Research Article

Approximation by Bézier Variant of Baskakov-Durrmeyer-Type Hybrid Operators

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We give a Bézier variant of Baskakov-Durrmeyer-type hybrid operators in the present article. First, we obtain the rate of convergence by using Ditzian-Totik modulus of smoothness and also for a class of Lipschitz function. Then, weighted modulus of continuity is investigated too. We study the rate of point-wise convergence for the functions having a derivative of bounded variation. Furthermore, we establish the quantitative Voronovskaja-type formula in terms of Ditzian-Totik modulus of smoothness at the end.

1. Introduction

To approximate continuous functions, many approximating operators have been introduced under certain conditions and with different parameters too. Many researchers have later generalized and modified these introduced operators and discussed various approximating properties of these operators. In 1957, Baskakov [1] introduced and studied such a class of positive linear operators, called Baskakov operators defined on the positive semiaxis. For $f \in \mathscr{C}[0,\infty)$, the sequence of Baskakov operators is given as

$$\mathscr{B}_n(f\,;y) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} y^k (1+y)^{-n-k} f\left(\frac{k}{n}\right), \quad (1)$$

for $y \in [0,\infty)$ and $n \in \mathbb{N}$. Later on, many authors have been considering the Baskakov operators; for instance, Aral in [2] defines the parametric generalization of Baskakov operators as

$$\mathscr{B}_n^v(f\,;x) = \sum_{k=0}^{\infty} \mathscr{P}_{n,k}^v(x) f\left(\frac{k}{n}\right), \quad (2)$$

where

$$\mathscr{P}_{n,k}^v(x) = \frac{x^{k-1}}{(1+x)^{n+k-1}}\left[\frac{vx}{1+x}\binom{n+k-1}{k} - (1-v)(1+x) \cdot \binom{n+k-3}{k-2} + (1-v)x\binom{n+k-1}{k}\right], \quad (3)$$

with $\binom{n+k-1}{k-2} = 0$ if $k = 0, 1$.

Among interesting studies realized in this context, we cite those based on the Baskakov-Kantorovitch-type operators in the generalized form (the original operator given by Kantorovich in [3]) defined as, for $f \in L^1([0, 1])$ (the class of Lebesgue integrable functions on $[0, 1]$),

$$\mathscr{B}K_n(f\,;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} \int_0^1 \chi_{n,k}(t) f(t)dt, \quad (4)$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k/n, k + 1/n]$.

It is well known that Bézier curves are the mathematically defined curves successively used in computer-aided geometric design (CAGD), image processing, and curve fitting. The miscellaneous Bézier variant of operators is crucial subject matter in approximation theory. In 1983, Chang [4] pioneered the Bernstein-Bézier operators. Afterwards, several researchers established the Bézier variant of various operators (c.f. [5, 6]). For more details on the approximation by Durrmeyer-type and Baskakov-Durrmeyer-type operators, one can refer to [7, 8], respectively. For more about Bézier variant of operators, one can refer to [9, 10].

We will be mainly interested to the Bézier variant operator type based on those of Baskakov-Durrmeyer defined as follows:

$$\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x) = \sum_{k=1}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x) \int_{0}^{\infty} \mathscr{J}_{n,k}^{\rho}(t) f(t) dt + \mathscr{X}_{n,0}^{v,\theta}(x) f(0),$$
(5)

where

$$\mathscr{X}_{n,k}^{v,\theta}(x) = \left[\xi_{n,k}^{v}(x)\right]^{\theta} - \left[\xi_{n,k+1}^{v}(x)\right]^{\theta}, \xi_{n,k}^{v}(x) = \sum_{j=k}^{\infty} \mathscr{P}_{n,j}^{v}(x)\, k = 0, 1, 2, \cdots$$

$$\mathscr{J}_{n,k}^{\rho}(t) = n\rho e^{-n\rho t} \frac{(n\rho t)^{k\rho-1}}{\Gamma(k\rho)}.$$
(6)

If we take $\theta = 1$, then operator (5) reduces to the following operator studied by [11].

$$\mathscr{B}_{n,\rho}^{v}(f\,;x) = \sum_{k=1}^{\infty} \mathscr{P}_{n,k}^{v}(x) \int_{0}^{\infty} \mathscr{J}_{n,k}^{\rho}(t) f(t) dt + \mathscr{P}_{n,0}^{v}(x) f(0).$$
(7)

Let us briefly summarize the outline of the paper. Next section is devoted to the computation of some auxiliary results which we need to prove our theorems in coming sections. In Section 3, we will prove some approximations of functions using Ditzian-Totik modulus and then we will deal to functions lie in the Lipschitz spaces. We treat in Section 4 the rate of convergence in the context of suitable weighted spaces and functions having a derivative of bounded variation. Finally, in Section 5, we state and prove the quantitative Voronovskaja-type theorem.

2. Preliminary Results

Lemma 1. $\xi_{n,k}^{v}(x)$ *satisfies the following important properties:*

(1) $\xi_{n,k}^{v}(x) - \xi_{n,k+1}^{v}(x) = \mathscr{P}_{n,k}^{v}(x)\, k = 0, 1, 2, \cdots$

(2) $\xi_{n,0}^{v}(x) > \xi_{n,1}^{v}(x) > \cdots \xi_{n,k}^{v}(x) > \xi_{n,k+1}^{v}(x) > \cdots$

$$(3)\ \left[\xi_{n,k}^{v}(x)\right]^{\theta} - \left[\xi_{n,k+1}^{v}(x)\right]^{\theta} \leq \begin{cases} \theta \mathscr{P}_{n,k}^{v}(x) & \text{if } \theta \geq 1 \\ \left(\mathscr{P}_{n,k}^{v}(x)\right)^{\theta} & \text{if } \theta \leq 1 \end{cases}$$

Proof. Since (1) and (2) are evident, we prove only the assertion (3).

If $\theta \geq 1$, it suffices to remark that by the mean value theorem, we have

$$b^{\theta} - a^{\theta} \leq \theta(b - a) \text{ for every } 0 < a < b < 1.$$
(8)

If $\theta < 1$, we shall prove that

$$b^{\theta} - a^{\theta} \leq (b - a)^{\theta} \text{ for every } 0 < a < b.$$
(9)

Dividing this inequality by a^{θ}, it is equivalent to prove that

$$f(r) = (r - 1)^{\theta} - r^{\theta} + 1 \geq 0 \text{ for every } r > 1.$$
(10)

We have $f'(r) = (\theta/(r - 1))e^{\theta \ln (r-1)} - (\theta/r)e^{\theta \ln (r)}$; then,

$$f'(r) > 0 \text{ if and only if } \ln\left(\frac{r}{r-1}\right) > \ln\left(\frac{e^{\theta \ln (r)}}{e^{\theta \ln (r-1)}}\right),$$
(11)

and this is true as $\theta < 1$.

We proved then f is increasing, so $f(r) > f(s)$ for all $r > s > 1$, letting s to 1, and we deduce that $f(r) \geq 0$. □

Remark 2. The operators $\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x)$ have the integral representation

$$\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x) = \int_{0}^{\infty} \mathscr{K}_{n,k}^{v,\theta}(x, u) f(u) du,$$
(12)

where $\mathscr{K}_{n,k}^{v,\theta}(x, u)$ is the kernal defined by

$$\mathscr{K}_{n,k}^{v,\theta}(x, u) = \sum_{k=1}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x) \mathscr{J}_{n,k}^{\rho}(u) + \mathscr{X}_{n,0}^{v,\theta}(x) \delta(u).$$
(13)

$\delta(u)$ is the Dirac-delta function.

Lemma 3. *Let $e_m(t) = t^m$ and $\varphi(t) = 1/(1 + t)^{n+2}$. For the operator $\mathscr{B}_{n,\rho}^{v}(f\,;x)$, we have*

(1) $\mathscr{B}_{n,\rho}^{v}(e_0\,;x) = \sum_{k=0}^{\infty} \mathscr{P}_{n,k}^{v}(x) = \sum_{k=0}^{\infty} \mathscr{P}_{n,k}^{0}(x) = \sum_{k=0}^{\infty} (-1)^k \varphi^{(k)}(x)/k! = \varphi(0) = 1$

(2) $\mathscr{B}_{n,\rho}^{v}(e_m\,;x) = \sum_{k=1}^{\infty} \mathscr{P}_{n,k}^{v}(x).(k\rho + m - 1).(k\rho + m - 2) \cdots (k\rho)/(n\rho)^m, m = 1, 2, 3, \cdots$

As an easy consequence of last lemma, we will prove the following result.

Lemma 4. *We have the following moments:*

(1) $\mathscr{B}_{n,\rho}^{v}(t\,;x) = x + 2x(v-1)/n$

(2) $\mathscr{B}_{n,\rho}^{v}(t^2\,;x) = x^2 + x^2(4v-3)/n + x(-2+n+2v+n\rho + 4\rho(v-1))/n^2\rho$

(3) $\mathscr{B}_{n,\rho}^{v}(t-x\,;x) = 2x(v-1)/n$

(4) $\mathscr{B}_{n,\rho}^{v}((t-x)^2\,;x) = x^2/n + (x/n^2\rho)n(1+\rho) + 2(v-1)(1+2\rho)$

(5) $n^2\mathscr{B}_{n,\rho}^{v}((t-x)^4\,;x) = (x^4/n^3)\alpha_1 + (6x^3/n^3\rho)\alpha_2 + (x^2(1+\rho)/n^3\rho^2)\alpha_3 + (x(1+\rho)/n^4\rho^3)\alpha_4$

where

$$\alpha_1 = 3n + 16v - 10,$$
$$\alpha_2 = n + 6v - 4 + \rho(n + 8v - 6), \tag{14}$$
$$\alpha_3 = 3n(1+\rho) + 4v(7+8\rho) - 25\rho - 17,$$

$$\alpha_4 = n(1+\rho)(3+\rho) + 4(v-1)(3+4\rho(2+\rho)). \tag{15}$$

Remark 5. We have

(1) $\lim_{n\longrightarrow\infty} n\mathscr{B}_{n,\rho}^{v}(t-x\,;x) = 2x(v-1)$

(2) $\lim_{n\longrightarrow\infty} n\mathscr{B}_{n,\rho}^{v}((t-x)^2\,;x) = x(1+\rho+\rho x)/\rho$

(3) $\lim_{n\longrightarrow\infty} n^2\mathscr{B}_{n,\rho}^{v}((t-x)^4\,;x) = 3(x(1+\rho+\rho x))^2/\rho^2$

Remark 6. For *n* large enough, we have the following inequalities:

(1) $|\mathscr{B}_{n,\rho}^{v}((t-x)^2\,;x)| \le C_1(x(1+\rho+\rho x)/n\rho)$

(2) $|\mathscr{B}_{n,\rho}^{v}((t-x)^4\,;x)| \le C_2((x(1+\rho+\rho x))^2/(n\rho)^2)$

Throughout this article, let $\mathscr{C}_B(\mathbb{R}_0^+)$ denote the space of all functions f on $\mathbb{R}_0^+$ which are bounded and continuous. We endowed it by the norm $\|f\| = \sup_{x\in\mathbb{R}_0^+} |f(x)|$.

Lemma 7. *Let $f \in \mathscr{C}_B(\mathbb{R}_0^+)$, and we have*

(1) $\|\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x)\| \le \mathscr{G}_{n,\rho}^{v,\theta}(e_0\,;x)\|f\|$ *and* $\mathscr{G}_{n,\rho}^{v,\theta}(e_0\,;x) = 1$

(2) $\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x) \le \theta\mathscr{B}_{n,\rho}^{v}(f\,;x) \le \theta\|f\|$

Proof.

(1) On the one hand, we have

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x)\right| = \left|\sum_{k=1}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x)\int_0^{\infty} \mathscr{J}_{n,k}^{\rho}(t)f(t)dt + \mathscr{X}_{n,0}^{v,\theta}(x)f(0)\right|$$
$$\le \left|\sum_{k=1}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x)\int_0^{\infty} \mathscr{J}_{n,k}^{\rho}(t)dt + \mathscr{X}_{n,0}^{v,\theta}(x)\right|\cdot\|f\|$$
$$\le \mathscr{G}_{n,\rho}^{v,\theta}(e_0\,;x)\|f\|. \tag{16}$$

On the other hand,

$$\mathscr{G}_{n,\rho}^{v,\theta}(e_0\,;x) = \sum_{k=0}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x) = \left(\xi_{n,0}^{v}(x)\right)^{\theta} = \left(\sum_{k=0}^{\infty} \mathscr{P}_{n,k}^{v}(x)\right)^{\theta} = 1^{\theta} = 1. \tag{17}$$

(2) We have

$$\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x) = \sum_{k=1}^{\infty} \mathscr{X}_{n,k}^{v,\theta}(x)\int_0^{\infty} \mathscr{J}_{n,k}^{\rho}(t)f(t)dt + \mathscr{X}_{n,0}^{v,\theta}(x)f(0)$$
$$= \sum_{k=1}^{\infty} \left(\left[\xi_{n,k}^{v}(x)\right]^{\theta} - \left[\xi_{n,k+1}^{v}(x)\right]^{\theta}\right)\int_0^{\infty} \mathscr{J}_{n,k}^{\rho}(t)f(t)dt$$
$$+ \left(\left[\xi_{n,0}^{v}(x)\right]^{\theta} - \left[\xi_{n,1}^{v}(x)\right]^{\theta}\right)f(0). \tag{18}$$

Using Lemma 1, it is easy to see that

$$\mathscr{G}_{n,\rho}^{v,\theta}(f\,;x) \le \theta\sum_{k=1}^{\infty} \mathscr{P}_{n,k}^{v}(x)\int_0^{\infty} \mathscr{J}_{n,k}^{\rho}(t)f(t)dt + \theta\mathscr{P}_{n,0}^{v}(x)f(0)$$
$$\le \theta\mathscr{B}_{n,\rho}^{v}(f\,;x). \tag{19}$$

□

3. Direct Approximation

Before we discuss the different approximations, we need some definitions. First, we recall the definition of the well-known Ditizian-Totik modulus of smoothness $w_{\varphi^{\tau}}(.,.)$ and Peetre's K-functional [12].

Definition 8. Let $\varphi(x) = \sqrt{x}$ and $f \in \mathscr{C}_B(\mathbb{R}_0^+)$. For $0 \le \tau \le 1$, we define

$$w_{\varphi^{\tau}}(f,\delta) = \sup_{0\le h\le\delta} \sup_{x\pm h\varphi^{\tau}(x)/2\in\mathbb{R}_0^+} \left|f\left(x + \frac{h\varphi^{\tau}(x)}{2}\right) - f\left(x - \frac{h\varphi^{\tau}(x)}{2}\right)\right|, \tag{20}$$

and the K-functional

$$K_{\varphi^\tau}(f,\delta) = \inf_{g \in W_\tau}\left\{\|f - g\| + \delta\|\varphi^\tau g'\|\right\}, \qquad (21)$$

where

$$W_\tau = \left\{g \in \text{AC}_{loc} : \|\varphi^\tau g'\| < \infty\right\}, \qquad (22)$$

with AC_{loc} is the set of all absolutely continuous function on every finite subinterval of $\mathbb{R}_0^+$.

Remark 9. $w_{\varphi^\tau}(f,\delta)$ and $K_{\varphi^\tau}(f,\delta)$ are equivalent, that is, there exists a constant $C > 0$ such that

$$C^{-1}w_{\varphi^\tau}(f,\delta) \leq K_{\varphi^\tau}(f,\delta) \leq Cw_{\varphi^\tau}(f,\delta). \qquad (23)$$

In the next definition, we cite Lipschitz-type functions:

Definition 10 [13]. For $a \geq 0$, $b > 0$ to be fixed, the class of two parametric Lipschitz-type functions is defined as

$$\text{Lip}_M^{a,b}(\beta) = \left\{g \in \mathscr{C}_B(\mathbb{R}_0^+): |f(y) - f(x)| \leq M\frac{|y - x|^\beta}{(y + ax^2 + bx)^{\beta/2}}, x, y > 0\right\}, \qquad (24)$$

where M is any positive constant and $0 < \beta \leq 1$.

The space $\text{Lip}_M^{0,1}(\beta)$ is the space $\text{Lip}_M^*(\beta)$ given by Szász [14].

We now proceed with the approximation results.

Theorem 11. *For $f \in \mathscr{C}_B(\mathbb{R}_0^+)$, we have*

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| \leq Cw_{\varphi^\tau}\left(f, \frac{\varphi^{2-\tau}(1 + x)}{\sqrt{n}}\right), \qquad (25)$$

where w_{φ^τ} is given by (20) and C is a constant free from the choice of n and x.

For the proof of this theorem, we use the following lemma proved in [15].

Lemma 12. *Let $\varphi(x) = \sqrt{x}$ and $0 \leq \tau \leq 1$; then, for $f \in W_\tau$ and $x, y > 0$, we have*

$$\left|\int_x^y f'(u)du\right| \leq 2^\tau x^{-\tau/2}\|x - y\|\|\varphi^\tau f'\|. \qquad (26)$$

Proof (Theorem 11). Let $g \in W_\tau$. Using Lemma 7, we have

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| &= \left|\mathscr{G}_{n,\rho}^{v,\theta}(f - g;x)\right| + |f(x) - g(x)| \\ &\quad + \left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x) - g(x)\right| \\ &\leq (1 + \theta)\|f(x) - g(x)\| \\ &\quad + \left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x) - g(x)\right|.\end{aligned} \qquad (27)$$

Since $g(y) = g(x) + \int_x^y g'(u)du$ and $\mathscr{G}_{n,\rho}^{v,\theta}(1;x) = 1$, we conclude that

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x) - g(x)\right| = \left|\mathscr{G}_{n,\rho}^{v,\theta}\left(\int_x^y g'(u)du;x\right)\right|. \qquad (28)$$

Therefore, Lemma 12 implies

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(g;x) - g(x)\right| \leq 2^\tau x^{-\tau/2}\|\varphi^\tau g'\|\mathscr{G}_{n,\rho}^{v,\theta}(|x - y|;x). \qquad (29)$$

By Cauchy-Schwarz inequality and Remark 6, it is easy to check that

$$\begin{aligned}\mathscr{G}_{n,\rho}^{v,\theta}(|x - y|;x) &= \sqrt{\mathscr{G}_{n,\rho}^{v,\theta}(1;x)}\sqrt{\mathscr{G}_{n,\rho}^{v,\theta}((x - y)^2;x)} \\ &\leq \sqrt{\frac{C_1 x\theta(1 + \rho + \rho x)}{n\rho}}.\end{aligned} \qquad (30)$$

Combining (27)-(30), we get

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| &\leq (1 + \theta)\|f(x) - g(x)\| \\ &\quad + C_3\|\varphi^\tau g'\|\frac{\varphi^{2-\tau}(1 + x)}{\sqrt{n}}.\end{aligned} \qquad (31)$$

Let now taking the infimum over $g \in W_\tau$, and we have

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| \leq C_4 K_{\varphi^\tau}\left(f, \frac{\varphi^{2-\tau}(1 + x)}{\sqrt{n}}\right). \qquad (32)$$

We thank to (26).

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| \leq Cw_{\varphi^\tau}\left(f, \frac{\varphi^{2-\tau}(1 + x)}{\sqrt{n}}\right). \qquad (33)$$

$\square$

Theorem 13. *For $f \in \text{Lip}_M^{a,b}(\beta)$, then for every $n \in \mathbb{N}, \rho > 0, \theta \geq 1$ and $x \in (0,+\infty)$, we have*

$$\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| \leq M\left(\frac{\theta\mathscr{B}_{n,\rho}^v((y - x)^2;x)}{ax^2 + bx}\right)^{\beta/2}, \qquad (34)$$

where $\mathscr{B}_{n,\rho}^v((y - x)^2;x)$ is given in Lemma 4.

Proof. Let $f \in \text{Lip}_M^{a,b}(\beta)$ and $x \in (0+\infty)$, and we have

$$\begin{aligned}\left|\mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x)\right| &= \left|\mathscr{G}_{n,\rho}^{v,\theta}(f(y) - f(x);x)\right| \\ &\leq \mathscr{G}_{n,\rho}^{v,\theta}(|f(y) - f(x)|;x) \\ &\leq \mathscr{G}_{n,\rho}^{v,\theta}\left(M\frac{|y - x|^\beta}{(y + ax^2 + bx)^{\beta/2}};x\right).\end{aligned} \qquad (35)$$

Let us consider the case $\beta = 1$. By the Cauchy-Schwarz inequality and the fact $\mathscr{G}_{n,\rho}^{v,\theta}(1;x) = 1$, we have immediately that

$$
\begin{aligned}
\left| \mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x) \right| &\leq \frac{M}{\sqrt{ax^2 + bx}} \left(\mathscr{G}_{n,\rho}^{v,\theta}\left((y-x)^2;x\right) \right)^{1/2} \\
&\leq \frac{M}{\sqrt{ax^2 + bx}} \left(\theta \mathscr{B}_{n,\rho}^{v}\left((y-x)^2;x\right) \right)^{1/2} \\
&\leq M \left(\frac{\theta \mathscr{B}_{n,\rho}^{v}\left((y-x)^2;x\right)}{ax^2 + bx} \right)^{1/2}.
\end{aligned}
\tag{36}
$$

This proves the result for $\beta = 1$.

If $0 < \beta < 1$, Holder's inequality with exponents $p = 1/\beta$ and $p' = 1/1 - \beta$, we get

$$
\left| \mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x) \right| \leq \frac{M}{(ax^2 + bx)^{\beta/2}} \left(\mathscr{G}_{n,\rho}^{v,\theta}(|y-x|;x) \right)^{\beta}.
\tag{37}
$$

Using again the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\left| \mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x) \right| &\leq \frac{M}{(ax^2 + bx)^{\beta/2}} \left(\mathscr{G}_{n,\rho}^{v,\theta}\left((y-x)^2;x\right) \right)^{\beta/2} \\
&\leq \frac{M}{(ax^2 + bx)^{\beta/2}} \left(\theta \mathscr{B}_{n,\rho}^{v}\left((y-x)^2;x\right) \right)^{\beta/2} \\
&\leq M \left(\frac{\theta \mathscr{B}_{n,\rho}^{v,\theta}\left((y-x)^2;x\right)}{ax^2 + bx} \right)^{\beta/2},
\end{aligned}
\tag{38}
$$

and this gives the result. $\qquad\square$

4. Rate of Convergence in Weighted Spaces

In this section, we focus about the rate of convergence of operators (5) in the context of suitable weighted function spaces and functions having a derivative of bounded variation. We will use the following spaces:

$$
\mathscr{B}_2(\mathbb{R}_0^+) = \left\{ f : |f(x)| \leq M_f\left(1 + x^2\right), M_f \text{ is a constant depend on } f \right\}.
\tag{39}
$$

Introduce also

$$
\begin{aligned}
\mathscr{C}_2(\mathbb{R}_0^+) &= \{ f \in \mathscr{B}_2(\mathbb{R}_0^+) : f \text{ is continuous } \}, \\
\mathscr{C}_2^*(\mathbb{R}_0^+) &= \left\{ f \in \mathscr{C}_2(\mathbb{R}_0^+) : \exists \lim_{x \to \infty} \frac{|f(x)|}{1 + x^2} < \infty \right\}.
\end{aligned}
\tag{40}
$$

These spaces are endowed with the norm

$$
\|f\|_2 = \sup_{x \in \mathbb{R}_0^+} \frac{|f(x)|}{1 + x^2}.
\tag{41}
$$

The weighted modulus of continuity is defined as (see [16])

$$
\Omega(f,\delta) = \sup_{x \geq 0} \sup_{|t| < \delta} \frac{|f(x+t) - f(x)|}{1 + (x+t)^2}.
\tag{42}
$$

Theorem 14. *Let $f \in \mathscr{C}_2^*(\mathbb{R}_0^+)$. Then, for $x \in \mathbb{R}_0^+, \rho, \delta > 0$, $\theta \geq 1$ and for large enough n, we have*

$$
\begin{aligned}
\left| \mathscr{G}_{n,\rho}^{v,\theta}(f;x) - f(x) \right| &\leq 2\left(1 + x^2\right) \Omega\left(f, \frac{1}{\sqrt{n}}\right) \\
&\times \left[1 + \theta C_1 \frac{x(1 + \rho + x\rho)}{n\rho} + \sqrt{\theta C_1} \left(\frac{x(1 + \rho + x\rho)}{\rho} \right)^{1/2} \right. \\
&\times \left. \left(1 + \sqrt{\theta C_2} \frac{x(1 + \rho + x\rho)}{n\rho} \right) \right],
\end{aligned}
\tag{43}
$$

where $C_1, C_2 > 1$ are constants independent of x and n.

Proof. Let $u, x \in \mathbb{R}_0^+, \delta > 0$. An immediate consequence of the definition of weighted modulus of continuity is

$$
|f(u) - f(x)| \leq 2\left(1 + x^2\right)\left(1 + (u-x)^2\right)\left(1 + \frac{|u-x|}{\delta}\right) \Omega(f,\delta).
\tag{44}
$$

Since $\mathscr{G}_{n,\rho}^{v,\theta}(f;x)$ is linear and increasing, we have from (44)

$$
\begin{aligned}
\left| \mathscr{G}_{n,\rho}^{v,\theta}(|f(u) - f(x)|;x) \right| &\leq 2\left(1 + x^2\right)\Omega(f,\delta) \\
&\cdot \left[\mathscr{G}_{n,\rho}^{v,\theta}\left(\left(1 + (u-x)^2\right);x\right) + \mathscr{G}_{n,\rho}^{v,\theta}\left(\frac{\left(1 + (u-x)^2\right)|u-x|}{\delta};x \right) \right].
\end{aligned}
\tag{45}
$$

Cauchy-Schwarz inequality was applied in the last term, and it gives us

$$
\begin{aligned}
\left| \mathscr{G}_{n,\rho}^{v,\theta}(|f(u) - f(x)|;x) \right| &\leq 2\left(1 + x^2\right)\Omega(f,\delta) \\
&\cdot \left[1 + \mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right) \frac{1}{\delta} \left(\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^4;x\right) \right)^{1/2} \right. \\
&\cdot \left(\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right) \right)^{1/2} + \frac{1}{\delta} \left(\mathscr{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right) \right)^{1/2} \left. \right].
\end{aligned}
\tag{46}
$$

Choosing $\delta = 1/\sqrt{n}$, we get the required result in virtue of Remark 6. $\qquad\square$

5. Rate of Convergence for Functions of Bounded Variation

Let $\mathrm{DBV}(\mathbb{R}_0^+)$ be the space of functions on $\mathbb{R}_0^+$ having a derivative of bounded variation on every finite subinterval of $\mathbb{R}_0^+$. Consider the space

$$\mathrm{DBV}_2(\mathbb{R}_0^+) = \left\{ f \in \mathrm{DBV}(\mathbb{R}_0^+) : |f(x)| \leq M_f (1 + x^2) \text{ for some constant } M_f > 0 \right\}. \tag{47}$$

It is known that every function f in $\mathrm{DBV}_2(\mathbb{R}_0^+)$ has a representation of the form

$$f(x) = \int_0^x g(u) du + f(0), \tag{48}$$

where g is a function of bounded variation on each finite subinterval of $\mathbb{R}_0^+$.

Lemma 15. *Let $x \in \mathbb{R}_0^+$, and let $\mathcal{K}_{n,\rho}^{v,\theta}(x, u)$ be the kernel defined by (13). Then, for $C_1 > 1$ and for n large enough, we have*

(1) $\xi_{n,\rho}^{v,\theta}(x\,;y) = \int_0^y \mathcal{K}_{n,\rho}^{v,\theta}(x,u) du \leq \theta C_1 (x(1 + \rho + x\rho)/n\rho)$
   $1/(x - y)^2, 0 \leq y < x$

(2) $1 - \xi_{n,\rho}^{v,\theta}(x\,;z) = \int_z^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) du \leq \theta C_1 (x(1 + \rho + x\rho)$
   $/n\rho)1/(z - x)^2, x < z < \infty$

Proof. Using Remark 6, we get

$$\xi_{n,\rho}^{v,\theta}(x\,;y) = \int_0^y \mathcal{K}_{n,\rho}^{v,\theta}(x,u) du \leq \int_0^y \left(\frac{u-y}{x-y}\right)^2 \mathcal{K}_{n,\rho}^{v,\theta}(x,u) du$$
$$\leq \frac{1}{(x-y)^2} \mathcal{G}_{n,\rho}^{v,\theta}((u-x)^2, x)$$
$$\leq \theta C_1 \frac{x(1 + \rho + x\rho)}{n\rho} \frac{1}{(x-y)^2}. \tag{49}$$

Similarly, we can show the second part; hence, the proof is omitted. $\square$

Theorem 16. *Let $f \in DBV_2(\mathbb{R}_0^+)$, and for every $x \in (0, \infty)$, consider the function f_x' defined by*

$$f_x'(u) = \begin{cases} f'(u) - f'(x^-), & \text{if } 0 \leq u < x, \\ 0, & \text{if } u = x, \\ f'(u) - f'(x^+), & \text{if } x < u < \infty. \end{cases} \tag{50}$$

Let us denote by $\vee_c^d f_x'$ the total variation of f_x' on $[c, d] \subset \mathbb{R}_0^+$. Then, for every $x \in (0, \infty)$ and large n,

$$\left| \mathcal{G}_{n,\rho}^{v,\theta}(f\,;x) - f(x) \right|$$
$$\leq \frac{\sqrt{\theta}}{1+\theta} |f'(x^+) + \theta f'(x^-)| \left(\frac{C_1 x(1 + \rho + x\rho)}{n\rho} \right)^{1/2}$$
$$+ \frac{\theta^{3/2}}{1+\theta} |f'(x^+) + \theta f'(x^-)| \left(\frac{C_1 x(1 + \rho + x\rho)}{n\rho} \right)^{1/2}$$
$$+ \theta \frac{C_1(1 + \rho + x\rho)}{n\rho} \sum_{k=1}^{[\sqrt{n}]} \left(\overset{x}{\underset{x-\frac{x}{k}}{\vee}} f_x' \right) + \frac{x}{\sqrt{n}} \left(\overset{x}{\underset{x-\frac{x}{\sqrt{n}}}{\vee}} f_x' \right)$$
$$+ \frac{x}{\sqrt{n}} \left(\overset{x+\frac{x}{\sqrt{n}}}{\underset{x}{\vee}} f_x' \right) + \theta \frac{C_1(1 + \rho + x\rho)}{n\rho} \sum_{0}^{[\sqrt{n}]} \left(\overset{x+\frac{x}{k}}{\underset{x}{\vee}} f_x' \right). \tag{51}$$

Proof. For any $f \in \mathrm{DBV}_2(\mathbb{R}_0^+)$, from the definition of $f_x'(u)$, we can write

$$f'(u) = \frac{1}{1+\theta} \left(f'(x^+) + \theta f'(x^-) \right) + \delta_x(u)$$
$$\cdot \left(f'(x) - \frac{1}{2} \left(f'(x^+) + f'(x^-) \right) \right)$$
$$+ f_x'(u) + \frac{1}{2} \left(f'(x^+) - f'(x^-) \right) \left(\mathrm{sgn}\ (u - x) + \frac{\theta - 1}{1+\theta} \right), \tag{52}$$

where

$$\delta_x(u) = \begin{cases} 1, & \text{if } u = x, \\ 0, & \text{if } u \neq x. \end{cases} \tag{53}$$

By the fact that $\mathcal{G}_{n,\rho}^{v,\theta}(1\,;x) = 1$, we have

$$\mathcal{G}_{n,\rho}^{v,\theta}(f\,;x) - f(x) = \mathcal{G}_{n,\rho}^{v,\theta}(f(u) - f(x)\,;x)$$
$$= \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)(f(u) - f(x)) du$$
$$= \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) \left(\int_x^u f'(v) dv \right) du. \tag{54}$$

From (52), we obtain

$$\mathcal{G}_{n,\rho}^{v,\theta}(f\,;x) - f(x) = \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) \left(\int_x^u \left\{ \frac{1}{1+\theta} \left(f'(x^+) + \theta f'(x^-) \right) \right\} dv \right) du$$
$$+ \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) \left(\int_x^u \left\{ \frac{1}{2} \left(f'(x^+) - f'(x^-) \right) \times \left(\mathrm{sgn}\ (v - x) + \frac{\theta - 1}{1+\theta} \right) \right\} dv \right) du$$
$$+ \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) \left(\int_x^u \delta_x(v) \left(f'(x) - \frac{1}{2} \left(f'(x^+) + f'(x^-) \right) \right) dv \right) du$$
$$+ \int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u) \left(\int_x^u f_x'(v) dv \right) du. \tag{55}$$

From the definition of $\delta_x(v)$, it is clear that

$$\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u \delta_x(v)\left(f'(x) - \frac{1}{2}\left(f'(x^+) + f'(x^-)\right)\right)dv\right)du = 0. \tag{56}$$

The first integral on the right hand side of (55) can be estimated as follows:

$$\left|\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u \left\{\frac{1}{1+\theta}\left(f'(x) + +\theta f'(x^-)\right)\right\}dv\right)du\right|$$
$$\leq \frac{1}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)|u-x|du. \tag{57}$$

Applying the Cauchy-Schwarz inequality and Remark 2, we have, for n large enough,

$$\left|\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u \left\{\frac{1}{1+\theta}\left(f'(x^+) + \theta f'(x^-)\right)\right\}dv\right)du\right|$$
$$\leq \frac{1}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\sqrt{\mathcal{G}_{n,\rho}^{v,\theta}((u-x)^2;x)}$$
$$\leq \frac{1}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\left(\frac{C_1\theta x(1+\rho+x\rho)}{n\rho}\right)^{1/2}$$
$$\leq \frac{\sqrt{\theta}}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\left(\frac{C_1 x(1+\rho+x\rho)}{n\rho}\right)^{1/2}. \tag{58}$$

Similarly, it is easy to find

$$\left|\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u \left\{\frac{1}{2}\left(f'(x^+) - f'(x^-)\right)\left(\mathrm{sgn}\ (v-x) + \frac{\theta-1}{1+\theta}\right)\right\}dv\right)du\right|$$
$$\leq \frac{\theta^{3/2}}{1+\theta}\left|f'(x^+) + \theta f'(x^-)\right|\left(\frac{C_1 x(1+\rho+x\rho)}{n\rho}\right)^{1/2}. \tag{59}$$

Write the last term of (55) as

$$\int_0^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f_x'(v)dv\right)du = \mathcal{A}_{n,\rho}^{v,\theta}\left(f_x';x\right) + \mathcal{B}_{n,\rho}^{v,\theta}\left(f_x';x\right), \tag{60}$$

where

$$\mathcal{A}_{n,\rho}^{v,\theta}\left(f_x';x\right) = \int_0^x \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f_x'(v)dv\right)du,$$
$$\mathcal{B}_{n,\rho}^{v,\theta}\left(f_x';x\right) = \int_x^\infty \mathcal{K}_{n,\rho}^{v,\theta}(x,u)\left(\int_x^u f_x'(v)dv\right)du. \tag{61}$$

Now, we estimate the terms $\mathcal{A}_{n,\rho}^{v,\theta}(f_x';x)$ and $\mathcal{B}_{n,\rho}^{v,\theta}(f_x';x)$.

Using the definition of $\xi_{n,\rho}^{v,\theta}(.;.)$ given in Lemma 15 and integrating by parts, we can write

$$\mathcal{A}_{n,\rho}^{v,\theta}\left(f_x';x\right) = \int_0^x \left(\int_x^u f_x'(v)dv\right)\frac{\partial \xi_{n,\rho}^{v,\theta}(x;u)}{\partial u}du$$
$$= \int_0^x f_x'(u)\xi_{n,\rho}^{v,\theta}(x;u)du. \tag{62}$$

Thus,

$$\left|\mathcal{A}_{n,\rho}^{v,\theta}\left(f_x';x\right)\right| \leq \int_0^{x-x/\sqrt{n}} \left|f_x'(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du$$
$$+ \int_{x-x/\sqrt{n}}^x \left|f_x'(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du. \tag{63}$$

Since $f_x'(x) = 0$ and $\xi_{n,\rho}^{v,\theta}(x;u) \leq 1$, we get

$$\int_{x-x/\sqrt{n}}^x \left|f_x'(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du$$
$$= \int_{x-x/\sqrt{n}}^x \left|f_x'(u) - f_x'(x)\right|\xi_{n,\rho}^{v,\theta}(x;u)du$$
$$\leq \int_{x-x/\sqrt{n}}^x \left|f_x'(u) - f_x'(x)\right|du \leq \int_{x-x/\sqrt{n}}^x \left(\bigvee_u^x f_x'\right)du \tag{64}$$
$$\leq \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f_x'\right).$$

Concerning the first integral on the right hand side of (63), using Lemma 15, we have

$$\int_0^{x-x/\sqrt{n}} \left|f_x'(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du$$
$$\leq \theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}} \frac{\left|f_x'(u)\right|}{(x-y)^2}du$$
$$= \theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}} \frac{\left|f_x'(u) - f_x'(x)\right|}{(x-y)^2}du \tag{65}$$
$$= \theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_0^{x-x/\sqrt{n}} \left(\bigvee_u^x f_x'\right)\frac{du}{(x-y)^2}.$$

By changing of variable $u = x - x/v$, we deduce that

$$\int_0^{x-x/\sqrt{n}} \left|f_x'(u)\right|\xi_{n,\rho}^{v,\theta}(x;u)du \leq \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\int_1^{\sqrt{n}} \left(\bigvee_{x-\frac{x}{v}}^x f_x'\right)dv$$
$$\leq \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^x f_x'\right). \tag{66}$$

Therefore,

$$\left| \mathcal{A}_{n,\rho}^{v,\theta}\left(f_x';x\right)\right| \le \theta \frac{C_1(1+\rho+x\rho)}{n\rho} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{k}}^{x} f_x'\right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x} f_x'\right).$$

(67)

What concerns the second term of the right hand side of (60), integrating by parts and Lemma 15 with $z = x + x/\sqrt{n}$, we can write

$$
\begin{aligned}
\left| \mathcal{B}_{n,\rho}^{v,\theta}\left(f_x';x\right)\right| &\le \left|\int_x^z f_x'(u)\left(1-\xi_{n,\rho}^{v,\theta}(x;u)\right)du\right| \\
&\quad + \left|\int_z^\infty f_x'(u)\left(1-\xi_{n,\rho}^{v,\theta}(x;u)\right)du\right| \\
&\le \int_x^z \bigvee_x^u f_x' du + \theta\frac{C_1 x(1+\rho+x\rho)}{n\rho}\int_z^\infty \bigvee_x^u f_x' \frac{1}{(u-x)^2} du \\
&\le \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \theta\frac{C_1 x(1+\rho+x\rho)}{n\rho} \\
&\quad \cdot \int_{x+x/\sqrt{n}}^\infty \bigvee_x^u f_x'(u-x)^{-2} du.
\end{aligned}
$$

(68)

Putting $u = x + x/v$, we get

$$
\begin{aligned}
\theta\frac{C_1 x(1+\rho+x\rho)}{n\rho} &\int_{x+x/\sqrt{n}}^\infty \bigvee_x^u f_x'(u-x)^{-2} du \\
&\le \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\int_0^{\sqrt{n}} \bigvee_x^{x+\frac{x}{v}} f_x' dv \\
&\le \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_0^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).
\end{aligned}
$$

(69)

Combining (68) and (69), we have

$$\left| \mathcal{B}_{n,\rho}^{v,\theta}\left(f_x';x\right)\right| \le \frac{x}{\sqrt{n}}\left(\bigvee_x^{x+\frac{x}{\sqrt{n}}} f_x'\right) + \theta\frac{C_1(1+\rho+x\rho)}{n\rho}\sum_0^{[\sqrt{n}]}\left(\bigvee_x^{x+\frac{x}{k}} f_x'\right).$$

(70)

Finally, by combining (52)-(70), we get (51). □

6. Quantitative Voronovskaja-Type Asymptotic Formula

In this last section, we deal with the Voronovskaja-type asymptotic theorem for $\mathcal{G}_{n,\rho}^{v,\theta}$. More precisely we will prove the following result:

Theorem 17. *For $f \in \mathcal{C}_B(\mathbb{R}_0^+)$ such that $f', f'' \in \mathcal{C}_B(\mathbb{R}_0^+)$. Then,*

$$
\begin{aligned}
\Big| n\Big\{ &\mathcal{G}_{n,\rho}^{v,\theta}(f;x) - f(x) - f'(x)\mathcal{G}_{n,\rho}^{v,\theta}(u-x;x) \\
&- \frac{1}{2}f''(x)\mathcal{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\Big\}\Big| \\
&\le C\theta\frac{x(1+\rho+x\rho)}{\rho} w_{\varphi^\tau}\left(f, \frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}}\right),
\end{aligned}
$$

(71)

where C is independent of n and x.

Proof. By Taylor's formula, we write

$$f(u) = f(x) + (u-x)f'(x) + \int_x^u (u-v)f''(v)dv.$$

(72)

It is clear that

$$
\begin{aligned}
&f(u) - f(x) - (u-x)f'(x) - \frac{1}{2}(u-x)^2 f''(x) \\
&= \int_x^u (u-v)\left(f''(v) - f''(x)\right)dv.
\end{aligned}
$$

(73)

On the one hand, we apply $\mathcal{G}_{n,\rho}^{v,\theta}(.;x)$ to both sides of the above equality, and we get

$$
\begin{aligned}
\Big| &\mathcal{G}_{n,\rho}^{v,\theta}(f;x) - f(x) - f'(x)\mathcal{G}_{n,\rho}^{v,\theta}(u-x;x) \\
&- \frac{1}{2}f''(x)\mathcal{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\Big| \\
&= \left| \mathcal{G}_{n,\rho}^{v,\theta}\left(\int_x^u (u-v)\left(f''(v)-f''(x)\right)dv;x\right)\right| \\
&\le \mathcal{G}_{n,\rho}^{v,\theta}\left(\left|\int_x^u (u-v)\left(f''(v)-f''(x)\right)dv\right|;x\right).
\end{aligned}
$$

(74)

On the other hand, for $g \in W_\tau$, we have

$$
\begin{aligned}
\left|\int_x^u (u-v)\left(f''(v)-f''(x)\right)dv\right| \\
\le \left\|f''-g\right\|(u-x)^2 + 2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\||u-x|^3,
\end{aligned}
$$

(75)

which implies, by (74),

$$
\begin{aligned}
\Big| &\mathcal{G}_{n,\rho}^{v,\theta}(f;x) - f(x) - f'(x)\mathcal{G}_{n,\rho}^{v,\theta}(u-x;x) \\
&- \frac{1}{2}f''(x)\mathcal{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right)\Big| \\
&\le \left\|f''-g\right\|\mathcal{G}_{n,\rho}^{v,\theta}\left((u-x)^2;x\right) \\
&\quad + 2^\tau\varphi^{-\tau}\left\|\varphi^\tau g'\right\|\mathcal{G}_{n,\rho}^{v,\theta}\left(|u-x|^3;x\right).
\end{aligned}
$$

(76)

After using the Cauchy-Schwarz inequality in the last term, we obtain

$$\left| \mathscr{G}_{n,\rho}^{\nu,\theta}(f\,;x) - f(x) - f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x\,;x) - \frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^2\,;x) \right|$$

$$\le 2^\tau \varphi^{-\tau} \|\varphi^\tau g'\| \left(\mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^4\,;x) \right)^{1/2} \left(\mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^2\,;x) \right)^{1/2}$$

$$+ \|f'' - g\| \mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^2\,;x). \tag{77}$$

In view of Remark 6, we have

$$\left| \mathscr{G}_{n,\rho}^{\nu,\theta}(f\,;x) - f(x) - f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x\,;x) - \frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^2\,;x) \right|$$

$$\le 2^\tau \varphi^{-\tau} \|\varphi^\tau g'\| \left(C_1\theta \frac{x(1+\rho+x\rho)}{n\rho} \right)^{1/2} \left(C_2\theta \left(\frac{x(1+\rho+x\rho)}{n\rho} \right)^2 \right)^{1/2}$$

$$+ \|f'' - g\| C_1\theta \frac{x(1+\rho+x\rho)}{n\rho}$$

$$\le C_1\theta \frac{x(1+\rho+x\rho)}{n\rho} \left\{ \|f'' - g\| + M^* \frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}} \|\varphi^\tau g'\| \right\}. \tag{78}$$

Taking the infimum on the right-hand side of the above inequality over $g \in W_\tau$, we get

$$\left| n \left\{ \mathscr{G}_{n,\rho}^{\nu,\theta}(f\,;x) - f(x) - f'(x)\mathscr{G}_{n,\rho}^{\nu,\theta}(u-x\,;x) \right. \right.$$

$$\left. \left. - \frac{1}{2}f''(x)\mathscr{G}_{n,\rho}^{\nu,\theta}((u-x)^2\,;x) \right\} \right| \tag{79}$$

$$\le C\theta \frac{x(1+\rho+x\rho)}{\rho} K_{\varphi^\tau} \left(f, M^* \frac{\varphi^{2-\tau}(1+x)}{\sqrt{n}} \right).$$

Recalling (23), the theorem is proved. □

Data Availability

We do not have any data supporting our results.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgments

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