Research Article

On the Equivalence between Two Fixed Point Theorems for Concave-Type Operators

Mohamed Jleli and Bessem Samet

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Mohamed Jleli; jleli@ksu.edu.sa

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In this note, we prove the equivalence between a fixed point theorem for φ -(h, e)-concave operators and a previous one for generalized concave operators.

1. Introduction

In this note, we show that the fixed point result established by Zhai and Wang [1] for φ -(h, e)-concave operators is equivalent to a previous result due to Zhai and Wang [2] for generalized concave operators. We first recall some notions regarding Banach spaces partially ordered by a cone. For the convenience of readers, we refer them to [3, 4] for more details.

Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty closed convex subset $P \subset E$ is said to be a cone in E , if the following two conditions are satisfied:

- (a) $\lambda \geq 0, x \in P \Rightarrow \lambda x \in P$
- (b) $x \in P, -x \in P \Rightarrow x = 0_E$, where 0_E is the zero vector of E

If P is a cone in E , then we can define a partial order $\leq_P$ in E as follows: for all $x, y \in E$,

$$x \leq_P y \Leftrightarrow y - x \in P. \quad (1)$$

A cone P in E is said to be normal, if there exists $K > 0$ such that for all $x, y \in E$,

$$0_E \leq_P x \leq_P y \Rightarrow \|x\| \leq K\|y\|. \quad (2)$$

Let P be a cone in E . We define the equivalence relation $\sim_P$ in E as follows: for all $x, y \in E$,

$$x \sim_P y \Leftrightarrow \text{there exist } \lambda, \mu > 0 \text{ such that } \lambda x \leq_P y \leq_P \mu y. \quad (3)$$

Given $h \in P$ with $h \neq 0_E$, we denote by P_h the equivalence class of h with respect to the equivalence relation $\sim_P$, i.e.,

$$P_h = \{x \in E \mid x \sim_P h\} = \{x \in E \mid \exists \lambda, \mu > 0 : \lambda h \leq_P x \leq_P \mu h\}. \quad (4)$$

It can be easily seen that $P_h \subset P$.

Definition 1. Let P be a cone in E and $A : D \rightarrow E$ be a given operator, where D is a nonempty subset of E . We say that A is increasing if

$$x, y \in D, x \leq_P y \Rightarrow Ax \leq_P Ay. \quad (5)$$

2. A Fixed Point Theorem for φ -(h, e)-Concave Operators

In this section, we recall the fixed point theorem established in [1] for φ -(h, e)-concave operators.

Let $(E, \|\cdot\|)$ be a real Banach space and P be a cone in E . Given $e, h \in P$ with $0_E \leq_P e \leq_P h$, we define the set

$$P_{h,e} = \{x \in E \mid x + e \in P_h\}, \quad (6)$$

that is,

$$P_{h,e} = \{x \in E \mid \exists \lambda, \mu > 0 : \lambda h \leq_P x + e \leq_P \mu h\}. \quad (7)$$

It can be easily seen that $P_h \subset P_{h,e}$. In particular, for $e = 0_E$, we have

$$P_{h,0_E} = P_h. \quad (8)$$

Zhai and Wang [1] introduced the notion of φ – (h, e) –concave operators as follows.

Definition 2 (φ – (h, e) –concave operators). Let $A : P_{h,e} \rightarrow E$ be a given operator. Suppose that for any $x \in P_{h,e}$ and $\lambda \in (0, 1)$, there exists $\varphi(\lambda) > \lambda$ such that

$$A(\lambda x + (\lambda - 1)e) \geq_P \varphi(\lambda)Ax + (\varphi(\lambda) - 1)e. \quad (9)$$

Then A is called a φ – (h, e) –concave operator.

Notice that for all $\lambda > 0$ and $x \in P_{h,e}$, we have $\lambda x + (\lambda - 1)e \in P_{h,e}$. In the special case $e = 0_E$, the above notion reduces to the generalized concavity concept.

Definition 3 (generalized concavity). Let $A : P_h \rightarrow E$ be a given operator. We say that A is a generalized concave operator, if for any $x \in P_h$ and $\lambda \in (0, 1)$, there exists $\varphi(\lambda) \in (\lambda, 1)$ such that

$$A(\lambda x) \geq_P \varphi(\lambda)Ax. \quad (10)$$

The main result obtained in [1] is the following fixed point theorem.

Theorem 4. Let P be a normal cone in E and $A : P_{h,e} \rightarrow E$ be an increasing φ – (h, e) –concave operator with $Ah \in P_{h,e}$. Then A admits a unique fixed point x^* in $P_{h,e}$. Moreover, for any $x_0 \in P_{h,e}$, the sequence $\{x_n\}$ defined by

$$x_n = Ax_{n-1}, \quad n = 1, 2, \dots, \quad (11)$$

converges to x^* , that is, $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$.

In the special case $e = 0_E$, we deduce from Theorem 4 the following fixed point theorem which was established by Zhai and Wang [2].

Theorem 5 (the case $e = 0_E$). Let P be a normal cone in E and $A : P_h \rightarrow E$ be an increasing φ – $(h, 0_E)$ –concave operator (i.e., generalized concave operator) with $Ah \in P_h$. Then A admits a unique fixed point x^* in P_h . Moreover, for any x_0

$\in P_h$, the sequence $\{x_n\}$ defined by

$$x_n = Ax_{n-1}, \quad n = 1, 2, \quad (12)$$

converges to x^* , that is, $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$.

In the next section, we will show that Theorem 5 $\Rightarrow$ Theorem 4. So, both theorems are equivalent.

3. The Equivalence between Theorems 4 and 5

We shall prove Theorem 4 using Theorem 5.

Proof of Theorem 4. Let P be a normal cone in E and $A : P_{h,e} \rightarrow E$ be an increasing φ – (h, e) –concave operator with $Ah \in P_{h,e}$. Let

$$h^* = h + e. \quad (13)$$

We claim that

$$P_{h^*} = P_h. \quad (14)$$

Let $y \in P_{h^*}$. Then there exist $\lambda, \mu > 0$ such that

$$\lambda h^* = \lambda(h + e) \leq_P y \leq_P \mu h^* = \mu(h + e). \quad (15)$$

Since $e \leq_P h$, we have

$$\mu(h + e) \leq_P 2\mu h. \quad (16)$$

Since $h + e \geq_P h$, we have

$$\lambda(h + e) \geq_P \lambda h. \quad (17)$$

It follows from (15), (16), and (17) that

$$\lambda h \leq_P y \leq_P 2\mu h, \quad (18)$$

which shows that $y \in P_h$. This proves that $P_{h^*} \subset P_h$. Let $z \in P_h$. Then there exist $\delta, \sigma > 0$ such that

$$\delta h \leq_P z \leq_P \sigma h. \quad (19)$$

Since $h \leq_P h^*$, we have

$$\sigma h \leq_P \sigma h^*. \quad (20)$$

Since $e \leq_P h$, we have

$$\delta h = \frac{\delta}{2}h + \frac{\delta}{2}h \geq_P \frac{\delta}{2}(h + e) = \frac{\delta}{2}h^*. \quad (21)$$

It follows from (19), (20), and (21) that

$$\frac{\delta}{2}h^* \leq_P z \leq_P \sigma h^*, \quad (22)$$

which shows that $z \in P_{h^*}$. This proves that $P_h \subset P_{h^*}$. Therefore, (14) is proved.

Now, we introduce the operator

$$B : P_{h^*} \longrightarrow E \quad (23)$$

defined by

$$By = A(y - e) + e, y \in P_{h^*}. \quad (24)$$

Notice that for all $y \in P_{h^*} = P_h$ (by (14)), we have $(y - e) + e = y \in P_h$, so $y - e \in P_{h,e}$. Hence, the operator B is well-defined.

Since A is increasing, for all $y_1, y_2 \in P_{h^*}$ with $y_1 \leq_P y_2$, we have $y_1 - e \leq_P y_2 - e$, which implies that

$$By_1 - e = A(y_1 - e) \leq_P A(y_2 - e) = By_2 - e, \quad (25)$$

and then $By_1 \leq_P By_2$. This shows that B is an increasing operator.

Next, we show that

$$Bh^* \in P_{h^*}. \quad (26)$$

We have

$$Bh^* = A(h^* - e) + e = Ah + e. \quad (27)$$

Since $Ah \in P_{h,e}$, then $Bh^* = Ah + e \in P_h = P_{h^*}$, which proves (26).

Now, we show that B is a generalized concave operator. Let $y \in P_{h^*} = P_h$ and $\lambda \in (0, 1)$. Since $x = y - e \in P_{h,e}$ and A is $\varphi - (h, e)$ -concave, then there exists $\varphi(\lambda) > \lambda$ such that

$$A(\lambda x + (\lambda - 1)e) \geq_P \varphi(\lambda)Ax + (\varphi(\lambda) - 1)e, \quad (28)$$

that is,

$$A(\lambda y - e) \geq_P \varphi(\lambda)A(y - e) + (\varphi(\lambda) - 1)e. \quad (29)$$

Hence, by the definition of the operator B , we obtain

$$B(\lambda y) - e \geq_P \varphi(\lambda)(By - e) + (\varphi(\lambda) - 1)e = \varphi(\lambda)By - e, \quad (30)$$

that is,

$$B(\lambda y) \geq_P \varphi(\lambda)By. \quad (31)$$

This shows that B is a generalized concave operator.

Next, by Theorem 5, we deduce that B has a unique fixed point y^* in P_{h^*} . Moreover, for any $y_0 \in P_{h^*}$, the sequence $\{y_n\}$ defined by

$$y_n = By_{n-1}, \quad n = 1, 2, \dots, \quad (32)$$

converges to y^* , that is, $\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0$. On the other hand, since $y^* \in P_{h^*} = P_h$ is a fixed point of B , then

$$y^* = By^* = A(y^* - e) + e, \quad (33)$$

that is,

$$x^* = y^* - e = A(y^* - e) = Ax^*. \quad (34)$$

This shows that $x^* \in P_{h,e}$ is a fixed point of A . Suppose now that $x^{**} \in P_{h,e}$ is another fixed point of A . Then we have $y^{**} = x^{**} + e \in P_h = P_{h^*}$ and

$$By^{**} = A(y^{**} - e) + e = Ax^{**} + e = x^{**} + e = y^{**}. \quad (35)$$

This shows that y^{**} is another fixed point of B in P_{h^*} , which contradicts the uniqueness of fixed points of B in P_{h^*} . Hence, we deduce that x^* is the unique fixed point of A in $P_{h,e}$. Finally, given $x_0 \in P_{h,e}$, we define the sequence $\{x_n\}$ by

$$x_n = Ax_{n-1}, \quad n = 1, 2, \dots. \quad (36)$$

Taking $y_0 = x_0 + e \in P_h = P_{h^*}$ and define the sequence $\{y_n\}$ by (32), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - y^*\| = 0 = \lim_{n \rightarrow \infty} \|x_n + e - (x^* + e)\| = \lim_{n \rightarrow \infty} \|x_n - x^*\|. \quad (37)$$

This concludes the proof of Theorem 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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