) \neq 0, \tilde{B}_1(t)$ is positive definite for all $x \in \mathbb{R}^n$, and $V, F : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$

satisfies (H_0) . Meanwhile, set

$$Z_2 = \begin{cases} \{x \in H^1([0, 1], \mathbb{R}^n) \mid x(1) = 0\}, & \alpha = 0, \beta \in (0, \pi), \\ \{x \in H^1([0, 1], \mathbb{R}^n) \mid x(0) = 0\}, & \alpha \in (0, \pi), \beta = \pi, \\ \{x \in H^1([0, 1], \mathbb{R}^n) \mid x(1) = x(0) = 0\}, & \alpha = 0, \beta = \pi, \\ H^1([0, 1], \mathbb{R}^n), & \alpha, \beta \in (0, \pi). \end{cases} \quad (117)$$

Thanks to Proposition 1.17 in [26], we have $Z_2 = D(|A_2|^{1/2})$ and the norm

$$\|x\|_{Z_2} = \left(\int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2] dt \right)^{1/2}, \quad (118)$$

for each $x \in Z_2$. Noticing that $i_{\alpha, \beta}(\bar{B}_0) = 0, \nu_{\alpha, \beta}(\bar{B}_0) \neq 0$, we can define another equivalent norm

$$\|x\| = \left(\int_0^1 [|\dot{x}(t)|^2 - (\bar{B}_0(t)x(t), x(t)) + |x(t)|^2] dt \right)^{1/2}, \quad (119)$$

for each $x \in Z_2$.

Let

$$\|x\|_{\bar{B}_1} = \left(\int_0^1 [|\dot{x}(t)|^2 - (\bar{B}_0(t)x(t), x(t)) + (\tilde{B}_1(t)x(t), x(t))] dt \right)^{1/2}, \quad \forall x \in Z_2. \quad (120)$$

Since $\bar{B}_1(t) \in L^\infty([0, 1], L_s(\mathbb{R}^n))$ is positive definite, there exist $\bar{\lambda} > \underline{\lambda} > 0$ such that $\bar{\lambda} \|x\|_{L^2}^2 \geq (B_1 x, x)_X \geq \underline{\lambda} \|x\|_{L^2}^2$ for all $x \in X$. Similar to Section 4.1, we know that (80)–(83) hold. Hence, the following four results also hold. Because of the similarities with the proofs of Theorems 17, 19, 21, and 23, we omit them here.

Theorem 25. Assume that $V(t, x)$ satisfies (H_0) , (H_1) , and (H_2) with $\bar{B}_0, \bar{B}_2$, and $\bar{B}_3$ replaced with $\bar{B}_0, \bar{B}_2$, and $\bar{B}_3$, respectively, then problem (115) has at least one solution with saddle character in Z_2 . In addition, assume that $V(t, x)$ satisfies (H_3) with $\bar{B}_{01}$ and $\bar{B}_{02}$ replaced with $\bar{B}_{01}$ and $\bar{B}_{02}$, respectively, then problem (115) has at least three distinct solutions in Z_2 .

Theorem 26. Assume that $V(t, x)$ satisfies (H_0) , (H_4) , and (H_5) with $\bar{B}_0$ replaced with $\bar{B}_0$, then problem (115) has at least one solution in Z_2 . In addition assume that $V(t, x)$ satisfies (H_3) with $\bar{B}_{01}$ and $\bar{B}_{02}$ replaced with $\bar{B}_{01}$ and $\bar{B}_{02}$, respectively, then problem (115) has at least three distinct solutions in Z_2 .

Theorem 27. Assume that $V(t, x)$ satisfies (H_0) , (H_6) , (H_7) , and (H_8) with $\bar{B}_0$ and A_1 replaced with $\bar{B}_0$ and A_2 , respectively, then there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive number r_0 with the following property: for each $\eta \in \Lambda$ and

for every $F(t, x): [0, 1] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (H_0) , there exists $\rho > 0$ such that, for each $\mu \in [0, \rho]$, problem (116) has at least three distinct solutions whose norms in Z_2 are less than r_0 .

Theorem 28. Assume that $V(t, x)$ satisfies (H_0) , (H_6) , (H_7) , and (H_9) with $\bar{B}_0$ and A_1 replaced with $\tilde{B}_0$ and A_2 , respectively, then for each $\eta \in E_2$ problem (116) admits at least three distinct solutions provided $\mu = 0$.

Remark 29. In particular, set $\alpha = 0$, $\beta = \pi$ and $\tilde{B}_0(t) \equiv \pi^2 I_n$. Then, we have $Z_2 = H_0^1$, $\sigma(A_2) = \{k^2 \pi^2 \mid k \in \mathbb{N} \setminus \{0\}\}$ and $\ker(A_2 - \tilde{B}_0) = \{a \sin t\pi \mid a \in \mathbb{R}^n\}$. Hence, the following problems

$$-\ddot{x}(t) - \pi^2 x = \nabla_x V(t, x(t)), x(0) = x(1) = 0, \quad (121)$$

$$\begin{cases} -\ddot{x} - \pi^2 x + \tilde{B}_1(t)x = \eta \nabla_x V(t, x) + \mu \nabla_x F(t, x), \text{ a.e. } t \in [0, 1], \\ x(0) = x(1) = 0, \end{cases} \quad (122)$$

have the conclusion corresponding to Theorems 25–28, respectively, where $\tilde{B}_{01}(t) \equiv (k\pi)^2 I_n$ and $\tilde{B}_{02}(t) = ((k+1)\pi)^2 I_n$ with $k \geq 2$. In (122), if $\tilde{B}_1(t) = \pi^2$, $\mu = 0$, then Theorem 27 corresponding to problem (122) reduces to Theorem 2 in [10] and Theorem 2 in [22], respectively, and our result requires $V(t, x)$ being measurable in t for every $x \in \mathbb{R}^n$ instead of continuous in t for every $x \in \mathbb{R}^n$.

4.3. Elliptic Partial Differential Equations Satisfying Dirichlet Boundary Value Conditions. As a third example, we discuss the solvability of the following two Dirichlet boundary value problems

$$\begin{cases} -\Delta x - b_0(u)x = f(u, x), & u \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \quad (123)$$

$$\begin{cases} -\Delta x - b_0(u)x + b_1(u)x = \eta f(u, x) + \mu g(u, x), & u \in \Omega, \\ x|_{\partial\Omega} = 0, \end{cases} \quad (124)$$

where $\Omega \in \mathbb{R}^n$ is a bounded open domain with smooth boundary, $\Delta x = \sum_{i=1}^n (\partial^2 / \partial u_i^2)x$, $b_0(u), b_1(u) \in L^\infty(\Omega, \mathbb{R})$ and $\eta, \mu \in [0, +\infty)$. As usual, $\bar{\Omega} = \Omega \cup \partial\Omega$.

Let $X = L^2(\bar{\Omega}, \mathbb{R})$. We define $A_3 : D(A_3) \longrightarrow X$ by $(A_3 x)(u) = -\Delta x(u)$ where $D(A_3) = H_0^2(\bar{\Omega})$. Set $(B_i x)(u) = b_i(u)x(u)$ ($i = 0, 1$). Then, $D(B_i) = X$. It is well known that A_3 is self-adjoint and $\sigma(A_1) = \sigma_d(A_1) \subset [0, +\infty)$. Define $i_\Delta(b_0) = i_{A_3}(B_0)$, $\nu_\Delta(b_0) = \nu_{A_3}(B_0)$ (see Definition 2.5.1 in [23]), that is, $\nu_\Delta(b_0)$ is the dimension of the solution subspace of (123) as $f(u, x) \equiv 0$ and $i_\Delta(b_0) = \sum_{\lambda < 0} \nu_\Delta(b_0 + \lambda)$.

Now, we assume that $b_1(u)$ is positive definite for all $x \in \mathbb{R}$, that is, there exists a positive constant $\underline{\lambda}$ such that $b_1(u)x^2 \geq \underline{\lambda}x^2$ for all $x \in \mathbb{R}$, and two Carathéodory functions $f, g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following condition:

(P_0) there exist $a_i > 0$ ($i = 1, 2$) such that

$$\begin{aligned} |f(u, x)| &\leq a_1(1 + |x|^{s_1}), \forall (u, x) \in \Omega \times \mathbb{R}, \\ |g(u, x)| &\leq a_2(1 + |x|^{s_2}), \forall (u, x) \in \Omega \times \mathbb{R}, \end{aligned} \quad (125)$$

where $s_1, s_2 \in (0, (n+2)/(n-2))$ if $n > 2$ and $s_1, s_2 \in (0, +\infty)$ if $n = 2$.

Further, assume that $i_\Delta(b_0) = 0$, $\nu_\Delta(b_0) \neq 0$. Meanwhile, set $Z_3 = H_0^1(\bar{\Omega}) = W_0^{1,2}(\Omega)$. Obviously, $Z_3 = D(|A_3|^{1/2})$ and the norm

$$\|x\|_{Z_3} = \left(\int_\Omega [|\nabla x(u)|^2 + x^2(u)] du \right)^{1/2}, \quad (126)$$

for each $x \in Z_3$. Noticing that $i_\Delta(b_0) = 0$, $\nu_\Delta(b_0) \neq 0$, we can define another equivalent norm

$$\|x\| = \left(\int_\Omega [|\nabla x(u)|^2 - b_0(u)x^2(u) + x^2(u)] du \right)^{1/2}, \quad (127)$$

for each $x \in Z_3$.

Let

$$\|x\|_{b_1} = \left(\int_\Omega [|\nabla x(u)|^2 - b_0(u)x^2(u) + b_1(u)x^2(u)] du \right)^{1/2}, \quad \forall x \in Z_3. \quad (128)$$

Since $b_1(u) \in L^\infty(\Omega, \mathbb{R})$ is positive definite, there exist $\bar{\lambda} > \underline{\lambda} > 0$ such that $\bar{\lambda}\|x\|_{L^2}^2 \geq \int_\Omega b_1(u)x^2(u) du \geq \underline{\lambda}\|x\|_{L^2}^2$ for all $x \in X$. So we have

$$\min \{1, \underline{\lambda}\} \|x\|^2 \leq \|x\|_{b_1}^2 \leq \max \{1, \bar{\lambda}\} \|x\|^2. \quad (129)$$

Observe that the embedding $Z_3 \hookrightarrow L^2 = X$ is compact. Let $\|\cdot\|_\infty$ be the norm of $L^\infty(\Omega, \mathbb{R})$. Then, there is a constant $\delta_0 > 0$ such that

$$|x| \leq \|x\|_\infty \leq \delta_0 \|x\|, \quad (130)$$

for any $x \in Z_3$. Let

$$k_2 = \sup_{x \in Z_3 \setminus \{0\}} \frac{\|x\|_\infty}{\|x\|_{b_1}}. \quad (131)$$

From (129) and (130), we have $0 < k_2 \leq \tilde{k} = \delta_0 (\min \{1, \underline{\lambda}\})^{-1/2}$.

For any $b_1(u), b_2(u) \in L^\infty(\Omega, \mathbb{R})$, we write $b_1 \leq b_2$ if $b_1(u) \leq b_2(u)$ for a.e. $u \in \Omega$ and define $b_1 < b_2$ if $b_1(u) < b_2(u)$ a.e. $u \in \Omega$. From Theorems 1–4, we have the following four results hold.

Theorem 30. Assume that $f(u, x)$ satisfies (P_0) and

(P₁) there exist $g \in L^1(\Omega, R^+)$ and $b_2 \in L^\infty(\Omega, R)$ with $b_2 \geq 0$, $v_\Delta(b_0 + b_2) = 0$ and $i_\Delta(b_0 + b_2) = v_\Delta(b_0)$ such that

$$|f(u, x)| \leq b_2(u)|x| + g(u), \quad (132)$$

for all $x \in R$ and a.e. $u \in \Omega$

(P₂) there exists a positive definite $b_3 \in L^\infty(\Omega, R)$ such that $\int_0^x f(u, t)dt - (1/2)b_3(u)x^2$ is convex in x for a.e. $u \in \Omega$

Then, problem (123) has at least one solution with saddle character in Z_3 .

Assume in addition that

(P₃) there exist $\delta_1 > 0$, $b_{01}, b_{02} \in L^\infty(\Omega, R)$ such that $b_{02} > b_{01} > b_0$ and $v_\Delta(b_{0i}) \neq 0 (i = 1, 2)$, $i_\Delta(b_{02}) = i_\Delta(b_{01}) + v_\Delta(b_{01})$, and for all $|x| \leq \delta_1$,

$$\frac{1}{2}(b_{01}(u) - b_0(u))x^2 \leq \int_0^x f(u, t)dt \leq \frac{1}{2}(b_{02}(u) - b_0(u))x^2, \quad (133)$$

for a.e. $u \in \Omega$. Then, problem (123) has at least three distinct solutions in Z_3 .

Proof. Set

$$\Phi(x) = \int_\Omega \left(\int_0^x f(u, t)dt \right) du, \quad \forall x \in Z_3. \quad (134)$$

Then, $\Phi \in C^1(Z_3, R)$ is weakly continuous with compact derivative, and for every $x \in Z_3$,

$$\Phi'(x)y = \int_\Omega f(u, x)ydu, \quad \forall y \in Z_3, \quad (135)$$

because of (P₀) and (132). Hence, (Φ_0) holds.

Thanks to (130) and (132) of (P₁), we have

$$\begin{aligned} |(\nabla\Phi(x), x)_{L^2}| &\leq \int_\Omega b_2(u)x^2(u)du + \int_\Omega g(u)|x(u)|du \\ &\leq (B_2x, x)_{L^2} + \|x\|_\infty \int_\Omega g(u)du \\ &\leq (B_2x, x)_{L^2} + c_0\|x\|, \end{aligned} \quad (136)$$

for all $x \in L^2$, where $c_0 = \delta_0 \int_\Omega g(u)du$. This shows that (Φ_1) holds.

Next, similar to the proof of Theorem 17, we can get the conclusion of Theorem 30 via Theorem 1.

Theorem 31. Assume that $f(u, x)$ satisfies (P₀) and

(P₄) there exist $f_1, g \in L^1(\Omega, R^+)$ and $\gamma_0 \in (0, 1)$ such that

$$|f(u, x)| \leq f_1(u)|x|^{\gamma_0} + g(u), \quad (137)$$

for all $x \in R$ and a.e. $u \in \Omega$

(P₅) there exists $c_3 > (\delta_0^2/2\varepsilon_2^2)(\int_\Omega f(u)du)^2 > 0$ such that

$$\limsup_{\|x\| \rightarrow +\infty} \|x\|^{-2\gamma_0} \int_\Omega \left(\int_0^x f(u, t)dt \right) du < -c_3, \quad (138)$$

for all $x \in \ker(A_3 - b_0)$, where the constants δ_0 and ε_2 are given by (130) and (49), respectively.

Then, problem (123) has at least one solution in Z_3 . Further, assume that $f(u, x)$ satisfies (P₃). Then, problem (123) has at least three distinct solutions in Z_3 .

Proof. Similar to the proof of Theorem 30, we need only to show that (Φ_4) follows from (P₄). By (P₄), we have

$$\begin{aligned} |(\nabla\Phi(x), y)_{L^2}| &\leq \int_\Omega f_1(u)|x(u)|^{\gamma_0} \cdot |y(u)|dt + \int_\Omega g(u)|y(u)|du \\ &\leq c_1\|x\|_\infty^{\gamma_0} \cdot \|y\|_\infty + c_2\|x\|_\infty, \end{aligned} \quad (139)$$

for all $x, y \in L^2$, where $c_1 = \int_\Omega f_1(u)du$, $c_2 = \int_\Omega g(u)du$, which shows that (Φ_4) holds. So, Theorem 31 follows from Theorem 2.

Theorem 32. Assume that $f(u, x)$ satisfies (P₀) and

(P₇) there exist $q(u) \in L^1(\Omega, R^+)$ and $\gamma \in (0, 2)$ such that $\int_0^x f(u, t)dt \leq q(u)(1 + |x|^\gamma)$ for all $x \in R$

(P₈) there exist a positive constant $\tilde{c}$ and a $x_0 \in \ker(A_3 - b_0)$ with $\tilde{c} < \tilde{k}\|x_0\|_{L^2}\sqrt{\tilde{\lambda}}$, such that

$$\sup_{(u,x) \in \Omega \times E_1} \int_0^x f(u, t)dt < \frac{\tilde{c}^2}{\tilde{\lambda}\tilde{k}^2\|x_0\|_{L^2}^2 \cdot \text{meas}(\Omega)} \int_\Omega \int_0^{x_0} f(u, t)dtdu, \quad (140)$$

where $E_1 = \{x \in R \mid |x| \leq \tilde{c}\}$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty)$ and a positive number r_0 with the following property: for each $\eta \in \Lambda$ and for every $g(u, x): \Omega \times R \rightarrow R$ satisfies (P₀), there exists $\rho > 0$ such that, for each $\mu \in [0, \rho]$, problem (124) has at least three distinct solutions whose norms in Z_3 are less than r_0 .

Proof. Clearly, $\Phi(\theta) = 0$ via the definition of $\Phi(x)$. Set $X_0 = L^\infty(\Omega, R)$, we have $Z_3 \subset X_0 \subset X = L^2$ and the embeddings $Z_3 \circ X_0$ and $X_0 \circ X = L^2$ are compact and continuous, respectively. Similar to the proofs of Theorem 21 and Theorem 30, we can get the conclusion of Theorem 32 via Theorem 3.

Remark 33. As $p = 2$ in [9], Theorem 32 reduces to Theorem 2.4 in [9], and the continuity of $f(u, x)$ in u and the condition (i) of Theorem 2.4 in [9] is discarded.

Theorem 34. Assume that $x_0, \tilde{c}, \gamma, q(u)$, and E_1 are the same as in Theorem 32, $f(u, x)$ satisfies (P_0) , (P_7) , and (P_9)

$$\sup_{(u,x) \in \Omega \times E_1} \int_0^x f(u, t) dt < \frac{\tilde{c}^2}{(\tilde{c}^2 + \bar{\lambda} \tilde{k}^2 \|x_0\|_{L^2}^2) \cdot \text{meas}(\Omega)} \int_{\Omega} \int_0^{x_0} f(u, t) dt du. \quad (141)$$

Then, for each

$$\eta \in \left(\frac{\bar{\lambda} \|x_0\|_{L^2}^2}{2\rho_0}, \frac{\tilde{c}^2}{2\tilde{k}^2 \rho_1} \right) = E_2, \quad (142)$$

problem (124) admits at least three distinct solutions provided $\mu = 0$, where

$$\begin{aligned} \rho_0 &= \int_{\Omega} \int_0^{x_0} f(u, t) dt du - \rho_1, \\ \rho_1 &= \text{meas}(\Omega) \sup_{(u,x) \in \Omega \times E_1} \int_0^x f(u, t) dt. \end{aligned} \quad (143)$$

Proof. Similar to the proof of Theorem 32, we need only to show that (Φ_9) follows from (P_9) . Noticing that $E \subset E_1$, we have

$$\sup_{x \in E} \Phi(x) = \sup_{x \in E} \int_{\Omega} \int_0^x f(u, t) dt du \leq \text{meas}(\Omega) \sup_{(u,x) \in \Omega \times E_1} \int_0^x f(u, t) dt, \quad (144)$$

which means that (Φ_9) holds and

$$\eta \in E_2 \subset \left(\frac{\bar{\lambda} \|x_0\|_{L^2}^2}{2 \left(\Phi(x_0) - \sup_{x \in E} \Phi(x) \right)}, \frac{\tilde{c}^2}{2\tilde{k}^2 \sup_{x \in E} \Phi(x)} \right). \quad (145)$$

Therefore, we can apply Theorem 4 to obtain Theorem 34.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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