Research Article

Convergence of Some Iterative Algorithms for System of Generalized Set-Valued Variational Inequalities

M. Akram ¹, Aysha Khan ², and M. Dilshad ³

¹Department of Mathematics, Faculty of Science, Islamic University of Madinah, Saudi Arabia

²College of Arts and Science, Wadi Addwasir, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia

³Department of Mathematics, Faculty of Science, University of Tabuk, Saudi Arabia

Correspondence should be addressed to Aysha Khan; aayshakhan055@gmail.com

Received 7 October 2020; Revised 24 November 2020; Accepted 27 November 2020; Published 5 January 2021

Academic Editor: Mohammad Mursaleen

Copyright © 2021 M. Akram et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this article, we consider and study a system of generalized set-valued variational inequalities involving relaxed cocoercive mappings in Hilbert spaces. Using the projection method and Banach contraction principle, we prove the existence of a solution for the considered problem. Further, we propose an iterative algorithm and discuss its convergence. Moreover, we establish equivalence between the system of variational inequalities and altering points problem. Some parallel iterative algorithms are proposed, and the strong convergence of the sequences generated by these iterative algorithms is discussed. Finally, a numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms.

1. Introduction

The theory of variational inequality was planted in the early 1960's by Stampacchia [1] in the framework of obstacle constraint minimization problems. The first evolutionary variational inequality was solved in the seminal paper of Lions and Stampacchia [2]. Since its inception, it has enjoyed a vigorous development for the last few decades. This subject has developed in multiple directions using innovative techniques to solve fundamental problems to be insurmountable previously. This field is influential and experiencing an explosive growth in theory as well as applications. Consequently, some of these developments in this area enriched other mutual areas of mathematical and engineering sciences such as economics, transportation, nonlinear programming, and operations research. It has been shown that this theory provides the most natural, direct, simple, unified, and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems; see, for example, [3–9] and the references cited therein.

Recently, fixed-point methods have been extensively investigated for solving monotone variational inequalities. Among the fixed point algorithms, Mann-like iterative algo-

gorithms are useful for solving several nonlinear problems. The proximal point algorithm is a widely used tool for solving a variety of (single objective) convex optimization problems such as finding zeros of maximal monotone operators and fixed points of nonexpansive mappings, as well as minimizing convex functions. In 2007, Agarwal et al. [10] introduced the following iteration process. Let C be a nonempty convex subset of a normed linear space X and let $T : C \rightarrow C$ be an operator. Then, for arbitrary $x_0 \in C$, estimate $x_n \in C$ by the following S-iterative scheme:

$$\begin{aligned}x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n, \\y_n &= (1 - \beta_n)x_n + \beta_nTx_n,\end{aligned}\tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying some suitable conditions. Sahu [11] proved that the S-iteration process is more applicable than the Picard [12], Mann [13], and Ishikawa [14] iteration algorithms because it converges faster than these iteration processes for contraction mappings and also works for nonexpansive mappings. Recently, Cholamjiak et al. [15] applied the S-iteration process for finding a minimizer of a convex function and fixed

points of nonexpansive mappings in CAT(0) spaces. In [16], Amir et al. studied the convex constraint multiobjective optimization problem as the constrained set of fixed points of the nonexpansive mapping.

On the other hand, a number of numerical methods such as auxiliary principle, projection method, Wiener-Hopf equations, dynamical systems, and decomposition have been developed for solving the variational inequalities and related optimization problems. Among these methods, the projection method and its variant forms have been proved innovative and important tools for finding the approximate solutions of variational inequalities. In this technique, the concept of projection is used for the fixed-point formulation of variational inequality. This alternative formulation has played a significant role in developing various projection-type methods for solving variational inequalities. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. It is also known that the relaxed cocoercive mappings are more general than strongly monotone mappings, and under some mild conditions, relaxed cocoercive mappings can be reduced to strongly monotone mappings. For some related works, see, [17–24].

Following the facts and discussion mentioned above, in this paper, we consider a system of generalized set-valued variational inequalities defined over closed and convex subsets of a real Hilbert space. We establish an equivalence between the system of generalized set-valued variational inequalities and nonlinear projection equations using the projection technique. Further, by virtue of the projection method and Banach contraction principle, we prove an existence result. Furthermore, we propose an iterative algorithm and show that the approximate solution generated by the proposed algorithm converges strongly to the unique solution of the system of generalized set-valued variational inequalities involving relaxed cocoercive and relaxed monotone mappings in Hilbert space. Moreover, we establish equivalence between the system of variational inequalities and altering points problem. Parallel Mann and parallel S-iterative algorithms [11] have been proposed for solving the considered system of variational inequalities. Finally, the convergence analysis of the proposed parallel iterative algorithms is discussed. A numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms. The results presented in this paper can be viewed as generalizations and refinements of several results existing in the literature and include general variational inequality and some other classes of variational inequalities as special cases.

Now, we enumerate some basic notions, definitions, and results that are worthwhile tools in succeeding analysis and will be utilized in the rest of this paper.

Throughout the paper, unless otherwise specified, let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$.

Definition 1 [25]. A mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be

- (i) γ -strongly monotone, if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \gamma \|x - y\|^2, \forall x, y \in \mathcal{H}. \quad (2)$$

- (ii) μ_1 -cocoercive, if there exists a constant $\mu_1 > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \mu_1 \|Tx - Ty\|^2, \forall x, y \in \mathcal{H}. \quad (3)$$

- (iii) Relaxed μ_2 -cocoercive, if there exists a constant $\mu_2 > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\mu_2) \|Tx - Ty\|^2, \forall x, y \in \mathcal{H}. \quad (4)$$

- (iv) Relaxed (μ, γ) -cocoercive, if there exist constants $\mu, \gamma > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\mu) \|Tx - Ty\|^2 + \gamma \|x - y\|^2, \forall x, y \in \mathcal{H}. \quad (5)$$

- (v) L -Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|, \forall x, y \in \mathcal{H}. \quad (6)$$

- (vi) α -contraction, if there exists a constant $0 < \alpha < 1$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \forall x, y \in \mathcal{H}. \quad (7)$$

T is called nonexpansive, if $\alpha = 1$.

Remark 2. We remark that every γ -strongly monotone mapping is relaxed (μ, γ) -cocoercive mapping and every γ -cocoercive mapping is $1/\gamma$ -Lipschitz continuous.

Define the norm $\|\cdot\|_*$ on $\mathcal{H} \times \mathcal{H}$ by

$$\|(p, q)\|_* = \|p\| + \|q\|, \forall p, q \in \mathcal{H}. \quad (8)$$

Note that $(\mathcal{H} \times \mathcal{H}, \|\cdot\|_*)$ is a Banach space.

Definition 3 [26]. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a single-valued mapping. A set-valued mapping $G : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is said to be a relaxed monotone with respect to T if and only if there exists a constant $c > 0$ such that

$$\langle Tu - Tv, x - y \rangle \geq (-c) \|x - y\|^2, \forall x, y \in \mathcal{H} \text{ and for some } u \in G(x), v \in G(y). \quad (9)$$

Let Ω be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. Then, for any $x \in \mathcal{H}$, there exists a unique

nearest point $P_\Omega(x)$ of Ω such that

$$\|x - P_\Omega(x)\| \leq \|x - y\|, \forall y \in \Omega. \quad (10)$$

The mapping P_Ω is called the metric projection [27] from $\mathcal{H}$ onto Ω . Note that the metric projection mapping P_Ω is nonexpansive from $\mathcal{H}$ onto Ω (see, Agarwal et al. [28]), i.e.,

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \forall x, y \in \mathcal{H}. \quad (11)$$

Lemma 4 [27]. *Let Ω be a closed and convex subset in $\mathcal{H}$. Then, for any $s \in \mathcal{H}$, the projection $P_\Omega : \mathcal{H} \rightarrow \Omega$ of $\mathcal{H}$ onto Ω satisfies*

$$\langle t - s, r - t \rangle \geq 0, \forall r \in \Omega, \quad (12)$$

if and only if

$$t = P_\Omega(s), \forall t \in \Omega. \quad (13)$$

Lemma 5. [29]. *Let $\{\omega_n\}$ be a nonnegative real sequence satisfying the following condition*

$$\omega_{n+1} \leq (1 - \alpha_n)\omega_n + \varepsilon_n, \forall n \geq n_0, \quad (14)$$

where $\alpha_n \in [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\varepsilon_n = o(\alpha_n)$. Then, $\lim_{n \rightarrow \infty} \omega_n = 0$.

2. System of Generalized Set-Valued Variational Inequalities

Let $\mathcal{H}$ be a real Hilbert space and Ω_1 and Ω_2 be the non-empty closed and convex subsets of $\mathcal{H}$. Let $I = \{1, 2\}$ be an index set; for each $i \in I$, let $\varphi_i, \psi_i : \mathcal{H} \rightarrow \mathcal{H}$; $M_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be nonlinear single-valued mappings and $P_i : \mathcal{H} \rightarrow CB(\mathcal{H})$ be a set-valued mapping. We consider the problem of finding $(x, y) \in \Omega_1 \times \Omega_2$ with $u \in P_1(x)$ and $v \in P_2(y)$ such that

$$\begin{cases} \langle Q_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y) \rangle \geq 0, \forall y^* \in \Omega_2, \\ \langle Q_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x) \rangle \geq 0, \forall x^* \in \Omega_1, \end{cases} \quad (15)$$

where Q_1 and Q_2 are positive constants. We call problem (15) a system of generalized set-valued variational inequalities. Some special cases of problem (15) are listed below.

- (i) If $P_i : \mathcal{H} \rightarrow \mathcal{H}$ are the single-valued mappings and $\varphi_i = I$ are the identity mappings, then problem (15) reduces to the equivalent problem of finding $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} \langle Q_1 M_1(y, x) + y - \psi_1(x), \psi_1(y^*) - y \rangle \geq 0, \forall y^* \in \Omega_2, \\ \langle Q_2 M_2(x, y) + x - \psi_2(y), \psi_2(x^*) - x \rangle \geq 0, \forall x^* \in \Omega_1. \end{cases} \quad (16)$$

Problem (16) is called the system of variational inequalities.

- (ii) If $M_i(\cdot, \cdot) = M_i(\cdot)$, then problem (16) coincides with the following problem of finding $(x, y) \in H \times H$ such that

$$\begin{cases} \langle Q_1 M_1(x) + y - \psi_1(x), \psi_1(y^*) - y \rangle \geq 0, \forall y^* \in \Omega_2, \\ \langle Q_2 M_2(y) + x - \psi_2(y), \psi_2(x^*) - x \rangle \geq 0, \forall x^* \in \Omega_1. \end{cases} \quad (17)$$

Problem (17) was studied by Sahu et al. [11].

- (iii) If $M_1 = M_2 = T$, $\psi_1 = \psi_2 = I$ and $\Omega_1 = \Omega_2 = C$, then problem (17) reduces to the following system of nonlinear variational inequalities of finding $x, y \in C$ such that

$$\begin{cases} \langle QT(y) + x - y, x^* - x \rangle \geq 0, \forall x^* \in C, \\ \langle \eta T(x) + y - x, x^* - y \rangle \geq 0, \forall x^* \in C. \end{cases} \quad (18)$$

Problem (18) was studied by Verma [30]. He extended the concept of variational inequalities to the system of nonlinear variational inequalities.

- (iv) If $M_i(\cdot, \cdot) = M_i(\cdot)$, $\psi_1(x) = \varphi_1(y)$, $\varphi_2(x) = \psi_2(y)$, $Q_1 = Q_2 = 1$ and $P_i : \mathcal{H} \rightarrow \mathcal{H}$ are the single-valued mappings, then problem (15) coincides to the following system of extended general variational inequalities of finding $x, y \in \mathcal{H}$ such that

$$\begin{cases} \langle M_1(x), \psi_1(y^*) - \varphi_1(y) \rangle \geq 0, \forall \psi_1(y^*) \in \Omega_1, \\ \langle M_2(y), \psi_2(x^*) - \varphi_2(x) \rangle \geq 0, \forall \psi_2(x^*) \in \Omega_2. \end{cases} \quad (19)$$

An equivalent form of problem (19) was studied by Noor et al. [31].

- (v) If $\psi_1(y^*) = \psi_2(x^*) = g(v)$, $\varphi_1 = \varphi_2 = \varphi$ and $\Omega_1 = \Omega_2 = \Omega$, then problem (19) reduces to the following system of variational inequalities of finding $x, y \in \mathcal{H}$, $\varphi(x), \varphi(y) \in \Omega$ such that

$$\begin{cases} \langle M_1(x), g(v) - \varphi(y) \rangle \geq 0, \forall g(v) \in \Omega, \\ \langle M_2(y), g(v) - \varphi(x) \rangle \geq 0, \forall g(v) \in \Omega. \end{cases} \quad (20)$$

The problem of type (20) is called a system of extended general variational inequalities with four nonlinear operators.

Next, we establish the following lemma which plays a crucial role to prove the existence result for the unique solution of the system of generalized set-valued variational inequalities (15) and to propose an iterative algorithm for studying convergence analysis.

Lemma 6. Let Ω_1 and Ω_2 be the nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. Let $I = \{1, 2\}$ be an index set; for each $i \in I$, let $\varphi_i, \psi_i : \mathcal{H} \rightarrow \mathcal{H}$ and $M_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be the nonlinear single-valued mappings. Let $P_i : \mathcal{H} \rightarrow CB(\mathcal{H})$ be a set-valued mapping. Then, the system of generalized set-valued variational inequalities (15) has a solution (x, y, u, v) , if and only if $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ satisfies

$$\varphi_1(y) = P_{\Omega_1}[\psi_1(x) - \mathbf{Q}_1 M_1(u, v)], \quad (21)$$

$$\varphi_2(x) = P_{\Omega_2}[\psi_2(y) - \mathbf{Q}_2 M_2(v, u)], \quad (22)$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are positive constants.

Proof. Let $(x, y) \in \Omega_1 \times \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ is a solution of the system of generalized set-valued variational inequalities (15). Then

$$\begin{cases} \langle \mathbf{Q}_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y) \rangle \geq 0, \forall y^* \in \Omega_2, \\ \langle \mathbf{Q}_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x) \rangle \geq 0, \forall x^* \in \Omega_1. \end{cases} \quad (23)$$

Then, from Lemma 4, we have

$$\begin{aligned} \varphi_1(y) &= P_{\Omega_1}[\psi_1(x) - \mathbf{Q}_1 M_1(u, v)], \\ \varphi_2(x) &= P_{\Omega_2}[\psi_2(y) - \mathbf{Q}_2 M_2(v, u)]. \end{aligned} \quad (24)$$

Conversely, suppose that $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ satisfies

$$\begin{aligned} \varphi_1(y) &= P_{\Omega_1}[\psi_1(x) - \mathbf{Q}_1 M_1(u, v)], \\ \varphi_2(x) &= P_{\Omega_2}[\psi_2(y) - \mathbf{Q}_2 M_2(v, u)]. \end{aligned} \quad (25)$$

Again, it follows from Lemma 4 that

$$\begin{aligned} \langle \mathbf{Q}_1 M_1(u, v) + \varphi_1(y) - \psi_1(x), \psi_1(y^*) - \varphi_1(y) \rangle &\geq 0, \forall y^* \in \Omega_2, \\ \langle \mathbf{Q}_2 M_2(v, u) + \varphi_2(x) - \psi_2(y), \psi_2(x^*) - \varphi_2(x) \rangle &\geq 0, \forall x^* \in \Omega_1. \end{aligned} \quad (26)$$

Thus, $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ is solution of the system of generalized set-valued variational inequalities (15).

Now, by virtue of the Banach contraction principle, we shall show the existence of the unique solution for the system of generalized set-valued variational inequalities (15).

Theorem 7. Let $\mathcal{H}$ be a real Hilbert space and $I = \{1, 2\}$ be an index set, for each $i \in I$; let Ω_i be a closed and convex subset of $\mathcal{H}$. Let $\varphi_i, \psi_i : \mathcal{H} \rightarrow \mathcal{H}$ and $M_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be nonlinear single-valued mappings such that φ_i is relaxed $(\epsilon_{\varphi_i}, \epsilon_{\varphi_i})$ -cocoercive and L_{φ_i} -Lipschitz type continuous, and ψ_i is relaxed $(\epsilon_{\psi_i}, \epsilon_{\psi_i})$ -cocoercive and L_{ψ_i} -Lipschitz type continuous. Let $P_i : \mathcal{H} \rightarrow CB(\mathcal{H})$ be a set-valued mapping such that

P_i is a relaxed monotone with respect to M_i with constant c_i , $\mathcal{D}$ -Lipschitz type continuous with constant ς_i , and M_i is L_{M_i} -Lipschitz type continuous in the first argument and L'_{M_i} -Lipschitz type continuous in the second argument. In addition, $\mu_i > 0$ satisfies

$$0 < \mu_1, \mu_2 < 1, \quad (27)$$

where $\mu_1 = \kappa_1 + \iota_1 + \omega_2, \mu_2 = \kappa_2 + \iota_2 + \omega_1$, and

$$\begin{aligned} \kappa_1 &= \sqrt{1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2}, \iota_1 = \sqrt{1 + 2\mathbf{Q}_1 c_1 + \mathbf{Q}_1^2 L_{M_1}^2 \varsigma_1^2}, \\ \kappa_2 &= \sqrt{1 - 2\epsilon_{\psi_2} + (1 + 2\epsilon_{\psi_2})L_{\psi_2}^2}, \iota_2 = \sqrt{1 + 2\mathbf{Q}_2 c_2 + \mathbf{Q}_2^2 L_{M_2}^2 \varsigma_2^2}, \\ \omega_1 &= \sqrt{1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2} + L'_{M_1} \varsigma_2, \omega_2 \\ &= \sqrt{1 - 2\epsilon_{\varphi_2} + (1 + 2\epsilon_{\varphi_2})L_{\varphi_2}^2} + L'_{M_2} \varsigma_1. \end{aligned} \quad (28)$$

Then, the system of generalized set-valued variational inequalities (15) admits unique solution.

Proof. For any given $x \in \Omega_1, y \in \Omega_2$, we define a mapping $\vartheta : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$ by

$$\vartheta(x, y) = (\Phi(x, y), Y(x, y)), \quad (29)$$

where $\Phi, Y : \Omega_1 \times \Omega_2 \rightarrow \mathcal{H}$ are the single-valued mappings defined by

$$\Phi(x, y) = x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) - \mathbf{Q}_2 M_2(v, u)], \quad (30)$$

$$Y(x, y) = y - \varphi_1(y) + P_{\Omega_1}[\psi_1(x) - \mathbf{Q}_1 M_1(u, v)], \quad (31)$$

where $u \in P_1(x), v \in P_2(y)$ and $\mathbf{Q}_1, \mathbf{Q}_2$ are positive constants. Utilizing the fact that the projection mapping P_{Ω_1} is nonexpansive, it follows from (31) that

$$\begin{aligned} &\|Y(x_1, y_1) - Y(x_2, y_2)\| \\ &= \|y_1 - \varphi_1(y_1) + P_{\Omega_1}[\psi_1(x_1) - \mathbf{Q}_1 M_1(u_1, v_1)] \\ &\quad - (y_2 - \varphi_1(y_2) + P_{\Omega_1}[\psi_1(x_2) - \mathbf{Q}_1 M_1(u_2, v_2)])\| \\ &\leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|P_{\Omega_1}[\psi_1(x_1) \\ &\quad - \mathbf{Q}_1 M_1(u_1, v_1)] - P_{\Omega_1}[\psi_1(x_2) - \mathbf{Q}_1 M_1(u_2, v_2)]\| \\ &\leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|\psi_1(x_1) - \mathbf{Q}_1 M_1(u_1, v_1) \\ &\quad - [\psi_1(x_2) - \mathbf{Q}_1 M_1(u_2, v_2)]\| \\ &\leq \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| + \|x_1 - x_2 - (\psi_1(x_1) \\ &\quad - \psi_1(x_2))\| + \|x_1 - x_2 - \mathbf{Q}_1 (M_1(u_1, v_1) - M_1(u_2, v_1))\| \\ &\quad + \mathbf{Q}_1 \|M_1(u_2, v_1) - M_1(u_2, v_2)\|. \end{aligned} \quad (32)$$

Since the mapping φ_1 is relaxed $(\epsilon_{\varphi_1}, \epsilon_{\varphi_1})$ -cocoercive and

L_{φ_1} -Lipschitz type continuous, then we have

$$\begin{aligned}
& \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\|^2 \\
&= \|y_1 - y_2\|^2 - 2\langle \varphi_1(y_1) - \varphi_1(y_2), y_1 - y_2 \rangle \\
&\quad + \|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\leq \|y_1 - y_2\|^2 + 2\epsilon_{\varphi_1} \|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\quad - 2\epsilon_{\varphi_1} \|y_1 - y_2\|^2 + \|\varphi_1(y_1) - \varphi_1(y_2)\|^2 \\
&\leq \left[1 - 2\epsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2\right] \|y_1 - y_2\|^2,
\end{aligned} \tag{33}$$

which implies that

$$\begin{aligned}
& \|y_1 - y_2 - (\varphi_1(y_1) - \varphi_1(y_2))\| \\
&\leq \sqrt{1 - 2\epsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2} \|y_1 - y_2\|.
\end{aligned} \tag{34}$$

Again, using the fact that the mapping ψ_1 is relaxed $(\epsilon_{\psi_1}, \epsilon_{\psi_1})$ -cocoercive and L_{ψ_1} -Lipschitz type continuous, then we have

$$\begin{aligned}
& \|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\|^2 \\
&= \|x_1 - x_2\|^2 - 2\langle \psi_1(x_1) - \psi_1(x_2), x_1 - x_2 \rangle \\
&\quad + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\leq \|x_1 - x_2\|^2 + 2\epsilon_{\psi_1} \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\quad - 2\epsilon_{\psi_1} \|x_1 - x_2\|^2 + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\
&\leq \left[1 - 2\epsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2\right] \|x_1 - x_2\|^2,
\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}
& \|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\| \\
&\leq \sqrt{1 - 2\epsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2} \|x_1 - x_2\|.
\end{aligned} \tag{36}$$

Since P_1 is relaxed monotone with respect to M_1 with constant c_1 in the first argument, $\mathcal{D}$ -Lipschitz type continuous with constant c_1 and M_1 is L_{M_1} -Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}
& \|x_1 - x_2 - \mathbf{Q}_1(M_1(u_1, v_1) - M_1(u_2, v_1))\|^2 \\
&= \|x_1 - x_2\|^2 - 2\mathbf{Q}_1 \langle M_1(u_1, v_1) - M_1(u_2, v_1), x_1 - x_2 \rangle \\
&\quad + \mathbf{Q}_1^2 \|M_1(u_1, v_1) - M_1(u_2, v_1)\|^2 \\
&\leq \|x_1 - x_2\|^2 + 2\mathbf{Q}_1 c_1 \|x_1 - x_2\|^2 + \mathbf{Q}_1^2 L_{M_1}^2 \|u_1 - u_2\|^2 \\
&\leq \|x_1 - x_2\|^2 + 2\mathbf{Q}_1 c_1 \|x_1 - x_2\|^2 + \mathbf{Q}_1^2 L_{M_1}^2 c_1^2 \|x_1 - x_2\|^2 \\
&= \left(1 + 2\mathbf{Q}_1 c_1 + \mathbf{Q}_1^2 L_{M_1}^2 c_1^2\right) \|x_1 - x_2\|^2,
\end{aligned} \tag{37}$$

which implies that

$$\begin{aligned}
& \|x_1 - x_2 - \mathbf{Q}_1(M_1(u_1, v_1) - M_1(u_2, v_1))\| \\
&\leq \sqrt{1 + 2\mathbf{Q}_1 c_1 + \mathbf{Q}_1^2 L_{M_1}^2 c_1^2} \|x_1 - x_2\|.
\end{aligned} \tag{38}$$

Since M_1 is L_{M_1}' -Lipschitz type continuous in the second argument and P_2 is $\mathcal{D}$ -Lipschitz type continuous with constant c_2 , then we have

$$\|M_1(u_2, v_1) - M_1(u_2, v_2)\| \leq L_{M_1}' \|v_1 - v_2\| \leq L_{M_1}' c_2 \|y_1 - y_2\|. \tag{39}$$

Thus, from (34), (36), (38), and (39), (32) becomes

$$\begin{aligned}
& \|Y(x_1, y_1) - Y(x_2, y_2)\| \\
&\leq \left[\sqrt{1 - 2\epsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2} + L_{M_1}' c_2 \right] \|y_1 - y_2\| \\
&\quad + \left[\sqrt{1 - 2\epsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2} \right. \\
&\quad \left. + \sqrt{1 + 2\mathbf{Q}_1 c_1 + \mathbf{Q}_1^2 L_{M_1}^2 c_1^2} \right] \|x_1 - x_2\|.
\end{aligned} \tag{40}$$

Again, utilizing the fact that the projection mapping P_{Ω_2} is nonexpansive, it follows from (30) that

$$\begin{aligned}
& \|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| \\
&= \|x_1 - \varphi_2(x_1) + P_{\Omega_2}[\psi_2(y_1) - \mathbf{Q}_2 M_2(v_1, u_1)] \\
&\quad - (x_2 - \varphi_2(x_2) + P_{\Omega_2}[\psi_2(y_2) - \mathbf{Q}_2 M_2(v_2, u_2)])\| \\
&\leq \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| + \\
&\quad \|P_{\Omega_2}[\psi_2(y_1) - \mathbf{Q}_2 M_2(v_1, u_1)] \\
&\quad - P_{\Omega_2}[\psi_2(y_2) - \mathbf{Q}_2 M_2(v_2, u_2)]\| \\
&\leq \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| \\
&\quad + \|\psi_2(y_1) - \mathbf{Q}_2 M_2(v_1, u_1) - [\psi_2(y_2) - \mathbf{Q}_2 M_2(v_2, u_2)]\| \\
&\leq \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| + \\
&\quad \|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\| + \\
&\quad \|y_1 - y_2 - \mathbf{Q}_2(M_2(v_1, u_1) - M_2(v_2, u_1))\| \\
&\quad + \mathbf{Q}_2 \|M_2(v_2, u_1) - M_2(v_2, u_2)\|.
\end{aligned} \tag{41}$$

Since the mapping φ_2 is relaxed $(\epsilon_{\varphi_2}, \epsilon_{\varphi_2})$ -cocoercive and L_{φ_2} -Lipschitz type continuous, then we have

$$\begin{aligned}
& \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\|^2 \\
&= \|x_1 - x_2\|^2 - 2\langle \varphi_2(x_1) - \varphi_2(x_2), x_1 - x_2 \rangle \\
&\quad + \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\leq \|x_1 - x_2\|^2 + 2\epsilon_{\varphi_2} \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\quad - 2\epsilon_{\varphi_2} \|x_1 - x_2\|^2 + \|\varphi_2(x_1) - \varphi_2(x_2)\|^2 \\
&\leq \left[1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2\right] \|x_1 - x_2\|^2,
\end{aligned} \tag{42}$$

which implies that

$$\begin{aligned}
& \|x_1 - x_2 - (\varphi_2(x_1) - \varphi_2(x_2))\| \\
&\leq \sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} \|x_1 - x_2\|.
\end{aligned} \tag{43}$$

Again, using the fact that the mapping ψ_2 is relaxed $(\epsilon_{\psi_2}, \epsilon_{\psi_2})$ -cocoercive and L_{ψ_2} -Lipschitz type continuous, then we have

$$\begin{aligned}
& \|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\|^2 \\
&= \|y_1 - y_2\|^2 - 2\langle \psi_2(y_1) - \psi_2(y_2), y_1 - y_2 \rangle \\
&\quad + \|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\leq \|y_1 - y_2\|^2 + 2\epsilon_{\psi_2} \|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\quad - 2\epsilon_{\psi_2} \|y_1 - y_2\|^2 + \|\psi_2(y_1) - \psi_2(y_2)\|^2 \\
&\leq \left[1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2\right] \|y_1 - y_2\|^2,
\end{aligned} \tag{44}$$

which implies that

$$\begin{aligned}
& \|y_1 - y_2 - (\psi_2(y_1) - \psi_2(y_2))\| \\
&\leq \sqrt{1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2} \|y_1 - y_2\|.
\end{aligned} \tag{45}$$

Since P_2 is relaxed monotone with respect to M_2 with constant c_2 in the first argument, $\mathcal{D}$ -Lipschitz type continuous with constant ς_2 and M_2 is L_{M_2} -Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}
& \|y_1 - y_2 - \mathcal{Q}_2(M_2(v_1, u_1) - M_2(v_2, u_1))\|^2 \\
&= \|y_1 - y_2\|^2 - 2\mathcal{Q}_2\langle M_2(v_1, u_1) - M_2(v_2, u_1), y_1 - y_2 \rangle \\
&\quad + \mathcal{Q}_2^2 \|M_2(v_1, u_1) - M_2(v_2, u_1)\|^2 \\
&\leq \|y_1 - y_2\|^2 + 2\mathcal{Q}_2 c_2 \|y_1 - y_2\|^2 + \mathcal{Q}_2^2 L_{M_2}^2 \|v_1 - v_2\|^2 \\
&\leq \|y_1 - y_2\|^2 + 2\mathcal{Q}_2 c_2 \|y_1 - y_2\|^2 + \mathcal{Q}_2^2 L_{M_2}^2 \varsigma_2^2 \|y_1 - y_2\|^2 \\
&= \left[1 + 2\mathcal{Q}_2 c_2 + \mathcal{Q}_2^2 L_{M_2}^2 \varsigma_2^2\right] \|y_1 - y_2\|^2,
\end{aligned} \tag{46}$$

which implies that

$$\begin{aligned}
& \|y_1 - y_2 - \mathcal{Q}_2(M_2(v_1, u_1) - M_2(v_2, u_1))\| \\
&\leq \sqrt{1 + 2\mathcal{Q}_2 c_2 + \mathcal{Q}_2^2 L_{M_2}^2 \varsigma_2^2} \|y_1 - y_2\|.
\end{aligned} \tag{47}$$

Since M_2 is L_{M_2}' -Lipschitz type continuous in the second argument and P_1 is $\mathcal{D}$ -Lipschitz type continuous with constant ς_1 , then we have

$$\|M_2(v_2, u_1) - M_2(v_2, u_2)\| \leq L_{M_2}' \|u_1 - u_2\| \leq L_{M_2}' \varsigma_1 \|x_1 - x_2\|. \tag{48}$$

Thus, from (43), (45), (47), and (48), (41) becomes

$$\begin{aligned}
& \|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| \\
&\leq \left[\sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} + L_{M_2}' \varsigma_1 \right] \\
&\quad \cdot \|x_1 - x_2\| + \left[\sqrt{1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2} \right. \\
&\quad \left. + \sqrt{1 + 2\mathcal{Q}_2 c_2 + \mathcal{Q}_2^2 L_{M_2}^2 \varsigma_2^2} \right] \|y_1 - y_2\|.
\end{aligned} \tag{49}$$

Now, it follows from (40) and (49) that

$$\begin{aligned}
& \|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| + \|Y(x_1, y_1) - Y(x_2, y_2)\| \\
&\leq \mu_1 \|x_1 - x_2\| + \mu_2 \|y_1 - y_2\| \\
&\leq \max \{\mu_1, \mu_2\} (\|x_1 - x_2\| + \|y_1 - y_2\|),
\end{aligned} \tag{50}$$

where $\mu_1 = \kappa_1 + \iota_1 + \omega_2$, $\mu_2 = \kappa_2 + \iota_2 + \omega_1$, and

$$\begin{aligned}
& \kappa_1 = \sqrt{1 - 2\epsilon_{\psi_1} + \left(1 + 2\epsilon_{\psi_1}\right)L_{\psi_1}^2}, \iota_1 = \sqrt{1 + 2\mathcal{Q}_1 c_1 + \mathcal{Q}_1^2 L_{M_1}^2 \varsigma_1^2}, \\
& \kappa_2 = \sqrt{1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}, \iota_2 = \sqrt{1 + 2\mathcal{Q}_2 c_2 + \mathcal{Q}_2^2 L_{M_2}^2 \varsigma_2^2}, \\
& \omega_1 = \sqrt{1 - 2\epsilon_{\varphi_1} + \left(1 + 2\epsilon_{\varphi_1}\right)L_{\varphi_1}^2} + L_{M_1}' \varsigma_2, \omega_2 \\
&= \sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} + L_{M_2}' \varsigma_1.
\end{aligned} \tag{51}$$

It follows from (8), (29), and (50) that

$$\|\vartheta(x_1, y_1) - \vartheta(x_2, y_2)\|_* \leq \max \{\mu_1, \mu_2\} \|(x_1, y_1) - (x_2, y_2)\|. \tag{52}$$

It follows from conditions (27) and (52) that ϑ is a contraction mapping. Therefore, there exists unique $(x, y) \in \Omega_1 \times \Omega_2$ such that $\vartheta(x, y) = (x, y)$. Thus, we have

$$\begin{aligned}
& \varphi_1(y) = P_{\Omega_1}[\psi_1(x) - \mathcal{Q}_1 M_1(u, v)], \\
& \varphi_2(x) = P_{\Omega_2}[\psi_2(y) - \mathcal{Q}_2 M_2(v, u)].
\end{aligned} \tag{53}$$

Therefore, by Lemma 6, one can conclude that $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$, and $v \in P_2(y)$ is the unique solution of the system of generalized set-valued variational inequalities (15).

3. Iterative Algorithm and Convergence Result

In this section, we suggest an iterative algorithm to analyse the convergence of the system of generalized set-valued variational inequalities (15).

By utilizing (21) and (22) of Lemma 6, we can offer the following iterative forms:

$$y = (1 - \gamma_n)y + \gamma_n(y - \varphi_1(y) + P_{\Omega_1}[\psi_1(x) - \mathbf{Q}_1 M_1(u, v)]), \quad (54)$$

$$x = (1 - \delta_n)x + \delta_n(x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) - \mathbf{Q}_2 M_2(v, u)]), \quad (55)$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are positive constants and the real sequences $\gamma_n, \delta_n \in [0, 1]$. Now, we propose the following iterative algorithm.

Algorithm 8. For any given $(x_0, y_0) \in \Omega_1 \times \Omega_2$, we choose $u_0 \in P_1(x_0)$ and $v_0 \in P_2(y_0)$. For $\mathbf{Q}_1, \mathbf{Q}_2 > 0$ and $\gamma_n, \delta_n \in [0, 1]$ and from (54) and (55), let

$$\begin{aligned} y_1 &= (1 - \gamma_n)y_0 + \gamma_n(y_0 - \varphi_1(y_0) + P_{\Omega_1}[\psi_1(x_0) - \mathbf{Q}_1 M_1(u_0, v_0)]), \\ x_1 &= (1 - \delta_n)x_0 + \delta_n(x_0 - \varphi_2(x_0) + P_{\Omega_2}[\psi_2(y_0) - \mathbf{Q}_2 M_2(v_0, u_0)]). \end{aligned} \quad (56)$$

Since $u_0 \in P_1(x_0)$ and $v_0 \in P_2(y_0)$, by Nadler's theorem [32], there exist $u_1 \in P_1(x_1)$ and $v_1 \in P_2(y_1)$ such that

$$\begin{aligned} \|u_0 - u_1\| &\leq (1 + 1)D(P_1(x_0), P_1(x_1)), \\ \|v_0 - v_1\| &\leq (1 + 1)D(P_2(y_0), P_2(y_1)), \end{aligned} \quad (57)$$

where D is the Hausdorff metric. Let

$$\begin{aligned} y_2 &= (1 - \gamma_n)y_1 + \gamma_n(y_1 - \varphi_1(y_1) + P_{\Omega_1}[\psi_1(x_1) - \mathbf{Q}_1 M_1(u_1, v_1)]), \\ x_2 &= (1 - \delta_n)x_1 + \delta_n(x_1 - \varphi_2(x_1) + P_{\Omega_2}[\psi_2(y_1) - \mathbf{Q}_2 M_2(v_1, u_1)]). \end{aligned} \quad (58)$$

Again, it follows from Nadler's theorem [32] that there exist $u_2 \in P_1(x_2)$ and $v_2 \in P_2(y_2)$ such that

$$\begin{aligned} \|u_1 - u_2\| &\leq (1 + 2^{-1})D(P_1(x_1), P_1(x_2)), \\ \|v_1 - v_2\| &\leq (1 + 2^{-1})D(P_2(y_1), P_2(y_2)). \end{aligned} \quad (59)$$

Continuing in the same manner, we can figure out the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ by the following iterative process:

$$y_{n+1} = (1 - \gamma_n)y_n + \gamma_n(y_n - \varphi_1(y_n) + P_{\Omega_1}[\psi_1(x_n) - \mathbf{Q}_1 M_1(u_n, v_n)]), \quad (60)$$

$$x_{n+1} = (1 - \delta_n)x_n + \delta_n(x_n - \varphi_2(x_n) + P_{\Omega_2}[\psi_2(y_n) - \mathbf{Q}_2 M_2(v_n, u_n)]), \quad (61)$$

and for $n = 0, 1, 2, \dots$, choose $u_{n+1} \in P_1(x_{n+1})$ and $v_{n+1} \in P_2(y_{n+1})$ such that

$$\|u_{n+1} - u_n\| \leq (1 + (n + 1)^{-1})D(P_1(x_{n+1}), P_1(x_n)), \quad (62)$$

$$\|v_{n+1} - v_n\| \leq (1 + (n + 1)^{-1})D(P_2(y_{n+1}), P_2(y_n)). \quad (63)$$

Now, we are accessible to study the convergence of the proposed iterative algorithm for the system of generalized set-valued variational inequalities (15).

Theorem 9. Let $\mathcal{H}$ be a real Hilbert space and $I = \{1, 2\}$ be an index set, for each $i \in I$; let Ω_i be a closed convex subset of $\mathcal{H}$. Let $\varphi_i, \psi_i : \mathcal{H} \rightarrow \mathcal{H}$ and $M_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be nonlinear single-valued mappings such that φ_i is relaxed $(\epsilon_{\varphi_i}, \epsilon_{\varphi_i})$ -cocoercive, L_{φ_i} -Lipschitz type continuous; ψ_i is relaxed $(\epsilon_{\psi_i}, \epsilon_{\psi_i})$ -cocoercive, L_{ψ_i} -Lipschitz type continuous; and M_i is L_{M_i} -Lipschitz type continuous in the first argument and L_{M_i}' -Lipschitz type continuous in the second argument. Let $P_i : \mathcal{H} \rightarrow \text{CB}(\mathcal{H})$ be a set-valued mapping such that P_i is a relaxed monotone with respect to M_i with constant c_i and D -Lipschitz type continuous with constant ς_i . In addition, for each $i = 1, 2$, $\rho_i, c_i > 0$ satisfy the following conditions:

- (i) $0 < n_i, l_i < 1$,
- (ii) $2\epsilon_{\psi_i} - (1 + 2\epsilon_{\psi_i})L_{\psi_i}^2 < 1$,
- (iii) $\delta_n(1 - l_1) - \gamma_n(m_2 + n_2) \geq 0, \gamma_n(1 - l_2) - \delta_n(m_1 + n_1) \geq 0$, such that

$$\begin{aligned} &\sum_{n=0}^{\infty} \delta_n(1 - l_1) - \gamma_n(m_2 + n_2) \\ &= \infty, \sum_{n=0}^{\infty} \gamma_n(1 - l_2) - \delta_n(m_1 + n_1) = \infty, \end{aligned} \quad (64)$$

where

$$\begin{aligned} l_1 &= \sqrt{1 - 2\epsilon_{\varphi_2} + (1 + 2\epsilon_{\varphi_2})L_{\varphi_2}^2} + \mathbf{Q}_2 L_{M_2}' \varsigma_1, m_1 \\ &= \sqrt{1 - 2\epsilon_{\psi_2} + (1 + 2\epsilon_{\psi_2})L_{\psi_2}^2}, \end{aligned}$$

$$\begin{aligned} l_2 &= \sqrt{1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2} + \mathbf{Q}_1 L_{M_1}' \varsigma_2, m_2 \\ &= \sqrt{1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2}, \end{aligned}$$

$$n_1 = \sqrt{1 + 2\mathbf{Q}_2 c_2 + \mathbf{Q}_2^2 L_{M_2}^2 \varsigma_2^2}, n_2 = \sqrt{1 + 2\mathbf{Q}_1 c_1 + \mathbf{Q}_1^2 L_{M_1}^2 \varsigma_1^2}. \quad (65)$$

Then, the approximate solution $\{(x_n, y_n, u_n, v_n)\}$

generated by Algorithm 8 converges strongly to the unique solution (x, y, u, v) of the system of generalized set-valued variational inequalities (15).

Proof. Let $(x, y, u, v), x \in \Omega_1, y \in \Omega_2, u \in P_1(x)$ and $v \in P_2(y)$ be a solution of (15); then, from (55) and (61) of Algorithm 8 and utilizing the fact that the projection mapping P_{Ω_2} is non-expansive, we have

$$\begin{aligned} \|x_{n+1} - x\| &= \|(1 - \delta_n)x_n + \delta_n(x_n - \varphi_2(x_n)) \\ &\quad + P_{\Omega_2}[\psi_2(y_n) - Q_2M_2(v_n, u_n)] \\ &\quad - [(1 - \delta_n)x + \delta_n(x - \varphi_2(x) + P_{\Omega_2}[\psi_2(y) \\ &\quad - Q_2M_2(v, u)])]\| \leq (1 - \delta_n)\|x_n - x\| + \delta_n\|x_n \\ &\quad - x - (\varphi_2(x_n) - \varphi_2(x))\| \\ &\quad + \delta_n\|P_{\Omega_2}[\psi_2(y_n) - Q_2M_2(v_n, u_n)] \\ &\quad - P_{\Omega_2}[\psi_2(y) - Q_2M_2(v, u)]\| \\ &\leq (1 - \delta_n)\|x_n - x\| + \delta_n\|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\| \\ &\quad + \delta_n\|\psi_2(y_n) - Q_2M_2(v_n, u_n) - (\psi_2(y) \\ &\quad - Q_2M_2(v, u))\| \leq (1 - \delta_n)\|x_n - x\| + \delta_n\|x_n \\ &\quad - x - (\varphi_2(x_n) - \varphi_2(x))\| + \delta_n\|y_n - y - (\psi_2(y_n) \\ &\quad - \psi_2(y))\| + \delta_n\|y_n - y - Q_2(M_2(v_n, u_n) \\ &\quad - M_2(v, u_n))\| + \delta_n Q_2\|M_2(v, u_n) - M_2(v, u)\|. \end{aligned} \quad (66)$$

Since the mapping φ_2 is relaxed $(\epsilon_{\varphi_2}, \epsilon_{\varphi_2})$ -cocoercive and L_{φ_2} -Lipschitz type continuous, then we have

$$\begin{aligned} \|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\|^2 &= \|x_n - x\|^2 - 2\langle \varphi_2(x_n) - \varphi_2(x), x_n - x \rangle \\ &\quad + \|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\leq \|x_n - x\|^2 + 2\epsilon_{\varphi_2}\|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\quad - 2\epsilon_{\varphi_2}\|x_n - x\|^2 + \|\varphi_2(x_n) - \varphi_2(x)\|^2 \\ &\leq \left[1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2\right]\|x_n - x\|^2, \end{aligned} \quad (67)$$

which implies that

$$\begin{aligned} \|x_n - x - (\varphi_2(x_n) - \varphi_2(x))\| &\leq \sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2}\|x_n - x\|. \end{aligned} \quad (68)$$

Again, using the fact that mapping ψ_2 is relaxed $(\epsilon_{\psi_2}, \epsilon_{\psi_2})$ -cocoercive and L_{ψ_2} -Lipschitz type continuous, then we have

$$\begin{aligned} \|y_n - y - (\psi_2(y_n) - \psi_2(y))\|^2 &= \|y_n - y\|^2 - 2\langle \psi_2(y_n) - \psi_2(y), y_n - y \rangle \\ &\quad + \|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\leq \|y_n - y\|^2 + 2\epsilon_{\psi_2}\|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\quad - 2\epsilon_{\psi_2}\|y_n - y\|^2 + \|\psi_2(y_n) - \psi_2(y)\|^2 \\ &\leq \left[1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2\right]\|y_n - y\|^2, \end{aligned} \quad (69)$$

which implies that

$$\begin{aligned} \|y_n - y - (\psi_2(y_n) - \psi_2(y))\| &\leq \sqrt{1 - 2\epsilon_{\psi_2} + \left(1 + 2\epsilon_{\psi_2}\right)L_{\psi_2}^2}\|y_n - y\|. \end{aligned} \quad (70)$$

Since P_2 is relaxed monotone with respect to M_2 with constant c_2 in the first argument, D -Lipschitz type continuous with constant ς_2 and M_2 is L_{M_2} -Lipschitz type continuous in the first argument, then we have

$$\begin{aligned} \|y_n - y - Q_2(M_2(v_n, u_n) - M_2(v, u_n))\|^2 &= \|y_n - y\|^2 - 2Q_2\langle M_2(v_n, u_n) - M_2(v, u_n), y_n - y \rangle \\ &\quad + Q_2^2\|M_2(v_n, u_n) - M_2(v, u_n)\|^2 \\ &\leq \|y_n - y\|^2 + 2Q_2c_2\|y_n - y\|^2 + Q_2^2L_{M_2}^2\|v_n - v\|^2 \\ &\leq \|y_n - y\|^2 + 2Q_2c_2\|y_n - y\|^2 + Q_2^2L_{M_2}^2\varsigma_2^2\|y_n - y\|^2 \\ &= \left(1 + 2Q_2c_2 + Q_2^2L_{M_2}^2\varsigma_2^2\right)\|y_n - y\|^2, \end{aligned} \quad (71)$$

which implies that

$$\begin{aligned} \|y_n - y - Q_2(M_2(v_n, u_n) - M_2(v, u_n))\| &\leq \sqrt{1 + 2Q_2c_2 + Q_2^2L_{M_2}^2\varsigma_2^2}\|y_n - y\|. \end{aligned} \quad (72)$$

Since M_2 is L_{M_2}' -Lipschitz type continuous in the second argument and P_1 is D -Lipschitz type continuous with constant ς_1 , then, we have

$$\|M_2(v, u_n) - M_2(v, u)\| \leq L_{M_2}'\|u_n - u\| \leq L_{M_2}'\varsigma_1\|x_n - x\|. \quad (73)$$

Thus, from (68), (70), (72), and (73), (66) becomes

$$\begin{aligned} \|x_{n+1} - x\| &\leq \left[1 - \delta_n\left(1 - \sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2} - Q_2L_{M_2}'\varsigma_1\right)\right] \\ &\quad \cdot \|x_n - x\| + \delta_n\left[\sqrt{1 - 2\epsilon_{\varphi_2} + \left(1 + 2\epsilon_{\varphi_2}\right)L_{\varphi_2}^2}\right. \\ &\quad \left.+ \sqrt{1 + 2Q_2c_2 + Q_2^2L_{M_2}^2\varsigma_2^2}\right]\|y_n - y\|. \end{aligned} \quad (74)$$

Again, from (54) and (60) of Algorithm 8 and utilizing

the fact that the projection mapping P_{Ω_1} is nonexpansive, we have

$$\begin{aligned}
\|y_{n+1} - y\| &= \|(1 - \gamma_n)y_n + \gamma_n(y_n - \varphi_1(y_n) + P_{\Omega_1}[\psi_1(x_n) \\
&\quad - Q_1 M_1(u_n, v_n)]) - [(1 - \gamma_n)y + \gamma_n(y - \varphi_1(y) \\
&\quad + P_{\Omega_1}[\psi_1(x) - Q_1 M_1(u, v)])]\| \\
&\leq (1 - \gamma_n)\|y_n - y\| + \gamma_n\|y_n - y - (\varphi_1(y_n) - \varphi_1(y))\| \\
&\quad + \gamma_n\|P_{\Omega_1}[\psi_1(x_n) - Q_1 M_1(u_n, v_n)] \\
&\quad - P_{\Omega_1}[\psi_1(x) - Q_1 M_1(u, v)]\| \\
&\leq (1 - \gamma_n)\|y_n - y\| + \gamma_n\|y_n - y - (\varphi_1(y_n) - \varphi_1(y))\| \\
&\quad + \gamma_n\|\psi_1(x_n) - Q_1 M_1(u_n, v_n) - [\psi_1(x) - Q_1 M_1(u, v)]\| \\
&\leq (1 - \gamma_n)\|y_n - y\| + \gamma_n\|y_n - y - (\varphi_1(y_n) - \varphi_1(y))\| \\
&\quad + \gamma_n\|x_n - x - (\psi_1(x_n) - \psi_1(x))\| + \gamma_n\|x_n - x \\
&\quad - Q_1(M_1(u_n, v_n) - M_1(u, v_n))\| \\
&\quad + \gamma_n Q_1\|M_1(u, v_n) - M_1(u, v)\|.
\end{aligned} \tag{75}$$

Since the mapping φ_1 is relaxed $(\epsilon_{\varphi_1}, \epsilon_{\varphi_1})$ -cocoercive and L_{φ_1} -Lipschitz type continuous, then we have

$$\begin{aligned}
&\|y_n - y - (\varphi_1(y_n) - \varphi_1(y))\|^2 \\
&= \|y_n - y\|^2 - 2\langle \varphi_1(y_n) - \varphi_1(y), y_n - y \rangle \\
&\quad + \|\varphi_1(y_n) - \varphi_1(y)\|^2 \\
&\leq \|y_n - y\|^2 + 2\epsilon_{\varphi_1}\|\varphi_1(y_n) - \varphi_1(y)\|^2 \\
&\quad - 2\epsilon_{\varphi_1}\|y_n - y\|^2 + \|\varphi_1(y_n) - \varphi_1(y)\|^2 \\
&\leq \left[1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2\right]\|y_n - y\|^2,
\end{aligned} \tag{76}$$

which implies that

$$\begin{aligned}
&\|y_n - y - (\varphi_1(y_n) - \varphi_1(y))\| \\
&\leq \sqrt{1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2}\|y_n - y\|.
\end{aligned} \tag{77}$$

Again, using the fact that mapping ψ_1 is relaxed $(\epsilon_{\psi_1}, \epsilon_{\psi_1})$ -cocoercive and L_{ψ_1} -Lipschitz type continuous, then we have

$$\begin{aligned}
&\|x_n - x - (\psi_1(x_n) - \psi_1(x))\|^2 \\
&= \|x_n - x\|^2 - 2\langle \psi_1(x_n) - \psi_1(x), x_n - x \rangle \\
&\quad + \|\psi_1(x_n) - \psi_1(x)\|^2 \\
&\leq \|x_n - x\|^2 + 2\epsilon_{\psi_1}\|\psi_1(x_n) - \psi_1(x)\|^2 \\
&\quad - 2\epsilon_{\psi_1}\|x_n - x\|^2 + \|\psi_1(x_n) - \psi_1(x)\|^2 \\
&\leq \left[1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2\right]\|x_n - x\|^2.
\end{aligned} \tag{78}$$

which implies that

$$\begin{aligned}
&\|x_n - x - (\psi_1(x_n) - \psi_1(x))\| \\
&\leq \sqrt{1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2}\|x_n - x\|.
\end{aligned} \tag{79}$$

Since P_1 is relaxed monotone with respect to M_1 with constant c_1 in the first argument, D -Lipschitz type continuous with constant ς_1 and M_1 is L_{M_1} -Lipschitz type continuous in the first argument, then we have

$$\begin{aligned}
&\|x_n - x - Q_1(M_1(u_n, v_n) - M_1(u, v_n))\|^2 \\
&= \|x_n - x\|^2 - 2Q_1\langle M_1(u_n, v_n) - M_1(u, v_n), x_n - x \rangle \\
&\quad + Q_1^2\|M_1(u_n, v_n) - M_1(u, v_n)\|^2 \\
&\leq \|x_n - x\|^2 + 2Q_1c_1\|x_n - x\|^2 + Q_1^2L_{M_1}^2\|u_n - u\|^2 \\
&\leq \|x_n - x\|^2 + 2Q_1c_1\|x_n - x\|^2 + Q_1^2L_{M_1}^2\varsigma_1^2\|x_n - x\|^2 \\
&= \left[1 + 2Q_1c_1 + Q_1^2L_{M_1}^2\varsigma_1^2\right]\|x_n - x\|^2,
\end{aligned} \tag{80}$$

which implies that

$$\begin{aligned}
&\|x_n - x - Q_1(M_1(u_n, v_n) - M_1(u, v_n))\| \\
&\leq \sqrt{1 + 2Q_1c_1 + Q_1^2L_{M_1}^2\varsigma_1^2}\|x_n - x\|.
\end{aligned} \tag{81}$$

Since M_1 is L_{M_1}' -Lipschitz type continuous in the second argument and P_2 is D -Lipschitz type continuous with constant ς_2 , then, we have

$$\|M_1(u, v_n) - M_1(u, v)\| \leq L_{M_1}'\|v_n - v\| \leq L_{M_1}'\varsigma_2\|y_n - y\|. \tag{82}$$

Thus, from (77), (79), (81), and (82), (75) becomes

$$\begin{aligned}
\|y_{n+1} - y\| &\leq \left[1 - \gamma_n\left(1 - \sqrt{1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2} - Q_1L_{M_1}'\varsigma_2\right)\right]\|y_n - y\| + \gamma_n \\
&\quad \cdot \left[\sqrt{1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2} + \sqrt{1 + 2Q_1c_1 + Q_1^2L_{M_1}^2\varsigma_1^2}\right]\|x_n - x\|.
\end{aligned} \tag{83}$$

For the sake of simplicity, we put

$$\begin{aligned}
I_1 &= \sqrt{1 - 2\epsilon_{\varphi_2} + (1 + 2\epsilon_{\varphi_2})L_{\varphi_2}^2} + Q_2L_{M_2}'\varsigma_1, m_1 \\
&= \sqrt{1 - 2\epsilon_{\psi_2} + (1 + 2\epsilon_{\psi_2})L_{\psi_2}^2}, \\
I_2 &= \sqrt{1 - 2\epsilon_{\varphi_1} + (1 + 2\epsilon_{\varphi_1})L_{\varphi_1}^2} + Q_1L_{M_1}'\varsigma_2, m_2 \\
&= \sqrt{1 - 2\epsilon_{\psi_1} + (1 + 2\epsilon_{\psi_1})L_{\psi_1}^2},
\end{aligned}$$

$$n_1 = \sqrt{1 + 2Q_2c_2 + Q_2^2L_{M_2}^2\varsigma_2^2}, n_2 = \sqrt{1 + 2Q_1c_1 + Q_1^2L_{M_1}^2\varsigma_1^2}. \tag{84}$$

Then, it follows from (74) and (83) that

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| &\leq [1 - \delta_n(1 - l_1)]\|x_n - x\| \\ &\quad + \delta_n(m_1 + n_1)\|y_n - y\| \\ &\quad + [1 - \gamma_n(1 - l_2)]\|y_n - y\| \\ &\quad + \gamma_n(m_2 + n_2)\|x_n - x\| \\ &= [1 - \delta_n(1 - l_1) + \gamma_n(m_2 + n_2)] \\ &\quad \cdot \|x_n - x\| + [1 - \gamma_n(1 - l_2) \\ &\quad + \delta_n(m_1 + n_1)]\|y_n - y\|, \end{aligned} \quad (85)$$

which implies that

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| &\leq \tau_1 \|x_n - x\| + \tau_2 \|y_n - y\| \\ &\leq \omega (\|x_n - x\| + \|y_n - y\|), \end{aligned} \quad (86)$$

where

$$\omega = \max \{\tau_1, \tau_2\}, \tau_1 = 1 - \delta_n(1 - l_1) + \gamma_n(m_2 + n_2) \quad \text{and} \\ \tau_2 = 1 - \gamma_n(1 - l_2) + \delta_n(m_1 + n_1).$$

Thus, from (86), Lemma 5, and conditions, we have

$$\lim_{n \rightarrow \infty} (\|x_n - x\| + \|y_n - y\|) = 0, \quad (87)$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = \lim_{n \rightarrow \infty} \|y_n - y\| = 0. \quad (88)$$

Hence, $x_n \rightarrow x$ and $y_n \rightarrow y$. It follows from (62) and (63) that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences; we can assume that $u_n \rightarrow u$ and $v_n \rightarrow v$, strongly. Next, we show that $u \in P_1(x)$ and $v \in P_2(y)$. Since $u_n \in P_1(x_n)$, then we have

$$\begin{aligned} d(u, P_1(x)) &\leq \|u - u_n\| + d(u_n, P_1(x)) \\ &\leq \|u - u_n\| + (1 + n^{-1})D(P_1(x_n), P_1(x)) \\ &\leq \|u - u_n\| + (1 + n^{-1})c_1\|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (89)$$

Hence, $d(u, P_1(x)) \rightarrow 0$, so $u \in P_1(x)$ as $P_1(x) \in CB(\mathcal{H})$. Similarly, it is easy to show that $v \in P_2(y)$. Thus, by Lemma 6, one can deduce that (x, y, u, v) , $x \in \Omega_1$, $y \in \Omega_2$, $u \in P_1(x)$, and $v \in P_2(y)$ is a solution to the system of generalized set-valued variational inequalities (15).

4. Altering Points Problem

In this section, the concept of altering points problem is used to find the solution of the considered system of variational inequalities (16). We propose parallel Mann and parallel S-iterative algorithms, and the strong convergence of the sequences generated by these parallel iterative algorithms is discussed.

Definition 10. [11]. Let Ω_1 and Ω_2 be nonempty subsets of a metric space X . Then, the points $x \in \Omega_1$ and $y \in \Omega_2$ are the altering points of mappings $A_1 : \Omega_1 \rightarrow \Omega_2$ and $A_2 : \Omega_2 \rightarrow$

Ω_1 , if

$$\begin{aligned} A_1(x) &= y, \\ A_2(y) &= x. \end{aligned} \quad (90)$$

Alt-
(A_1, A_2) = $\{(x, y) \in \Omega_1 \times \Omega_2 : A_1(x) = y \text{ and } A_2(y) = x\}$ is called the set of altering points of the mappings A_1 and A_2 .

Lemma 11 [33]. Let Ω_1 and Ω_2 be nonempty closed subsets of a complete metric space X . Let $A_1 : \Omega_1 \rightarrow \Omega_2$ and $A_2 : \Omega_2 \rightarrow \Omega_1$ be Lipschitz continuous mappings with constants κ_1 and κ_2 , respectively, such that $\kappa_1\kappa_2 < 1$. Then, the following conditions hold:

- (i) There exists unique point $(x, y) \in \Omega_1 \times \Omega_2$ such that x and y are altering points of the mappings A_1 and A_2 .
- (ii) For arbitrary $x_1 \in \Omega_1$, the sequence $\{(x_n, y_n)\} \in \Omega_1 \times \Omega_2$ generated by

$$\begin{cases} y_n = A_1 x_n, \\ x_{n+1} = A_2 y_n, n \in \mathbb{N}, \end{cases} \quad (91)$$

converges to (x, y) .

Lemma 12. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. Let $I = \{1, 2\}$ be an index set, for each $i \in I$; let $\varphi_i, \psi_i : \mathcal{H} \rightarrow \mathcal{H}$ and $M_i : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ be non-linear single-valued mappings. Then, $(x, y) \in \Omega_1 \times \Omega_2$ is the solution to the system of variational inequalities (16), if and only if (x, y) satisfies

$$\begin{aligned} y &= P_{\Omega_2}[\psi_1(x) - Q_1 M_1(y, x)], \\ x &= P_{\Omega_1}[\psi_2(y) - Q_2 M_2(x, y)], \end{aligned} \quad (92)$$

where Q_1 and Q_2 are positive constants.

Define the mappings $A_1 : \Omega_1 \rightarrow \Omega_2$ and $A_2 : \Omega_2 \rightarrow \Omega_1$ as follows:

$$A_1 := P_{\Omega_2}[\psi_1 - Q_1 M_1(y, \cdot)], \quad (93)$$

$$A_2 := P_{\Omega_1}[\psi_2 - Q_2 M_2(x, \cdot)], \quad (94)$$

where Q_1, Q_2 are constants in $(0, 1]$. By virtue of Lemma 4, we can formulate the system of variational inequalities (16) into the following equivalent altering points problem.

Find $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\begin{cases} x = P_{\Omega_1}[\psi_2 - Q_2 M_2(x, \cdot)](y), \\ y = P_{\Omega_2}[\psi_1 - Q_1 M_1(y, \cdot)](x). \end{cases} \quad (95)$$

Now, we propose the following parallel Mann iterative algorithm to solve the system of variational inequalities (16) as follows.

Algorithm 13. Let Ω_1, Ω_2 be nonempty closed convex subsets of a real Hilbert space H . For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_2(y_n), \quad (96)$$

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n A_1(x_n), \quad (97)$$

where $\alpha_n \in [0, 1]$, $n \in \mathbb{N}$ and A_1, A_2 are the mappings defined by (93) and (94), respectively.

Also, we propose the following parallel S-iterative algorithm to solve the system of variational inequalities (16). Note that the parallel S-iterative algorithm is more general than the parallel Mann iterative algorithm.

Algorithm 14. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space H . For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ defined by

$$x_{n+1} = A_2[(1 - \alpha_n)y_n + \alpha_n A_1(x_n)], \quad (98)$$

$$y_{n+1} = A_1[(1 - \alpha_n)x_n + \alpha_n A_2(y_n)], \quad (99)$$

where $\alpha_n \in (0, 1)$, $n \in \mathbb{N}$ and A_1, A_2 are the mappings defined by (93) and (94), respectively.

Theorem 15 [33]. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a Banach space X . Let $A_1 : \Omega_1 \rightarrow \Omega_2$ and $A_2 : \Omega_2 \rightarrow \Omega_1$ be Lipschitz continuous mappings with constants $\kappa_1 < 1$ and $\kappa_2 < 1$, respectively. Then, the sequence $\{(x_n, y_n)\} \in \Omega_1 \times \Omega_2$ generated by the parallel S-iterative algorithm (98)–(99) converges strongly to the unique point $(x, y) \in \Omega_1 \times \Omega_2$ such that x and y are altering points of the mappings A_1 and A_2 .

Next, we prove the following proposition, which plays a crucial role to prove the convergence of parallel iterative algorithms.

Proposition 16. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. For each $i = 1, 2$, let $\psi_i : \mathcal{H} \rightarrow \mathcal{H}$ be the single valued mappings such that ψ_i is δ_{ψ_i} -strongly monotone and L_{ψ_i} -Lipschitz type continuous. Let $M_1 : \Omega_2 \times \Omega_1 \rightarrow \mathcal{H}$ and $M_2 : \Omega_1 \times \Omega_2 \rightarrow \mathcal{H}$ be the single-valued mappings such that M_i is δ_{M_i} -strongly monotone and L_{M_i} -Lipschitz type continuous with respect to the second argument. Then, the mappings A_1 and A_2 defined by (93) and (94) are Lipschitz type continuous with constant $(\mathcal{U}_1 + \Theta_1)$ and $(\mathcal{U}_2 + \Theta_2)$, respectively, where $\mathcal{U}_i = \sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ and $\Theta_i = \sqrt{1 - 2\mathcal{Q}_i\delta_{M_i} + \rho_i^2 L_{M_i}^2}$.

Proof. For any given $x_1, x_2 \in \Omega_1$, (93), and using the nonexpansiveness of the projection mapping P_{Ω_2} , we have

$$\begin{aligned} \|A_1(x_1) - A_1(x_2)\| &= \|P_{\Omega_2}[\psi_1 - \mathcal{Q}_1 M_1(y, \cdot)](x_1) \\ &\quad - P_{\Omega_2}[\psi_1 - \mathcal{Q}_1 M_1(y, \cdot)](x_2)\| \\ &\leq \|\psi_1(x_1) - \mathcal{Q}_1 M_1(y, (x_1))\| \\ &\quad - \|\psi_1(x_2) - \mathcal{Q}_1 M_1(y, (x_2))\| \quad (100) \\ &\leq \|\psi_1(x_1) - \psi_1(x_2) - (x_1 - x_2)\| \\ &\quad + \|(x_1 - x_2) - \mathcal{Q}_1(M_1(y, (x_1)) \\ &\quad - M_1(y, (x_2)))\|. \end{aligned}$$

Since ψ_1 is δ_{ψ_1} -strongly monotone and L_{ψ_1} -Lipschitz type continuous, then, we get

$$\begin{aligned} \|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\|^2 &= \|x_1 - x_2\|^2 - 2\langle \psi_1(x_1) - \psi_1(x_2), x_1 - x_2 \rangle \\ &\quad + \|\psi_1(x_1) - \psi_1(x_2)\|^2 \\ &\leq \|x_1 - x_2\|^2 - 2\delta_{\psi_1}\|x_1 - x_2\|^2 + L_{\psi_1}^2\|x_1 - x_2\|^2 \quad (101) \\ &= (1 - 2\delta_{\psi_1} + L_{\psi_1}^2)\|x_1 - x_2\|^2 \\ &= \mathcal{U}_1^2\|x_1 - x_2\|^2, \end{aligned}$$

which implies that

$$\|x_1 - x_2 - (\psi_1(x_1) - \psi_1(x_2))\| \leq \mathcal{U}_1\|x_1 - x_2\|. \quad (102)$$

Again, using the fact that M_1 is δ_{M_1} -strongly monotone and L_{M_1} -Lipschitz type continuous with respect to the second argument, we have

$$\begin{aligned} \|x_1 - x_2 - \mathcal{Q}_1(M_1(y, (x_1)) - M_1(y, (x_2)))\|^2 &= \|x_1 - x_2\|^2 - 2\mathcal{Q}_1\langle M_1(y, (x_1)) - M_1(y, (x_2)), x_1 - x_2 \rangle \\ &\quad + \rho_1^2\|M_1(y, (x_1)) - M_1(y, (x_2))\|^2 \\ &\leq \|x_1 - x_2\|^2 - 2\mathcal{Q}_1\delta_{M_1}\|x_1 - x_2\|^2 + \rho_1^2 L_{M_1}^2\|x_1 - x_2\|^2 \\ &= (1 - 2\mathcal{Q}_1\delta_{M_1} + \rho_1^2 L_{M_1}^2)\|x_1 - x_2\|^2 \\ &= \Theta_1^2\|x_1 - x_2\|^2, \end{aligned} \quad (103)$$

which implies that

$$\|x_1 - x_2 - \mathcal{Q}_1(M_1(y, (x_1)) - M_1(y, (x_2)))\| \leq \Theta_1\|x_1 - x_2\|. \quad (104)$$

Utilizing (102) and (104), (100) becomes

$$\|A_1(x_1) - A_1(x_2)\| \leq (\mathcal{U}_1 + \Theta_1)\|x_1 - x_2\|, \quad (105)$$

i.e., A_1 is $(\mathcal{U}_1 + \Theta_1)$ -Lipschitz continuous, where $\mathcal{U}_1 =$

$\sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ and $\Theta_i = \sqrt{1 - 2Q_i\delta_{\psi_i} + Q_i^2L_{M_i}^2}$. Similarly, one can prove that A_2 is $(\mathfrak{U}_2 + \Theta_2)$ -Lipschitz continuous.

Now, we prove the convergence of the parallel Mann iterative scheme (96)–(97).

Theorem 17. Let Ω_1 and Ω_2 be nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. For each $i = 1, 2$, let $\psi_i : \mathcal{H} \rightarrow \mathcal{H}$ be the single-valued mapping such that ψ_i is δ_{ψ_i} -strongly monotone and L_{ψ_i} -Lipschitz type continuous. Let $M_1 : \Omega_2 \times \Omega_1 \rightarrow \mathcal{H}$ and $M_2 : \Omega_1 \times \Omega_2 \rightarrow \mathcal{H}$ be the single-valued mappings such that M_i is δ_{M_i} -strongly monotone and L_{M_i} -Lipschitz type continuous with respect to the second argument. For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ generated by Algorithm 13, where $\alpha_n \in [0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$ and A_1, A_2 are the mappings defined by (93) and (94), respectively. In addition, the following condition holds:

$$0 < \mathfrak{U}_i + \Theta_i < 1, \quad (106)$$

where $\mathfrak{U}_i = \sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ and $\Theta_i = \sqrt{1 - 2Q_i\delta_{M_i} + Q_i^2L_{M_i}^2}$, for $i = 1, 2$. Then

- (i) There exists unique point $(x, y) \in \Omega_1 \times \Omega_2$, which is the solution of the system of variational inequalities (16).
- (ii) The sequence $\{(x_n, y_n)\}$ generated by parallel Mann iterative Algorithm 13 converges strongly to the point (x, y) .

Proof.

- (i) Conclusion follows from Lemma 11 and (95).
- (ii) It follows from (94), (95), (96), and Lipschitz continuity of mapping A_2 that

$$\begin{aligned} \|x_{n+1} - x\| &= \|(1 - \alpha_n)x_n + \alpha_n A_2(y_n) - x\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n A_2(y_n) - (1 - \alpha_n)x - \alpha_n x\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n\|A_2(y_n) - x\| \\ &= (1 - \alpha_n)\|x_n - x\| + \alpha_n\|A_2(y_n) - A_2(y)\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n(\mathfrak{U}_2 + \Theta_2)\|y_n - y\|. \end{aligned} \quad (107)$$

Again, utilizing (93), (95), (97), and Lipschitz continuity of mapping A_1 , we get

$$\begin{aligned} \|y_{n+1} - y\| &= \|(1 - \alpha_n)y_n + \alpha_n A_1(x_n) - y\| \\ &= \|(1 - \alpha_n)y_n + \alpha_n A_1(x_n) - (1 - \alpha_n)y - \alpha_n y\| \\ &\leq (1 - \alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - y\| \\ &= (1 - \alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - A_1(x)\| \\ &\leq (1 - \alpha_n)\|y_n - y\| + \alpha_n(\mathfrak{U}_1 + \Theta_1)\|x_n - x\|. \end{aligned} \quad (108)$$

Letting $\Theta = \max\{\mathfrak{U}_1 + \Theta_1, \mathfrak{U}_2 + \Theta_2\}$; then, from (107) and (108), we have

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n(\mathfrak{U}_2 + \Theta_2)\|y_n - y\| \\ &\quad + (1 - \alpha_n)\|y_n - y\| + \alpha_n(\mathfrak{U}_1 + \Theta_1)\|x_n - x\| \\ &\leq (1 - \alpha_n)\|x_n - x\| + \alpha_n\Theta\|y_n - y\| \\ &\quad + (1 - \alpha_n)\|y_n - y\| + \alpha_n\Theta\|x_n - x\| \\ &= (1 - \alpha_n(1 - \Theta))(\|x_n - x\| + \|y_n - y\|). \end{aligned} \quad (109)$$

Thus, from (8) and (109), we have

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (x, y)\|_* &= \|(x_{n+1} - x, y_{n+1} - y)\|_* \\ &= \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\leq (1 - \alpha_n(1 - \Theta))(\|x_n - x\| + \|y_n - y\|) \\ &\leq (1 - \alpha_n(1 - \Theta))\|(x_n, y_n) - (x, y)\|_*. \end{aligned} \quad (110)$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$ and from condition (106), we have $\Theta \in (0, 1)$. Thus, from Lemma 5, we can conclude that $\lim_{n \rightarrow \infty} \|(x_n, y_n) - (x, y)\|_* = 0$. Therefore, $\lim_{n \rightarrow \infty} \|x_n - x\| = 0 = \lim_{n \rightarrow \infty} \|y_n - y\|$. Hence, the sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel Mann iterative algorithm converge strongly to x and y , respectively.

Now, we prove the convergence of the parallel S-iterative algorithm (98)–(99).

Theorem 18. Let Ω_1 and Ω_2 be nonempty closed and convex subsets of a real Hilbert space $\mathcal{H}$. For each $i = 1, 2$, let $\psi_i : \mathcal{H} \rightarrow \mathcal{H}$ be the single-valued mapping such that ψ_i is δ_{ψ_i} -strongly monotone and L_{ψ_i} -Lipschitz type continuous. Let $M_1 : \Omega_2 \times \Omega_1 \rightarrow \mathcal{H}$ and $M_2 : \Omega_1 \times \Omega_2 \rightarrow \mathcal{H}$ be the single-valued mappings such that M_i is δ_{M_i} -strongly monotone and L_{M_i} -Lipschitz type continuous with respect to the second argument. For any $(x_0, y_0) \in \Omega_1 \times \Omega_2$, let $\{(x_n, y_n)\}$ be an iterative sequence in $\Omega_1 \times \Omega_2$ generated by Algorithm 14, where $\alpha_n \in (0, 1)$, A_1 , and A_2 are the mappings defined by (93) and (94), respectively. In addition, the following condition holds:

$$0 < \mathfrak{U}_i + \Theta_i < 1, \quad (111)$$

where $\mathfrak{U}_i = \sqrt{1 - 2\delta_{\psi_i} + L_{\psi_i}^2}$ and $\Theta_i = \sqrt{1 - 2\rho_i\delta_{M_i} + \rho_i^2 L_{M_i}^2}$, for $i = 1, 2$. Then

- (i) There exists unique point $(x, y) \in \Omega_1 \times \Omega_2$, which is the solution of the system of variational inequalities (16).
- (ii) The sequence $\{(x_n, y_n)\}$ generated by parallel S-iterative Algorithm 14 converges strongly to the point (x, y) .

Proof.

- (i) The conclusion follows from Lemma 11 and (95).
- (ii) It follows from (94), (95), (98), and Lipschitz continuity of A_2 , we get

$$\begin{aligned} \|x_{n+1} - x\| &= \|A_2[(1 - \alpha_n)y_n + \alpha_n A_1(x_n)] - x\| \\ &= \|A_2[(1 - \alpha_n)y_n + \alpha_n A_1(x_n)] - A_2(y)\| \\ &\leq (\mathfrak{U}_2 + \Theta_2)\|(1 - \alpha_n)y_n + \alpha_n A_1(x_n) - y\| \\ &\leq (\mathfrak{U}_2 + \Theta_2)[(1 - \alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - y\|] \\ &= (\mathfrak{U}_2 + \Theta_2)[(1 - \alpha_n)\|y_n - y\| + \alpha_n\|A_1(x_n) - A_1(x)\|] \\ &\leq (\mathfrak{U}_2 + \Theta_2)[(1 - \alpha_n)\|y_n - y\| + \alpha_n(\mathfrak{U}_1 + \Theta_1)\|x_n - x\|]. \end{aligned} \quad (112)$$

Using the same facts, we have

$$\|y_{n+1} - y\| \leq (\mathfrak{U}_1 + \Theta_1)[(1 - \alpha_n)\|x_n - x\| + \alpha_n(\mathfrak{U}_2 + \Theta_2)\|y_n - y\|]. \quad (113)$$

Letting $\Theta = \max\{\mathfrak{U}_1 + \Theta_1, \mathfrak{U}_2 + \Theta_2\}$, then from (112) and (113), we have

$$\begin{aligned} \|x_{n+1} - x\| + \|y_{n+1} - y\| &\leq (\mathfrak{U}_2 + \Theta_2)[(1 - \alpha_n)\|y_n - y\| + \alpha_n \\ &\quad \cdot (\mathfrak{U}_1 + \Theta_1)\|x_n - x\|] + (\mathfrak{U}_1 + \Theta_1) \\ &\quad \cdot [(1 - \alpha_n)\|x_n - x\| + \alpha_n(\mathfrak{U}_2 + \Theta_2)\|y_n - y\|] \\ &\leq \Theta[(1 - \alpha_n)\|y_n - y\| + \alpha_n\Theta\|x_n - x\|] + \Theta \\ &\quad \cdot [(1 - \alpha_n)\|x_n - x\| + \alpha_n\Theta\|y_n - y\|] \\ &= \Theta[(1 - \alpha_n(1 - \Theta))][\|x_n - x\| + \|y_n - y\|]. \end{aligned} \quad (114)$$

Thus, from (8) and (114), we have

$$\begin{aligned} \|(x_{n+1}, y_{n+1}) - (x, y)\|_* &= \|(x_{n+1} - x, y_{n+1} - y)\|_* \\ &= \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ &\leq \Theta[(1 - \alpha_n(1 - \Theta))][\|x_n - x\| + \|y_n - y\|] \\ &= \Theta[(1 - \alpha_n(1 - \Theta))]\|(x_n, y_n) - (x, y)\|_*. \end{aligned} \quad (115)$$

Notice that $\Theta[(1 - \alpha_n(1 - \Theta))] < \Theta$ and from condition (111), we have $\Theta \in (0, 1)$. Thus, we can conclude that $\lim_{n \rightarrow \infty} \|$

$(x_n, y_n) - (x, y)\|_* = 0$. Therefore, $\lim_{n \rightarrow \infty} \|x_n - x\| = 0 = \lim_{n \rightarrow \infty} \|y_n - y\|$. Hence, the sequences $\{x_n\}$ and $\{y_n\}$ generated by parallel S-iterative algorithm converge strongly to x and y , respectively.

Finally, we discuss an example which illustrates the convergence analysis of parallel Mann and parallel S-iterative algorithms.

Example 1. Let $\mathcal{H} = \mathbb{R}$, $\Omega_1 = (-\infty, 0]$ and $\Omega_2 = [0, \infty)$. Let $\psi_1, \psi_2 : \mathcal{H} \rightarrow \mathcal{H}$ be the single-valued mappings defined by

$$\psi_1(x) = \frac{2x - 1}{3}, \psi_2(x) = \frac{5x - 2}{6}, \forall x \in \mathcal{H}. \quad (116)$$

Suppose that the mappings $M_1, M_2 : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ are defined by

$$M_1(x, y) = \frac{4x + 3y - 2}{3}, M_2(x, y) = \frac{17x + 17y + 1}{12}, \forall x, y \in \mathcal{H}. \quad (117)$$

Then, it is easy to check that for each $i = 1, 2$, ψ_i is δ_{ψ_i} -strongly monotone and L_{ψ_i} -Lipschitz continuous with $\delta_{\psi_1} = 2/3 = L_{\psi_1}$ and $\delta_{\psi_2} = 5/6 = L_{\psi_2}$ and M_i is δ_{M_i} -strongly monotone and L_{M_i} -Lipschitz continuous in the second argument with $\delta_{M_1} = 4/3 = L_{M_1}$ and $\delta_{M_2} = 17/12 = L_{M_2}$. Let the mappings $A_1 : \Omega_1 \rightarrow \Omega_2$ and $A_2 : \Omega_2 \rightarrow \Omega_1$ be defined as follows:

$$\begin{aligned} A_1 &:= P_{\Omega_2}[\psi_1 - \mathfrak{Q}_1 M_1(y, \cdot)], \\ A_2 &:= P_{\Omega_1}[\psi_2 - \mathfrak{Q}_2 M_2(x, \cdot)]. \end{aligned} \quad (118)$$

Then, for $\mathfrak{Q}_1, \mathfrak{Q}_2 = 1$,

$$\begin{aligned} A_1(x) &= P_{\Omega_2}[\psi_1(x) - M_1(y, x)] = P_{\Omega_2}\left[\frac{2x - 1}{3} - \frac{4x + 3y - 2}{3}\right] \\ &= \frac{-2x - 3y + 1}{3}, \forall x, y \in \mathcal{H}, \\ A_2(y) &= P_{\Omega_1}[\psi_2(y) - M_2(x, y)] = P_{\Omega_1}\left[\frac{5y - 2}{6} - \frac{17x + 17y + 1}{12}\right] \\ &= \frac{-17x - 7y - 5}{12}, \forall x, y \in \mathcal{H}. \end{aligned} \quad (119)$$

It is easy to verify that A_1 is $2/3$ -Lipschitz continuous and A_2 is $7/12$ -Lipschitz continuous. Also,

$$\begin{aligned}
\mathcal{U}_1 + \Theta_1 &= \sqrt{1 - 2\delta_{\psi_1} + L_{\psi_1}^2} + \sqrt{1 - 2\rho_1\delta_{M_1} + \rho_1^2 L_{M_1}^2} \\
&= \sqrt{1 - \frac{4}{3} + \frac{4}{9}} + \sqrt{1 - \frac{8}{3} + \frac{16}{9}} = \frac{2}{3} < 1, \\
\mathcal{U}_2 + \Theta_2 &= \sqrt{1 - 2\delta_{\psi_2} + L_{\psi_2}^2} + \sqrt{1 - 2\rho_2\delta_{M_2} + \rho_2^2 L_{M_2}^2} \\
&= \sqrt{1 - \frac{10}{6} + \frac{25}{36}} + \sqrt{1 - \frac{34}{12} + \frac{289}{144}} = \frac{7}{12} < 1.
\end{aligned} \tag{120}$$

Thus, all the conditions of Theorem 17 and Theorem 18 are satisfied. Hence, by using Proposition 16, Algorithm 13, and Algorithm 14, the conclusions of Theorem 17 and Theorem 18 follow.

5. Concluding Remarks

In this paper, a system of generalized set-valued variational inequalities involving relaxed cocoercive mappings in Hilbert spaces is considered. Using the projection method and Banach contraction principle, an existence result is proved. Also, we proposed an iterative algorithm, and its convergence is discussed. Moreover, we established an equivalence between the system of variational inequalities and altering points problem. Some parallel iterative algorithms are proposed, and the strong convergence of the sequences generated by these iterative algorithms is discussed. Finally, a numerical example is constructed to illustrate the convergence analysis of the proposed parallel iterative algorithms.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally to the writing of this manuscript. All authors read and approve the final version.

Acknowledgments

The second author would like to thank the Deanship of Scientific Research, Prince Sattam bin Abdulaziz University for supporting this work.

References

- [1] G. Stampacchia, "Formes bilineaires coercitives sur les ensembles convexes," *Académie des Sciences de Paris*, vol. 258, pp. 4413–4416, 1964.
- [2] J.-L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 493–519, 1967.
- [3] M. Akram and M. Dilshad, "New generalized system of variational inclusions involving $H(\dots)$ -co-accretive mapping," *Afrika Matematika*, vol. 29, pp. 1173–1187, 2018.
- [4] M. Akram, "Existence and iterative approximation of solution for generalized Yosida inclusion problem," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 15, no. 2, pp. 147–161, 2020.
- [5] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, pp. 117–136, 2005.
- [6] K. S. Kim, "System of extended general variational inequalities for relaxed cocoercive mappings in Hilbert space," *Mathematics*, vol. 6, no. 10, 2018.
- [7] C. E. Lemke, "Bimatrix equilibrium points and mathematical programming," *Management science*, vol. 11, no. 7, pp. 681–689, 1965.
- [8] J.-F. Rodrigues, "Obstacle problems in mathematical physics," in *North-Holland Mathematics Studies*, vol. 134, Elsevier, 1987.
- [9] M. J. Smith, "The existence, uniqueness and stability of traffic equilibria," *Transportation Research Part B: Methodological*, vol. 13, no. 4, pp. 295–304, 1979.
- [10] R. P. Agarwal, D. ÓRegan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *Journal of Nonlinear and convex Analysis*, vol. 8, no. 1, pp. 61–79, 2007.
- [11] D. R. Sahu, S. M. Kang, and A. Kumar, "Convergence analysis of parallel S-iteration process for system of generalized variational inequalities," *Journal of Function Spaces*, vol. 2017, Article ID 5847096, 10 pages, 2017.
- [12] E. Picard, "Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives," *Journal de Mathématiques pures et appliquées*, vol. 16, pp. 145–210, 1890.
- [13] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.
- [14] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.
- [15] P. Chulamjiak, A. A. Abdou, and Y. J. Cho, "Proximal point algorithms involving fixed points of nonexpansive mappings in $CAT(0)$ spaces," *Fixed Point Theory and Applications*, vol. 2015, Article ID 227, 2015.
- [16] F. Amir, A. Farajzadeh, and N. Petrot, "Hybrid proximal point algorithm for solution of convex multiobjective optimization problem over fixed point constraint," *Thai Journal of Mathematics*, vol. 18, no. 3, pp. 841–850, 2020.
- [17] M. Alansari, M. Akram, and M. Dilshad, "Iterative algorithms for a generalized system of mixed variational-like inclusion problems and altering points problem," *Statistics, Optimization & Information Computing*, vol. 8, no. 2, pp. 549–564, 2020.
- [18] T. T. Anh, L. B. Long, and T. V. Anh, "A projection method for bilevel variational inequalities," *Journal of Inequalities and Applications*, vol. 2014, Article ID 205, 2014.
- [19] A. Farajzadeh, M. Mursaleen, and A. Shafie, "On mixed vector equilibrium problems," *Azerbaijan Journal of Mathematics*, vol. 6, no. 2, pp. 87–102, 2016.
- [20] P. Lohawech, A. Kaewcharoen, and A. Farajzadeh, "Algorithms for the common solution of the split variational

- inequality problems and fixed point problems with applications,” *Journal of Inequalities and Applications*, vol. 2015, Article ID 358, 2018.
- [21] N. Nimana, A. Farajzadeh, and N. Petrot, “Adaptive subgradient method for the split quasi-convex feasibility problems,” *Optimization*, vol. 65, no. 10, pp. 1885–1898, 2016.
 - [22] M. A. Noor, K. I. Noor, and A. Bnouhachem, “Some new iterative methods for solving variational inequalities,” *Canadian Journal of Applied Mathematics*, vol. 2, no. 2, pp. 1–17, 2020.
 - [23] Y. Shehu, O. S. Iyiola, X. Li, and Q. Dong, “Convergence analysis of projection method for variational inequalities,” *Computational and Applied Mathematics*, vol. 38, 2019.
 - [24] Y. Yao, X. Qin, and J.-C. Yao, “Projection methods for firmly type nonexpansive operators,” *Journal of Nonlinear and Convex Analysis*, vol. 19, pp. 407–415, 2018.
 - [25] R. U. Verma, “Generalized system for relaxed cocoercive variational inequalities and Projection methods,” *Journal of Optimization Theory and Applications*, vol. 121, pp. 203–210, 2004.
 - [26] R. Ahmad, M. Akram, and J. C. Yao, “Generalized monotone mapping with an application for solving a variational inclusion problem,” *Journal of Optimization Theory and Applications*, vol. 157, no. 2, pp. 324–346, 2013.
 - [27] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
 - [28] R. P. Agarwal, D. ÓRegan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, 2009.
 - [29] X. Weng, “Fixed point iteration for local strictly pseudocontractive mappings,” *Proceedings of the American Mathematical Society*, vol. 113, pp. 727–731, 1991.
 - [30] R. U. Verma, “Projection methods, algorithms and a new system of nonlinear variational inequalities,” *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 1025–1031, 2001.
 - [31] M. A. Noor, K. I. Noor, A. G. Khan, and R. Latif, “Some new algorithms for solving a system of general variational inequalities,” *Applied Mathematics & Information Sciences*, vol. 10, no. 2, pp. 557–564, 2016.
 - [32] S. B. Nadlar, “Multivalued contraction mapping,” *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
 - [33] D. R. Sahu, “Altering points and applications,” *Nonlinear Studies*, vol. 21, no. 2, pp. 349–365, 2014.