

## Research Article

# On a Nonhomogeneous Timoshenko System with Nonlocal Constraints

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Our main concern in this paper is to prove the well posedness of a nonhomogeneous Timoshenko system with two damping terms. The system is supplemented by some initial and nonlocal boundary conditions of integral type. The uniqueness and continuous dependence of the solution on the given data follow from some established a priori bounds, and the proof of the existence of the solution is based on some density arguments.

## 1. Introduction

Timoshenko [1] was the first who introduced a model describing the transverse vibration of a beam. More precisely, his research work concerns with the correction for shear of a differential equation for transverse vibrations of prismatic bars. This model was given by a system of two coupled hyperbolic partial differential equations complemented with some boundary conditions.

$$\begin{cases} \rho_1 u_{tt} = \kappa(u_x - v)_x & (x, t) \in (0, L) \times (0, \infty), \\ \rho_2 v_{tt} = \kappa^* v_{xx} + \kappa(u_x - v) & (x, t) \in (0, L) \times (0, \infty), \\ (u_x - v)|_{x=0} = 0, v_x|_{x=0} = 0, & \end{cases} \quad (1)$$

where  $L$  is the length of the beam in its equilibrium configuration. The function  $u$  models the transverse displacement of the beam, and  $v$  models the rotation angle of its filament. The coefficients  $\rho_1, \rho_2, \kappa$ , and  $\kappa^*$  are, respectively, the density, the polar moment of inertia of a cross section, the shear modulus, and the Young's modulus of elasticity. In [2], the authors considered and proved some exponential decay results for a linear homogeneous Timoshenko system with a memory term of the form

$$\begin{cases} \rho_1 U_{tt} - \kappa_1(U_x + V)_x = 0, \\ \rho_2 V_{tt} - \kappa_2 V_{xx} + \kappa_1(U_x + V) + h * V_{xx}(x, t) = 0, \\ U(0, t) = U(L, t) = V(0, t) = V(L, t) = 0, \\ U(x, 0) = U_0, U_t(x, 0) = U_1, V(x, 0) = V_0, V_t(x, 0) = V_1, \end{cases} \quad (2)$$

where  $(x, t) \in (0, L) \times (0, \infty)$ . The same problem (2) was considered in [3] where the authors discussed the decay properties of the semigroup generated by a linear Timoshenko system with fading memory. In paper [4], the authors studied the exponential stability for the following Timoshenko system with two weak dampings.

$$\begin{cases} \rho_1 u_{tt} = \kappa(u_x - v)_x - u_t, & \text{in } (0, L) \times (0, \infty), \\ \rho_2 v_{tt} = \kappa^* v_{xx} - \kappa(u_x - v)_x - v_t, & \text{in } (0, L) \times (0, \infty), \\ u(0, t) = u(L, t) = v(0, t) = v(L, t) = 0, & t > 0. \end{cases} \quad (3)$$

In [5], the authors investigated the effect of both frictional and viscoelastic dampings. They considered in the domain  $(0, L) \times (0, \infty)$  the following system

$$\begin{cases} U_{tt} - (U_x + V)_x = 0, \\ V_{tt} - V_{xx} + U_x + V + \int_0^t g(t-s)(a(x)v_x(x,s))_x ds + b(x)h(v_t) = 0, \\ u(0, t) = u(1, t) = v(0, t) = v(1, t) = 0, \quad t > 0, \end{cases} \quad (4)$$

and proved some exponential and polynomial decay results. For more results concerning Timoshenko systems, we refer the reader to [6–15].

Motivated by the above systems, we consider a nonlocal initial boundary value problem for a nonhomogeneous Timoshenko system with memory term of type (2), complemented with boundary integral boundary conditions. The study of mixed problems with nonlocal conditions such as integral conditions goes back to the year 1963, when Cannon [16] used the potential method to investigate the existence and uniqueness of the solution of the heat equation subject to the specification of energy (integral constraint). This type of conditions arises mainly when the data cannot be measured directly on the boundary, but only their averages (weighted averages) are known. Due to their importance, physical significance (mean, total flux, total energy, etc.), and numerous applications in different fields of science and engineering, several authors extensively studied this type of problems, and we can cite, for example, [17–24]. Some recent new results on this direction were obtained (see [25, 26]). In this work, a functional analysis method based on some a priori bounds and on the density of the range of the unbounded operator corresponding to the abstract formulation of the given problem is used to prove the well posedness of the posed problem. This work can be considered as a contribution to the development of the energy inequality method used to prove the well posedness of mixed problems with nonlocal conditions such as integral boundary conditions (see, for example, [17, 18, 27–31]).

## 2. Formulation of the Problem and Function Spaces

In the bounded domain  $Q^T = (0, L) \times (0, T)$ , we consider the initial boundary value problem for a nonhomogeneous Timoshenko system with a viscoelastic term of the form

$$\begin{cases} \mathcal{L}_1(u, v) = F(x, t), \text{ in } (0, L) \times [0, T], \\ \mathcal{L}_2(u, v) = G(x, t), \text{ in } (0, L) \times [0, T], \end{cases} \quad (5)$$

where

$$\begin{cases} \mathcal{L}_1(u, v) = \rho_1 u_{tt} - \kappa_1 (u_x + v)_x + u_t, \\ \mathcal{L}_2(u, v) = \rho_2 v_{tt} - \kappa_2 v_{xx} + \kappa_1 (u_x + v) + \int_0^t h(t-s)v_{xx}(x, s) ds, \end{cases} \quad (6)$$

$\rho_1, \rho_2, \kappa_1$ , and  $\kappa_2$  are positive constants,  $f, g, \varphi, \psi, F$ , and  $G$  are given functions, and  $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a twice differentiable function such that

$$\kappa_2 - \int_0^T h(t) dt = l > 0, \quad h'(t) < 0, \quad \forall t \geq 0. \quad (7)$$

The convolution term

$$h * v_{xx}(x, t) = \int_0^t h(t-s)v_{xx}(x, s) ds, \quad (8)$$

represents the memory effect with a real valued function  $h$  of class  $C^2$ .

System (5) is supplemented with the initial conditions

$$\begin{cases} \ell_1 u = u(x, 0) = \varphi(x), \quad \ell_2 u = u_t(x, 0) = \psi(x), \\ \ell_1 v = v(x, 0) = f(x), \quad \ell_2 v = v_t(x, 0) = g(x), \end{cases} \quad (9)$$

and the boundary integral conditions

$$\int_0^L u dx = 0, \quad \int_0^L x u dx = 0, \quad \int_0^L v dx = 0, \quad \int_0^L x v dx = 0. \quad (10)$$

This system of coupled hyperbolic equations represents a Timoshenko model for a thick beam of length  $L$ , where  $u$  is the transverse displacement of the beam and  $v$  is the rotation angle of the filament of the beam. The coefficients  $\rho_1, \rho_2, \kappa_1$ , and  $\kappa_2$  are, respectively, the density, the polar moment of inertia of a cross section, the shear modulus, and the Young modulus of elasticity. The integral conditions represent the averages (weighted averages) of the total transverse displacement of the beam and the rotation angle of the filament of the beam.

Our aim is to study the well posedness of the solution of problems (5), (9), and (10). That is, on the basis of some a priori bounds and on the density of the range of the operator generated by the problem under consideration, we prove the existence, uniqueness, and continuous dependence of the solution on the given data of problems (5), (9), and (10). We now introduce some function spaces needed throughout the sequel. Let  $L^2(Q^T)$  be the Hilbert space of square integrable functions on  $Q^T = (0, 1) \times (0, T)$ ,  $T < \infty$ , with scalar product and norm, respectively.

$$(Z, S)_{L^2(Q^T)} = \int_{Q^T} Z S dx dt, \quad \|Z\|_{L^2(Q^T)}^2 = \int_{Q^T} Z^2 dx dt. \quad (11)$$

We also use the space  $L^2((0, 1))$  on the interval  $(0, 1)$ , whose definition is analogous to the space on  $Q$ . Let  $B_2^1(0, L)$  be the space obtained by completion of the space  $C_0(0, L)$  of real continuous functions with compact support in the interval  $(0, L)$  with respect to the inner product

$$(\theta, \theta^*)_{B_2^1(0, L)} = \int_0^L \mathfrak{F}_x \theta \cdot \mathfrak{F}_x \theta^* dx, \quad (12)$$

where  $\mathfrak{F}_x \theta = \int_0^x \theta(\zeta) d\zeta$  for every fixed  $x \in (0, L)$ . The associated norm is  $\|\theta\|_{B_2^1(0, L)}^2 = \sqrt{(\theta, \theta)_{B_2^1(0, L)}} = \int_0^L (\mathfrak{F}_x \theta)^2 dx$ . We

denote by  $C(\bar{J}; L^2(0, L))$ , with  $J = (0, T)$  the set of all continuous functions  $\theta(\cdot, t): J \rightarrow L^2(0, L)$  with norm

$$\|\theta\|_{C(\bar{J}; L^2(0, L))}^2 = \sup_{0 \leq t \leq T} \|\theta(\cdot, t)\|_{L^2(0, L)}^2 < \infty, \quad (13)$$

and  $C(\bar{J}; B_2^1(0, L))$  the set of functions  $\theta(\cdot, t): \bar{J} \rightarrow B_2^1(0, L)$  with norm

$$\begin{aligned} \|\theta\|_{C(\bar{J}; B_2^1(0, L))}^2 &= \sup_{0 \leq t \leq T} \|\mathfrak{F}_x \theta(\cdot, t)\|_{L^2(0, L)}^2 \\ &= \sup_{0 \leq t \leq T} \|\theta(\cdot, t)\|_{B_2^1(0, L)}^2 < \infty. \end{aligned} \quad (14)$$

To obtain a priori estimates for the solution, we write down our problems (5), (9), and (10) in its operator form:  $\mathcal{A}\mathcal{U} = H$  with  $\mathcal{U} = (u, v)$ ,  $\mathcal{A}\mathcal{U} = (L_1(u, v), L_2(u, v))$ , and  $H = (H_1, H_2)$ , where

$$\begin{cases} L_1(u, v) = \{\mathcal{L}_1(u, v), \ell_1 u, \ell_2 u\}, \\ L_2(u, v) = \{\mathcal{L}_2(u, v), \ell_1 v, \ell_2 v\}, \\ H_1 = \{F, \varphi, \psi\}, H_2 = \{G, f, g\}. \end{cases} \quad (15)$$

The operator  $\mathcal{A}$  is an unbounded operator of domain of definition  $D(\mathcal{A})$  consisting of elements  $(u, v) \in (L^2(\bar{J}; L^2(0, L)))^2$  such that  $u_x, v_x, u_t, v_t, u_{tt}, v_{tt}, u_{xx}, v_{xx}$  belong to  $L^2(\bar{J}; L^2(0, L))$  and verify initial and boundary conditions (9) and (10). The operator  $\mathcal{A}$  is acting from the Banach space  $\mathcal{B}$  into the Hilbert space  $\mathcal{E}$ , where  $\mathcal{B}$  is the Banach space obtained by completing  $D(\mathcal{A})$  with respect to the norm

$$\begin{aligned} \|\mathcal{U}\|_{\mathcal{B}}^2 &= \|u_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))}^2 + \|v_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))}^2 \\ &\quad + \|u(\cdot, t)\|_{C(\bar{J}; L^2(0, L))}^2 + \|v(\cdot, t)\|_{C(\bar{J}; L^2(0, L))}^2, \end{aligned} \quad (16)$$

and  $\mathcal{E} = [L^2(Q^T) \times (L^2(0, L))^2] \times [L^2(Q^T) \times (L^2(0, L))^2]$  is the Hilbert space consisting of vector-valued functions  $H = (\{F, \varphi, \psi\}, \{G, f, g\})$  for which the norm

$$\begin{aligned} \|H\|_{\mathcal{E}}^2 &= \|F\|_{L^2(Q^T)}^2 + \|\varphi\|_{L^2(0, L)}^2 + \|\psi\|_{L^2(0, L)}^2 + \|G\|_{L^2(Q^T)}^2 \\ &\quad + \|f\|_{L^2(0, L)}^2 + \|g\|_{L^2(0, L)}^2. \end{aligned} \quad (17)$$

is finite. The functions  $\mathcal{U} = (u, v)$  are continuous on the interval  $\bar{J}$  with values in  $L^2(0, L)$  and having continuous derivatives  $\mathcal{U}_t = (u_t, v_t)$  on  $\bar{J}$  with values in  $B_2^1(0, L)$ . Hence, the mappings

$$\begin{cases} \ell_1 : \mathcal{B} \ni \mathcal{U} = (u, v) \rightarrow \ell_1 \mathcal{U}|_{t=0} = (\ell_1 u, \ell_1 v) = (u, v)|_{t=0} \in L^2(0, L) \times L^2(0, L), \\ \ell_2 : \mathcal{B} \ni \mathcal{U}_t = (u_t, v_t) \rightarrow \ell_2 \mathcal{U}|_{t=0} = (\ell_2 u, \ell_2 v) = (u_t, v_t)|_{t=0} \in B_2^1(0, L) \times B_2^1(0, L), \end{cases} \quad (18)$$

are defined and continuous on the Banach space  $\mathcal{B}$ .

### 3. A Priori Estimate and Its Consequences

In this section, we establish an energy inequality from which we deduce the uniqueness and continuous dependence of solution of problems (5), (9), and (10) on the given data.

**Theorem 1.** For any function  $\mathcal{U} = (u, v) \in D(\mathcal{A})$ , the following a priori estimate holds

$$\begin{aligned} &\|u_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))} + \|v_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))} \\ &\quad + \|u(\cdot, t)\|_{C(\bar{J}; L^2(0, L))} + \|v(\cdot, t)\|_{C(\bar{J}; L^2(0, L))} \\ &\leq C \left( \|\varphi\|_{L^2(0, L)}^2 + \|\psi\|_{L^2(0, L)}^2 + \|g\|_{L^2(0, L)}^2 \right. \\ &\quad \left. + \|f\|_{L^2(0, L)}^2 + \|F\|_{L^2(Q^T)}^2 + \|G\|_{L^2(Q^T)}^2 \right), \end{aligned} \quad (19)$$

where  $C = \mathcal{D}e^{\mathcal{D}T}$  with  $\mathcal{D}$  is a positive constant independent of  $\mathcal{U} \ll (u, v)$  given by equation (41) below.

*Proof.* Define the integrodifferential operators  $\mathcal{M}_1 u = -\mathfrak{F}_x^2 u_t$  and  $\mathcal{M}_2 v = -\mathfrak{F}_x^2 v_t$ , where

$$\begin{aligned} \mathfrak{F}_x^2 u(x, t) &= \int_0^x \int_0^\xi u(\eta, t) d\eta d\xi, \quad \mathfrak{F}_x^2 v(x, t) \\ &= \int_0^x \int_0^\xi v(\eta, t) d\eta d\xi, \end{aligned} \quad (20)$$

and consider the identity

$$\begin{aligned} &(\rho_1 u_{tt}, \mathcal{M}_1 u)_{L^2(Q^T)} - \kappa_1 ((u_x + v)_x, \mathcal{M}_1 u)_{L^2(Q^T)} \\ &\quad - (u_t, \mathcal{M}_1 u)_{L^2(Q^T)} + (\rho_2 v_{tt}, \mathcal{M}_2 v)_{L^2(Q^T)} \\ &\quad - \kappa_2 ((v_{xx}, \mathcal{M}_2 v)_{L^2(Q^T)} + \kappa_1 ((u_x + v), \mathcal{M}_2 v)_{L^2(Q^T)}) \\ &\quad + \left( \int_0^t h(t-s) v_{xx}(x, s) ds, \mathcal{M}_2 v \right)_{L^2(Q^T)} \\ &= (F(x, t), \mathcal{M}_1 u)_{L^2(Q^T)} + (G(x, t), \mathcal{M}_2 v)_{L^2(Q^T)}, \end{aligned} \quad (21)$$

where  $Q^T = (0, L) \times (0, \tau)$ .

The standard integration by parts of each term in (21) and conditions (9) and (10) give

$$\begin{aligned} -\rho_1 (u_{tt}, \mathfrak{F}_x^2 u_t)_{L^2(Q^T)} &= -\rho_1 \int_0^\tau \int_0^L u_{tt} \mathfrak{F}_x^2 u_t dx dt \\ &= -\rho_1 \int_0^\tau \left[ \mathfrak{F}_x u_{tt} \mathfrak{F}_x^2 u_t \right]_0^L dt \\ &\quad + \rho_1 \int_0^\tau \int_0^L \mathfrak{F}_x u_{tt} \mathfrak{F}_x u_t dx dt \\ &= \frac{\rho_1}{2} \left( \|\mathfrak{F}_x u_t(\cdot, \tau)\|_{L^2(0, L)}^2 - \|\mathfrak{F}_x \psi\|_{L^2(0, L)}^2 \right), \end{aligned} \quad (22)$$

$$-\rho_2(v_{tt}, \mathfrak{F}_x^2 v_t)_{L^2(Q^r)} = \frac{\rho_2}{2} \left( \|\mathfrak{F}_x v_t(\cdot, \tau)\|_{L^2(0,L)}^2 - \|\mathfrak{F}_x g\|_{L^2(0,L)}^2 \right), \quad (23)$$

$$-(u_t, \mathcal{M}_1 u)_{L^2(Q^r)} = \|\mathfrak{F}_x u_t\|_{L^2(Q^r)}^2, \quad (24)$$

$$\begin{aligned} \kappa_1(u_{xx}, \mathfrak{F}_x^2 u_t)_{L^2(Q^r)} &= \kappa_1 \int_0^\tau \int_0^L u_{xx} \cdot \mathfrak{F}_x^2 u_t dx dt \\ &= \kappa_1 \int_0^\tau \left[ \mathfrak{F}_x^2 u_t \cdot u_x \right]_0^L dt \\ &\quad - \kappa_1 \int_0^\tau \int_0^L \mathfrak{F}_x u_t \cdot u_x dx dt \\ &= \kappa_1 \int_0^\tau \int_0^L u_t u dx dt \\ &= \frac{\kappa_1}{2} \left( \|u(\cdot, \tau)\|_{L^2(0,L)}^2 - \|\varphi\|_{L^2(0,L)}^2 \right), \end{aligned} \quad (25)$$

and in the same manner, we have

$$\kappa_2(v_{xx}, \mathfrak{F}_x^2 v_t)_{L^2(Q^r)} = \frac{\kappa_2}{2} \left( \|v(\cdot, \tau)\|_{L^2(0,L)}^2 - \|f\|_{L^2(0,L)}^2 \right), \quad (26)$$

$$\begin{aligned} \kappa_1(v_x, \mathfrak{F}_x^2 u_t)_{L^2(Q^r)} &= \kappa_1 \int_0^\tau \int_0^L v_x \cdot \mathfrak{F}_x^2 u_t dx dt \\ &= \kappa_1 \int_0^\tau \left[ \mathfrak{F}_x^2 u_t \cdot v \right]_0^L dt \\ &\quad - \kappa_1 \int_0^\tau \int_0^L \mathfrak{F}_x u_t \cdot v dx dt \\ &= -\kappa_1 \int_0^\tau \int_0^L \mathfrak{F}_x u_t \cdot v dx dt, \end{aligned} \quad (27)$$

$$-\kappa_1(u_x, \mathfrak{F}_x^2 v_t)_{L^2(Q^r)} = \kappa_1 \int_0^\tau \int_0^L \mathfrak{F}_x v_t \cdot u dx dt, \quad (28)$$

$$\begin{aligned} -\kappa_1(v, \mathfrak{F}_x^2 v_t)_{L^2(Q^r)} &= -\kappa_1 \int_0^\tau \int_0^L v \cdot \mathfrak{F}_x^2 v_t dx dt \\ &= -\kappa_1 \int_0^\tau \left[ \mathfrak{F}_x^2 v_t \cdot \mathfrak{F}_x v \right]_0^L dt \\ &\quad + \kappa_1 \int_0^\tau \int_0^L \mathfrak{F}_x v_t \cdot \mathfrak{F}_x v dx dt \\ &= \|\mathfrak{F}_x v(\cdot, \tau)\|_{L^2(0,L)}^2 - \|\mathfrak{F}_x f\|_{L^2(0,L)}^2, \end{aligned} \quad (29)$$

$$\begin{aligned} &-\left( \int_0^t h(t-s) \cdot v_{xx}(x, s) ds, \mathfrak{F}_x^2 v_t \right)_{L^2(Q^r)} \\ &= -\int_0^\tau \int_0^L \left( \int_0^t h(t-s) \cdot v_{xx}(x, s) ds \right) \mathfrak{F}_x^2 v_t dx dt \\ &= -\int_0^\tau \left( \int_0^t h(t-s) \cdot v_x(x, s) ds \right) \cdot \mathfrak{F}_x^2 v_t \Big|_0^L dx dt \\ &\quad + \int_0^\tau \int_0^L \left( \int_0^t h(t-s) \cdot v_x(x, s) ds \right) \mathfrak{F}_x v_t dx dt \\ &= \int_0^\tau \left( \int_0^t h(t-s) \cdot v(x, s) ds \right) \mathfrak{F}_x v_t \Big|_0^L dx dt \\ &\quad - \int_0^\tau \int_0^L \left( \int_0^t h(t-s) \cdot v(x, s) ds \right) \cdot v_t dx dt \\ &= -\int_0^L \left( \int_0^\tau h(t-s) \cdot v(x, s) ds \right) v dx \Big|_0^\tau \\ &\quad + \int_0^\tau \int_0^L h(0) v^2 dx dt + \int_0^\tau \int_0^L \\ &\quad \cdot \left( \int_0^t h'(t-s) \cdot v(x, s) ds \right) v(x, t) dx dt \\ &= -\int_0^L \left( \int_0^\tau h(\tau-s) \cdot v(x, s) ds \right) v(x, \tau) dx + h(0) \cdot \|v\|_{L^2(Q^r)}^2 \\ &\quad + \int_0^\tau \int_0^L \left( \int_0^t h'(t-s) \cdot v(x, s) ds \right) v dx dt. \end{aligned} \quad (30)$$

Substituting equalities (22)-(30) into (21), we obtain

$$\begin{aligned} &\frac{\rho_1}{2} \|\mathfrak{F}_x u_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|u(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &\quad + \frac{\rho_2}{2} \|\mathfrak{F}_x v_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \frac{\kappa_2}{2} \|v(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &\quad + \frac{\kappa_1}{2} \|\mathfrak{F}_x v(\cdot, \tau)\|_{L^2(0,L)}^2 + h(0) \|v\|_{L^2(Q^r)}^2 \\ &= \frac{\rho_1}{2} \|\mathfrak{F}_x \psi\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|\varphi\|_{L^2(0,L)}^2 + \frac{\rho_2}{2} \|\mathfrak{F}_x g\|_{L^2(0,L)}^2 \\ &\quad + \frac{\kappa_2}{2} \|f\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|\mathfrak{F}_x f\|_{L^2(0,L)}^2 \\ &\quad + \kappa_1 \int_0^\tau \int_0^L v \mathfrak{F}_x u_t dx dt - \kappa_1 \int_0^\tau \int_0^L u \mathfrak{F}_x v_t dx dt \\ &\quad + \int_0^L \left( \int_0^\tau h(\tau-s) v(x, s) ds \right) v(x, \tau) dx \\ &\quad - \int_0^\tau \int_0^L \left( \int_0^t h'(t-s) \cdot v(x, s) ds \right) v(x, t) dx dt \\ &\quad - \int_0^\tau \int_0^L F \mathfrak{F}_x^2 u_t dx dt - \int_0^\tau \int_0^L G \mathfrak{F}_x^2 v_t dx dt. \end{aligned} \quad (31)$$

By using the Cauchy  $\varepsilon$ -inequality, the last six terms in the right-hand side of (31) can be estimated as follows:

$$\begin{aligned} \kappa_1 \int_0^\tau \int_0^L v \mathfrak{F}_x u_t dx dt &\leq \frac{\kappa_1 \varepsilon_1}{2} \int_0^\tau \int_0^L v^2 dx dt \\ &+ \frac{\kappa_1}{2\varepsilon_1} \int_0^\tau \int_0^L (\mathfrak{F}_x u_t)^2 dx dt, \end{aligned} \tag{32}$$

$$\begin{aligned} -\kappa_1 \int_0^\tau \int_0^L u \mathfrak{F}_x v_t dx dt &\leq \frac{\kappa_1 \varepsilon_2}{2} \int_0^\tau \int_0^L u^2 dx dt \\ &+ \frac{\kappa_1}{2\varepsilon_2} \int_0^\tau \int_0^L (\mathfrak{F}_x v_t)^2 dx dt, \end{aligned} \tag{33}$$

$$\begin{aligned} &\int_0^L \left( \int_0^\tau h(\tau-s)v(x,s)ds \right) v(x,\tau) dx \\ &\leq \frac{\varepsilon_3}{2} \int_0^L v^2(x,\tau) dx + \frac{1}{2\varepsilon_3} \int_0^L \\ &\quad \cdot \left( \int_0^\tau h(\tau-s)v(x,s)ds \right)^2 dx \end{aligned} \tag{34}$$

$$\begin{aligned} &\leq \frac{\varepsilon_3}{2} \int_0^L v^2(x,\tau) dx + \frac{1}{2\varepsilon_3} \int_0^L \\ &\quad \cdot \left( \int_0^\tau h^2(\tau-s)ds \right) \left( \int_0^\tau v^2(x,s)ds \right) dx \\ &\leq \frac{\varepsilon_3}{2} \int_0^L v^2(x,\tau) dx + \frac{T}{2\varepsilon_3} \sup_{0 \leq \tau \leq T} h^2(\tau) \int_0^\tau \int_0^L v^2 dx dt, \end{aligned}$$

$$\begin{aligned} &-\int_0^\tau \int_0^L \left( \int_0^t h'(t-s) \cdot v(x,s)ds \right) v dx dt \\ &\leq \frac{\varepsilon_4}{2} \int_0^\tau \int_0^L v^2 dx dt + \frac{1}{2\varepsilon_4} \int_0^\tau \int_0^L \\ &\quad \cdot \left( \int_0^t h'2(t-s)ds \right) \left( \int_0^t v^2(x,s)ds \right) dx dt \\ &\leq \frac{\varepsilon_4}{2} \int_0^\tau \int_0^L v^2 dx dt + \frac{T}{2\varepsilon_4} \sup_{0 \leq t \leq T} h'2(t) \int_0^\tau \int_0^L \\ &\quad \cdot \left( \int_0^t v^2(x,s)ds \right) dx dt \\ &= \frac{\varepsilon_4}{2} \int_0^\tau \int_0^L v^2 dx dt + \frac{T}{2\varepsilon_4} \sup_{0 \leq t \leq T} h'2(t) \int_0^\tau \\ &\quad \cdot \left[ \left( t \int_0^t v^2(x,s)ds \right)_0^\tau - \int_0^\tau t v^2 dt \right] dx \\ &= \frac{\varepsilon_4}{2} \int_0^\tau \int_0^L v^2 dx dt + \frac{T}{2\varepsilon_4} \sup_{0 \leq t \leq T} h'2(t) \int_0^\tau \\ &\quad \cdot \left[ \int_0^\tau (\tau-t)v^2(x,t)dt \right] dx \\ &\leq \frac{\varepsilon_4}{2} \int_0^\tau \int_0^L v^2 dx dt + \frac{T^2}{2\varepsilon_4} \sup_{0 \leq t \leq T} h'2(t) \int_0^\tau \int_0^L v^2 dt dx \\ &= \left( \frac{\varepsilon_4}{2} + \frac{T^2}{2\varepsilon_4} \sup_{0 \leq t \leq T} h'2(t) \right) \int_0^\tau \int_0^L v^2 dx dt, \end{aligned} \tag{35}$$

$$\begin{aligned} \int_0^\tau \int_0^L F \mathfrak{F}_x^2 u_t dx dt &\leq \frac{\varepsilon_5}{2} \|F\|_{L^2(Q^\tau)}^2 + \frac{1}{2\varepsilon_5} \|\mathfrak{F}_x^2 u_t\|_{L^2(Q^\tau)}^2 \\ &\leq \frac{\varepsilon_5}{2} \|F\|_{L^2(Q^\tau)}^2 + \frac{L}{4\varepsilon_5} \|\mathfrak{F}_x u_t\|_{L^2(Q^\tau)}^2, \end{aligned} \tag{36}$$

$$\begin{aligned} \int_0^\tau \int_0^L G \mathfrak{F}_x^2 v_t dx dt &\leq \frac{\varepsilon_6}{2} \|G\|_{L^2(Q^\tau)}^2 + \frac{1}{2\varepsilon_6} \|\mathfrak{F}_x^2 v_t\|_{L^2(Q^\tau)}^2 \\ &\leq \frac{\varepsilon_6}{2} \|G\|_{L^2(Q^\tau)}^2 + \frac{L}{4\varepsilon_6} \|\mathfrak{F}_x v_t\|_{L^2(Q^\tau)}^2. \end{aligned} \tag{37}$$

If we let

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 1, \varepsilon_3 = \frac{\kappa_2}{2}, \tag{38}$$

we then infer from (31)-(37) that

$$\begin{aligned} &\frac{\rho_1}{2} \|\mathfrak{F}_x u_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|u(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &+ \frac{\rho_2}{2} \|\mathfrak{F}_x v_t(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &+ \frac{\kappa_2}{4} \|v(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &+ \frac{\kappa_1}{2} \|\mathfrak{F}_x v(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &+ h(0) \|v\|_{L^2(Q^\tau)}^2 + \|\mathfrak{F}_x u_t\|_{L^2(Q^\tau)}^2 \\ &\leq \frac{\rho_2}{2} \|\mathfrak{F}_x g\|_{L^2(0,L)}^2 + \frac{\kappa_2}{2} \|f\|_{L^2(0,L)}^2 \\ &+ \frac{\kappa_1}{2} \|\mathfrak{F}_x f\|_{L^2(0,L)}^2 + \frac{\rho_1}{2} \|\mathfrak{F}_x \psi\|_{L^2(0,L)}^2 \\ &+ \frac{\kappa_1}{2} \|\varphi\|_{L^2(0,L)}^2 + \left( \frac{L}{4} + \frac{\kappa_1}{2} \right) \|\mathfrak{F}_x v_t\|_{L^2(Q^\tau)}^2 \\ &+ \left( \frac{L}{4} + \frac{\kappa_1}{2} \right) \|\mathfrak{F}_x u_t\|_{L^2(Q^\tau)}^2 + \frac{\kappa_1}{2} \|u\|_{L^2(Q^\tau)}^2 \\ &+ \left( \frac{\kappa_1}{2} + \frac{T}{\kappa_2} \sup_{0 \leq t \leq T} h^2(t) + \frac{1}{2} + \frac{T^2}{4} \sup_{0 \leq t \leq T} h'2(t) \right) \|v\|_{L^2(Q^\tau)}^2 \\ &+ \frac{1}{2} \|F\|_{L^2(Q^\tau)}^2 + \frac{1}{2} \|G\|_{L^2(Q^\tau)}^2. \end{aligned} \tag{39}$$

By discarding the last three terms on the left-hand side of (39) and using the poincare ' type inequality [27], we have

$$\begin{aligned} &\|u(\cdot, \tau)\|_{L^2(0,L)}^2 + \|v(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &+ \|\mathfrak{F}_x u_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \|\mathfrak{F}_x v_t(\cdot, \tau)\|_{L^2(0,L)}^2 \\ &\leq \mathfrak{D} \left\{ \|\psi\|_{L^2(0,L)}^2 + \|\varphi\|_{L^2(0,L)}^2 + \|g\|_{L^2(0,L)}^2 + \|f\|_{L^2(0,L)}^2 \right. \\ &+ \|F\|_{L^2(Q^\tau)}^2 + \|G\|_{L^2(Q^\tau)}^2 + \|u\|_{L^2(Q^\tau)}^2 \\ &\left. + \|v\|_{L^2(Q^\tau)}^2 + \|\mathfrak{F}_x u_t\|_{L^2(Q^\tau)}^2 + \|\mathfrak{F}_x v_t\|_{L^2(Q^\tau)}^2 \right\}, \end{aligned} \tag{40}$$

where

$$\mathcal{D} = \max \frac{\left( \rho_1 L/4, \rho_2 L/4, \kappa_1/2 + \kappa_1/\kappa_2 \sup_{0 \leq t \leq T} h^2(t) + 1/2 + T^2/4 \sup_{0 \leq t \leq T} h'(t), L/4 + \kappa_1/2, \kappa_2/2 + \kappa_1 L/4 \right)}{\rho_1/2, \rho_1/2, \kappa_1/2, \kappa_2/4} \quad (41)$$

Application of Gronwall's lemma (see [28]) to (40) gives

$$\begin{aligned} & \|\mathfrak{F}_x u_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \|u(\cdot, \tau)\|_{L^2(0,L)}^2 \\ & + \|\mathfrak{F}_x v_t(\cdot, \tau)\|_{L^2(0,L)}^2 + \|v(\cdot, \tau)\|_{L^2(0,L)}^2 \\ & \leq e^{\mathcal{D}\tau} \left( \|\psi\|_{L^2(0,L)}^2 + \|\varphi\|_{L^2(0,L)}^2 + \|f\|_{L^2(0,L)}^2 + \|g\|_{L^2(0,L)}^2 \right. \\ & \quad \left. + \|F\|_{L^2(Q^T)}^2 + \|G\|_{L^2(Q^T)}^2 \right) \\ & \leq e^{\mathcal{D}T} \left( \|\psi\|_{L^2(0,L)}^2 + \|\varphi\|_{L^2(0,L)}^2 + \|f\|_{L^2(0,L)}^2 \right. \\ & \quad \left. + \|g\|_{L^2(0,L)}^2 + \|F\|_{L^2(Q^T)}^2 + \|G\|_{L^2(Q^T)}^2 \right). \end{aligned} \quad (42)$$

The right-hand side of inequality (42) does not depend on  $\tau$ . By taking the supremum with respect to  $\tau$  over  $[0, T]$ , we get

$$\begin{aligned} & \|u_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0,L))} + \|v_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0,L))} \\ & + \|u(\cdot, t)\|_{C(\bar{J}; L^2(0,L))} + \|v(\cdot, t)\|_{C(\bar{J}; L^2(0,L))} \\ & \leq \mathcal{D}e^{\mathcal{D}T} \left( \|\psi\|_{L^2(0,L)}^2 + \|\varphi\|_{L^2(0,L)}^2 + \|f\|_{L^2(0,L)}^2 \right. \\ & \quad \left. + \|g\|_{L^2(0,L)}^2 + \|F\|_{L^2(Q^T)}^2 + \|G\|_{L^2(Q^T)}^2 \right). \end{aligned} \quad (43)$$

Then, the a priori estimate (19) follows with  $C = \mathcal{D}e^{\mathcal{D}T}$ .

At the moment, we do not have any information about the range  $R(\mathcal{A})$  of the operator  $\mathcal{A}$  except that  $R(\mathcal{A}) \subset \mathcal{E}$ , and we must extend  $\mathcal{A}$  so that inequality (43) holds for the extension and its range is the whole space  $\mathcal{E}$ . In this regard, we prove the following.

**Proposition 2.** *The unbounded operator  $\mathcal{A} : \mathcal{B} \rightarrow \mathcal{E}$  admits a closure  $\bar{\mathcal{A}}$  with domain of definition  $D(\bar{\mathcal{A}})$ .*

*Proof.* Let  $\mathcal{U}_n = (u_n, v_n) \in D(\mathcal{A})$  be a sequence such that

$$\mathcal{U}_n = (u_n, v_n) \xrightarrow{n \rightarrow \infty} (0, 0) \text{ in } \mathcal{B}, \quad (44)$$

$$\begin{aligned} \mathcal{A}\mathcal{U}_n &= \{L_1(u_n, v_n), L_2(u_n, v_n)\} \xrightarrow{n \rightarrow \infty} H = H_1 \times H_2 \\ &= (\{F, \varphi, \psi\}, \{G, f, g\}) \text{ in } \mathcal{E}, \end{aligned} \quad (45)$$

where

$$L_1(u_n, v_n) = \{\mathcal{L}_1(u_n, v_n), \ell_1 u_n, \ell_2 u_n\},$$

$$L_2(u_n, v_n) = \{\mathcal{L}_2(u_n, v_n), \ell_1 v_n, \ell_2 v_n\},$$

$$H_1 = \{F, \varphi, \psi\},$$

$$H_2 = \{G, f, g\}. \quad (46)$$

Then, we must show that  $H_1 = \{0\}$  and  $H_2 = \{0\}$ . That is,  $F = 0, \varphi = 0, \psi = 0, G = 0, f = 0$ , and  $g = 0$ .

Equality ((44)) implies that

$$\mathcal{U}_n = (u_n, v_n) \xrightarrow{n \rightarrow \infty} (0, 0) \text{ in } \mathcal{D}'(Q^T) \times \mathcal{D}'(Q^T), \quad (47)$$

where  $\mathcal{D}'(Q^T)$  is the space of distributions on  $Q^T$ . By the continuity of derivation of  $\mathcal{D}'(Q^T) \times \mathcal{D}'(Q^T) \rightarrow \mathcal{D}'(Q^T) \times \mathcal{D}'(Q^T)$ , then (47) implies

$$\begin{cases} \lim_{n \rightarrow \infty} [\rho_1(u_n)_{tt} - \kappa_1((u_n)_x + v_n)_x + (u_n)_t] = 0, \\ \lim_{n \rightarrow \infty} [\rho_2(v_n)_{tt} - \kappa_2(v_n)_{xx} + \kappa_1((u_n)_x + v_n) + \int_0^t h(t-s)(v_n)_{xx}(x, s) ds] = 0, \end{cases} \quad (48)$$

in  $\mathcal{D}'(Q^T) \times \mathcal{D}'(Q^T)$ . Then, from (45) it follows that

$$\begin{cases} \lim_{n \rightarrow \infty} [\rho_1(u_n)_{tt} - \kappa_1((u_n)_x + v_n)_x + (u_n)_t] = F, \\ \lim_{n \rightarrow \infty} [\rho_2(v_n)_{tt} - \kappa_2(v_n)_{xx} + \kappa_1((u_n)_x + v_n) + \int_0^t h(t-s)(v_n)_{xx}(x, s) ds] = G, \end{cases} \quad (49)$$

in  $L^2(Q^T) \times L^2(Q^T)$ . Therefore,

$$\begin{cases} \lim_{n \rightarrow \infty} [\rho_1(u_n)_{tt} - \kappa_1((u_n)_x + v_n)_x + (u_n)_t] = F, \\ \lim_{n \rightarrow \infty} [\rho_2(v_n)_{tt} - \kappa_2(v_n)_{xx} + \kappa_1((u_n)_x + v_n) + \int_0^t h(t-s)(v_n)_{xx}(x, s) ds] = G, \end{cases} \quad (50)$$

in  $\mathcal{D}'(Q^T) \times \mathcal{D}'(Q^T)$ . By virtue of the uniqueness of the limit in  $\mathcal{D}'(Q^T)$ , the identities (48) and (50) lead to  $F = 0$ , and  $G = 0$ .

Similarly, we have from (45)

$$(u_n(\cdot, 0), v_n(\cdot, 0)) \xrightarrow{n \rightarrow \infty} (\varphi, f) \text{ in } L^2(0, L) \times L^2(0, L). \quad (51)$$

We observe from (44) and the obvious inequalities

$$\begin{cases} \|u_n(\cdot, 0)\|_{L^2(0,L)} \leq \|u_n\|_{\mathcal{B}}, \quad \forall n \in \mathbb{N}, \\ \|v_n(\cdot, 0)\|_{L^2(0,L)} \leq \|v_n\|_{\mathcal{B}}, \quad \forall n \in \mathbb{N}, \end{cases} \quad (52)$$

that

$$\begin{cases} \lim_{n \rightarrow \infty} u_n(\cdot, 0) = 0, \text{ in } L^2(0, L), \\ \lim_{n \rightarrow \infty} v_n(\cdot, 0) = 0, \text{ in } L^2(0, L). \end{cases} \quad (53)$$

We conclude from (51) and (53) and the uniqueness of the limit in  $L^2(0, L)$  that  $\varphi = 0$ , and  $f = 0$ . In the same manner, we show that  $\psi = 0$  and  $g = 0$ .

**Definition 3.** The solution of the equation  $\bar{\mathcal{A}}\mathcal{U} \ll H = (\{F, \varphi, \psi\}, \{G, f, g\})$  is called a strong solution of problems (5), (9), and (10).

The energy inequality (19) can be extended to

$$\|\mathcal{U}\|_{\mathcal{B}}^2 \leq C \|\bar{\mathcal{A}}\mathcal{U}\|_{\mathcal{E}}^2, \quad \forall \mathcal{U} \in D(\bar{\mathcal{A}}). \quad (54)$$

The previous a priori bound shows that the operator  $\bar{\mathcal{A}}$  is injective and that  $\bar{\mathcal{A}}^{-1}$  is continuous from the range  $R(\bar{\mathcal{A}})$  onto  $\mathcal{B}$  from which we assert that if a strong solution of problems (5), (9), and (10) exists, it is unique and depends continuously on the initial data  $(\varphi, \psi)$ ,  $(f, g)$  and the free terms  $F$  and  $G$ .

**Corollary 4.** The set  $R(\bar{\mathcal{A}}) \subset \mathcal{E}$  is closed and  $R(\bar{\mathcal{A}}) = R(\bar{\mathcal{A}})$ .

#### 4. Solvability of the Posed Problem

Here is the main result of the paper.

**Theorem 5.** Problems (5), (9), and (10) admit a unique strong solution satisfying

$$\begin{aligned} \mathcal{U} &= (u, v) \in C(\bar{J}; L^2(0, L)), \\ \mathcal{U}_t &= (u_t, v_t) \in C(\bar{J}; B_2^1(0, L)). \end{aligned} \quad (55)$$

Moreover, the solution  $\mathcal{U} = (u, v)$  and its time derivative  $\mathcal{U}_t = (u_t, v_t)$  depend continuously on the data  $F, \varphi, \psi, G, f, g$ , that is,

$$\begin{aligned} &\|u(\cdot, t)\|_{C(\bar{J}; L^2(0, L))}^2 + \|v(\cdot, t)\|_{C(\bar{J}; L^2(0, L))}^2 \\ &\quad + \|u_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))}^2 + \|v_t(\cdot, t)\|_{C(\bar{J}; B_2^1(0, L))}^2 \\ &\leq \mathcal{D}e^{\mathcal{D}T} \left( \|\psi\|_{L^2(0, L)}^2 + \|\varphi\|_{L^2(0, L)}^2 + \|f\|_{L^2(0, L)}^2 \right. \\ &\quad \left. + \|g\|_{L^2(0, L)}^2 + \|F\|_{L^2(Q^T)}^2 + \|G\|_{L^2(Q^T)}^2 \right). \end{aligned} \quad (56)$$

*Proof.* It follows from Corollary 4 that in order to prove the existence of the strongly generalized solution of problems (5), (9), and (10), it is sufficient to show that the range  $R(\mathcal{A})$  of the operator  $\mathcal{A}$  is everywhere dense in the space  $\mathcal{E}$ ; that is, the operator  $\mathcal{A}$  is injective. To this end, we first prove the density in the following special case.

**Theorem 6.** If for some function  $W = (\omega_1, \omega_2) \in (L^2(Q^T))^2$  and for elements  $\mathcal{U} \in D_0(\mathcal{A}) = \{\mathcal{U} : \mathcal{U} \in D(\mathcal{A}) \text{ and } l_i u = l_i v = 0, i = 1, 2\}$ , we have

$$(L_1(u, v), \omega_1)_{L^2(Q^T)} + (L_2(u, v), \omega_2)_{L^2(Q^T)} = 0, \quad (57)$$

then  $W = 0$  a.e in  $Q^T$ .

*Proof.* Since relation (57) holds for any element of  $D_0(\mathcal{A})$ , we take an element  $\mathcal{U} = (u, v)$  with special form given by

$$\mathcal{U} = (u, v) = \begin{cases} (0, 0), & \text{if } 0 \leq t \leq s, \\ \left( \int_s^t (\tau - t) u_{\tau\tau}(x, \tau) d\tau, \int_s^t v_{\tau}(x, \tau) d\tau \right), & \text{if } s \leq t \leq T, \end{cases} \quad (58)$$

and consider the system

$$\begin{cases} E_1(xt) = \mathfrak{F}_x^2 u_{tt}(xt) = \int_T^t \omega_1(x, \tau) d\tau, \\ E_2(xt) = \mathfrak{F}_x^2 v_{tt}(xt) = \int_T^t \omega_2(x, \tau) d\tau. \end{cases} \quad (59)$$

It follows from the above relations that

$$\mathfrak{F}_x^2 u_{ttt}(xt) = \omega_1(x, t), \quad \mathfrak{F}_x^2 v_{ttt}(xt) = \omega_2(x, t). \quad (60)$$

**Lemma 7.** The function  $W = (\omega_1, \omega_2)$ , defined by (60), belongs to  $(L^2(Q^T))^2$ .

*Proof* (of Lemma). We use the  $t$ -averaging operator  $\rho_\varepsilon$  introduced in [32]. By applying the operators  $\rho_\varepsilon$  and  $\partial/\partial t$  to the first equation in (59), we obtain

$$\frac{\partial}{\partial t} (\mathfrak{F}_x^2(u_{tt})) = \frac{\partial}{\partial t} [\mathfrak{F}_x^2(u_{tt}) - \rho_\varepsilon(\mathfrak{F}_x^2(u_{tt}))] + \frac{\partial}{\partial t} \rho_\varepsilon(E_1(xt)), \quad (61)$$

then

$$\begin{aligned} \left\| \frac{\partial}{\partial t} (\mathfrak{F}_x^2(u_{tt})) \right\|_{L^2(Q^T)}^2 &\leq 2 \left\| \frac{\partial}{\partial t} [\mathfrak{F}_x^2(u_{tt}) - \rho_\varepsilon(\mathfrak{F}_x^2(u_{tt}))] \right\|_{L^2(Q^T)}^2 \\ &\quad + 2 \left\| \frac{\partial}{\partial t} \rho_\varepsilon(E_1(xt)) \right\|_{L^2(Q^T)}^2. \end{aligned} \quad (62)$$

Using the  $t$ -averaging operator  $\rho_\varepsilon$  properties, we infer from the above inequality that

$$\left\| \frac{\partial}{\partial t} (\mathfrak{F}_x^2(u_{tt})) \right\|_{L^2(Q^T)}^2 \leq 2 \left\| \frac{\partial}{\partial t} \rho_\varepsilon(E_1(xt)) \right\|_{L^2(Q^T)}^2. \quad (63)$$

Since  $\rho_\varepsilon u \rightarrow u$  as  $\varepsilon \rightarrow 0$  in  $L^2(Q^T)$  and the norm of  $\partial/\partial t$  ( $\mathfrak{F}_x^2(u_{tt})$ ) in  $L^2(Q^T)$  is bounded, we conclude that  $\omega_1 \in L^2(Q^T)$ . Similarly, applying  $\partial/\partial t \rho_\varepsilon$  to the second equation in (59), we conclude that  $\omega_2 \in L^2(Q^T)$ . Consequently,  $W = (\omega_1, \omega_2) \in (L^2(Q^T))^2$ .

We now continue the proof of Theorem 6. Replacing the functions  $\omega_1$  and  $\omega_2$  given by (60) in (57), we obtain

$$\begin{aligned} & (\rho_1 u_{tt}, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} - \kappa_1 (u_{xx}, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} \\ & - \kappa_1 (v_x, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} + (u_t, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} \\ & + (\rho_2 v_{tt}, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} - \kappa_2 (v_{xx}, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} \\ & + \kappa_1 \left( (u_x, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} + \kappa_1 (v, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} \right. \\ & \left. + \left( \int_0^t h(t-s) v_{xx}(x, s) ds, \mathfrak{F}_x^2 v_{tt} \right)_{L^2(Q^T)} \right) = 0. \end{aligned} \quad (64)$$

Invoking the boundary integral conditions and carrying out appropriate integrations by parts of each term, we have

$$\begin{aligned} (\rho_1 u_{tt}, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} &= \int_0^T \int_0^L \rho_1 u_{tt} \mathfrak{F}_x^2 u_{tt} dx dt \\ &= \int_s^T \int_0^L \rho_1 u_{tt} \mathfrak{F}_x^2 u_{tt} dx dt \\ &= \frac{\rho_1}{2} \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0, L)}^2, \end{aligned} \quad (65)$$

$$(\rho_2 v_{tt}, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} = \frac{\rho_2}{2} \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0, L)}^2, \quad (66)$$

$$(u_t, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} = \|\mathfrak{F}_x u_{tt}\|_{L^2(Q^T)}^2, \quad (67)$$

$$\begin{aligned} -\kappa_1 (u_{xx}, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} &= \kappa_1 (u_{xx}, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} \\ &= -\kappa_1 (u_x, \mathfrak{F}_x u_{tt})_{L^2(Q^T)} \\ &= \kappa_1 (u, u_{tt})_{L^2(Q^T)} \\ &= -\kappa_1 (u_t, u_{tt})_{L^2(Q^T)} \\ &= \frac{\kappa_1}{2} \|u_t(\cdot, T)\|_{L^2(0, L)}^2, \end{aligned} \quad (68)$$

$$-\kappa_2 (v_{xx}, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} = \frac{\kappa_2}{2} \|v_t(\cdot, T)\|_{L^2(0, L)}^2, \quad (69)$$

$$\begin{aligned} -\kappa_1 (v_x, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} &= -\kappa_1 (v_x, \mathfrak{F}_x^2 u_{tt})_{L^2(Q^T)} \\ &= \kappa_1 (v, \mathfrak{F}_x u_{tt})_{L^2(Q^T)} \\ &= -\kappa_1 (v_t, \mathfrak{F}_x u_{tt})_{L^2(Q^T)}, \end{aligned} \quad (70)$$

$$\kappa_1 (u_x, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} = \kappa_1 (u_t, \mathfrak{F}_x v_{tt})_{L^2(Q^T)}, \quad (71)$$

$$\begin{aligned} & \left( \int_0^t h(t-\eta) v_{xx}(x, \eta) d\eta, \mathfrak{F}_x^2 v_{tt} \right)_{L^2(Q^T)} \\ &= \left( \int_0^t h(t-\eta) v_{xx}(x, \eta) d\eta, \mathfrak{F}_x^2 v_{tt} \right)_{L^2(Q^T)} \\ &= - \left( \int_0^t h(t-\eta) v_x(x, \eta) d\eta, \mathfrak{F}_x v_{tt} \right)_{L^2(Q^T)} \\ &= \left( \int_0^t h(t-\eta) v(x, \eta) d\eta, v_{tt} \right)_{L^2(Q^T)} \\ &= -h(0) \int_s^T \int_0^L v_{tt} v dx dt - \int_s^T \int_0^L v_{tt} \\ & \quad \cdot \left( \int_0^t h'(t-\eta) v(x, \eta) d\eta \right) dx dt \\ &= -h(0) \int_0^L \left[ v v_t \right]_s^T dx + h(0) \int_s^T \int_0^L v_t^2 dx dt \\ & \quad - \int_0^L \left( \int_0^t h'(t-\eta) v(x, \eta) d\eta \right) v_t \Big|_s^T dx \\ & \quad + h'(0) \int_s^T \int_0^L v_t v dx dt + \int_s^T \int_0^L v_t \\ & \quad \cdot \left( \int_0^t h''(t-\eta) v(x, \eta) d\eta \right) dx dt \\ &= -h(0) \int_0^L v(x, T) v_t(x, T) dx + h(0) \int_s^T \int_0^L v_t^2 dx dt \\ & \quad - \int_0^L v_t(x, T) \left( \int_0^T h'(T-\eta) v(x, \eta) d\eta \right) dx \\ & \quad + \frac{h'(0)}{2} \int_0^L v^2(x, T) dx + \int_s^T \int_0^L v_t \\ & \quad \cdot \left( \int_0^t h''(t-\eta) v(x, \eta) d\eta \right) dx dt, \end{aligned} \quad (72)$$

$$\begin{aligned} \kappa_1 (v, \mathfrak{F}_x^2 v_{tt})_{L^2(Q^T)} &= \kappa_1 \int_s^T \int_0^L v \mathfrak{F}_x^2 v_{tt} dx dt \\ &= -\kappa_1 \int_s^T \int_0^L v_t \mathfrak{F}_x^2 v_{tt} dx dt \\ &= \kappa_1 \int_s^T \int_0^L \mathfrak{F}_x v_t \mathfrak{F}_x v_{tt} dx dt \\ &= \frac{\kappa_1}{2} \int_0^L \left( \mathfrak{F}_x v_t \right)_s^T dx \\ &= \frac{\kappa_1}{2} \|\mathfrak{F}_x v_t(\cdot, T)\|_{L^2(0, L)}^2. \end{aligned} \quad (73)$$



Substitution of equations (65)-(73) into (64) gives

$$\begin{aligned}
& \frac{\rho_1}{2} \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|u_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& + \frac{\rho_2}{2} \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \frac{\kappa_2}{2} \|v_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& + \frac{\kappa_1}{2} \|\mathfrak{F}_x v_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& + \frac{h'(0)}{2} \|v(\cdot, T)\|_{L^2(0,L)}^2 + h(0) \|v_t\|_{Q^s}^2 \\
= & -\kappa_1 \int_{Q^s} u_t \mathfrak{F}_x v_{tt} dx dt + \kappa_1 \int_{Q^s} v_t \mathfrak{F}_x u_{tt} dx dt \quad (74) \\
& + h(0) \int_0^L v_t(\cdot, T) v(x, T) dx + \int_0^L v_t(\cdot, T) \\
& \cdot \left( \int_0^T h'(T-\eta) v(x, \eta) d\eta \right) dx \\
& - \int_{Q^s} v_t \left( \int_0^t h'(t-\eta) v(x, \eta) d\eta \right) dx dt.
\end{aligned}$$

By using the Cauchy  $\varepsilon$ -inequality, we estimate each term of the right-hand side of the previous relations to get

$$\kappa_1 \int_{Q^s} v_t \mathfrak{F}_x u_{tt} dx dt \leq \frac{\kappa_1 \varepsilon_1}{2} \|v_t\|_{L^2(Q^s)}^2 + \frac{\kappa_1}{2\varepsilon_1} \|\mathfrak{F}_x u_{tt}\|_{L^2(Q^s)}^2, \quad (75)$$

$$-\kappa_1 \int_{Q^s} u_t \mathfrak{F}_x v_{tt} dx dt \leq \frac{\kappa_1 \varepsilon_2}{2} \|u_t\|_{L^2(Q^s)}^2 + \frac{\kappa_1}{2\varepsilon_2} \|\mathfrak{F}_x v_{tt}\|_{L^2(Q^s)}^2, \quad (76)$$

$$\begin{aligned}
& h(0) \int_0^L v_t(x, T) v(x, T) dx \\
& \leq \frac{h(0)\varepsilon_3}{2} \|v_t(\cdot, T)\|_{L^2(0,L)}^2 + \frac{h(0)}{2\varepsilon_3} \|v(\cdot, T)\|_{L^2(0,L)}^2 \\
& \leq \frac{h(0)\varepsilon_3}{2} \|v_t(\cdot, T)\|_{L^2(0,L)}^2 + \frac{h(0)}{2\varepsilon_3} \|v\|_{L^2(Q^s)}^2 + \frac{h(0)}{2\varepsilon_3} \|v_t\|_{L^2(Q^s)}^2, \quad (77)
\end{aligned}$$

$$\begin{aligned}
& \int_0^L v_t(x, T) \left( \int_0^T h'(T-\eta) v(x, \eta) d\eta \right) dx \\
& \leq \frac{\varepsilon_4}{2} \int_0^L v_t^2(x, T) dx + \frac{1}{2\varepsilon_4} \int_0^L \int_0^T h'2 \\
& \cdot (T-\eta) d\eta \int_0^T v^2(x, \eta) d\eta dx \quad (78) \\
& \leq \frac{\varepsilon_4}{2} \int_0^L v_t^2(x, T) dx + \frac{T \sup_{0 \leq t \leq T} h'2(t)}{2\varepsilon_4} \int_0^L \int_s^T v^2 dt dx,
\end{aligned}$$

$$\begin{aligned}
& - \int_s^T \int_0^L \left( \int_0^t h'(t-\eta) v(x, \eta) d\eta \right) v_t(x, t) dx dt \\
& \leq \frac{\varepsilon_5}{2} \int_s^T \int_0^L v_t^2 dx dt + \frac{1}{2\varepsilon_5} \int_s^T \int_0^L \\
& \cdot \left( \int_0^t h'(t-\eta) v(x, \eta) d\eta \right)^2 dx \\
& \leq \frac{\varepsilon_5}{2} \int_s^T \int_0^L v_t^2 dx \\
& + \frac{1}{2\varepsilon_5} \int_s^T \int_0^L \underbrace{\left( \left( \int_0^t h'2(t-\eta) d\eta \right)^{1/2} \left( \int_0^t v^2(x, \eta) d\eta \right)^{1/2} \right)^2}_{\text{using Holder's inequality}} dx \\
& \leq \frac{\varepsilon_5}{2} \|v_t\|_{L^2(Q^s)}^2 + \frac{1}{2\varepsilon_5} \int_s^T \int_0^L \\
& \cdot \left( \int_0^t h'2(t-\eta) d\eta \right) \\
& \cdot \left( \int_0^t v^2(x, \eta) d\eta \right) dx dt \\
& \leq \frac{\varepsilon_5}{2} \|v_t\|_{L^2(Q^s)}^2 + \frac{T \sup_{0 \leq t \leq T} h'2(t)}{2\varepsilon_5} \\
& \cdot \int_s^T \int_0^L \int_0^t v^2(x, \eta) d\eta dx dt \\
& = \frac{\varepsilon_5}{2} \|v_t\|_{L^2(Q^s)}^2 + \frac{T \sup_{0 \leq t \leq T} h'2(t)}{2\varepsilon_5} \\
& \cdot \int_0^L \int_0^T (T-t) v^2 dt dx \\
& \leq \frac{\varepsilon_5}{2} \|v_t\|_{L^2(Q^s)}^2 + \frac{T^2 \sup_{0 \leq t \leq T} h'2(t)}{2\varepsilon_5} \int_0^L \int_s^T v^2 dt dx. \quad (79)
\end{aligned}$$

By combining equality (74) and inequalities (75)-(79) and taking

$$\varepsilon_1 = \frac{2h(0)}{\kappa_1},$$

$$\varepsilon_2 = \frac{1}{2},$$

$$\varepsilon_3 = \frac{\kappa_2}{2h(0)},$$

$$\varepsilon_4 = \frac{\kappa_2}{4},$$

$$\varepsilon_5 = 1, \quad (80)$$

we obtain

$$\begin{aligned}
& \frac{\rho_1}{2} \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2} \|u_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& + \frac{\rho_2}{2} \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \frac{\kappa_2}{8} \|v_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& + \frac{\kappa_2}{2} \|\mathfrak{F}_x v_t(\cdot, T)\|_{L^2(0,L)}^2 + \frac{h'(0)}{2} \|v(\cdot, T)\|_{L^2(0,L)}^2 \\
& \leq \left( \frac{h^2(0)}{\kappa_2} + \frac{1}{2} \right) \|v_t\|_{L^2(Q^s)}^2 + \frac{\kappa_1^2}{4h(0)} \|\mathfrak{F}_x u_{tt}\|_{L^2(Q^s)}^2 \\
& + \frac{\kappa_1}{4} \|u_t\|_{L^2(Q^s)}^2 + \kappa_1 \|\mathfrak{F}_x v_{tt}\|_{L^2(Q^s)}^2 \\
& + \left( \frac{2T \sup_{0 \leq t \leq T} h'2(t)}{\kappa_2} + \frac{h^2(0)}{\kappa_2} + \frac{T^2 \sup_{0 \leq t \leq T} h''2(t)}{2} \right) \|v\|_{L^2(Q^s)}^2.
\end{aligned} \tag{81}$$

By discarding the terms  $\kappa_2/2 \|\mathfrak{F}_x v_t(x, T)\|_{L^2(0,L)}^2$ ,  $\|v(x, T)\|_{L^2(0,L)}^2$  from the left-hand side of (81) and using the Frederick's inequality for the norm of  $v$  obtained from the norm of  $v_t$ , it follows that

$$\begin{aligned}
& \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0,L)}^2 \\
& + \|u_t(\cdot, T)\|_{L^2(0,L)}^2 + \|v_t(\cdot, T)\|_{L^2(0,L)}^2 \\
& \leq C^* \int_s^T \left( \|\mathfrak{F}_x u_{tt}\|_{L^2(0,L)}^2 + \|\mathfrak{F}_x v_{tt}\|_{L^2(0,L)}^2 \right. \\
& \left. + \|u_t\|_{L^2(0,L)}^2 + \|v_t\|_{L^2(0,L)}^2 \right) dt,
\end{aligned} \tag{82}$$

where

$$C^* = \frac{\max \left\{ 1/2 + h^2(0)/\kappa_2, \kappa_1, \mathcal{M}^* \left( 2T \sup_{0 \leq t \leq T} h'2(t)/\kappa_2 + T^2 \sup_{0 \leq t \leq T} h''2(t)/2 + h^2(0)/\kappa_2 \right) \right\}}{\min \left\{ \rho_1/2, \kappa_1/2, \rho_2/2, \kappa_2/8, h'(0/2) \right\}}, \tag{83}$$

and  $\mathcal{M}^*$  is the Frederick's constant.

Inequality (82) is important and fundamental in the proof; to use it, we introduce the new functions  $\xi$ ,  $\mu$  defined by

$$\mu(x, t) = \int_t^T v_{\tau\tau}(x, \tau) d\tau, \quad \xi(x, t) = \int_t^T u_{\tau\tau}(x, \tau) d\tau. \tag{84}$$

Then,  $\mu(x, s) = v_t(x, T)$ ,  $v_t(x, t) = \mu(x, s) - \mu(x, t)$  and  $\xi(x, s) = u_t(x, T)$ ,  $u_t(x, t) = \xi(x, s) - \xi(x, t)$ , and we have

$$\begin{aligned}
& \int_s^T \|u_t\|_{L^2(0,L)}^2 dt \leq 2 \int_s^T \|\xi(x, t)\|_{L^2(0,L)}^2 dt \\
& + 2(T-s) \|\xi(x, s)\|_{L^2(0,L)},
\end{aligned}$$

$$\begin{aligned}
& \int_s^T \|v_t\|_{L^2(0,L)}^2 dt \leq 2 \int_s^T \|\mu(x, t)\|_{L^2(0,L)}^2 dt \\
& + 2(T-s) \|\mu(x, s)\|_{L^2(0,L)}.
\end{aligned} \tag{85}$$

Consequently, inequality (82) reduces to

$$\begin{aligned}
& \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + (1 - 2C^*(T-s)) \|\xi(\cdot, s)\|_{L^2(0,L)}^2 \\
& + \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + (1 - 2C^*(T-s)) \|\mu(\cdot, s)\|_{L^2(0,L)}^2 \\
& \leq 2C^* \int_s^T \left( \|\mathfrak{F}_x u_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\xi(\cdot, t)\|_{L^2(0,L)}^2 \right. \\
& \left. + \|\mathfrak{F}_x v_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\mu(\cdot, t)\|_{L^2(0,L)}^2 \right) dt.
\end{aligned} \tag{86}$$

Choose  $s_0 \geq 0$  such that  $s \in [T - s_0, T]$ . and  $2C^*(T - s_0) = 1/2$ ; then, inequality (86) implies

$$\begin{aligned}
& \|\mathfrak{F}_x u_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \|\xi(\cdot, s)\|_{L^2(0,L)}^2 \\
& + \|\mathfrak{F}_x v_{tt}(\cdot, s)\|_{L^2(0,L)}^2 + \|\mu(\cdot, s)\|_{L^2(0,L)}^2 \\
& \leq 4C^* \int_s^T \left( \|\mathfrak{F}_x u_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\xi(\cdot, t)\|_{L^2(0,L)}^2 \right. \\
& \left. + \|\mathfrak{F}_x v_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\mu(\cdot, t)\|_{L^2(0,L)}^2 \right) dt,
\end{aligned} \tag{87}$$

for all  $s \in [T - s_0, T]$ . If in (86) we let

$$\begin{aligned}
\Lambda(s) = & \int_s^T \left( \|\mathfrak{F}_x u_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\xi(\cdot, t)\|_{L^2(0,L)}^2 \right. \\
& \left. + \|\mathfrak{F}_x v_{tt}(\cdot, t)\|_{L^2(0,L)}^2 + \|\mu(\cdot, t)\|_{L^2(0,L)}^2 \right) dt,
\end{aligned} \tag{88}$$

then it follows from (87) that

$$-\frac{d\Lambda(s)}{2ds} \leq C^* \Lambda(s). \tag{89}$$

Thus,

$$-\frac{d}{ds} \left( \Lambda(s) e^{2C^*s} \right) \leq 0. \tag{90}$$

It then follows from inequality (90) that  $\Lambda(T) = 0$ , and hence,  $W = (\omega_1, \omega_2) = 0$  almost everywhere in  $Q^{T-s_0}$ . Proceeding in this way step by step along a rectangle of side  $s_0$ , we prove that  $W \equiv 0$  almost everywhere in  $Q^T$ .

We now consider the general case for density.

Since  $\mathcal{E}$  is a Hilbert space, then  $R(\bar{\mathcal{A}}) = \mathcal{E}$  is equivalent to the orthogonality of the vector  $\mathcal{N} \ll (\mathcal{N}_1, \mathcal{N}_2) = \{(\omega_1, \omega_3, \omega_4), (\omega_2, \omega_5, \omega_6)\} \in \mathcal{E}$  to the set  $R(\mathcal{A})$ , that is, if and only if the relation

$$\begin{aligned} (\mathcal{A}\mathcal{U}, \mathcal{N})_{\mathcal{E}} &= (\{L_1(u, v), L_2(u, v)\}, \{\mathcal{N}_1, \mathcal{N}_2\})_{\mathcal{E}} \\ &= (\{L_1(u, v), \ell_1 u, \ell_2 u\}, \{L_2(u, v), \ell_1 v, \ell_2 v\})_{\mathcal{E}} \\ &\quad \cdot \{(\omega_1, \omega_2, \omega_3), (\omega_4, \omega_5, \omega_6)\}_{\mathcal{E}} \\ &= (L_1(u, v), \omega_1)_{L^2(Q^T)} + (\ell_1 u, \omega_2)_{L^2(0,L)} \\ &\quad + (\ell_2 u, \omega_3)_{L^2(0,L)} + (L_2(u, v), \omega_4)_{L^2(Q^T)} \\ &\quad + (\ell_1 v, \omega_5)_{L^2(0,L)} + (\ell_2 v, \omega_6)_{L^2(0,L)} \\ &= 0, \end{aligned} \tag{91}$$

where  $\mathcal{U}$  runs over the space  $\mathcal{B}$  and  $\mathcal{N} \in \mathcal{E}$ , which implies that  $\mathcal{N} \ll (\mathcal{N}_1, \mathcal{N}_2) = (0, 0)$ . That is,  $\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_5 = \omega_6 = 0$ .

Let  $\mathcal{U} \in D_0(\mathcal{A})$ ; then, equation (91) becomes

$$(L_1(u, v), \omega_1)_{L^2(Q^T)} + L_1(u, v), \omega_4)_{L^2(Q^T)} = 0. \tag{92}$$

Hence, by virtue of Theorem 6, it follows from (92) that  $\omega_1 = \omega_4 = 0$ . Consequently, equation (91) takes the form

$$(\ell_1 u, \omega_2)_{L^2(0,L)} + (\ell_2 u, \omega_3)_{L^2(0,L)} + (\ell_1 v, \omega_5)_{L^2(0,L)} + (\ell_2 v, \omega_6)_{L^2(0,L)} = 0. \tag{93}$$

Since the four terms in (93) vanish independently and since the ranges  $R(\ell_1), R(\ell_2)$  of the trace operators  $\ell_1, \ell_2$  are everywhere dense in the space  $L^2(0, L)$ , then it follows from (93) that  $\omega_2 = \omega_3 = \omega_5 = \omega_6 = 0$ . Consequently,  $\mathcal{N} = 0$ , that is,  $R(\mathcal{A})^\perp = \{0\}$ . Thus,  $R(\bar{\mathcal{A}}) = \mathcal{E}$ .

## 5. Conclusion

In this article, we proved the well posedness of a nonhomogeneous Timoshenko system with a viscoelastic damping term. The coupled two hyperbolic equations were associated with initial conditions and nonlocal boundary conditions. The proofs of the results are mainly based on some energy and a priori estimates and on some density arguments. The method uses functional analysis tools such as operator theory and density arguments. It is found that the method is efficient and powerful for solving initial boundary value problems with nonlocal constraints. The a priori estimate for the solution can be provided by constructing suitable multipliers and from which it is also possible to establish the solvability of the stated problem. We note, here, that no previous works were done for Timoshenko systems with nonlocal conditions of integral type.

## Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

## Conflicts of Interest

The authors declare that they have no competing interests.

## Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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