Research Article

Inequalities for Unified Integral Operators via Strongly $(\alpha, h-m)$ -Convexity

Zhongyi Zhang¹, Ghulam Farid², and Kahkashan Mahreen²

¹Nanjing Vocational College of Information Technology, Nanjing, Jiangsu 210023, China

²Department of Mathematics, COMSATS University Islamabad, Attock Campus, Pakistan

Correspondence should be addressed to Zhongyi Zhang; zhangzhongyi80@sina.com and Ghulam Farid; faridphdsms@hotmail.com

Received 21 December 2020; Revised 14 April 2021; Accepted 8 May 2021; Published 4 June 2021

Academic Editor: Fu-Dong Ge

Copyright © 2021 Zhongyi Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we give a generalized definition namely strongly $(\alpha, h-m)$ -convex function that unifies many known definitions. By applying this new definition, we present inequalities for unified integral operators which have connection with many of the well-known results for different kinds of convex functions. Moreover, this paper at once provides refinements and generalizations of a lot of fractional integral inequalities which are identified in remarks.

1. Introduction

There are many applications of convexity in diverse fields of mathematics including operation research, mathematical statistics, optimization theory, and graph theory. In mathematical inequalities' point of view, convex functions are very important. They are extended and generalized in different ways to obtain corresponding generalizations and extensions of well-known inequalities. For the detail study of different kinds of convex functions, we refer the readers to [1–7].

In recent years, the researchers are working on fractional versions of mathematical inequalities by utilizing classical and new kinds fractional integral/derivative operators, see [8–11]. Also, several kinds of convex functions are applied to obtain these fractional versions, for example, see [1, 12–17] and references therein. The inequalities for fractional integrals and derivatives are very useful in the study of fractional differential equations. Using fractional differential and integral inequalities, qualitative properties of fractional differential equations involving the Riemann-Liouville and the Caputo derivatives can be found frequently in literature.

Our objective is to investigate integral inequalities for newly defined function called strongly $(\alpha, h-m)$ -convex function. Integral operators (5) and (6) are used to establish

these inequalities, and they have interesting consequences for distinctive fractional inequalities for various types of functions. Next, we give some definitions of known generalized fractional integral operators which can be directly obtained from operators (5) and (6).

Definition 1 (see [18]). Let $\psi : [a, b] \rightarrow \mathbb{R}$ be an integrable function and ξ be an increasing positive function defined on $[a, b]$ has a continuous derivative ξ' on (a, b) . The fractional integrals of a function ψ with respect to another function ξ on $[a, b]$ of order μ ($R(\mu) > 0$) are defined by

$$\begin{aligned} {}^{\mu}I_{a^+} \psi(x) &= \frac{1}{\Gamma(\mu)} \int_a^x (\xi(x) - \xi(t))^{\mu-1} \xi'(t) \psi(t) dt, x > a, \\ {}^{\mu}I_{b^-} \psi(x) &= \frac{1}{\Gamma(\mu)} \int_x^b (\xi(t) - \xi(x))^{\mu-1} \xi'(t) \psi(t) dt, x < b, \end{aligned} \quad (1)$$

here, $\Gamma(\cdot)$ represents the gamma function.

One can see a k -analogue of Definition 1 in [16]. The following generalized integral operator is given in [19].

Definition 2. Let $\psi, \xi : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$ be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and increasing. Also let ϕ/x be an increasing function on $[a, \infty)$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$\begin{aligned} (F_{a^+}^{\phi, \xi} \psi)(x) &= \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)} \xi'(t) \psi(t) dt, \quad x > a, \\ (F_{b^-}^{\phi, \xi} \psi)(x) &= \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)} \xi'(t) \psi(t) dt, \quad x < b. \end{aligned} \quad (2)$$

The Mittag-Leffler function was introduced in 1903, which is generalization of the exponential function just by a single parameter with a convergence condition. The further generalization by another parameter was given by Wiman; further, it was extended by Prabhakar and then by other authors; to see the importance of these extensions, we suggest the reader to [20, 21]. By utilizing an extended generalized Mittag-Leffler function, we have defined a fractional integral operator.

Definition 3 (see [12]). Let $\alpha, \gamma, \mu, c, l, w \in \mathbb{C}$, $\Re(\alpha), \Re(\mu), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ and $p \geq 0$, $\delta > 0$ with $0 < k \leq \delta + \Re(\mu)$. Let $\psi \in L_1[a, b]$ and $x \in [a, b]$. The generalized fractional integral operators $\varepsilon_{\mu, \alpha, l, w, a^+}^{\gamma, \delta, k, c} \psi$ and $\varepsilon_{\mu, \alpha, l, w, b^-}^{\gamma, \delta, k, c} \psi$ are defined by:

$$\begin{aligned} (\varepsilon_{\mu, \alpha, l, w, a^+}^{\gamma, \delta, k, c} \psi)(x; p) &= \int_a^x (x-t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(w(x-t)^\mu; p) \psi(t) dt, \\ (\varepsilon_{\mu, \alpha, l, w, b^-}^{\gamma, \delta, k, c} \psi)(x; p) &= \int_x^b (t-x)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(w(t-x)^\mu; p) \psi(t) dt, \end{aligned} \quad (3)$$

where

$$E_{\mu, \alpha, l}^{\gamma, \delta, k, c}(t; p) = \sum_{n=0}^{\infty} \frac{\eta_p(\gamma + nk, c - \gamma)}{\eta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \quad (4)$$

is the extended generalized Mittag-Leffler function.

Unified integral operator is based on a kernel which also involves a real valued strictly increasing function along with two variables. This integral operator also unifies the above definitions.

Definition 4 (see [22]). Let $\psi, \xi : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$ be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and strictly increasing. Also, let ϕ/x be differentiable and strictly increasing. Also, let $[a, \infty)$ and $\tau, l, \gamma, c \in \mathbb{C}$, $\Re(\tau), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$(\xi F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi)(x, w; p) = \int_a^x K_x^\gamma(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi) \psi(y) d(\xi(y)), \quad (5)$$

$$(\xi F_{\mu, \tau, l, b^-}^{\phi, \gamma, \delta, k, c} \psi)(x, w; p) = \int_x^b K_x^\gamma(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi) \psi(y) d(\xi(y)), \quad (6)$$

where we have

$$K_x^\gamma(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi) = \frac{\phi(\xi(x) - \xi(y))}{\xi(x) - \xi(y)} E_{\mu, \tau, l}^{\gamma, \delta, k, c}(w(\xi(x) - \xi(y))^\mu; p). \quad (7)$$

Several recently defined fractional integrals studied in [12, 14, 18, 21, 23–30] can be reproduced from the above definition, see ([31], Remarks 6 and 7). The following results are obtained for strongly convex functions in [32].

Theorem 5. Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be a positive, integrable, and strongly convex function with modulus $\lambda \geq 0$. Let $\xi : [a, b] \longrightarrow \mathbb{R}$ be a strictly increasing and differentiable function, also let ϕ/x be an increasing function on $[a, b]$. If $\alpha, \gamma, \eta, c, l \in \mathbb{R}_+$, $c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, the following inequality holds:

$$\begin{aligned} &(\xi F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} \psi)(x, w; p) + (\xi F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi)(x, w; p) \\ &\leq K_x^a(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi)((\xi(x) - \xi(a))(\psi(x) + \psi(a)) \\ &\quad - \lambda(x-a)(2I(a, x, I_d \xi) - (a+x)I(a, x, \xi))) \\ &\quad + K_b^x(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi)((\xi(b) - \xi(x))(\psi(b) + \psi(x)) \\ &\quad - \lambda(b-x)(2I(x, b, I_d \xi) - (x+b)I(x, b, \xi))), \end{aligned} \quad (8)$$

where I_d is the identity function and $I(a, b, \psi) := \int_a^b \psi(t) dt$.

Theorem 6. Under the assumptions of Theorem 5, in addition, if $\psi(x) = \psi(a+b-x)$, then, we have

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) \left(\xi F_{\mu, \alpha, l, b^-}^{\phi, \gamma, \delta, k, c} I \right)(a, w; p) + \frac{\lambda}{4} \left(\xi F_{\mu, \alpha, l, b^-}^{\phi, \gamma, \delta, k, c} (a+b-2x)^2 \right) \\ &\quad \times (a, w; p) + f\left(\frac{a+b}{2}\right) \left(\xi F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} I \right)(b, w; p) + \frac{\lambda}{4} \\ &\quad \times \left(\xi F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} (a+b-2x)^2 \right)(b, w; p) \\ &\leq \left(\xi F_{\mu, \alpha, l, b^-}^{\phi, \gamma, \delta, k, c} \psi \right)(a, w; p) + \left(\xi F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right)(b, w; p) \\ &\leq \left(K_b^a(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, \xi; \phi) + K_b^a(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi) \right) \\ &\quad \times ((\xi(b) - \xi(a))(\psi(b) + \psi(a)) - (b-a)\lambda(2I(a, b, I_d \xi) \\ &\quad - (a+b)I(a, b, \xi))). \end{aligned} \quad (9)$$

Theorem 7. Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function. If $|\psi'|$ is strongly convex with modulus $\lambda \geq 0$ and $\xi : [a, b] \longrightarrow \mathbb{R}$ be strictly increasing and differentiable, also let ϕ/x be an increasing function on $[a, b]$. If $\alpha, \gamma, \eta, c, l \in \mathbb{R}_+$, $c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, the following

inequality holds:

$$\begin{aligned} & \left| \left({}_{\xi} F_{\mu, \alpha, l, a^+}^{\phi, \gamma, \delta, k, c} \psi * \xi \right) (x, w; p) + \left({}_{\xi} F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi * \xi \right) (x, w; p) \right| \\ & \leq K_x^a \left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \times \left((\xi(x) - \xi(a)) (|\psi'(x)| + |\psi'(a)|) \right. \\ & \quad \left. - \lambda(x - a)(2I(a, x, I_d \xi) - (a + x)I(a, x, \xi)) \right) \\ & \quad + K_b^x \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \left((\xi(b) - \xi(x)) (|\psi'(b)| + |\psi'(x)|) \right. \\ & \quad \left. - \lambda(b - x)(2I(x, b, I_d \xi) - (x + b)I(x, b, \xi)) \right). \end{aligned} \quad (10)$$

Next, we give some definitions of convex functions. The definition of $(h-m)$ -convex function is given as follows:

Definition 8 (see [5]). Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, b] \rightarrow \mathbb{R}$ is called $(h-m)$ -convex function if

$$\psi(\zeta x + m(1 - \zeta)y) \leq h(\zeta)\psi(x) + mh(1 - \zeta)\psi(y), \quad (11)$$

holds for all $x, y \in [0, b]$, $m \in [0, 1]$, and $\zeta \in (0, 1)$.

Remark 9.

- (i) For $m = 1$, (11) gives the definition of h -convex function
- (ii) For $h(\zeta) = \zeta$, (11) gives the definition of m -convex function
- (iii) For $h(\zeta) = \zeta$ and $m = 1$, (11) gives the definition of convex function
- (iv) For $h(\zeta) = 1$ and $m = 1$, (11) gives the definition of p -function
- (v) For $h(\zeta) = \zeta^s$ and $m = 1$, (11) gives the definition of s -convex function
- (vi) For $h(\zeta) = 1/\zeta$ and $m = 1$, (11) gives the definition of Godunova-Levin function
- (vii) For $h(\zeta) = 1/\zeta^s$ and $m = 1$, (11) gives the definition of s -Godunova-Levin function of second

Definition 10 (see [4]). A function $\psi : [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be (α, m) -convex if

$$\psi(\zeta x + m(1 - \zeta)y) \leq \zeta^\alpha \psi(x) + m(1 - \zeta^\alpha) \psi(y), \quad (12)$$

holds for all $x, y \in [0, b]$, $\zeta \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Remark 11.

- (i) For $(\alpha, m) = (1, m)$, (12) provides m -convex function
- (ii) For $(\alpha, m) = (1, 1)$, (12) provides convex function

- (iii) For $(\alpha, m) = (1, 0)$, (12) provides star-shaped function

Definition 12 (see ([33])). A function $\psi : [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be (s, m) -convex, where $(s, m) \in [0, 1]^2$ if

$$\psi(\zeta x + m(1 - \zeta)y) \leq \zeta^s \psi(x) + m(1 - \zeta)^s \psi(y), \quad (13)$$

holds for all $x, y \in [0, b]$ and $\zeta \in [0, 1]$.

The following definition unifies $(h-m)$ -convex, (s, m) -convex, and (α, m) -convex functions in a single inequality.

Definition 13. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, b] \rightarrow \mathbb{R}$ is said to be $(\alpha, h-m)$ -convex function if

$$\psi(\zeta x + m(1 - \zeta)y) \leq h(\zeta^\alpha) \psi(x) + mh(1 - \zeta^\alpha) \psi(y), \quad (14)$$

holds for all $x, y \in [0, b]$, $\zeta \in (0, 1)$, $(\alpha, m) \in [0, 1]^2$.

Next, we give definitions of strongly convex, strongly (s, m) -convex, and strongly (α, m) -convex functions.

Definition 14 (see [34]). Let I be a nonempty convex subset of normed space $(X, \|\cdot\|)$. A real valued function ψ is said to be strongly convex, with modulus $\lambda \geq 0$, on I if for each $a, b \in I$ and $\zeta \in [0, 1]$,

$$\psi(\zeta x + (1 - \zeta)y) \leq \zeta \psi(x) + (1 - \zeta) \psi(y) - \lambda \zeta(1 - \zeta) \|b - a\|^2. \quad (15)$$

Definition 15 (see [35]). A function $\psi : [0, +\infty) \rightarrow \mathbb{R}$ is said to be strongly (s, m) -convex function, with modulus $\lambda \geq 0$, for $(s, m) \in [0, 1]^2$, if

$$\psi(\zeta x + m(1 - \zeta)y) \leq \zeta^s \psi(x) + m(1 - \zeta)^s \psi(y) - \lambda m \zeta(1 - \zeta)(y - x)^2, \quad (16)$$

holds for all $x, y \in [0, +\infty)$ and $\zeta \in [0, 1]$.

Definition 16 (see [36]). A function $\psi : [0, b] \rightarrow \mathbb{R}$, $b > 0$ is said to be strongly (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$ if

$$\psi(\zeta x + m(1 - \zeta)y) \leq \zeta^\alpha \psi(x) + m(1 - \zeta^\alpha) \psi(y) - \lambda m \zeta^\alpha (1 - \zeta^\alpha) |y - x|^2, \quad (17)$$

holds for all $x, y \in [0, b]$ and $\zeta \in [0, 1]$.

Next, we give a property of the kernel given in (7), which will be useful for finding the results of this paper.

P: Let ξ and ϕ/I be increasing functions. Then, for $u < t < v$, $u, v \in [a, b]$, the kernel $K_u^\nu(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi)$ satisfies the following inequality:

$$K_t^u \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t) \leq K_v^u \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t). \quad (18)$$

It is easy to prove by using the following inequalities:

$$\begin{aligned} \frac{\phi(\xi(t) - \xi(u))}{\xi(t) - \xi(u)} \xi'(t) &\leq \frac{\phi(\xi(v) - \xi(u))}{\xi(v) - \xi(u)} \xi'(t), \\ E_{\mu,\tau,l}^{\gamma,\delta,k,c} (\omega(\xi(t) - \xi(u))^\mu; p) &\leq E_{\mu,\tau,l}^{\gamma,\delta,k,c} (\omega(\xi(v) - \xi(u))^\mu; p). \end{aligned} \quad (19)$$

If ξ and ϕ/I are of opposite monotonicities, then (18) holds in reverse direction. For further properties, see [37].

In Section 2, we will define a new notion of strongly $(\alpha, h-m)$ -convex function which unifies several kinds of convex functions in a single inequality. By applying this new definition, we give generalizations of results for strongly convex functions. The results established here will produce generalizations and refinements of fractional integral inequalities for different kinds of convex and strongly convex functions which have been published in various papers.

2. Main Results

We give the definition of a function will be called strongly $(\alpha, h-m)$ -convex function.

Definition 17. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, b] \rightarrow \mathbb{R}$ is said to be strongly $(\alpha, h-m)$ -convex function with modulus $\lambda \geq 0$ if

$$\psi(\zeta x + m(1-\zeta)y) \leq h(\zeta^\alpha)\psi(x) + mh(1-\zeta^\alpha)\psi(y) - m\lambda h(\zeta^\alpha)h(1-\zeta^\alpha)|y-x|^2, \quad (20)$$

holds for all $x, y \in [0, b]$, $\zeta \in (0, 1)$, $(\alpha, m) \in [0, 1]^2$.

The definition of strongly $(h-m)$ -convexity can be achieved by taking $\alpha = 1$ in (20).

Definition 18. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \rightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, b] \rightarrow \mathbb{R}$ is said to be strongly $(h-m)$ -convex function with modulus $\lambda \geq 0$ if

$$\psi(\zeta x + m(1-\zeta)y) \leq h(\zeta)\psi(x) + mh(1-\zeta)\psi(y) - m\lambda h(\zeta)h(1-\zeta)|y-x|^2, \quad (21)$$

holds for all $x, y \in [0, b]$, $\zeta \in (0, 1)$, $m \in [0, 1]$.

One can obtain from (20) definitions of strongly convex, strongly s -convex, strongly m -convex, strongly h -convex, strongly (α, m) -convex, and strongly (s, m) -convex functions.

Theorem 19. Let $\psi \in L_1[a, b]$ be a positive strongly $(\alpha, h-m)$ -convex function with modulus $\lambda \geq 0$, $m \in (0, 1]$, $0 < a < mb$. Let ξ be strictly increasing and differentiable function, also

let ϕ/x be an increasing function on $[a, b]$ and $h(x)h(y) \leq h(x+y)$. If $\tau, \eta, l, \gamma, c \in \mathbb{R}_+$, $c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, the following inequality holds:

$$\begin{aligned} &\left({}_{\xi}F_{\mu,\tau,l,a^+}^{\phi,\gamma,\delta,k,c} \psi \right)(x, w; p) + \left({}_{\xi}F_{\mu,\eta,l,b^-}^{\phi,\gamma,\delta,k,c} \psi \right)(x, w; p) \\ &\leq K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) (x-a) \left(\psi(a) X_x^a(r^\alpha, h; \xi') \right) \\ &\quad + m\psi \left(\frac{x}{m} \right) X_x^a \left(1-r^\alpha, h; \xi' \right) - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \\ &\quad + K_b^x \left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi \right) (b-x) \times \left(\psi(b) X_x^b(r^\alpha, h; \xi') \right) \\ &\quad + m\psi \left(\frac{x}{m} \right) X_x^b \left(1-r^\alpha, h; \xi' \right) - \frac{\lambda(bm-x)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)}, \end{aligned} \quad (22)$$

while $X_x^a(r^\alpha, h; \xi') = \int_0^1 h(r^\alpha) \xi'(x-r(x-a)) dr$, $X_x^a(1-r^\alpha, h; \xi') = \int_0^1 h(1-r^\alpha) \xi'(x-r(x-a)) dr$.

Proof. Using (P), we can write the following inequalities

$$K_x^t \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t) \leq K_x^x \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t), \quad t \in (a, x), \quad (23)$$

$$K_t^x \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t) \leq K_b^x \left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(t), \quad t \in (x, b). \quad (24)$$

Using strongly $(\alpha, h-m)$ -convexity of ψ , we have

$$\begin{aligned} \psi(t) &\leq h \left(\frac{x-t}{x-a} \right)^\alpha \psi(a) + mh \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \psi \left(\frac{x}{m} \right) \\ &\quad - \frac{\lambda(x-am)^2}{m} h \left(\frac{x-t}{x-a} \right)^\alpha h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right), \end{aligned} \quad (25)$$

$$\begin{aligned} \psi(t) &\leq h \left(\frac{t-x}{b-x} \right)^\alpha \psi(b) + mh \left(1 - \left(\frac{t-x}{b-x} \right)^\alpha \right) \psi \left(\frac{x}{m} \right) \\ &\quad - \frac{\lambda(bm-x)^2}{m} h \left(\frac{t-x}{b-x} \right)^\alpha h \left(1 - \left(\frac{t-x}{b-x} \right)^\alpha \right). \end{aligned} \quad (26)$$

From (23) and (25), the following inequality is obtained:

$$\begin{aligned} &\int_a^x K_x^t \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \psi(t) d(\xi(t)) \\ &\leq \psi(a) K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \times \int_a^x h \left(\frac{x-t}{x-a} \right)^\alpha d(\xi(t)) \\ &\quad + m\psi \left(\frac{x}{m} \right) K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \times \int_a^x h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) d(\xi(t)) \\ &\quad - \frac{\lambda(x-am)^2}{m} K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \times \int_a^x h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) h \left(\frac{x-t}{x-a} \right)^\alpha d(\xi(t)). \end{aligned} \quad (27)$$

By setting $r = (x-t)/(x-a)$ on the right side and using (5) on left side of above inequality, we get

$$\begin{aligned}
\left({}_{\xi} F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right) (x, w; p) &\leq K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (x-a) \\
&\times \left(\psi(a) \int_0^1 h(r^\alpha) \xi'(x-r(x-a)) dr \right. \\
&+ m\psi \left(\frac{x}{m} \right) \int_0^1 h(1-r^\alpha) \xi'(x-r(x-a)) dr \\
&\left. - \frac{\lambda(x-am)^2}{m} \int_0^1 h(1-r^\alpha) h(r^\alpha) \xi'(x-r(x-a)) dr \right). \quad (28)
\end{aligned}$$

The inequality (28) can take the following form:

$$\begin{aligned}
\left({}_{\xi} F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right) (x, w; p) &\leq K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (x-a) \\
&\times \left(\psi(a) X_x^a \left(r^\alpha, h; \xi' \right) + m\psi \left(\frac{x}{m} \right) X_x^a \left(1-r^\alpha, h; \xi' \right) \right. \\
&\left. - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \right). \quad (29)
\end{aligned}$$

On the other hand, multiplying (24) and (26), and adopting the same pattern as we did for (23) and (25), the following inequality holds true:

$$\begin{aligned}
\left({}_{\xi} F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi \right) (x, w; p) &\leq K_b^x \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (b-x) \\
&\times \left(\psi(b) \int_0^1 h(r^\alpha) \xi'(x-r(x-b)) dr \right. \\
&+ m\psi \left(\frac{x}{m} \right) \int_0^1 h(1-r^\alpha) \xi'(x-r(x-b)) dr \\
&\left. - \frac{\lambda(bm-x)^2}{m} \int_0^1 h(1-r^\alpha) h(r^\alpha) \xi'(x-r(x-b)) dr \right). \quad (30)
\end{aligned}$$

The inequality (30) can take the following form:

$$\begin{aligned}
\left({}_{\xi} F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi \right) (x, w; p) &\leq K_b^x \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (b-x) \times \left(\psi(b) X_x^b \left(r^\alpha, h; \xi' \right) \right. \\
&+ m\psi \left(\frac{x}{m} \right) X_x^b \left(1-r^\alpha, h; \xi' \right) \\
&\left. - \frac{\lambda(bm-x)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)} \right). \quad (31)
\end{aligned}$$

By adding (29) and (31), (22) can be achieved. $\square$

Corollary 20. For $w = p = 0$, (22) gives the following inequality obtained for fractional integral operator defined in [19]:

$$\begin{aligned}
\frac{\left(F_{a^+}^{\phi, \xi} \psi \right) (x)}{\Gamma(\tau)} + \frac{\left(F_{b^-}^{\phi, \xi} \psi \right) (x)}{\Gamma(\eta)} &\leq \frac{\phi(\xi(x) - \xi(a))(x-a)}{\Gamma(\tau)(\xi(x) - \xi(a))} \left(\psi(a) X_x^a \left(r^\alpha, h; \xi' \right) \right. \\
&+ m\psi \left(\frac{x}{m} \right) X_x^a \left(1-r^\alpha, h; \xi' \right) - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \Big) \\
&+ \frac{\phi(\xi(b) - \xi(x))(b-x)}{\Gamma(\eta)(\xi(b) - \xi(x))} \times \left(\psi(b) X_x^b \left(r^\alpha, h; \xi' \right) + m\psi \left(\frac{x}{m} \right) X_x^b \left(1-r^\alpha, h; \xi' \right) \right. \\
&\left. - \frac{\lambda(bm-x)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)} \right). \quad (32)
\end{aligned}$$

Remark 21.

- (i) For $\lambda = 0$, (22) gives ([38], Theorem 1)
- (ii) For $\tau = \eta$ and $h(\zeta) = \zeta$, (22) gives ([39], Theorem 1)
- (iii) For $\lambda = 0$, $\tau = \eta$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (22) gives ([31], Theorem 8)
- (iv) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau) \zeta^{\tau/k} / k \Gamma_k(\tau)$, $h(\zeta) = \xi(t) = \zeta$, $\alpha = m = 1$, and $w = p = 0$, (22) gives ([40], Theorem 1)
- (v) For $\lambda = 0$, $\tau = \eta$, the result of (iv) gives ([40], Corollary 1)
- (vi) For $\lambda = 0$, $k = 1$, and $x = a$ or $x = b$, the result of (v) gives ([40], Corollary 2)
- (vii) For $\lambda = 0$, $k = 1$, and $x = (a+b)/2$, the result of (v) gives ([40], Corollary 3)
- (viii) For $\lambda = 0$, $\phi(\zeta) = \zeta^\tau$, $\alpha = 1$, and $\xi(t) = \zeta$, (22) gives ([41], Theorem 1)
- (ix) For $\lambda = 0$, $\phi(\zeta) = \zeta^\tau$, $\alpha = m = 1$, and $h(\zeta) = \xi(t) = \zeta$, (22) gives ([41], Corollary 1)
- (x) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $\alpha = 1$, $w = p = 0$, and $\xi(\zeta) = \zeta$, (22) gives ([42], Theorem 2.1)
- (xi) For $\lambda = 0$, $\tau = \eta$, the result of (x) gives ([42], Corollary 2.2)
- (xii) For $\lambda = 0$, $\tau = \eta$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $w = p = 0$, $\xi(\zeta) = \zeta$, and $\alpha = 1$ and using ([12], Remark 11), (22) gives ([42], Corollary 2.3)
- (xiii) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $\alpha = 1$, $w = p = 0$, $\xi(\zeta) = \zeta$, and $h(\zeta) = 1$, (22) gives inequality (26) of ([42], Corollary 2.4) similarly, under the same assumptions along with $h(\zeta) = \zeta^p$, (22) gives inequality (27) of ([42], Corollary 2.4)
- (xiv) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $w = p = 0$, $\alpha = m = 1$, and $\xi(t) = h(\zeta) = \zeta$, (22) gives ([43], Theorem 1)
- (xv) For $\lambda = 0$, $\tau = \eta$, the result of (xiv) gives ([43], Corollary 1)
- (xvi) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $w = p = 0$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (22) gives ([44], Theorem 1)
- (xvii) For $\lambda = 0$, $\tau = \eta$, $\phi(\zeta) = \Gamma(\tau) \zeta^\tau$, $w = p = 0$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (22) gives ([44], Corollary 1)
- (xviii) For $\lambda = 0$, $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta^s$, $\alpha = m = 1$, (22) gives ([45], Theorem 2.1)
- (xix) For $\lambda = 0$, $\tau = \eta$, $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, $\alpha = m = 1$, and $h(\zeta) = \zeta^s$, (22) gives ([45], Corollary 2.1)
- (xx) For $\lambda = 0$, $w = p = 0$, $\alpha = 1$, and $h(\zeta) = \zeta^s$, (22) gives ([46], Theorem 1)

- (xxi) For $\lambda = 0$, $\phi(\zeta) = \zeta^\tau$, $h(\zeta) = \zeta^s$, $\alpha = 1$, and $\xi(\zeta) = \zeta$, (22) gives ([47], Theorem 1)
- (xxii) For $\lambda = 0$, $\tau = \eta$, the result of (xxi) gives ([47], Corollary 1)
- (xxiii) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau/k}/k\Gamma_k(\tau)$, $h(\zeta) = z$, and $w = p = 0$, (22) gives ([15], Theorem 1)
- (xxiv) For $\lambda = 0$, $\tau = \eta$, the result of (xxiii) gives ([15], Corollary 1)
- (xxv) For $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (22) gives ([36], Theorem 4)
- (xxvi) For $\tau = \eta$, the result of (xxiv) gives ([36], Corollary 1)
- (xxvii) For $\psi \in L_\infty[a, b]$, the result of (xxiv) gives ([36], Corollary 2)
- (xxviii) For $\tau = \eta$, the result of (xxvii) gives ([36], Corollary 3)
- (xxix) For $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (22) gives ([48], Theorem 1)
- (xxx) For $\tau = \eta$, the result of (xxix) gives ([48], Corollary 1)
- (xxxi) For $\psi \in L_\infty[a, b]$, the result of (xxix) gives ([48], Corollary 2)
- (xxxii) For $\tau = \eta$, the result of (xxxi), gives ([48], Corollary 3)

For the proof of next theorem, we need the following lemma.

Lemma 22. Let $\psi : [a, b] \longrightarrow \mathbb{R}$, be a strongly $(\alpha, h-m)$ -convex function with modulus $\lambda \geq 0$, $m \in (0, 1]$, $0 \leq a < mb$. If $\psi(x) = \psi((a + mb - x)/m)$, then the following inequality holds:

$$\begin{aligned} \psi\left(\frac{a+mb}{2}\right) &\leq \left(h\left(\frac{1}{2^\alpha}\right) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\psi(x) \\ &\quad - \frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(a-x+mb-mx)^2. \end{aligned} \quad (33)$$

Proof. As ψ is strongly $(\alpha, h-m)$ -convex function, we have

$$\begin{aligned} \psi\left(\frac{a+mb}{2}\right) &\leq h\left(\frac{1}{2^\alpha}\right)\psi((1-t)a+mtb) + mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\psi\left(\frac{t+m(1-t)b}{m}\right) \\ &\quad - \frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(t(1+m)(a-mb)+mb-ma)^2. \end{aligned} \quad (34)$$

Let $x = a(1-t) + mtb$. Then, we have $a + mb - x = ta + m(1-t)b$, and using $\psi((a + mb - x)/m) = \psi(x)$, the inequality (33) is obtained. $\square$

The upcoming theorem provides the Hadamard inequality for strongly $(\alpha, h-m)$ -convex function.

Theorem 23. Under the assumptions of Theorem 19, in addition, if $\psi(x) = \psi((a + mb - x)/m)$, then, we have

$$\begin{aligned} &\frac{1}{h(1/2^\alpha) + mh((2^\alpha-1)/2^\alpha)} \left(\psi\left(\frac{a+mb}{2}\right) \left(\left({}_\xi F_{\mu,\tau,l,b}^{\phi,\gamma,\delta,k,c} 1 \right)(a, w; p) \right) \right. \\ &\quad + \left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c} 1 \right)(b, w; p) + \frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right) \\ &\quad \times \left(\left({}_\xi F_{\mu,\tau,l,b^-}^{\phi,\gamma,\delta,k,c} (a-x+mb-mx)^2 \right)(a, w; p) \right. \\ &\quad \left. + \left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c} (a-x+mb-mx)^2 \right)(b, w; p) \right) \Big) \\ &\leq \left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c} \psi \right)(b, w; p) + \left({}_\xi F_{\mu,\tau,l,b^-}^{\phi,\gamma,\delta,k,c} \psi \right)(a, w; p) \\ &\leq (b-a) \left(K_b^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) + K_b^a \left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \right) \\ &\quad \times \left(\psi(b) X_b^a(r^\alpha, h; \xi') + m\psi\left(\frac{a}{m}\right) X_b^a(1-r^\alpha, h; \xi') \right) \\ &\quad - \frac{\lambda(b-ma)^2 h(1)(\xi(b) - \xi(a))}{m(b-a)}. \end{aligned} \quad (35)$$

Proof. Using (P), we can write the following inequalities:

$$K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(x) \leq K_b^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(x), x \in (a, b), \quad (36)$$

$$K_b^x \left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(x) \leq K_b^a \left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \xi'(x), x \in (a, b). \quad (37)$$

Using strongly $(\alpha, h-m)$ -convexity of ψ , we have

$$\begin{aligned} \psi(x) &\leq h\left(\frac{x-a}{b-a}\right)^\alpha \psi(b) + mh\left(1 - \left(\frac{x-a}{b-a}\right)^\alpha\right) \psi\left(\frac{a}{m}\right) \\ &\quad - \frac{\lambda(a-bm)^2}{m} h\left(\frac{x-a}{b-a}\right)^\alpha h\left(1 - \left(\frac{x-a}{b-a}\right)^\alpha\right). \end{aligned} \quad (38)$$

Multiplying (36) and (38) and integrating the resulting inequality over $[a, b]$, we obtain

$$\begin{aligned} &\int_a^b K_x^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \psi(x) d(\xi(x)) \leq \psi(b) K_b^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \\ &\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)^\alpha d(\xi(x)) + m\psi\left(\frac{a}{m}\right) K_b^a \left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi \right) \\ &\quad \times \int_a^b h\left(1 - \left(\frac{x-a}{b-a}\right)^\alpha\right) d(\xi(x)) - \frac{\lambda(a-bm)^2}{m} \\ &\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)^\alpha h\left(1 - \left(\frac{x-a}{b-a}\right)^\alpha\right) d(\xi(x)). \end{aligned} \quad (39)$$

By setting $r = ((x-a)/(b-a))$ on the right side and using

(5) on left side of above inequality, we get

$$\begin{aligned} \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} \psi \right) (a, w; p) &\leq K_b^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (b-a) \\ &\quad \times \left(\psi(b) \int_0^1 h(r^\alpha) \xi'(a+r(b-a)) dr + m \psi \left(\frac{a}{m} \right) \right. \\ &\quad \cdot \int_0^1 h(1-r^\alpha) \xi'(a+r(b-a)) dr - \frac{\lambda(a-bm)^2}{m} \\ &\quad \cdot \left. \int_0^1 h(r^\alpha) h(1-r^\alpha) \xi'(a+r(b-a)) dr \right). \end{aligned} \quad (40)$$

The inequality (40) can take the following form:

$$\begin{aligned} \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} \psi \right) (a, w; p) &\leq K_b^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (b-a) \\ &\quad \times \left(\psi(b) X_b^a \left(r^\alpha, h; \xi' \right) + m \psi \left(\frac{a}{m} \right) X_b^a \left(1-r^\alpha, h; \xi' \right) \right. \\ &\quad \cdot \left. - \frac{\lambda(a-bm)^2 h(1)(\xi(b)-\xi(a))}{m(b-a)} \right). \end{aligned} \quad (41)$$

Adopting the same pattern of simplification as we did for (36) and (38), the following inequality can be observed for (38) and (37):

$$\begin{aligned} \left({}_{\xi} F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right) (b, w; p) &\leq K_b^a \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (b-a) \times \left(\psi(b) X_b^a \left(r^\alpha, h; \xi' \right) \right. \\ &\quad + m \psi \left(\frac{a}{m} \right) X_b^a \left(1-r^\alpha, h; \xi' \right) \\ &\quad \cdot \left. - \frac{\lambda(a-bm)^2 h(1)(\xi(b)-\xi(a))}{m(b-a)} \right). \end{aligned} \quad (42)$$

By adding (41) and (42), following inequality can be achieved:

$$\begin{aligned} \left({}_{\xi} F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right) (b, w; p) &+ \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} \psi \right) (a, w; p) \\ &\leq (b-a) \left(K_b^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) + K_b^a \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi \right) \right) \left(\psi(b) X_b^a \left(r^\alpha, h; \xi' \right) \right. \\ &\quad + m \psi \left(\frac{a}{m} \right) X_b^a \left(1-r^\alpha, h; \xi' \right) - \frac{\lambda(a-bm)^2 h(1)(\xi(b)-\xi(a))}{m(b-a)} \left. \right). \end{aligned} \quad (43)$$

Multiplying both sides of (33) by $K_x^a(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi)d(\xi(x))$ and integrating over $[a, b]$, one can get

$$\begin{aligned} \psi \left(\frac{a+mb}{2} \right) \int_a^b K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) d(\xi(x)) \\ \leq \left(h \left(\frac{1}{2^\alpha} \right) + mh \left(\frac{2^\alpha-1}{2^\alpha} \right) \right) \int_a^b K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) \psi(x) d(\xi(x)) \\ - \frac{\lambda}{m} h \left(\frac{1}{2^\alpha} \right) h \left(\frac{2^\alpha-1}{2^\alpha} \right) \int_a^b K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right) (a-x+mb-mx)^2 d(\xi(x)). \end{aligned} \quad (44)$$

By using the Definition 4, one can obtain the following inequality:

$$\begin{aligned} \frac{1}{h(1/2^\alpha) + mh((2^\alpha-1)/2^\alpha)} \left(\psi \left(\frac{a+mb}{2} \right) \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} 1 \right) (a, w; p) \right. \\ \left. + \frac{\lambda}{m} h \left(\frac{1}{2^\alpha} \right) \times h \left(\frac{2^\alpha-1}{2^\alpha} \right) \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} (a-x+mb-mx)^2 \right) (a, w; p) \right) \\ \leq \left({}_{\xi} F_{\mu, \tau, l, b}^{\phi, \gamma, \delta, k, c} \psi \right) (a, w; p). \end{aligned} \quad (45)$$

Now, multiplying by $K_b^x(E_{\mu, \eta}^{\gamma, \delta, k, c}, \xi; \phi)d(\xi(x))$ on both sides of (33), then integrating over $[a, b]$, we get

$$\begin{aligned} \frac{1}{h(1/2^\alpha) + mh((2^\alpha-1)/2^\alpha)} \left(\psi \left(\frac{a+mb}{2} \right) \left({}_{\xi} F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} 1 \right) (b, w; p) \right. \\ \left. + \frac{\lambda}{m} h \left(\frac{1}{2^\alpha} \right) \times h \left(\frac{2^\alpha-1}{2^\alpha} \right) \left({}_{\xi} F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} (a-x+mb-mx)^2 \right) (b, w; p) \right) \\ \leq \left({}_{\xi} F_{\mu, \eta, l, a^+}^{\phi, \gamma, \delta, k, c} \psi \right) (b, w; p). \end{aligned} \quad (46)$$

From (43), (45), and (46), inequality (35) can be achieved. $\square$

Corollary 24. For $w = p = 0$, (35) gives the following inequality obtained for the fractional integral operator that defined in [19]:

$$\begin{aligned} \frac{1}{h(1/2^\alpha) + mh((2^\alpha-1)/2^\alpha)} \left(\psi \left(\frac{a+mb}{2} \right) \left(\frac{{}_{\xi} F_{b^-}^{\phi, \xi} 1(a)}{\Gamma(\tau)} + \frac{{}_{\xi} F_{a^+}^{\phi, \xi} 1(b)}{\Gamma(\eta)} \right) \right. \\ \left. + \frac{\lambda}{m} h \left(\frac{1}{2^\alpha} \right) h \left(\frac{2^\alpha-1}{2^\alpha} \right) \left(\frac{{}_{\xi} F_{b^-}^{\phi, \xi} (a-x+mb-mx)^2(a)}{\Gamma(\tau)} \right. \right. \\ \left. \left. + \frac{{}_{\xi} F_{a^+}^{\phi, \xi} (a-x+mb-mx)^2(b)}{\Gamma(\eta)} \right) \right) \leq \frac{{}_{\xi} F_{b^-}^{\phi, \xi} \psi(b)}{\Gamma(\eta)} + \frac{{}_{\xi} F_{a^+}^{\phi, \xi} \psi(a)}{\Gamma(\tau)} \\ \leq \frac{(b-a)(\xi(b)-\xi(a))}{\xi(b)-\xi(a)} \left(\frac{1}{\Gamma(\tau)} + \frac{1}{\Gamma(\eta)} \right) \times \left(\psi(b) X_b^a \left(r^\alpha, h; \xi' \right) \right. \\ \left. + m \psi \left(\frac{a}{m} \right) X_b^a \left(1-r^\alpha, h; \xi' \right) - \frac{\lambda(b-ma)^2 h(1)(\xi(b)-\xi(a))}{m(b-a)} \right). \end{aligned} \quad (47)$$

Remark 25.

- (i) For $\lambda = 0$, $\tau = \eta$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (35) gives ([31], Theorem 22)
- (ii) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{(\tau/k)+1}$, $h(\zeta) = \xi(\zeta) = \zeta$, $\alpha = m = 1$, and $w = p = 0$, (35) gives ([40], Theorem 3)
- (iii) For $\lambda = 0$, $\tau = \eta$, the result of (ii) gives ([40], Corollary 6)
- (iv) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau+1}$, $w = p = 0$, $\alpha = m = 1$, and $\xi(\zeta) = h(\zeta) = \zeta$, (35) gives ([43], Theorem 3)
- (v) For $\lambda = 0$, $\tau = \eta$, and $\phi(\zeta) = \Gamma(\tau)\zeta^\tau$ in the result of (v) gives ([43], Corollary 6)
- (vi) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau+1}$, $w = p = 0$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (35) gives ([44], Theorem 3)

- (vii) For $\lambda = 0$, $\tau = \eta$, the result of (vii) gives ([44], Corollary 3)
- (viii) For $\lambda = 0$, $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta^s$, $\alpha = m = 1$, (35) gives ([45], Theorem 2.4)
- (ix) For $\lambda = 0$, $\tau = \eta$, the result of (ix) gives ([45], Corollary 2.6)
- (x) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (35) gives ([36], Theorem 6)
- (xi) For $\tau = \eta$, the result of (xi) gives ([36], Corollary 5)
- (xii) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (35) gives ([48], Theorem 4)
- (xiii) For $\tau = \eta$, the result of (xiii) gives ([48], Corollary 5)

Theorem 26. Let $\psi : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that $|\psi'|$ is strongly $(\alpha, h-m)$ -convex with modulus $\lambda \geq 0$, $m \in (0, 1]$, $0 < a < mb$. Let $\xi : [a, b] \rightarrow \mathbb{R}$ be strictly increasing and differentiable function, also let ϕ/x be a function which is increasing on the interval $[a, b]$ and $h(x)h(y) \leq h(x+y)$. If $c > \gamma$, $p, \mu, \delta \geq 0$, $\tau, \eta, l, \gamma, c \in \mathbb{R}_+$, and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$ the following inequality holds:

$$\begin{aligned} & \left| \left({}_{\xi}F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c}(\psi * \xi) \right)(x, w; p) + \left({}_{\xi}F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c}(\psi * \xi) \right)(x, w; p) \right| \\ & \leq K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \times (x-a) \left(|\psi'(a)| X_x^a(r^\alpha, h; \xi') \right) \\ & \quad + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^a \left(1 - r^\alpha, h; \xi' \right) - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \\ & \quad + K_b^x \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}; \xi; \phi \right) (b-x) \times \left(|\psi'(b)| X_x^b(r^\alpha, h; \xi') \right) \\ & \quad + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^b \left(1 - r^\alpha, h; \xi' \right) - \frac{\lambda(b-mb)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)}, \end{aligned} \quad (48)$$

where

$$\begin{aligned} \left({}_{\xi}F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) &:= \int_a^x K_x^t \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \psi'(t) d(\xi(t)), \\ \left({}_{\xi}F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) &:= \int_x^b K_t^x \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \psi'(t) d(\xi(t)). \end{aligned} \quad (49)$$

Proof. Since $|\psi'|$ is strongly $(\alpha, h-m)$ -convex function, one can have

$$\begin{aligned} |\psi'(t)| &\leq h \left(\frac{x-t}{x-a} \right)^\alpha |\psi'(a)| + mh \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \left| \psi' \left(\frac{x}{m} \right) \right| \\ &\quad - \frac{\lambda(x-am)^2}{m} h \left(\frac{x-t}{x-a} \right)^\alpha h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right). \end{aligned} \quad (50)$$

The inequality (50) can take the following form:

$$\begin{aligned} & - \left(h \left(\frac{x-t}{x-a} \right)^\alpha |\psi'(a)| + mh \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \left| \psi' \left(\frac{x}{m} \right) \right| \right. \\ & \quad \left. - \frac{\lambda(x-am)^2}{m} h \left(\frac{x-t}{x-a} \right)^\alpha h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \right) \\ & \leq \psi'(t) \leq \left(h \left(\frac{x-t}{x-a} \right)^\alpha |\psi'(a)| + mh \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \right. \\ & \quad \left. \cdot \left| \psi' \left(\frac{x}{m} \right) \right| - \frac{\lambda(x-am)^2}{m} h \left(\frac{x-t}{x-a} \right)^\alpha h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \right). \end{aligned} \quad (51)$$

From inequality (51), we have

$$\begin{aligned} \psi'(t) &\leq h \left(\frac{x-t}{x-a} \right)^\alpha |\psi'(a)| + mh \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) \left| \psi' \left(\frac{x}{m} \right) \right| \\ &\quad - \frac{\lambda(x-am)^2}{m} h \left(\frac{x-t}{x-a} \right)^\alpha h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right). \end{aligned} \quad (52)$$

Multiplying (23) and (52) and integrating over $[a, x]$, we obtain

$$\begin{aligned} & \int_a^x K_x^t \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \psi'(t) d(\xi(t)) \\ & \leq |\psi'(a)| K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \times \int_a^x h \left(\frac{x-t}{x-a} \right)^\alpha d(\xi(t)) \\ & \quad + m \left| \psi' \left(\frac{x}{m} \right) \right| K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \\ & \quad \times \int_a^x h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) d(\xi(t)) \\ & \quad - \frac{\lambda(bm-x)^2}{m} K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) \\ & \quad \times \int_a^x h \left(1 - \left(\frac{x-t}{x-a} \right)^\alpha \right) h \left(\frac{x-t}{x-a} \right)^\alpha d(\xi(t)), \end{aligned} \quad (53)$$

which gives

$$\begin{aligned} & \left({}_{\xi}F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) \\ & \leq K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}; \xi; \phi \right) (x-a) \times \left(|\psi'(a)| X_x^a(r^\alpha, h; \xi') \right) \\ & \quad + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^a \left(1 - r^\alpha, h; \xi' \right) \\ & \quad - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)}. \end{aligned} \quad (54)$$

Using the other inequality of (51) and doing on the same

way as adopted for the right hand inequality, one can get

$$\begin{aligned} & \left({}_{\xi} F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) \\ & \geq -K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right)(x-a) \times \left(\left| \psi'(a) \right| X_x^a \left(r^\alpha, h; \xi' \right) \right. \\ & \quad \left. + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^a \left(1-r^\alpha, h; \xi' \right) \right. \\ & \quad \left. - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \right). \end{aligned} \quad (55)$$

From (54) and (55), the following inequality is observed:

$$\begin{aligned} & \left| \left({}_{\xi} F_{\mu, \tau, l, a^+}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) \right| \\ & \leq K_x^a \left(E_{\mu, \tau, l}^{\gamma, \delta, k, c}, \xi; \phi \right)(x-a) \times \left(\left| \psi'(a) \right| X_x^a \left(r^\alpha, h; \xi' \right) \right. \\ & \quad \left. + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^a \left(1-r^\alpha, h; \xi' \right) \right. \\ & \quad \left. - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \right). \end{aligned} \quad (56)$$

By applying strongly $(\alpha, h-m)$ -convexity of $|\psi'|$, one can get

$$\begin{aligned} |\psi'(t)| & \leq h \left(\frac{t-x}{b-x} \right)^\alpha |\psi'(b)| + mh \left(1 - \left(\frac{t-x}{b-x} \right)^\alpha \right) \left| \psi' \left(\frac{x}{m} \right) \right| \\ & \quad - \frac{\lambda(bm-x)^2}{m} h \left(\frac{t-x}{b-x} \right)^\alpha h \left(1 - \left(\frac{t-x}{b-x} \right)^\alpha \right). \end{aligned} \quad (57)$$

By following the same steps as we did for (23) and (50), from (24) and (57), one can get the following inequality:

$$\begin{aligned} & \left| \left({}_{\xi} F_{\mu, \eta, l, b^-}^{\phi, \gamma, \delta, k, c} \psi * \xi \right)(x, w; p) \right| \\ & \leq K_b^x \left(E_{\mu, \eta, l}^{\gamma, \delta, k, c}, \xi; \phi \right)(b-x) \times \left(\left| \psi'(b) \right| X_x^b \left(r^\alpha, h; \xi' \right) \right. \\ & \quad \left. + m \left| \psi' \left(\frac{x}{m} \right) \right| X_x^b \left(1-r^\alpha, h; \xi' \right) \right. \\ & \quad \left. - \frac{\lambda(bm-x)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)} \right). \end{aligned} \quad (58)$$

By adding (56) and (58), inequality (48) can be achieved. $\square$

Corollary 27. For $w = p = 0$, (48) gives the following inequality obtained for the fractional integral operator that defined in [19]:

$$\begin{aligned} & \left| \frac{\left(F_{a^+}^{\phi, \xi} \psi * \xi \right)(x)}{\Gamma(\tau)} + \frac{\left(F_{b^-}^{\phi, \xi} \psi * \xi \right)(x)}{\Gamma(\eta)} \right| \\ & \leq \frac{\phi(\xi(x) - \xi(a))(x-a)}{\Gamma(\tau)(\xi(x) - \xi(a))} \times \left(\left| \psi'(a) \right| X_x^a \left(r^\alpha, h; \xi' \right) \right. \\ & \quad \left. + m \left| \psi'(x/m) \right| X_x^a \left(1-r^\alpha, h; \xi' \right) \right. \\ & \quad \left. - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)} \right) \\ & \quad + \frac{\phi(\xi(b) - \xi(x))(b-x)}{\Gamma(\eta)(\xi(b) - \xi(x))} \times \left(\left| \psi'(b) \right| X_x^b \left(r^\alpha, h; \xi' \right) \right. \\ & \quad \left. + m \left| \psi'(x/m) \right| X_x^b \left(1-r^\alpha, h; \xi' \right) \right. \\ & \quad \left. - \frac{\lambda(b-mb)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)} \right). \end{aligned} \quad (59)$$

Remark 28.

- (i) For $\lambda = 0$, (48) gives ([38], Theorem 4)
- (ii) For $\lambda = 0$, $\tau = \eta$, and $h(\zeta) = \zeta$, ((48) gives ([39], Theorem 3)
- (iii) For $\lambda = 0$, $\tau = \eta$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (48) gives ([31], Theorem 25)
- (iv) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{(\tau/k)+1}$, $h(\zeta) = \xi(\zeta) = \zeta$, $\alpha = m = 1$, and $w = p = 0$, (48) gives ([40], Theorem 2)
- (v) For $\lambda = 0$, $\tau = \eta$, the result of (iv) gives ([40], Corollary 4)
- (vi) For $\lambda = 0$, $\phi(\zeta) = \zeta^\tau$, $\alpha = 1$, and $\xi(\zeta) = \zeta$, (48) gives ([41], Theorem 2)
- (vii) For $\lambda = 0$, $m = 1$, and $h(\zeta) = \zeta$, the result of (vi) gives ([41], Corollary 2)
- (viii) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau+1}$, $\alpha = 1$, $w = p = 0$, and $\xi(\zeta) = \zeta$, (48) gives ([42], Theorem 2.6)
- (ix) For $\lambda = 0$, $\tau = \eta$, the result of (viii) gives ([42], Corollary 2.7)
- (x) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau+1}$, $w = p = 0$, $\alpha = m = 1$, and $\xi(\zeta) = h(\zeta) = \zeta$, (48) gives ([43], Theorem 2)
- (xi) For $\lambda = 0$, $\tau = \eta$, and $\phi(\zeta) = \Gamma(\tau)\zeta^\tau$, the result of (x) gives ([43], Corollary 4)
- (xii) For $\lambda = 0$, $w = p = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau+1}$, $\alpha = m = 1$, and $h(\zeta) = \zeta$, (48) gives ([44], Theorem 2)
- (xiii) For $\lambda = 0$, $\tau = \eta$, the result of (xii) gives ([44], Corollary 2).
- (xiv) For $\lambda = 0$, $\xi(\zeta) = \zeta$ and $h(\zeta) = \zeta^s$, $\phi(\zeta) = \zeta^{\tau+1}$, $\alpha = m = 1$, (48) gives ([45], Theorem 2.3)

- (xv) For $\lambda = 0$, $\tau = \eta$, the result of (xiv) gives ([45], Corollary 2.5)
- (xvi) For $\lambda = 0$, $w = p = 0$, $\alpha = 1$ and $h(\zeta) = \zeta^s$, (48) gives ([46], Theorem 2)
- (xvii) For $\lambda = 0$, $h(\zeta) = \zeta^s$, $\phi(\zeta) = \Gamma(\tau)\zeta^\tau$, $\alpha = 1$, and $\xi(\zeta) = \zeta$, (48) gives ([47], Theorem 3)
- (xviii) For $\lambda = 0$, $\tau = \eta$, the result of (xvii) gives ([47], Corollary 5)
- (xix) For $\lambda = 0$, $h(\zeta) = \zeta$, $\phi(\zeta) = \Gamma(\tau)\zeta^{(\tau/k)+1}$, and $w = p = 0$, (48) gives ([15], Theorem 2)
- (xx) For $\lambda = 0$, $\tau = \eta$, the result of (xix) gives ([15], Corollary 2)
- (xxi) For $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (48) gives ([36], Theorem 5)
- (xxii) For $\tau = \eta$ in the result of (xxi) gives ([36], Corollary 4)
- (xxiii) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (48) gives ([48], Theorem 3)
- (xxiv) For $\tau = \eta$, the result of (xxiii) gives ([48], Corollary 4)

3. Concluding Remarks

A new definition is given and utilized to obtain some integral inequalities via a unified integral operator. The established results provide generalizations of many well-known inequalities. They also give refinements of recently published results for convex, m -convex, h -convex, s -convex, (α, m) -convex, (s, m) -convex, $(h-m)$ -convex and $(\alpha, h-m)$ -convex functions. The reader also can obtain more fractional integral inequalities by setting appropriate functions and parameters involved in the kernel of unified integral operators.

Data Availability

There is no data required for preparation of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research has been financed by the 2018 Nanjing Vocational College of Information Technology Doctoral special fund project "Study on the Climate Ensemble Forecast Model Based on the East Asian Monsoon Region" (YB20180903).

References

- [1] G. Farid, A. U. Rehman, and Q. U. Ain, "k-fractional integral inequalities of Hadamard type for $(h-m)$ -convex functions," *Computational Methods for Differential Equations*, vol. 7, no. 5, pp. 1–22, 2019.
- [2] D. A. Ion, "Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, annals of University of Craiova," *Mathematics for Computer Science*, vol. 34, pp. 82–87, 2007.
- [3] Y. C. Kwun, M. S. Saleem, M. Ghafoor, W. Nazeer, and S. M. Kang, "Hermite Hadamard-type inequalities for functions whose derivatives are η -convex via fractional integrals," *Journal of Inequalities and Applications*, vol. 2019, 2019.
- [4] V. G. Mihesan, "A generalization of the convexity," in *Proceedings of the Seminar on Functional Equations Approximation*, Cluj-Napoca, Romania, 1993.
- [5] M. E. Özdemir, A. O. Akdemri, and E. Set, "On $(h-m)$ -convexity and Hadamard-type inequalities," *Transylvanian Journal of Mathematics and Mechanics*, vol. 8, no. 1, pp. 51–58, 2016.
- [6] X. Qiang, G. Farid, J. Pečarić, and S. B. Akbar, "Generalized fractional integral inequalities for exponentially (s, m) -convex functions," *Journal of Inequalities and Applications*, vol. 2020, 2020.
- [7] S. Varošanec, "On h -convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.
- [8] G. A. Anastassiou, "Abstract generalized fractional Landau inequalities over $\mathbb{R}$," *Constructive Mathematical Analysis*, vol. 4, no. 1, pp. 34–47, 2021.
- [9] G. A. Anastassiou, "General multivariate lyengar type inequalities," *Constructive Mathematical Analysis*, vol. 2, no. 2, pp. 64–80, 2019.
- [10] S. S. Dragomir, "Inequalities for synchronous functions and applications," *Constructive Mathematical Analysis*, vol. 2, no. 3, pp. 109–123, 2019.
- [11] S. S. Dragomir, "Ostrowski's type inequalities for the complex integral on paths," *Constructive Mathematical Analysis*, vol. 3, no. 4, pp. 125–138, 2020.
- [12] M. Andrić, G. Farid, and J. Pečarić, "A further extension of Mittag-Leffler function," *Fractional Calculus and Applied Analysis*, vol. 21, no. 5, pp. 1377–1395, 2018.
- [13] H. Bai, M. S. Saleem, W. Nazeer, M. S. Zahoor, and T. Zhao, "Hermite-Hadamard and Jensen-type inequalities for interval nonconvex function," *Journal of Mathematics*, vol. 2020, Article ID 3945384, 2020.
- [14] S. Habib, S. Mubeen, and M. N. Naeem, "Chebyshev type integral inequalities for generalized k-fractional conformable integrals," *Journal of Inequalities and Special Functions*, vol. 9, no. 4, pp. 53–65, 2018.
- [15] S. M. Kang, G. Farid, M. Waseem, S. Ullah, W. Nazeer, and S. Mehmood, "Generalized k-fractional integral inequalities associated with (α, m) -convex functions," *Journal of Inequalities and Applications*, vol. 2019, 2019.
- [16] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, and S. M. Kang, "Generalized Riemann-Liouville k -fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities," *IEEE Access*, vol. 6, pp. 64946–64953, 2018.
- [17] S. Mehmood and G. Farid, "Fractional integrals inequalities for exponentially m -convex functions," *Open Journal of Mathematical Sciences*, vol. 2020, no. 4, pp. 78–85, 2020.
- [18] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations," in *North-Holland Mathematics Studies*, vol. 204, Elsevier, New York-London, 2006.

- [19] G. Farid, "Existence of an integral operator and its consequences in fractional and conformable integrals," *Open Journal of Mathematical Science*, vol. 3, no. 3, pp. 210–216, 2019.
- [20] H. J. Houbold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 2011.
- [21] T. R. Parbhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel," *Yokohama Mathematical Journal*, vol. 19, pp. 7–15, 1971.
- [22] G. Farid, "A unified integral operator and its consequences," *Open Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 1–7, 2020.
- [23] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Advances in Difference Equations*, vol. 2017, 2017.
- [24] T. U. Khan and M. A. Khan, "Generalized conformable fractional operators," *Journal of Computational and Applied Mathematics*, vol. 346, pp. 378–389, 2019.
- [25] S. Mubeen and G. M. Habibullah, " k -fractional integrals and applications," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 2, pp. 89–94, 2012.
- [26] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen, and M. Arshad, "The extended Mittag-Leffler function via fractional calculus," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 4244–4253, 2013.
- [27] T. O. Salim and A. W. Faraj, "A generalization of Mittag-Leffler function and integral operator associated with integral calculus," *Journal of Fractional Calculus and Applications*, vol. 3, no. 5, pp. 1–13, 2012.
- [28] M. Z. Sarikaya, M. Dahmani, M. E. Kiris, and F. Ahmad, " (k, s) -Riemann-Liouville fractional integral and applications," *Haceteppe Journal of Mathematics and Statistics*, vol. 45, no. 1, pp. 77–89, 2016.
- [29] H. M. Srivastava and Z. Tomovski, "Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernel," *Applied Mathematics and Computation*, vol. 211, no. 1, pp. 198–210, 2009.
- [30] T. Tunc, H. Budak, F. Usta, and M. Z. Sarikaya, "On new generalized fractional integral operators and related fractional inequalities," <https://www.researchgate.net/publication/313650587>.
- [31] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and S. M. Kang, "Inequalities for a unified integral operator and associated results in fractional calculus," *IEEE Access*, vol. 7, pp. 126283–126292, 2019.
- [32] C. Y. Jung, G. Farid, M. Andrić, J. Pečarić, and Y.-M. Chu, "Refinements of some integral inequalities for unified integral operators," *Journal of Inequalities and Applications*, vol. 2021, Article ID 7, 2021.
- [33] N. Eftekhari, "Some remarks on (s, m) -convexity in the second sense," *Journal of Mathematical Inequalities*, vol. 8, no. 3, pp. 489–495, 2014.
- [34] B. T. Polyak, "Existence theorems and convergence of minimizing sequences in extremum problems with restrictions," *Soviet Mathematics Doklady*, vol. 7, pp. 72–75, 1966.
- [35] M. Bracamonte, J. Giménez, and M. Vivas-Cortez, "Hermite-Hadamard-Fejér type inequalities for strongly (s, m) -convex functions with modulus c , in second sense," *Applied Mathematics & Information Sciences*, vol. 10, pp. 2045–2053, 2016.
- [36] Y. Dong, M. Saddiqa, S. Ullah, and G. Farid, "Study of fractional integral operators containing Mittag-Leffler functions via strongly (α, m) -convex functions," *Mathematical Problems in Engineering*, vol. 2021, Article ID 6693914, 2021.
- [37] Z. He, G. Farid, A. U. Haq, and K. Mahreen, "Bounds of a unified integral operator for (s, m) -convex functions and their consequences," *AIMS Mathematics*, vol. 5, no. 6, pp. 5510–5520, 2020.
- [38] G. Farid, K. Mahreen, and Y.-M. Chu, "Study of inequalities for unified integral operators of generalized convex functions," *Open Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 80–93, 2021.
- [39] G. Farid, B. Ni, and K. Mahreen, "Inequalities for a unified integral operator via (α, m) -convex functions," *Journal of Mathematics*, vol. 2020, Article ID 2345416, 2020.
- [40] G. Farid, "Estimation of Riemann-Liouville k -fractional integrals via convex functions," *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol. 23, no. 1, pp. 71–78, 2019.
- [41] Z. Chen, G. Farid, A. U. Rehman, and N. Latif, "Estimation of fractional integral operators via convex functions and related results," *Advances in Difference Equations*, vol. 2020, 2020.
- [42] G. Farid, "Bounds of Riemann-Liouville fractional integral operators," *Computational Methods for Differential Equations*, vol. 9, no. 2, pp. 637–648, 2021.
- [43] G. Farid, "Some Riemann-Liouville fractional integral inequalities for convex functions," *Journal of Analysis*, vol. 27, no. 4, pp. 1095–1102, 2019.
- [44] G. Farid, W. Nazeer, M. S. Saleem, S. Mehmood, and S. M. Kang, "Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications," *Mathematics*, vol. 6, no. 11, p. 248, 2018.
- [45] L. Chen, G. Farid, S. I. But, and S. B. Akbar, "Boundedness of fractional integral operators containing Mittag-Leffler functions," *Turkish Journal of Inequalities*, vol. 4, no. 1, pp. 14–24, 2020.
- [46] Y. C. Kwun, G. Farid, S. M. Kang, B. K. Bangash, and S. Ullah, "Derivation of bounds of several kinds of operators via (s, m) -convexity," *Advances in Difference Equations*, vol. 2020, 2020.
- [47] G. Farid, S. B. Akbar, S. U. Rehman, and J. Pečarić, "Boundedness of fractional integral operators containing Mittag-Leffler functions via (s, m) -convexity," *AIMS Mathematics*, vol. 5, no. 2, pp. 966–978, 2020.
- [48] Z. Chen, G. Farid, M. Saddiqa, S. Ullah, and N. Latif, "Study of fractional integral inequalities involving Mittag-Leffler functions via convexity," *Journal of Inequalities and Applications*, vol. 2020, Article ID 206, 2020.