Research Article

On Unique and Nonunique Fixed Points and Fixed Circles in $\mathcal{M}_\nu^b$ -Metric Space and Application to Cantilever Beam Problem

Meena Joshi ¹, Anita Tomar ², Hossam A. Nabwey ^{3,4} and Reny George ^{3,5}

¹S.G.R.R. (P.G.) College, Dehradun, India

²Government Degree College Thatyur (Tehri Garhwal), Uttarakhand, India

³Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

⁴Department of Basic Engineering Sciences, Faculty of Engineering, Menofia University, Menofia, Egypt

⁵Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Durg, India

Correspondence should be addressed to Reny George; renygeorge02@yahoo.com

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We introduce $\mathcal{M}_\nu^b$ -metric to generalize and improve $\mathcal{M}_\nu$ -metric and unify numerous existing distance notions. Further, we define topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of the space to discuss topology on $\mathcal{M}_\nu^b$ -metric space and to create an environment for the survival of a unique fixed point. Also, we introduce a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map and establish fixed circle and fixed disc theorems. Further, we verify all these results by illustrative examples to demonstrate the authenticity of the postulates. Towards the end, we solve a fourth order differential equation arising in the bending of an elastic beam.

1. Introduction

A Greek mathematician Euclid of Alexandria (323-283 BC) was the first to communicate the notion of distance. Indeed, Euclidean distance is a noteworthy measure of closeness among two quantities, which is one of the initial conceptions appreciated by mankind. Fréchet [1] deliberated the general and more axiomatic form of distance as “ L -space.” Hausdorff [2] reexamined it as a metric space in the setting of points which has been refined, discussed, and generalized in numerous ways. Bakhtin [3], Branciari [4], Asadi et al. [5], George et al. [6], Mitrović and Radenović [7], Özgür et al. [8], Karahan and Isik [9], and Asim et al. [10] introduced the notions of a b -metric, a rectangular metric and a ν -generalized metric, an M -metric, a rectangular b -metric, a $b_\nu(s)$ -metric, a rectangular M -metric, a generalized p_ν^b -partial metric, and an M_ν -metric, respectively. Further, one may also refer to Fernandez et al. [11] and Kanwal et al. [12, 13] for work in N_b -cone metric spaces over Banach algebra, orthogonal F -metric spaces, and weak partial b -metric spaces, respec-

tively. One may allude to Kirk and Shahzed [14], to study in detail about the generalizations of the metric notion.

The aim of this work is to introduce a novel distance structure called an M_ν^b -metric space which is an improvement and generalization of an M_ν -metric space. Also, we introduce topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of the space to discuss topology and to create an environment for the survival of a unique fixed point in an M_ν^b -metric space. Further, we demonstrate that the collection of open balls, which forms a basis on M_ν^b -metric space, generates a $\mathcal{T}_0$ topology on it. In the sequel, with the help of examples and remarks, we demonstrate that an M_ν^b -metric space unifies and combines numerous distance conceptions and marks supremacy over all those spaces wherein the continuity of a map is required for the survival of a fixed point. We also introduce a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map in an M_ν^b -metric space and establish fixed

circle and greatest fixed disc theorems. We verify these results by illustrative examples with geometric interpretations to establish the authenticity of the postulates. Towards the end, we solve fourth order differential equation arising in the Cantilever beam problem.

2. Preliminaries

In this section, we use notations $m_{v_{u,w}} = \min \{m_v(u, u), m_v(w, w)\}$ and $M_{v_{u,w}} = \max \{m_v(u, u), m_v(w, w)\}$.

In 2017, Mitrović and Radenović [7] introduced b_v^s -metric space.

Definition 1. A $b_v(s)$ -metric on a nonempty set $\mathcal{M}$ with $s \geq 1$ is a map $b_v(s): \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(b_v(s)(i)) \quad b_v(s)(u, v) = 0 \text{ iff } u = v,$$

$$(b_v(s)(ii)) \quad b_v(s)(u, v) \geq 0,$$

$$(b_v(s)(iii)) \quad b_v(s)(u, w) = b_v(s)(w, u),$$

$$(b_v(s)(iv)) \quad b_v(s)(u, w) \leq s[b_v(s)(u, z_1) + b_v(s)(z_1, z_2) + \dots + b_v(s)(z_n, w)], \quad (1)$$

$u, z_1, z_2, \dots, z_n, w \in \mathcal{M}$ and are distinct. A pair $(\mathcal{M}, b_v(s))$ is a $b_v(s)$ -metric space.

Remark 2. A $b_v(s)$ -metric may be reduced to a v -generalized metric [4] for $s = 1$, a rectangular metric [4] for $v = 2$ and $s = 1$, a rectangular b -metric [6] for $v = 2$, a b -metric [3] for $v = 1$, and a usual metric [1] for $v = s = 1$.

In 2018, Karahan and Isik [9] introduced the pb_v^b -partial metric.

Definition 3. A partial $b_v(s)$ -metric on a nonempty set $\mathcal{M}$ with $s \geq 1$ is a map $pb_v(s): \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(pb_v(s)(i)) \quad pb_v(s)(u, u) = pb_v(s)(w, w) = pb_v(s)(u, w) \text{ iff } u = w,$$

$$(pb_v(s)(ii)) \quad pb_v(s)(u, u) \leq pb_v(s)(u, w),$$

$$(pb_v(s)(iii)) \quad pb_v(s)(u, w) = pb_v(s)(w, u),$$

$$(pb_v(s)(iv)) \quad pb_v(s)(u, w) \leq s[pb_v(s)(u, z_1) + pb_v(s)(z_1, z_2) + \dots + pb_v(s)(z_n, w)] - \sum_{i=1}^n pb_v(s)(z_i, z_i), \quad (2)$$

$u, z_1, z_2, \dots, z_n, w \in \mathcal{M}$ and are distinct. A pair $(\mathcal{M}, pb_v(s))$ is a partial $b_v(s)$ -metric space.

Remark 4. A partial $b_v(s)$ -metric may be reduced to a rectangular partial metric [15] for $v = 2$ and $s = 1$, a rectangular partial b -metric for $v = 2$, a partial b -metric [16] for $v = 1$, and a partial metric [17] for $v = s = 1$.

In 2019, Asim et al. [10] have familiarized the notion of an M_v -metric space.

Definition 5. An M_v -metric on a nonempty set $\mathcal{M}$ is a map $m_v: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(m_v i) \quad m_v(u, u) = m_v(w, w) = m_v(u, w) \text{ iff } u = w,$$

$$(m_v ii) \quad m_{v_{u,w}} \leq m_v(u, w),$$

$$(m_v iii) \quad m_v(u, w) = m_v(w, u),$$

$$(m_v iv) \quad \left(m_v(u, w) - m_{v_{u,w}} \right) \leq \left(m_v(u, z_1) - m_{v_{u,z_1}} \right) + \left(m_v(z_1, z_2) - m_{v_{z_1,z_2}} \right) + \dots + \left(m_v(z_n, w) - m_{v_{z_n,w}} \right), \quad (3)$$

$u, z_1, z_2, \dots, z_n, w \in \mathcal{M}$ and are distinct. A pair $(\mathcal{M}, m_v)$ is an M_v -metric space.

Example 6 (see [10]). Let $\mathcal{M} = \mathbb{R}$. Define $m_v: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ by $m_v(u, w) = |u| + |w|/2$, $u, w \in \mathcal{M}$, then $(\mathcal{M}, m_v)$ is an M_v -metric space.

Remark 7. In the above example, if $v = 1$, m_v will be an M -metric [5]. In this case, for $1, 2, 3 \in \mathcal{M}$,

$$(m_v(1, 3) - m_{v_{(1,3)}}) > (m_v(1, 2) - m_{v_{(1,2)}}) + (m_v(2, 3) - m_{v_{(2,3)}}) - m_v(2, 2). \quad (4)$$

If $v = 2$, it will be a rectangular M -metric [18]. Again, in this case, for $1, 2, 3, 4 \in \mathcal{M}$,

$$(m_v(1, 4) - m_{v_{(1,4)}}) > (m_v(1, 2) - m_{v_{(1,2)}}) + (m_v(2, 3) - m_{v_{(2,3)}}) + (m_v(3, 4) - m_{v_{(3,4)}}) - m_v(2, 2) - m_v(3, 3). \quad (5)$$

This suggests that if in an M_v -metric, we subtract the term $\sum_{i=1}^n m_v(z_i, z_i)$, then the function m_v will no longer be a metric. However, the terms like $\sum_{i=1}^n m_v(z_i, z_i)$ are indispensable in the analogous generalizations of a partial metric as a self-distance of a point in this space is not essentially zero.

For instance, if $m_v(u, w) = 3^{-n}$, then for a finite n and $u = w$, $m_v(u, u)$ is not equal to zero.

Definition 8 (see [19]). A function $\psi: [0, \infty) \longrightarrow [0, \infty)$ is a subadditive altering distance function if

- (i) ψ is an altering distance function (i.e., ψ is continuous, strictly increasing, and $\psi(t) = 0$ iff $t = 0$)

$$(ii) \quad \psi(\mathbf{u} + \mathbf{w}) \leq \psi(\mathbf{u}) + \psi(\mathbf{w}), \mathbf{u}, \mathbf{w} \in [0, \infty)$$

3. Main Results

We use notations $m_{v,u,w}^b = \min \{m_v^b(\mathbf{u}, \mathbf{u}), m_v^b(\mathbf{w}, \mathbf{w})\}$ and $M_{v,u,w}^b = \max \{m_v^b(\mathbf{u}, \mathbf{u}), m_v^b(\mathbf{w}, \mathbf{w})\}$. First, we introduce an M_v^b -metric space.

Definition 9. An M_v^b -metric on a nonempty set $\mathcal{M}$ with $s \geq 1$ is a map $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ satisfying

$$(m_{vi}^b) \quad m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{w}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{w}) \text{ iff } \mathbf{u} = \mathbf{w},$$

$$(m_{vii}^b) \quad m_{v,u,w}^b \leq m_v^b(\mathbf{u}, \mathbf{w}),$$

$$(m_{viii}^b) \quad m_v^b(\mathbf{u}, \mathbf{w}) = m_v^b(\mathbf{w}, \mathbf{u}),$$

$$(m_{vix}^b) \quad \left(m_v^b(\mathbf{u}, \mathbf{w}) - m_{v,u,w}^b \right) \leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v,z_v,w}^b \right) \right] - \sum_{i=1}^v m_v^b(\mathbf{z}_i, \mathbf{z}_i), \quad (6)$$

$\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_v, \mathbf{w} \in \mathcal{M}$ and are distinct. A pair $(\mathcal{M}, m_v^b)$ is called an M_v^b -metric space.

Example 10. Let $\mathcal{M} = \mathbb{R}^+$ and $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ be defined by $m_v^b(\mathbf{u}, \mathbf{w}) = |\mathbf{u} - \mathbf{w}|^\alpha + \max \{\mathbf{u}, \mathbf{w}\}^\alpha$, $\alpha > 1$. By routine calculation, we verify that $(\mathcal{M}, m_v^b)$ is an M_v^b -metric space with $s \geq 3^{\alpha-1}$. But $(\mathcal{M}, m_v^b)$ is not an M_v -metric space. Since, for $\mathbf{u} = 1, \mathbf{w} = n$ and $\mathbf{z}_1 = 2, \mathbf{z}_2 = 3, \dots, \mathbf{z}_v = n-1$, we attain

$$m_v^b(1, n) - m_{v,1,n}^b = |1 - n|^\alpha + \max \{1, n\}^\alpha - 1^\alpha = 1 + n^\alpha - 1 = n^\alpha,$$

$$m_v^b(1, 2) - m_{v,1,2}^b = |1 - 2|^\alpha + \max \{1, 2\}^\alpha - 1^\alpha = 1 + 2^\alpha - 1 = 2^\alpha,$$

$$m_v^b(2, 3) - m_{v,2,3}^b = |2 - 3|^\alpha + \max \{2, 3\}^\alpha - 2^\alpha = 1 + 3^\alpha - 2^\alpha, \dots$$

$$m_v^b(n-2, n-1) - m_{v,n-2,n-1}^b = |n-2 - n+1|^\alpha + \max \{n-2, n-1\}^\alpha - (n-2)^\alpha = 1 + (n-1)^\alpha - (n-2)^\alpha. \quad (7)$$

$$\text{Therefore, } m_v^b(1, n) - m_{v,1,n}^b > m_v^b(1, 2) - m_{v,1,2}^b + m_v^b(2, 3) - m_{v,2,3}^b + \cdots + m_v^b(n-2, n-1) - m_{v,n-2,n-1}^b.$$

Also, if $s = 1$, $(\mathcal{M}, m_v^b)$ is an improvement and extension of an M_v -metric space [10]. Inclusion of the terms containing nonzero self-distances demonstrate that it is a proper generalization of a notion of an M_v -metric space and consequently partial metric space as well. In particular, $(\mathcal{M}, m_v^b)$ is an M_v -metric [18] for $v = 1$ and rectangular M_b -metric space [20] for $v = 2$.

Theorem 11. Let m_v^b be an M_v^b -metric on a set $\mathcal{M}$. Let $(pb_v(s))^*(\mathbf{u}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{w}) + M_{v,u,w}^b - m_{v,u,w}^b$, then $(pb_v(s))^*$ is a generalized partial $b_v(s)$ -metric on $\mathcal{M}$.

Proof. Let $\mathbf{u}, \mathbf{w} \in \mathcal{M}$:

$$(pb_v(s)(i))$$

$$\text{Let } (pb_v(s))^*(\mathbf{u}, \mathbf{u}) = (pb_v(s))^*(\mathbf{w}, \mathbf{w}) = (pb_v(s))^*(\mathbf{u}, \mathbf{w}) \Leftrightarrow m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{w}, \mathbf{w}) = (pb_v(s))^*(\mathbf{u}, \mathbf{w}).$$

$$\text{So, } (pb_v(s))^*(\mathbf{u}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{w}), \text{ i.e., } m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{w}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{w}) \implies \mathbf{u} = \mathbf{w}.$$

$$\text{Hence, } (pb_v(s))^*(\mathbf{u}, \mathbf{u}) = (pb_v(s))^*(\mathbf{v}, \mathbf{v}) = (pb_v(s))^*(\mathbf{u}, \mathbf{v}) \Leftrightarrow \mathbf{u} = \mathbf{v}.$$

$$(pb_v(s)(ii)) \text{ For } \mathbf{u}, \mathbf{w} \in \mathcal{M}, (pb_v(s))^*(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{u}, \mathbf{u}) + M_{v,u,u}^b - m_{v,u,u}^b = m_v^b(\mathbf{u}, \mathbf{u}) \text{ and } m_v^b(\mathbf{u}, \mathbf{w}) - m_{v,u,w}^b \geq 0.$$

$$\text{Now, } (pb_v(s))^*(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{u}, \mathbf{u}) \leq M_{v,u,u}^b \leq M_{v,u,w}^b + m_v^b(\mathbf{u}, \mathbf{w}) - m_{v,u,w}^b = (pb_v(s))^*(\mathbf{u}, \mathbf{w}).$$

$$(pb_v(s)(iii)) \text{ Symmetric condition is trivially satisfied.}$$

$$(pb_v(s)(iv)) \text{ For } \mathbf{u}, \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_v, \mathbf{w} \in \mathcal{M},$$

$$\begin{aligned} (pb_v(s))^*(\mathbf{u}, \mathbf{w}) &= m_v^b(\mathbf{u}, \mathbf{w}) + M_{v,u,w}^b - m_{v,u,w}^b \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v,z_v,w}^b \right) \right] - \sum_{i=1}^v m_v^b(\mathbf{z}_i, \mathbf{z}_i) \\ &= s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v,z_v,w}^b \right) \right] - \sum_{i=1}^v m_v^b(\mathbf{z}_i, \mathbf{z}_i) + M_{v,u,w}^b \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v,z_v,w}^b \right) \right] + M_{v,u,z_1}^b + M_{v,z_1,z_2}^b + \cdots + M_{v,z_v,w}^b - \sum_{i=1}^v m_v^b(\mathbf{z}_i, \mathbf{z}_i) \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v,z_v,w}^b \right) \right] \\ &\quad + s \left[M_{v,u,z_1}^b + M_{v,z_1,z_2}^b + \cdots + M_{v,z_v,w}^b \right] - \sum_{i=1}^v (pb_v(s))^*(\mathbf{z}_i, \mathbf{z}_i) \\ &= s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) + M_{v,u,z_1}^b - m_{v,u,z_1}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) + M_{v,z_1,z_2}^b - m_{v,z_1,z_2}^b \right) + \cdots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) + M_{v,z_v,w}^b - m_{v,z_v,w}^b \right) \right] - \sum_{i=1}^v (pb_v(s))^*(\mathbf{z}_i, \mathbf{z}_i) \\ &= s[(pb_v(s))^*(\mathbf{u}, \mathbf{z}_1) + (pb_v(s))^*(\mathbf{z}_1, \mathbf{z}_2) + \cdots + (pb_v(s))^*(\mathbf{z}_v, \mathbf{w})] - \sum_{i=1}^v (pb_v(s))^*(\mathbf{z}_i, \mathbf{z}_i). \end{aligned} \quad (8)$$

Thus, $(\mathcal{M}, (pb_v(s))^*)$ is a generalized partial $b_v(s)$ -metric space.

Theorem 12. Let m_v^b be an M_v^b -metric on a set $\mathcal{M}$. Let $(b_v(s))^*(\mathbf{u}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{w}) - m_{v_{\mathbf{u}, \mathbf{w}}}^b$, then $(b_v(s))^*$ is a $b_v(s)$ -metric on $\mathcal{M}$.

Proof. $(b_v(s)(i))$ For $\mathbf{u}, \mathbf{w} \in \mathcal{M}$,

$$\begin{aligned} \text{let } (b_v(s))^*(\mathbf{u}, \mathbf{w}) = 0 &\Leftrightarrow m_v^b(\mathbf{u}, \mathbf{w}) = m_{v_{\mathbf{u}, \mathbf{w}}}^b \Leftrightarrow m_v^b(\mathbf{u}, \mathbf{w}) \\ &= m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{w}, \mathbf{w}) \Leftrightarrow \mathbf{u} = \mathbf{w}. \end{aligned} \quad (9)$$

$(b_v(s)(ii))$ Since, $m_v^b(\mathbf{u}, \mathbf{w}) \geq m_{v_{\mathbf{u}, \mathbf{w}}}^b, m_v^b(\mathbf{u}, \mathbf{w}) - m_{v_{\mathbf{u}, \mathbf{w}}}^b \geq 0$, i.e., $(b_v(s))^*(\mathbf{u}, \mathbf{w}) \geq 0, \mathbf{u}, \mathbf{w} \in \mathcal{M}$.

$(b_v(s)(iii))$ Symmetric condition is trivially satisfied.

$(b_v(s)(iv))$ For $\mathbf{u}, \mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_v, \mathbf{w} \in \mathcal{M}$,

$$\begin{aligned} (b_v(s))^*(\mathbf{u}, \mathbf{w}) &= m_v^b(\mathbf{u}, \mathbf{w}) - m_{v_{\mathbf{u}, \mathbf{w}}}^b, \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v_{\mathbf{u}, \mathbf{z}_1}}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v_{\mathbf{z}_1, \mathbf{z}_2}}^b \right) \right. \\ &\quad \left. + \dots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v_{\mathbf{z}_v, \mathbf{w}}}^b \right) \right] - \sum_{i=1}^v m_v^b(\mathbf{z}_i, \mathbf{z}_i), \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{z}_1) - m_{v_{\mathbf{u}, \mathbf{z}_1}}^b \right) + \left(m_v^b(\mathbf{z}_1, \mathbf{z}_2) - m_{v_{\mathbf{z}_1, \mathbf{z}_2}}^b \right) \right. \\ &\quad \left. + \dots + \left(m_v^b(\mathbf{z}_v, \mathbf{w}) - m_{v_{\mathbf{z}_v, \mathbf{w}}}^b \right) \right] \\ &= s[(b_v(s))^*(\mathbf{u}, \mathbf{z}_1) + (b_v(s))^*(\mathbf{z}_1, \mathbf{z}_2) \\ &\quad + \dots + (b_v(s))^*(\mathbf{z}_v, \mathbf{w})]. \end{aligned} \quad (10)$$

Thus, $(\mathcal{M}, (b_v(s))^*)$ is a $b_v(s)$ -metric.

Remark 13. Let $pb_v(s)$ be a partial $b_v(s)$ -metric on a set $\mathcal{M}$, then $(b_v(s))_1^* = pb_v(s)(\mathbf{u}, \mathbf{w}) - pb_v(s)(\mathbf{u}, \mathbf{u}) - pb_v(s)(\mathbf{w}, \mathbf{w})$ is $b_v(s)$ -metric space.

To discuss the topology corresponding to M_v^b -metric, the open ball with centre at $\mathbf{u}$ and radius $\mathbf{r} \in (0, \infty)$ is defined as $\mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon) = \{\mathbf{w} \in \mathcal{M} : m_v^b(\mathbf{u}, \mathbf{w}) < m_{v_{\mathbf{u}, \mathbf{w}}}^b + \varepsilon/s\}$.

Similarly, the closed ball with centre at $\mathbf{u}$ and radius $\mathbf{r} \in (0, \infty)$ is defined as $\mathcal{U}_{m_v^b}[\mathbf{u}, \varepsilon] = \{\mathbf{w} \in \mathcal{M} : m_v^b(\mathbf{u}, \mathbf{w}) \leq m_{v_{\mathbf{u}, \mathbf{w}}}^b + \varepsilon/s\}$.

Lemma 14. In an M_v^b -metric space $(\mathcal{M}, m_v^b)$, every open ball is an open set.

Proof. Let $\mathbf{w}_0 \in \mathcal{U}_{m_v^b}(\mathbf{u}, \mathbf{r})$, then $m_v^b(\mathbf{u}, \mathbf{w}_0) < m_{v_{\mathbf{u}, \mathbf{w}_0}}^b + \mathbf{r}/s$. Choose $\varepsilon/s = m_{v_{\mathbf{u}, \mathbf{w}_0}}^b + \mathbf{r}/s - m_v^b(\mathbf{u}, \mathbf{w}_0) > 0$.

Again, let $\mathbf{w}_1 \in \mathcal{U}_{m_v^b}(\mathbf{w}_0, \varepsilon)$, so $m_v^b(\mathbf{w}_1, \mathbf{w}_0) < m_{v_{\mathbf{w}_1, \mathbf{w}_0}}^b + \varepsilon/s$ and choose $\varepsilon_1/s = m_{v_{\mathbf{w}_1, \mathbf{w}_0}}^b + \varepsilon/s - m_v^b(\mathbf{w}_1, \mathbf{w}_0) > 0$.

Proceeding as above, let $\mathbf{w}_v \in \mathcal{U}_{m_v^b}(\mathbf{w}_{v-1}, \varepsilon_v)$, so $m_v^b(\mathbf{w}_v, \mathbf{w}_{v-1}) < m_{v_{\mathbf{w}_v, \mathbf{w}_{v-1}}}^b + \varepsilon_{v-1}/s$, choose $\varepsilon_v/s = m_{v_{\mathbf{w}_v, \mathbf{w}_{v-1}}}^b + \varepsilon_{v-1} - m_v^b(\mathbf{w}_v, \mathbf{w}_{v-1}) > 0$.

Now, for $\mathbf{u}, \mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_v$,

$$\begin{aligned} m_v^b(\mathbf{u}, \mathbf{w}_v) - m_{v_{\mathbf{u}, \mathbf{w}_v}}^b &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{w}_0) - m_{v_{\mathbf{u}, \mathbf{w}_0}}^b \right) + \left(m_v^b(\mathbf{w}_0, \mathbf{w}_1) - m_{v_{\mathbf{w}_0, \mathbf{w}_1}}^b \right) \right. \\ &\quad \left. + \dots + \left(m_v^b(\mathbf{w}_{v-1}, \mathbf{w}_v) - m_{v_{\mathbf{w}_{v-1}, \mathbf{w}_v}}^b \right) \right] - m_v^b(\mathbf{w}_1, \mathbf{w}_1) \\ &\quad - m_v^b(\mathbf{w}_2, \mathbf{w}_2) - \dots - m_v^b(\mathbf{w}_{v-1}, \mathbf{w}_{v-1}) \\ &\leq s \left[\left(m_v^b(\mathbf{u}, \mathbf{w}_0) - m_{v_{\mathbf{u}, \mathbf{w}_0}}^b \right) + \left(m_v^b(\mathbf{w}_0, \mathbf{w}_1) - m_{v_{\mathbf{w}_0, \mathbf{w}_1}}^b \right) \right. \\ &\quad \left. + \dots + \left(m_v^b(\mathbf{w}_{v-1}, \mathbf{w}_v) - m_{v_{\mathbf{w}_{v-1}, \mathbf{w}_v}}^b \right) \right] \\ &= s \left[\left(\frac{\mathbf{r}}{s} - \frac{\varepsilon}{s} \right) + \left(\frac{\varepsilon}{s} - \frac{\varepsilon_1}{s} \right) + \dots + \left(\frac{\varepsilon_{v-1}}{s} - \frac{\varepsilon_v}{s} \right) \right] = \mathbf{r} - \varepsilon_v. \end{aligned} \quad (11)$$

Hence, $\mathcal{U}_{m_v^b}(\mathbf{w}_0, \varepsilon) \subseteq \mathcal{U}_{m_v^b}(\mathbf{u}, \mathbf{r})$.

Theorem 15. If $(\mathcal{M}, m_v^b)$ is an M_v^b -metric space and τ_v^b a topology generated by the open balls $\mathcal{U}_{m_v^b}(\mathbf{u}, \mathbf{r})$, then $(\mathcal{M}, \tau_v^b)$ is a $\mathcal{T}_0$ -space.

Proof. Let $(\mathcal{M}, \tau_v^b)$ be an M_v^b -metric space and $\mathbf{u}, \mathbf{w} \in \mathcal{M}$ are two distinct points. Then,

$$m_{v_{\mathbf{u}, \mathbf{w}}}^b \leq m_v^b(\mathbf{u}, \mathbf{w}) \implies \min \{ m_v^b(\mathbf{u}, \mathbf{u}), m_v^b(\mathbf{w}, \mathbf{w}) \} \leq m_v^b(\mathbf{u}, \mathbf{w}), \quad (12)$$

i.e., $m_v^b(\mathbf{u}, \mathbf{u}) \leq m_v^b(\mathbf{u}, \mathbf{w})$ or $m_v^b(\mathbf{w}, \mathbf{w}) \leq m_v^b(\mathbf{u}, \mathbf{w})$.

Firstly, assume that $m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{u}, \mathbf{w})$,

$$\begin{aligned} \text{then } m_{v_{\mathbf{u}, \mathbf{w}}}^b &= m_v^b(\mathbf{u}, \mathbf{u}) = m_v^b(\mathbf{w}, \mathbf{w}) < m_v^b(\mathbf{u}, \mathbf{w}) \implies m_{v_{\mathbf{u}, \mathbf{w}}}^b - m_v^b(\mathbf{u}, \mathbf{w}) \\ &= m_{v_{\mathbf{u}, \mathbf{w}}}^b - m_v^b(\mathbf{u}, \mathbf{u}) < 0. \end{aligned} \quad (13)$$

For $s \geq 1$, if we chose $\varepsilon > 0$ such that $m_v^b(\mathbf{u}, \mathbf{w}) = m_{v_{\mathbf{u}, \mathbf{w}}}^b + \varepsilon/s$, so $\mathbf{w} \notin \mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon)$.

Next, assume that $m_v^b(\mathbf{u}, \mathbf{u}) < m_v^b(\mathbf{w}, \mathbf{w})$, then $m_{v_{\mathbf{u}, \mathbf{w}}}^b = m_v^b(\mathbf{u}, \mathbf{u}) < m_v^b(\mathbf{u}, \mathbf{w})$:

$$\implies m_{v_{\mathbf{u}, \mathbf{w}}}^b - m_v^b(\mathbf{u}, \mathbf{w}) = m_v^b(\mathbf{u}, \mathbf{u}) - m_v^b(\mathbf{u}, \mathbf{w}) < 0. \quad (14)$$

Again, for $s \geq 1$, if we chose $\varepsilon_1 > 0$ such that $m_v^b(\mathbf{u}, \mathbf{w}) = m_{v_{\mathbf{u}, \mathbf{w}}}^b + \varepsilon/s$, so $\mathbf{w} \notin \mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon_1)$.

Similarly, for $m_v^b(\mathbf{w}, \mathbf{w}) < m_v^b(\mathbf{u}, \mathbf{u})$, one may easily find an open ball so that $\mathbf{u} \in \mathcal{U}_{m_v^b}(\mathbf{w}, \varepsilon_1)$ and $\mathbf{w} \notin \mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon_1)$, i.e., for two distinct points $\mathbf{u}, \mathbf{w} \in \mathcal{M}$, there is a ball including the point $\mathbf{u}$ but not including the other point $\mathbf{w}$. Thus, $(\mathcal{M}, m_v^b)$ is a $\mathcal{T}_0$ -space.

Now, we discuss the convergence of the sequence and introduce definitions related to it.

Definition 16.

- (i) A sequence $\{\mathbf{u}_n\}$ in $(\mathcal{M}, m_v^b)$ is m_v^b -convergent to $\mathbf{u} \in \mathcal{M}$ iff $\lim_{n \rightarrow \infty} m_v^b(\mathbf{u}_n, \mathbf{u}) - m_{v_{\mathbf{u}_n, \mathbf{u}}}^b = 0$

In other words, a sequence $\{\mathbf{u}_n\}$ in a topological space $(\mathcal{M}, \tau_v^b)$ converges to a point $\mathbf{u}$ in $\mathcal{M}$ if for each open ball $\mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon)$ containing $\mathbf{u}$ there exists k so that for each $n > k$, $\mathbf{u}_n \in \mathcal{U}_{m_v^b}(\mathbf{u}, \varepsilon)$.

- (ii) A sequence $\{\mathbf{u}_n\}$ in $(\mathcal{M}, m_v^b)$ is m_v^b -Cauchy sequence iff $\lim_{n, m \rightarrow \infty} (m_v^b(\mathbf{u}_n, \mathbf{u}_m) - m_{v_{\mathbf{u}_n, \mathbf{u}_m}}^b)$ and $\lim_{n, m \rightarrow \infty} (M_{v_{\mathbf{u}_n, \mathbf{u}_m}}^b - m_{v_{\mathbf{u}_n, \mathbf{u}_m}}^b)$ exist and are finite

- (iii) An M_v^b -metric space is m_v^b -complete if each m_v^b -Cauchy sequence $\{\mathbf{u}_n\}$ converges to a point $\mathbf{u} \in \mathcal{M}$ so that $\lim_{n, m \rightarrow \infty} (m_v^b(\mathbf{u}_n, \mathbf{u}) - m_{v_{\mathbf{u}_n, \mathbf{u}}}^b) = 0$ and $\lim_{n, m \rightarrow \infty} (M_{v_{\mathbf{u}_n, \mathbf{u}}}^b - m_{v_{\mathbf{u}_n, \mathbf{u}}}^b) = 0$

Lemma 17. Let $(\mathcal{M}, m_v^b)$ be an M_v^b -metric space and $\mathcal{A}: \mathcal{M} \rightarrow \mathcal{M}$ a self-map on $\mathcal{M}$ so that there exists $\eta \in [0, 1/s]$, satisfying

$$m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w}) \leq \eta m_v^b(\mathbf{u}, \mathbf{w}). \quad (15)$$

Consider the sequence $\{\mathbf{u}_n\}$ defined by $\mathbf{u}_{n+1} = \mathcal{A}\mathbf{u}_n$. If $\mathbf{u}_n \rightarrow \mathbf{u}$ as $n \rightarrow \infty$, then $\mathcal{A}\mathbf{u}_n \rightarrow \mathcal{A}\mathbf{u}$ as $n \rightarrow \infty$.

Proof. If $m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) = 0$,

$$m_{v_{\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}}}^b \leq m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) = 0 \implies m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) - m_{v_{\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}}}^b \rightarrow 0, \text{ i.e., } \mathcal{A}\mathbf{u}_n \rightarrow \mathcal{A}\mathbf{u} \text{ as } n \rightarrow \infty.$$

Now, if $m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) > 0$, then by (15),

$$m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) \leq \eta m_v^b(\mathbf{u}_n, \mathbf{u}). \quad (16)$$

There are two cases.

Case 1. If $m_v^b(\mathbf{u}, \mathbf{u}) \leq m_v^b(\mathbf{u}_n, \mathbf{u}_n)$, then $m_v^b(\mathbf{u}_n, \mathbf{u}_n) \rightarrow 0$ as $n \rightarrow \infty \implies m_v^b(\mathbf{u}, \mathbf{u}) = 0$.

Since $m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{u}) \leq m_v^b(\mathbf{u}, \mathbf{u})$, we have $m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{u}) = 0$. Again, $\mathbf{u}_n \rightarrow \mathbf{u}$ implies that $\lim_{n \rightarrow \infty} (m_v^b(\mathbf{u}_n, \mathbf{u}) - m_{v_{\mathbf{u}_n, \mathbf{u}}}^b) = 0$.

Since, $m_{v_{\mathbf{u}_n, \mathbf{u}}}^b = \min \{m_v^b(\mathbf{u}_n, \mathbf{u}_n), m_v^b(\mathbf{u}, \mathbf{u})\} = 0$, so $\lim_{n \rightarrow \infty} m_v^b(\mathbf{u}_n, \mathbf{u}) = 0$.

Now, from equation (16),

$$\lim_{n \rightarrow \infty} m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) = 0, \text{ i.e., } \mathcal{A}\mathbf{u}_n \rightarrow \mathcal{A}\mathbf{u} \text{ as } n \rightarrow \infty. \quad (17)$$

Case 2. If $m_v^b(\mathbf{u}_n, \mathbf{u}_n) \leq m_v^b(\mathbf{u}, \mathbf{u})$, then $m_v^b(\mathbf{u}_n, \mathbf{u}_n) \rightarrow 0$ as $n \rightarrow \infty$.

So, as above $\lim_{n \rightarrow \infty} m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}) = 0$, i.e., $\mathcal{A}\mathbf{u}_n \rightarrow \mathcal{A}\mathbf{u}$ as $n \rightarrow \infty$.

Now, we prove the first main result for a Kannan type contraction.

Theorem 18. Let $(\mathcal{M}, m_v^b)$ be an M_v^b -complete metric space with coefficient $s \geq 1$. Suppose, a self-map $\mathcal{A}: \mathcal{M} \rightarrow \mathcal{M}$ satisfies

$$m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w}) \leq \mu \left[m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u}) + m_v^b(\mathbf{w}, \mathcal{A}\mathbf{w}) \right], \quad \mu < \frac{1}{2s}, \mathbf{u}, \mathbf{w} \in \mathcal{M}. \quad (18)$$

Then, $\mathcal{A}$ has a unique fixed point $\mathbf{u}^*$ so that $m_v^b(\mathbf{u}^*, \mathbf{u}^*) = 0$ and the sequence of iterates $\{\mathcal{A}^n \mathbf{u}_0\} \subseteq \mathcal{M}$ converges to $\mathbf{u}^* \in \mathcal{M}$.

Proof. Starting from the given element $\mathbf{u}_0 \in \mathcal{M}$, form the sequence $\{\mathbf{u}_n\}$, where $\mathbf{u}_n = \mathcal{A}\mathbf{u}_{n-1}$, $n \in \mathbb{N}$. If $m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) = 0, n \geq 0$, then $\mathcal{A}\mathbf{u}_n = \mathbf{u}_{n+1} = \mathbf{u}_n$ and $m_v^b(\mathbf{u}_n, \mathbf{u}_n) = 0$, and this completes the proof.

Further, take $m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) > 0, n \geq 0$. For $\mathbf{u} = \mathbf{u}_n, \mathbf{w} = \mathbf{u}_{n+1}$, utilizing condition (18),

$$\begin{aligned} m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) &= m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}_{n+1}) \\ &\leq \mu \left[m_v^b(\mathbf{u}_n, \mathcal{A}\mathbf{u}_n) + m_v^b(\mathbf{u}_{n+1}, \mathcal{A}\mathbf{u}_{n+1}) \right] \\ &= \mu \left[m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) + m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) \right], \end{aligned}$$

$$\text{i.e., } (1 - \mu) m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) \leq \mu m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}),$$

$$\text{i.e., } m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) \leq \frac{\mu}{1 - \mu} m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}). \quad (19)$$

Let, $\xi_v^b(n) = m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}), n \geq 0, \lambda = \mu/(1 - \mu) < 1$, then $\xi_v^b(n+1) \leq \lambda \xi_v^b(n)$.

On repeating these steps, we get $\xi_v^b(n+1) \leq \lambda^n \xi_v^b(0) \rightarrow 0, n \rightarrow \infty$.

Now,

$$\begin{aligned} m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}) &= m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}_n) \leq \mu \left[m_v^b(\mathbf{u}_n, \mathcal{A}\mathbf{u}_n) + m_v^b(\mathbf{u}_n, \mathcal{A}\mathbf{u}_n) \right] \\ &= 2\mu m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) = 2\mu \xi_v^b(n) \\ &\leq 2\lambda^{n-1} \xi_v^b(0) \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \quad (20)$$

First, we show that $\mathbf{u}_n \neq \mathbf{u}_m$, for $n \neq m$. Suppose, $\mathbf{u}_n = \mathbf{u}_m$, for $n > m$, then $\mathcal{A}\mathbf{u}_n = \mathbf{u}_{n+1} = \mathcal{A}\mathbf{u}_m = \mathbf{u}_{m+1}$.

Now, using inequality (18), for $\mathbf{u} = \mathbf{u}_n$ and $\mathbf{w} = \mathbf{u}_{n+1}$,

$$\begin{aligned}\xi_v^b(m) &= m_v^b(\mathbf{u}_m, \mathbf{u}_{m+1}) = m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) = \xi_v^b(n) \leq \lambda \xi_v^b(n-1) \\ &\leq \lambda^2 \xi_v^b(n-2) \leq \dots \leq \lambda^{-m} \xi_v^b(m) < \xi_v^b(m),\end{aligned}\quad (21)$$

a contradiction. Thus, $\mathbf{u}_n \neq \mathbf{u}_m$, for $n \neq m$.

Now, we assert that $\{\mathbf{u}_n\}$ is a Cauchy sequence in $(\mathcal{M}, m_v^b)$. We discuss two cases.

Case 1. First, let l be odd, i.e., $l = 2m + 1$, for $n, m \in \mathbb{N}$. Now, by using $(m_v^b iv)$ for $n \leq v \leq n + l$,

$$\begin{aligned}m_v^b(\mathbf{u}_n, \mathbf{u}_{n+l}) &= m_v^b(\mathbf{u}_n, \mathbf{u}_{n+2m+1}) \\ &\leq s \left[m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) + m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) + \dots + m_v^b(\mathbf{u}_{n+v-1}, \mathbf{u}_{n+v}) \right. \\ &\quad \left. + m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m+1}) \right] - m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}) - m_v^b(\mathbf{u}_{n+2}, \mathbf{u}_{n+2}) \\ &\quad - \dots - m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+v}) \leq s \left(\lambda^{n-1} + \lambda^n + \dots + \lambda^{n+v-2} \right) \xi_v^b(0) \\ &\quad - 2(\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+v-1}) \xi_v^b(0) + s m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m+1}) \\ &= s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - 2 \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m+1}) \\ &\leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left[m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+v+1}) \right. \\ &\quad \left. + m_v^b(\mathbf{u}_{n+v+1}, \mathbf{u}_{n+v+2}) + \dots + m_v^b(\mathbf{u}_{n+2v-1}, \mathbf{u}_{n+2v}) + m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2m+1}) \right] \\ &\quad - s \left[m_v^b(\mathbf{u}_{n+v+1}, \mathbf{u}_{n+v+1}) + m_v^b(\mathbf{u}_{n+v+2}, \mathbf{u}_{n+v+2}) + \dots + m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2v}) \right] \\ &\leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 (\lambda^{n+v-1} + \lambda^{n+v} \\ &\quad + \dots + \lambda^{n+2v-2}) \xi_v^b(0) + s^2 m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2m+1}) - s (\lambda^{n+v} + \lambda^{n+v+1} \\ &\quad + \dots + \lambda^{n+2v-1}) \xi_v^b(0) \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \dots \\ &\quad + s^{2m/v-1} m_v^b(\mathbf{u}_{n+2m-v}, \mathbf{u}_{n+2m+1}) \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) \\ &\quad - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + \dots + s^{2m/v} \left[m_v^b(\mathbf{u}_{n+2m-v}, \mathbf{u}_{n+2m-v+1}) + m_v^b(\mathbf{u}_{n+2m-v+1}, \mathbf{u}_{n+2m-v+2}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+2m}, \mathbf{u}_{n+2m+1}) \right] - s^{2m/v-1} \left[m_v^b(\mathbf{u}_{n+2m-v+1}, \mathbf{u}_{n+2m-v+1}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+2m}, \mathbf{u}_{n+2m}) \right] \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) \\ &\quad - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + \dots + s^{2m/v} \left(\frac{\lambda^{n+2m-v-2}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s^{2m/v-1} \frac{\lambda^{n+2m-v-1}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad \cdot (0) \longrightarrow 0, \text{ as } n \longrightarrow \infty,\end{aligned}\quad (22)$$

i.e., $\lim_{n,m \rightarrow \infty} m_v^b(\mathbf{u}_n, \mathbf{u}_{n+2m+1}) = 0$.

Case 2. Now, let l is even, i.e., $l = 2m$ for $n, m \in \mathbb{N}$. Now, by using $(m_v^b iv)$ for $n \leq v \leq n + l$,

$$\begin{aligned}m_v^b(\mathbf{u}_n, \mathbf{u}_{n+l}) &= m_v^b(\mathbf{u}_n, \mathbf{u}_{n+2m}) \leq s \left[m_v^b(\mathbf{u}_n, \mathbf{u}_{n+1}) + m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+2}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+v-1}, \mathbf{u}_{n+v}) + m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m}) \right] - m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}) \\ &\quad - m_v^b(\mathbf{u}_{n+2}, \mathbf{u}_{n+2}) - \dots - m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+v}) \\ &\leq s (\lambda^{n-1} + \lambda^n + \dots + \lambda^{n+v-2}) \xi_v^b(0) - (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+v-1}) \xi_v^b(0) \\ &\quad + s m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m}) = s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + s m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+2m}) \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + s^2 \left[m_v^b(\mathbf{u}_{n+v}, \mathbf{u}_{n+v+1}) + m_v^b(\mathbf{u}_{n+v+1}, \mathbf{u}_{n+v+2}) + \dots + m_v^b(\mathbf{u}_{n+2v-1}, \mathbf{u}_{n+2v}) \right. \\ &\quad \left. + m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2m+1}) \right] - s \left[m_v^b(\mathbf{u}_{n+v+1}, \mathbf{u}_{n+v+1}) + m_v^b(\mathbf{u}_{n+v+2}, \mathbf{u}_{n+v+2}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2v}) \right] \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + s^2 (\lambda^{n+v-1} + \lambda^{n+v} + \dots + \lambda^{n+2v-2}) m_v^b(0) + s^2 m_v^b(\mathbf{u}_{n+2v}, \mathbf{u}_{n+2m}) \\ &\quad - s (\lambda^{n+v} + \lambda^{n+v+1} + \dots + \lambda^{n+2v-1}) \xi_v^b(0) \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) \\ &\quad - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + \dots + s^{2m/v-1} m_v^b(\mathbf{u}_{n+2m-1-v}, \mathbf{u}_{n+2m}) \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) \\ &\quad - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \dots \\ &\quad + s^{2m/v} \left[m_v^b(\mathbf{u}_{n+2m-1-v}, \mathbf{u}_{n+2m-v}) + m_v^b(\mathbf{u}_{n+2m-v}, \mathbf{u}_{n+2m-v+1}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+2m-1}, \mathbf{u}_{n+2m}) \right] - s^{2m/v-1} \left[m_v^b(\mathbf{u}_{n+2m-v}, \mathbf{u}_{n+2m-v}) \right. \\ &\quad \left. + \dots + m_v^b(\mathbf{u}_{n+2m-1}, \mathbf{u}_{n+2m-1}) \right] \leq s \left(\frac{\lambda^{n-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) \\ &\quad - \frac{\lambda^n(1-\lambda^v)}{1-\lambda} \xi_v^b(0) + s^2 \left(\frac{\lambda^{n+v-1}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s \frac{\lambda^{n+v}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad + \dots + s^{2m/v} \left(\frac{\lambda^{n+2m-v-2}(1-\lambda^v)}{1-\lambda} \right) \xi_v^b(0) - s^{2m/v-1} \frac{\lambda^{n+2m-v-1}(1-\lambda^v)}{1-\lambda} \xi_v^b(0) \\ &\quad \cdot (0) \longrightarrow 0, \text{ as } n \longrightarrow \infty,\end{aligned}\quad (23)$$

i.e., $\lim_{n,m \rightarrow \infty} m_v^b(\mathbf{u}_n, \mathbf{u}_{n+2m}) = 0$.

So, $\lim_{n,m \rightarrow \infty} (m_v^b(\mathbf{u}_n, \mathbf{u}_m) - m_{v_{\mathbf{u}_n, \mathbf{u}_m}}^b) = 0$.

Let

$$M_v^b(\mathbf{u}_n, \mathbf{u}_m) = m_v^b(\mathbf{u}_n, \mathbf{u}_n).$$

Now,

$$M_v^b(\mathbf{u}_n, \mathbf{u}_m) - m_v^b(\mathbf{u}_n, \mathbf{u}_m) \leq M_v^b(\mathbf{u}_n, \mathbf{u}_m) = m_v^b(\mathbf{u}_n, \mathbf{u}_n) \leq 2 \lambda^{n-1} \xi_v^b(0) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Hence, $\lim_{n,m \rightarrow \infty} M_v^b(\mathbf{u}_n, \mathbf{u}_m) - m_v^b(\mathbf{u}_n, \mathbf{u}_m) = 0$.

Therefore, the sequence $\{\mathbf{u}_n\}$ is m_v^b -Cauchy in $\mathcal{M}$.

Since, $\mathcal{M}$ is m_v^b -complete, there exists $\mathbf{u}^* \in \mathcal{U}$ so that $\mathbf{u}_n \longrightarrow \mathbf{u}^*$. Now, we assert that $\mathcal{A}\mathbf{u}^* = \mathbf{u}^*$:

$$\begin{aligned}\lim_{n \rightarrow \infty} (m_v^b(\mathbf{u}_n, \mathbf{u}^*) - m_{v_{\mathbf{u}_n, \mathbf{u}^*}}^b) \\ &= 0 \implies \lim_{n \rightarrow \infty} (m_v^b(\mathbf{u}_{n+1}, \mathbf{u}^*) - m_{v_{\mathbf{u}_{n+1}, \mathbf{u}^*}}^b) \\ &= 0 \implies \lim_{n \rightarrow \infty} (m_v^b(\mathcal{A}\mathbf{u}_n, \mathbf{u}^*) - m_{v_{\mathcal{A}\mathbf{u}_n, \mathbf{u}^*}}^b) \\ &= 0 \implies m_v^b(\mathcal{A}\mathbf{u}^*, \mathbf{u}^*) - m_{v_{\mathcal{A}\mathbf{u}^*, \mathbf{u}^*}}^b = 0 \text{ (using Lemma 17),}\end{aligned}\quad (24)$$

i.e., $m_v^b(\mathcal{A}u^*, u^*) = \min \{m_v^b(\mathcal{A}u^*, \mathcal{A}u^*), m_v^b(u, u^*)\} \implies m_v^b(\mathcal{A}u^*, u^*) = m_v^b(\mathcal{A}u^*, \mathcal{A}u^*)$ or $m_v^b(\mathcal{A}u^*, u^*) = m_v^b(u^*, u^*)$.

Hence, $\mathcal{A}u^* = u^*$, i.e., u^* is a fixed point of $\mathcal{A}$.

Now, we assert that $m_v^b(u^*, u^*) = 0$:

$$\begin{aligned} m_v^b(u^*, u^*) &= m_v^b(\mathcal{M}u^*, \mathcal{M}u^*) \leq \mu [m_v^b(u^*, \mathcal{A}u^*) + m_v^b(u^*, \mathcal{A}u^*)] \\ &= 2\mu m_v^b(u^*, \mathcal{A}u^*) = 2\mu m_v^b(u^*, u^*) < m_v^b(u^*, u^*), \end{aligned} \quad (25)$$

a contradiction. Hence, $m_v^b(u^*, u^*) = 0$.

Suppose, u^* and w^* are two different fixed points of $\mathcal{A}$, so

$$\begin{aligned} m_v^b(u^*, w^*) &= m_v^b(\mathcal{A}u^*, \mathcal{A}w^*) \leq \mu [m_v^b(u^*, \mathcal{A}u^*) + m_v^b(w^*, \mathcal{A}w^*)] \\ &= \mu [m_v^b(u^*, u^*) + m_v^b(w^*, w^*)] = 0 \implies m_v^b(u^*, w^*) = 0. \end{aligned} \quad (26)$$

Hence, $u^* = w^*$.

Now, we furnish two examples (one of continuous and another of discontinuous self-map) to validate Theorem 18 and demonstrate the fact that the continuity of a self-map is not indispensable for the survival of a unique fixed point in an M_v^b -metric space for the Kannan type contraction. Also, the self-distance of a fixed point is zero, and the sequence of iterates $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$ converges to a fixed point.

Example 19. Let $\mathcal{M} = [-10, 10]$ and an M_v^b -metric $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined by $m_v^b(u, w) = ((|u| + |w|)/2)^2$, $u, w \in \mathcal{M}$. Then, $(\mathcal{M}, m_v^b)$ is a complete M_v^b -metric space for $s = 3$. Define a self-map $\mathcal{A}$ on $\mathcal{M}$ as $\mathcal{A}u = 3/19u$, $u \in \mathcal{M}$. Observe that, for all $u, w \in \mathcal{M}$, we obtain

$$\begin{aligned} m_v^b(\mathcal{A}u, \mathcal{A}w) &= \left(\frac{|\mathcal{A}u| + |\mathcal{A}w|}{2} \right)^2 = \left(\frac{3/19|u| + 3/19|w|}{2} \right)^2 \\ &= \frac{9}{361} \left(\frac{|u| + |w|}{2} \right)^2, \end{aligned}$$

$$\begin{aligned} m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w) &= \left(\frac{|u| + |\mathcal{A}u|}{2} \right)^2 + \left(\frac{|w| + |\mathcal{A}w|}{2} \right)^2 \\ &= \left(\frac{|u| + 3/19|u|}{2} \right)^2 + \left(\frac{|w| + 3/19|w|}{2} \right)^2 \\ &= \frac{484}{361} \left(\frac{|u|}{2} \right)^2 + \frac{484}{361} \left(\frac{|w|}{2} \right)^2 \\ &\leq \frac{484}{361} \left(\frac{|u| + |w|}{2} \right)^2, \end{aligned} \quad (27)$$

i.e., $m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \mu [m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w)]$, $\mu = 9/484 < 1/2s$.

Thus, all the postulates of Theorem 18 are verified and $\mathcal{A}$ has a unique fixed point 0 in $\mathcal{M}$ and clearly, $m_v^b(0, 0) = 0$. Further, there exists a sequence $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$, $\mathcal{A}^n u_0 = -n/(n^2 + n)$, $n \in \mathbb{N}$, which m_v^b -converges to 0.

Example 20. Consider $\mathcal{M} = [0, 1]$ and an M_v^b -metric $m_v^b : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ defined by $m_v^b(u, w) = (|u| + |w|/2)^2$, $u, w \in \mathcal{M}$. Then, $(\mathcal{M}, m_v^b)$ is a complete M_v^b -metric space for $s = 3$. Define a self-map $\mathcal{A}$ on $\mathcal{M}$ as

$$\mathcal{A}u = \begin{cases} \frac{u}{3}, & u \in [0, 1) \\ \frac{1}{9}, & u = 1 \end{cases}, \quad u \in \mathcal{M}. \quad (28)$$

Observe that, for all $u, w \in \mathcal{M}$, we obtain the following.

Case 21. When $u, w \in [0, 1)$,

$$m_v^b(\mathcal{A}u, \mathcal{A}w) = \left(\frac{\mathcal{A}u + \mathcal{A}w}{2} \right)^2 = \left(\frac{u/3 + w/3}{2} \right)^2 = \left(\frac{u + w}{6} \right)^2,$$

$$\begin{aligned} m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w) &= \left(\frac{u + \mathcal{A}u}{2} \right)^2 + \left(\frac{w + \mathcal{A}w}{2} \right)^2 \\ &= \left(\frac{u + u/3}{2} \right)^2 + \left(\frac{w + w/3}{2} \right)^2 \\ &= \left(\frac{4u}{6} \right)^2 + \left(\frac{4w}{6} \right)^2 \leq \frac{4}{9} (u + w)^2. \end{aligned} \quad (29)$$

Case 22. When $u = w = 1$,

$$\begin{aligned} m_v^b(\mathcal{A}u, \mathcal{A}w) &= \left(\frac{1/9 + 1/9}{2} \right)^2 = \frac{1}{81}, \\ m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w) &= \left(\frac{1 + 1/9}{2} \right)^2 + \left(\frac{1 + 1/9}{2} \right)^2 = \frac{50}{81}. \end{aligned} \quad (30)$$

Case 23. When $u = 0, w = 1$,

$$\begin{aligned} m_v^b(\mathcal{A}u, \mathcal{A}w) &= \left(\frac{0 + 1/9}{2} \right)^2 = \frac{1}{324}, \\ m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w) &= \left(\frac{0 + 0}{2} \right)^2 + \left(\frac{1 + 1/9}{2} \right)^2 = \frac{25}{81}, \end{aligned} \quad (31)$$

i.e., $m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \mu [m_v^b(u, \mathcal{A}u) + m_v^b(w, \mathcal{A}w)]$, $\mu = 1/16 < 1/2s$.

Thus, all the postulates of Theorem 18 are verified and $\mathcal{A}$ has a unique fixed point 0 in $\mathcal{M}$, and clearly, $m_v^b(0, 0) = 0$. Further, there exists a sequence $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$, $\mathcal{A}^n u_0 = 1/(3n^2 + 2n)$, $n \in \mathbb{N}$, which m_v^b -converges to 0.

Remark 24. Examples 19 and 20 demonstrate that Theorem 18 is an extension, generalization, and improvement of Asim et al. [10], Banach [21], Branciari [4], Kannan [22], Karahan and Isik [9], Mitrović and Radenović [7], Özgür et al. [8], Shukla [15, 16], and references therein to an M_v^b -metric space. It is fascinating to see that the continuity of a self-map is not an indispensable requirement for the survival of a unique fixed point satisfying (18) in this novel space.

Our next result extends the result of Reich [23] to M_v^b -metric space.

Theorem 25. Let $(\mathcal{M}, m_v^b)$ be an M_v^b -complete metric space with coefficient $s \geq 1$. Suppose a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ satisfies

$$\begin{aligned} m_v^b(\mathcal{A}u, \mathcal{A}w) &\leq \alpha m_v^b(u, \mathcal{A}u) + \beta m_v^b(w, \mathcal{A}w) + \gamma m_v^b(u, w), \alpha, \beta, \gamma \\ &\geq 0, \alpha + \beta + \gamma < \frac{1}{2s+1}, u, w \in \mathcal{M}. \end{aligned} \quad (32)$$

Then, $\mathcal{A}$ has a unique fixed point u^* so that $m_v^b(u^*, u^*) = 0$ and the sequence of iterates $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$ converges to $u^* \in \mathcal{M}$.

Proof. It follows the pattern of Theorem 18.

Theorem 25 is an extension, improvement, and generalization of Banach [21] ($\alpha = \beta = 0$), Kannan [22] ($\alpha = \beta, \gamma = 0$), Reich [23], and Asim et al. [10] ($\alpha = \beta = 0$) to an M_v^b -metric space.

Now, we prove the result for a Hardy and Rogers type contraction [24], which includes all the results stated above as a special case.

Theorem 26. Let $(\mathcal{M}, m_v^b)$ be an M_v^b -complete metric space. Suppose, a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ satisfies

$$\begin{aligned} m_v^b(\mathcal{A}u, \mathcal{A}w) &\leq \alpha m_v^b(u, \mathcal{A}w) + \beta m_v^b(w, \mathcal{A}u) + \gamma m_v^b(u, w) \\ &\quad + \delta m_v^b(u, \mathcal{A}u) + \kappa m_v^b(w, \mathcal{A}w), \alpha, \beta, \gamma, \delta, \kappa \\ &\geq 0, \alpha + \beta + \gamma + \delta + \kappa < \frac{1}{2s+1}, u, w \in \mathcal{M}. \end{aligned} \quad (33)$$

Then, $\mathcal{A}$ has a unique fixed point u^* so that $m_v^b(u^*, u^*) = 0$ and the sequence of iterates $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$ converges to $u^* \in \mathcal{M}$.

Proof. It follows the pattern of Theorem 18.

Theorem 26 is an extension, improvement, and generalization of Banach [21] ($\alpha = \beta = \delta = \kappa = 0$), Kannan [22] ($\alpha = \beta = \gamma = 0, \delta = \kappa$), Chatterjee [25] ($\gamma = \delta = \kappa = 0, \alpha = \beta$), Reich [23] ($\alpha = \beta = 0$), Hardy and Rogers [24] ($s = 1$), Asim et al. [10] ($\alpha = \beta = \delta = \kappa = 0$), and references therein to an M_v^b -metric space.

The following result is more fascinating as it is proved by altering the distances between the points, exploiting subadditive altering distance function [19].

Theorem 27. Let $(\mathcal{M}, m_v^b)$ be a M_v^b -complete metric space. Suppose, we have a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ satisfying

$$\begin{aligned} \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) &\leq \mu \left[\psi\left(m_v^b(u, w)\right) + \psi\left(m_v^b(u, \mathcal{A}u)\right) \right. \\ &\quad \left. + \psi\left(m_v^b(w, \mathcal{A}w)\right) \right], \quad \mu \\ &< \frac{1}{2s+1}, u, w \in \mathcal{M}. \end{aligned} \quad (34)$$

Then, $\mathcal{A}$ has a unique fixed point u^* so that $m_v^b(u^*, u^*) = 0$, and the sequence of iterates $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$ converges to $u^* \in \mathcal{M}$ and for $\kappa = 2\mu/(1-\mu) < 1$,

$$M_v^b(\mathcal{A}^{n+1} u^*, \mathcal{A}^n u^*) \leq \kappa^n M_v^b(u^*, \mathcal{A}u^*), \quad n = 0, 1, 2, \dots \quad (35)$$

Proof. For an arbitrary $u \in \mathcal{M}$, let $w = \mathcal{A}u$. Then,

$$\begin{aligned} \psi\left(m_v^b(w, \mathcal{A}w)\right) &= \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) \\ &\leq \mu \left[\psi\left(m_v^b(u, w)\right) + \psi\left(m_v^b(u, \mathcal{A}u)\right) \right. \\ &\quad \left. + \psi\left(m_v^b(w, \mathcal{A}w)\right) \right] \\ &= \mu \left[\psi\left(m_v^b(u, \mathcal{A}u)\right) + \psi\left(m_v^b(u, \mathcal{A}u)\right) \right. \\ &\quad \left. + \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) \right] (1-\mu) \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) \\ &\leq 2\mu \psi\left(m_v^b(u, \mathcal{A}u)\right) \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) \\ &\leq \frac{2\mu}{1-\mu} \psi\left(m_v^b(u, \mathcal{A}u)\right) \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) \\ &\leq \kappa \psi\left(m_v^b(u, w)\right), \quad \text{where } \kappa = \frac{2\mu}{1-\mu} \\ &< 1, \implies m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \kappa m_v^b(u, w). \end{aligned} \quad (36)$$

Now, starting from the given element $u_0 \in \mathcal{M}$, form the sequence $\{u_n\}$, where $u_n = \mathcal{A}u_{n-1}, n \in \mathbb{N}$. If $m_v^b(u_n, u_{n+1}) = 0, n \geq 0$, then $\mathcal{A}u_n = u_{n+1} = u_n$ and $m_v^b(u_n, u_n) = 0$, and this completes the proof.

Further, take $m_v^b(u_n, u_{n+1}) > 0, n \geq 0$. For $u = u_n, w = u_{n+1}$, utilizing condition (36),

$$\begin{aligned} m_v^b(u_{n+1}, u_{n+2}) &= m_v^b(\mathcal{A}u_n, \mathcal{A}u_{n+1}) \leq \kappa m_v^b(u_n, u_{n+1}) \\ &\leq \kappa^n m_v^b(u_0, u_1) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \quad (37)$$

Also,

$$\begin{aligned} m_v^b(\mathbf{u}_{n+1}, \mathbf{u}_{n+1}) &= m_v^b(\mathcal{A}\mathbf{u}_n, \mathcal{A}\mathbf{u}_n) \leq \kappa m_v^b(\mathbf{u}_n, \mathbf{u}_n) \\ &\leq \kappa^n m_v^b(\mathbf{u}_0, \mathbf{u}_0) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \quad (38)$$

So, following a similar pattern as that of Theorem 18, we may easily conclude that $\mathcal{A}$ has a unique fixed point $\mathbf{u} \in \mathcal{M}$, where $m_v^b(\mathbf{u}, \mathbf{u}) = 0$.

Finally, by taking $\mathbf{u} = \mathcal{A}^{n-1}\mathbf{u}$, from (36),

$$\begin{aligned} m_v^b(\mathcal{A}^n \mathbf{u}, \mathcal{A}^{n+1} \mathbf{u}) &\leq \kappa m_v^b(\mathcal{A}^{n-1} \mathbf{u}, \mathcal{A}^n \mathbf{u}) \leq \kappa^2 m_v^b(\mathcal{A}^{n-2} \mathbf{u}, \mathcal{A}^{n-1} \mathbf{u}) \dots \\ &\leq \kappa^n m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u}), \quad n = 1, 2, 3, \dots \end{aligned} \quad (39)$$

Now, we furnish two examples of a discontinuous self-map to validate Theorem 27.

Example 1. Let $\mathcal{M} = \mathbb{R}$ and an M_v^b -metric $m_v^b: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as $m_v^b(\mathbf{u}, \mathbf{w}) = \max\{|\mathbf{u}|^2, |\mathbf{w}|^2\} + |\mathbf{u} - \mathbf{w}|^2$, $\mathbf{u}, \mathbf{w} \in \mathcal{M}$ with $s = 3$.

Define a self-map $\mathcal{A}: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$ by

$$\mathcal{A}\mathbf{u} = \begin{cases} \frac{\mathbf{u}}{9}, & \mathbf{u} \in [-9, 9], \\ \frac{3\mathbf{u}}{5}, & \text{otherwise,} \end{cases} \quad (40)$$

and $\psi: [0, \infty) \longrightarrow [0, \infty)$ as $\psi(\mathbf{u}) = \mathbf{u}/7$, then observe that, for all $\mathbf{u}, \mathbf{w} \in \mathcal{M}$, we obtain

$$\begin{aligned} \psi(m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w})) &= \psi(\max\{|\mathcal{A}\mathbf{u}|^2, |\mathcal{A}\mathbf{w}|^2\} + |\mathcal{A}\mathbf{u} - \mathcal{A}\mathbf{w}|^2) \\ &\leq \psi\left(\frac{9}{25} \max\{|\mathbf{u}|^2, |\mathbf{w}|^2\} + |\mathbf{u} - \mathbf{w}|^2\right) \\ &< \frac{9}{25} \left[\psi(m_v^b(\mathbf{u}, \mathbf{w})) + \psi(m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u})) \right. \\ &\quad \left. + \psi(m_v^b(\mathbf{w}, \mathcal{A}\mathbf{w})) \right]. \end{aligned} \quad (41)$$

Thus, all the postulates of Theorem 27 are verified and $\mathcal{A}$ has a unique fixed point 0 in $\mathcal{M}$, and clearly, $m_v^b(0, 0) = 0$. Further, there exists a sequence $\{\mathcal{A}^n \mathbf{u}_0\} \subseteq \mathcal{M}$, $\mathcal{A}^n \mathbf{u}_0 = n/(n^2 + 1)$, $n \in \mathbb{N}$, which m_v^b -converges to 0 .

Example 2. Let $\mathcal{M} = [-2, 2]$ and an M_v^b -metric $m_v^b: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}^+$ be defined as

$$m_v^b(\mathbf{u}, \mathbf{w}) = \left(\frac{|\mathbf{u}| + |\mathbf{w}|}{2} \right)^2, \quad \mathbf{u}, \mathbf{w} \in \mathcal{M} \text{ with } s = 3. \quad (42)$$

Define a self-map $\mathcal{A}: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$ by

$$\mathcal{A}\mathbf{u} = \begin{cases} 0, & \mathbf{u} \in [-2, 0], \\ -1, & \text{otherwise,} \end{cases} \quad (43)$$

and $\psi: [0, \infty) \longrightarrow [0, \infty)$ as $\psi(\mathbf{u}) = \mathbf{u}/2$, then observe that, for all $\mathbf{u}, \mathbf{w} \in \mathcal{M}$, we obtain

$$\begin{aligned} &\psi(m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w})) \\ &< \mu \left[\psi(m_v^b(\mathbf{u}, \mathbf{w})) + \psi(m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u})) \right. \\ &\quad \left. + \psi(m_v^b(\mathbf{w}, \mathcal{A}\mathbf{w})) \right], \quad \mu = \frac{2}{11} < \frac{1}{2s+1}. \end{aligned} \quad (44)$$

Thus, all the postulates of Theorem 27 are verified and $\mathcal{A}$ has a unique fixed point 0 in $\mathcal{M}$, and clearly, $m_v^b(0, 0) = 0$. Further, there exists a sequence $\{\mathcal{A}^n \mathbf{u}_0\} \subseteq \mathcal{M}$, $\mathcal{A}^n \mathbf{u}_0 = -3n/(n^2 + n)$, $n \in \mathbb{N}$, which m_v^b -converges to 0 .

Remark 28. Examples 1 and 2 demonstrate that Theorem 27 is an extension, generalization, and improvement of Asim et al. [10], Banach [21], Kannan [22], Hardy and Rogers [24], Karahan and Isik [9], Khan et al. [19], Mlaiki et al. [18], Mitrović and Radenović [7], Reich [23], and references therein to M_v^b -metric space. It is fascinating to see that continuity of a self-map is not indispensable for the existence of a fixed point satisfying (34) in this novel space.

Corollary 29. *The conclusion of Theorem 27 is true even if we replace (34) by*

$$\begin{aligned} m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w}) &\leq \mu \left[m_v^b(\mathbf{u}, \mathbf{w}) + m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u}) + m_v^b(\mathbf{w}, \mathcal{A}\mathbf{w}) \right], \\ \mu &< \frac{1}{2s+1}, \quad \mathbf{u}, \mathbf{w} \in \mathcal{M}. \end{aligned} \quad (45)$$

Proof. The proof follows, if we take, $\psi(\mathbf{u}) = \mathbf{u}$ in (34).

Corollary 30. *The conclusion of Theorem 27 is true even if we replace (34) by*

$$\begin{aligned} \psi(m_v^b(\mathcal{A}^j \mathbf{u}, \mathcal{A}^j \mathbf{w})) &\leq \mu \left[\psi(m_v^b(\mathbf{u}, \mathbf{w})) + \psi(m_v^b(\mathbf{u}, \mathcal{A}^j \mathbf{u})) \right. \\ &\quad \left. + \psi(m_v^b(\mathbf{w}, \mathcal{A}^j \mathbf{w})) \right], \\ \mu &< \frac{1}{2s+1}, \quad \mathbf{u}, \mathbf{w} \in \mathcal{M}. \end{aligned} \quad (46)$$

Proof. Applying Theorem 27 to the self-map $\mathcal{T} = \mathcal{A}^j$, we get $\mathcal{T}$ has a unique fixed point, say $\mathbf{u}$, i.e., $\mathcal{T}\mathbf{u} = \mathcal{A}^j \mathbf{u} = \mathbf{u}$.

Now,

$$\mathcal{T}\mathcal{A}\mathbf{u} = \mathcal{A}^j \mathcal{A}\mathbf{u} = \mathcal{A}^{j+1} \mathbf{u} = \mathcal{A}^j \mathbf{u} = \mathcal{A}\mathbf{u}, \quad (47)$$

so $\mathcal{A}\mathbf{u}$ is a unique fixed point of $\mathcal{T}$.

Corollary 31. *The conclusion of Theorem 27 remains true even if we replace (34) by*

$$\begin{aligned} \left(1 + m_v^b(\mathcal{A}u, \mathcal{A}w)\right)^{1/\mu} &\leq \left(1 + m_v^b(u, w)\right) \left(1 + m_v^b(u, \mathcal{A}u)\right) \\ &\quad \cdot \left(1 + m_v^b(w, \mathcal{A}w)\right), \\ \mu &< \frac{1}{2s+1}, \quad u, w \in \mathcal{M}. \end{aligned} \quad (48)$$

Proof. The proof follows the pattern of Theorem 27 if we take $\psi(u) = \ln(1+u)$ in (34).

Theorem 32. *Let $(\mathcal{M}, m_v^b)$ be an M_v^b -complete metric space with coefficient $s\beta < 1$. Suppose $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ satisfies*

$$\begin{aligned} \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) &\leq \alpha\psi\left(m_v^b(u, \mathcal{A}u)\right) + \beta\psi\left(m_v^b(w, \mathcal{A}w)\right) \\ &\quad + \gamma\psi\left(m_v^b(u, w)\right), \quad \alpha + \beta + \gamma \\ &< 1, \quad u, w \in \mathcal{M}. \end{aligned} \quad (49)$$

Then, $\mathcal{A}$ has a unique fixed point u^ so that $m_v^b(u^*, u^*) = 0$, and the sequence of iterates $\{\mathcal{A}^n u_0\} \subseteq \mathcal{M}$ converges to $u^* \in \mathcal{M}$ and for $\delta = (\alpha + \beta)/(1 - \gamma)$,*

$$m_v^b(\mathcal{A}^{n+1}u, \mathcal{A}^n u) \leq \delta^n m_v^b(u, \mathcal{A}u), \quad n = 0, 1, 2, \dots \quad (50)$$

Proof. The proof follows the pattern of Theorem 27.

Theorem 33. *The conclusion of Theorem 27 remains true even if we replace inequality (34) by*

$$\begin{aligned} \psi\left(m_v^b(\mathcal{A}u, \mathcal{A}w)\right) &\leq \mu \left[\psi\left(m_v^b(w, \mathcal{A}w)\right) + \psi\left(m_v^b(u, \mathcal{A}u)\right) \right], \\ \mu &< \frac{1}{2s}, \quad u, w \in \mathcal{M}. \end{aligned} \quad (51)$$

Proof. The proof follows the pattern of Theorem 27.

We note that when $\psi(u) = u$, the above theorem is the same as Theorem 18.

4. Existence of a Fixed Circle/Disc

Following Özgür and Tas [26], we first familiarize fixed circle in an M_v^b -metric space to establish fixed circle theorems. Next, we exploit m_v^b -metric version (m_v^b -Caristi map) of the classical Caristi map [27] to establish that the set of non-unique fixed points of a map includes a circle. Also, following Aydi et al. [28], we familiarize fixed disc to establish the greatest fixed disc in an M_v^b -metric space.

Definition 34. We define a circle centred at u_0 and radius r in an M_v^b -metric space $(\mathcal{M}, m_v^b)$ as

$$\mathcal{C}(u_0, r) = \left\{ u \in \mathcal{M} : m_v^b(u_0, u) = r + m_v^b(u_0, u_0), \quad u_0 \in \mathcal{M}, r \in [0, \infty) \right\}. \quad (52)$$

Definition 35. We define a disc centred at u_0 and radius r in an M_v^b -metric space $(\mathcal{M}, m_v^b)$ as

$$\mathcal{D}(u_0, r) = \left\{ u \in \mathcal{M} : m_v^b(u_0, u) \leq r + m_v^b(u_0, u_0), \quad u_0 \in \mathcal{M}, r \in [0, \infty) \right\}. \quad (53)$$

Geometrically, it is not necessary that a circle/disc defined in a M_v^b -metric space is the same as the circle/disc in a Euclidean space.

Definition 36. Let $\mathcal{C}(u_0, r) / \mathcal{D}(u_0, r)$ be a circle/disc centred at u_0 and radius r in an M_v^b -metric space $(\mathcal{M}, m_v^b)$. For a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ in an M_v^b -metric space $(\mathcal{M}, m_v^b)$, if $m_v^b(u, \mathcal{A}u) = m_v^b(u, u) = m_v^b(\mathcal{A}u, \mathcal{A}u)$, $u \in \mathcal{C}(u_0, r) / \mathcal{D}(u_0, r)$, then $\mathcal{C}(u_0, r) / \mathcal{D}(u_0, r)$ is called the fixed circle/fixed disc of $\mathcal{A}$.

Theorem 37. *Let $\mathcal{C}(u_0, r)$ be a circle in an M_v^b -metric space $(\mathcal{M}, m_v^b)$. Define $\varsigma : \mathcal{M} \rightarrow [0, \infty)$ as*

$$\varsigma(u) = m_v^b(u, u_0) + m_v^b(u, u), \quad u \in \mathcal{M}. \quad (54)$$

If there exists a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ so that

$$\begin{aligned} (i) \quad &m_v^b(u, \mathcal{A}u) \leq \varsigma(\mathcal{A}u) - \varsigma(u) + m_v^b(u, u), \\ (ii) \quad &m_v^b(\mathcal{A}u, u_0) \leq r + m_v^b(u_0, u_0), \\ (iii) \quad &m_v^b(\mathcal{A}u, \mathcal{A}u) \leq m_v^b(u, u), \end{aligned} \quad (55)$$

(iv) If $m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \mu m_v^b(u, w)$, $u \in \mathcal{C}(u_0, r)$, $w \in \mathcal{M} \setminus \mathcal{C}(u_0, r)$, $\mu \in [0, 1)$, then $\mathcal{C}(u_0, r)$ is a unique fixed circle of $\mathcal{M}$

Proof. Let $u \in \mathcal{C}(u_0, r)$ be any arbitrary point. Using (i) and equation (54),

$$\begin{aligned} m_v^b(u, \mathcal{A}u) &\leq m_v^b(\mathcal{A}u, u_0) + m_v^b(\mathcal{A}u, \mathcal{A}u) - m_v^b(u, u_0) \\ &\quad - m_v^b(u, u) + m_v^b(u, u) \\ &= m_v^b(\mathcal{A}u, u_0) + m_v^b(\mathcal{A}u, \mathcal{A}u) - m_v^b(u, u_0) \\ &\leq r + m_v^b(u_0, u_0) + m_v^b(\mathcal{A}u, \mathcal{A}u) - r \\ &\quad - m_v^b(u_0, u_0) \text{ (using (ii))}, \end{aligned} \quad (56)$$

i.e., $m_v^b(u, \mathcal{A}u) \leq m_v^b(\mathcal{A}u, \mathcal{A}u)$.

But by the definition of an M_v^b -metric space, $m_v^b(\mathcal{A}u, \mathcal{A}u) \leq m_v^b(u, \mathcal{A}u)$. So, $m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u)$. Using (iii) $m_v^b(u, \mathcal{A}u) \leq m_v^b(u, u)$. Again, by the definition of an M_v^b -metric space, $m_v^b(u, u) \leq m_v^b(u, \mathcal{A}u)$. Hence,

$$m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u) = m_v^b(u, u), \quad (57)$$

i.e., u is a fixed point of $\mathcal{A}$, $\forall u \in \mathcal{C}(u_0, r)$. So, the self-map $\mathcal{A}$ fixes the circle $\mathcal{C}(u_0, r)$, i.e., the set of nonunique fixed points of a map $\mathcal{A}$ includes a circle.

Let there exist two fixed circles, $\mathcal{C}(u_0, r_0)$ and $\mathcal{C}(u_1, r_1)$, $r \neq r_1$ of $\mathcal{A}$, i.e., $\mathcal{A}$ satisfies all the conditions (i) to (iii) for each of the circles $\mathcal{C}(u_0, r_0)$ and $\mathcal{C}(u_1, r_1)$. Let $u \in \mathcal{C}(u_0, r_0)$ and $w \in \mathcal{C}(u_1, r_1)$. Using (iv), $m_v^b(u, w) = m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \mu m_v^b(u, w)$, a contradiction.

Hence, $\mathcal{C}(u_0, r_0)$ is a unique fixed circle of $\mathcal{A}$.

The following example illustrates Theorem 37.

Example 38. Let $\mathcal{M} = \mathbb{R}$ and an M_v^b -metric $m_v^b : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+$ be defined as $m_v^b(u, w) = \max\{|u|^\alpha, |w|^\alpha\} + |u - w|^\alpha$, $\alpha > 1$, $u, w \in \mathcal{M}$ with $s = 3^{\alpha-1}$. The circle

$$\begin{aligned} \mathcal{C}(-2, 6) &= \{u \in \mathcal{M} : m_v^b(-2, u) = 6 + m_v^b(-2, -2)\} \\ &= \{u \in \mathcal{M} : \max\{|-2|^\alpha, |u|^\alpha\} + |-2 - u|^\alpha \\ &= 6 + |-2|^\alpha\} = \{u \in \mathcal{M} : \max\{2^\alpha, |u|^\alpha\} + |2 + u|^\alpha = 6 + 2^\alpha\}. \end{aligned} \quad (58)$$

Now, we discuss circles for different values of α :

(i) When $\alpha = 2$, $\max\{4, |u|^2\} + (2 + u)^2 = 6 + 4$, then

$$\begin{aligned} \text{either } 4 + (2 + u)^2 &= 10 \text{ or } u^2 + (2 + u)^2 = 10, \\ (2 + u)^2 &= 6 \text{ or } u^2 + 4 + u^2 + 4u = 10, \\ 2 + u &= \pm\sqrt{6} \text{ or } 2u^2 + 4u - 6 = 0, \\ u &= -2 \pm \sqrt{6} \text{ or } u = -3, u = 1 \end{aligned} \quad (59)$$

(ii) When $\alpha = 3$, $\max\{4, |u|^3\} + (2 + u)^3 = 6 + 8$, then

$$\begin{aligned} \text{either } 8 + |2 + u|^3 &= 14 \text{ or } |u|^3 + |2 + u|^3 = 14 \pm (2 + u)^3 \\ &= 6 \text{ or } \pm u^3 \pm (2 + u)^3 = 14, \end{aligned}$$

$$\begin{aligned} 2 + u &= \pm(6)^{1/3} \text{ or } u = -2.40629, 0.6097, -1 \pm \sqrt{2}, \\ u &= -2 \pm 6^{1/3} \text{ or } u = -2.40629, 0.6097, -1 \pm \sqrt{2} \end{aligned} \quad (60)$$

Now, define a self-map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ as

$$\mathcal{A}u = \begin{cases} u, & u \in \mathcal{C}(-2, 6), \\ -1 + \sqrt{2}, & u \notin \mathcal{C}(-2, 6). \end{cases} \quad (61)$$

Then, the self-map $\mathcal{A}$ verifies all the postulates of Theorem 37 except (iv) and fixes the circle $\mathcal{C}(-2, 6)$. Clearly, the set of fixed points of $\mathcal{A}$ contains a circle $\mathcal{C}(-2, 6)$. However, one may notice that there are more than one circle corresponding to each value of α . For instance, $\mathcal{C}(2, 6)$ is also a fixed circle of $\mathcal{A}$.

It is obvious that geometrically (i) implies that $\mathcal{A}u$ is in the exterior of a circle and (ii) implies that $\mathcal{A}u$ is in the interior of a circle. It means $\mathcal{A}(\mathcal{C}(u_0, r)) \subseteq \mathcal{C}(u_0, r)$ (see Example 38).

Theorem 39. The conclusion of Theorem 37 is true even if we substitute (i) by (i)' and (ii) by (ii)':

$$\begin{aligned} (i)' \quad m_v^b(u, \mathcal{A}u) &\leq \varsigma(u) - \varsigma(\mathcal{A}u) + m_v^b(\mathcal{A}u, \mathcal{A}u), \\ (ii)' \quad m_v^b(\mathcal{A}u, u_0) &\geq r + m_v^b(u_0, u_0). \end{aligned} \quad (62)$$

Proof. Let $u \in \mathcal{C}(u_0, r)$ be any arbitrary point. Using (i)' and equation (54)

$$\begin{aligned} m_v^b(u, \mathcal{A}u) &\leq m_v^b(u, u_0) + m_v^b(u, u) - m_v^b(\mathcal{A}u, u_0) \\ &\quad - m_v^b(\mathcal{A}u, \mathcal{A}u) + m_v^b(\mathcal{A}u, \mathcal{A}u) = r + m_v^b(u_0, u_0) \\ &\quad + m_v^b(u, u) - m_v^b(\mathcal{A}u, u_0) \leq r + m_v^b(u_0, u_0) + m_v^b(u, u) \\ &\quad - r - m_v^b(u_0, u_0) \quad (\text{using (ii)'}), \end{aligned} \quad (63)$$

$$\text{i.e., } m_v^b(u, \mathcal{A}u) \leq m_v^b(u, u). \quad (64)$$

But by the definition of an M_v^b -metric space, $m_v^b(u, u) \leq m_v^b(u, \mathcal{A}u)$.

So, $m_v^b(u, \mathcal{A}u) = m_v^b(u, u)$. Using (63) and (iii), $m_v^b(u, \mathcal{A}u) \leq m_v^b(\mathcal{A}u, \mathcal{A}u)$. Also, by definition of an M_v^b -metric $m_v^b(\mathcal{A}u, \mathcal{A}u) \leq m_v^b(u, \mathcal{A}u)$. So, $m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u)$, i.e., $m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u) = m_v^b(u, u) \implies u$ is a fixed point of $\mathcal{A}$, $\forall u \in \mathcal{C}(u_0, r)$, i.e., the self-map $\mathcal{A}$ fixes the circle $\mathcal{C}(u_0, r)$, i.e., the set of fixed points of a map $\mathcal{A}$ includes a circle.

Uniqueness of a fixed circle may be proved as in Theorem 37.

Example 40. Let $\mathcal{M} = \mathbb{R}$ and an M_v^b -metric $m_v^b : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+$ be defined as $m_v^b(u, w) = (|u| + |w|/2)^2 + |u - w|^2$, $u, w \in \mathcal{M}$ with $s = 4$:

$$\begin{aligned} \mathcal{C}(-4, 11) &= \{u \in \mathcal{M} : m_v^b(-4, u) = 11 + m_v^b(-4, -4)\} \\ &= \left\{u \in \mathcal{M} : \left(\frac{4 + |u|}{2}\right)^2 + |-4 - u|^2 = 11 + \left(\frac{4 + 4}{2}\right)^2\right\}, \end{aligned}$$

$$\begin{aligned} &\implies \left(\frac{4+|u|}{2}\right)^2 + |4+u|^2 = 11+16, \\ &\implies (4+|u|)^2 + 4(4+u)^2 = 108. \end{aligned} \quad (65)$$

There arises two cases:

- (i) $(4+u)^2 + 4(4+u)^2 = 108, (4+u)^2 = 108/5 = 21.6$
 $, 4+u = \pm 4.65 \implies u = 0.65, u = -8.65$
- (ii) $(4-u)^2 + 4(4+u)^2 = 108, 4+u^2 - 8u + 4(16+u^2 + 8u) = 108, 5u^2 + 24u + 68 = 108$
 $, 5u^2 + 24u - 40 = 0 \implies u = -6.1, u = -1.3$

Define a self-map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ as

$$\mathcal{A}u = \begin{cases} u, & u \in \mathcal{C}(-4, 11), \\ 0.65, & u \notin \mathcal{C}(-4, 11). \end{cases} \quad (66)$$

Then, map $\mathcal{A}$ verifies all the postulates of Theorem 39 and fixes the unique circle $\mathcal{C}(-4, 11)$, i.e., the set of non-unique fixed points of $\mathcal{A} = \{-8.65, -6.1, -1.3, 0.65\}$ contains a unique circle $\mathcal{C}(-4, 11)$.

It is clear that geometrically (i)' implies that $\mathcal{A}u$ is in the interior of a circle and (ii)' implies that $\mathcal{A}u$ is in the exterior of a circle. It means $\mathcal{A}(\mathcal{C}(u_0, r)) \subseteq \mathcal{C}(u_0, r)$ (see Example 40).

In the above theorems, we have assumed Banach contraction to demonstrate the uniqueness of fixed circle. So, it is significant to establish the uniqueness of the fixed circle using different contractive conditions. In the following, we establish uniqueness using a more general contractive condition.

Theorem 41. Let $(\mathcal{M}, m_v^b)$ be an M_v^b -metric space and $\mathcal{C}(u_0, r)$ be a circle on $\mathcal{M}$. Let $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ be a self-map satisfying (i)-(iii), (i)' and (ii)' of Theorem 37 or 39 along with the contraction condition

$$m_v^b(\mathcal{M}u, \mathcal{M}w) \leq \mu \max \left\{ m_v^b(u, w), m_v^b(u, \mathcal{A}w), m_v^b(w, \mathcal{M}u), \frac{1}{2} \left(m_v^b(u, \mathcal{M}u) + m_v^b(w, \mathcal{M}w) \right), \frac{1}{2} \left(m_v^b(u, \mathcal{M}w) + m_v^b(w, \mathcal{M}u) \right) \right\}, \quad (67)$$

$u \in \mathcal{C}(u_0, r), w \in \mathcal{M} \setminus \mathcal{C}(u_0, r)$, where $\varsigma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous nondecreasing function and $\varsigma(t) < t$, then $\mathcal{C}(u_0, r)$ is a unique fixed circle of $\mathcal{M}$.

Proof. Let $\mathcal{C}(u_0, r_0)$ and $\mathcal{C}(u_1, r_1)$ be two fixed circles of $\mathcal{M}$, i.e., $\mathcal{M}$ satisfies the conditions (i)-(iii), (i)' and (ii)' (Theorem 37 and 39) for both the circles $\mathcal{C}(u_0, r_0)$ and $\mathcal{C}(u_1, r_1)$. Let $u \in \mathcal{C}(u_0, r_0)$ and $w \in \mathcal{C}(u_1, r_1)$. Using inequality (67),

$$\begin{aligned} m_v^b(u, w) &= m_v^b(\mathcal{M}u, \mathcal{M}w) \leq \mu \left(m_v^b(u, w), m_v^b(u, w), m_v^b(u, w), \frac{1}{2} \right. \\ &\quad \cdot \left. \left(m_v^b(u, w) + m_v^b(u, w) \right), \frac{1}{2} \left(m_v^b(u, w) + m_v^b(u, w) \right) \right) \\ &\leq \mu \left(m_v^b(u, w) \right) < m_v^b(u, w), \end{aligned} \quad (68)$$

a contradiction. Hence, $\mathcal{C}(u_0, r_0)$ is a unique fixed circle of $\mathcal{M}$.

Next, we establish the existence of a greatest fixed disc.

Theorem 42. Let $\mathcal{D}(u_0, r)$ be a disc in an M_v^b -metric space $(\mathcal{M}, m_v^b)$. Define $\varsigma : \mathcal{M} \longrightarrow [0, \infty)$ as in (54). If there exists a self-map $\mathcal{A} : \mathcal{M} \longrightarrow \mathcal{M}$ so that

$$\begin{aligned} m_v^b(\mathcal{A}u, u_0) &\leq r + m_v^b(u_0, u_0), \\ m_v^b(\mathcal{A}u, \mathcal{A}u) &\leq m_v^b(u, u), \end{aligned} \quad (69)$$

- (i) $m_v^b(u, \mathcal{A}u) \leq \varsigma(\mathcal{A}u) + \varsigma(\mathcal{A}u) - 2r - 2m_v^b(u_0, u_0)$,
then $\mathcal{D}(u_0, r)$ is a fixed disc of $\mathcal{M}$

- (ii) If $m_v^b(\mathcal{A}u, \mathcal{A}w) \leq \mu m_v^b(u, w), u \in \mathcal{D}(u_0, r), w \in \mathcal{M} \setminus \mathcal{D}(u_0, r), \mu \in [0, 1)$, then $\mathcal{D}(u_0, r)$ is a fixed disc of maximum radius r , i.e., there is no fixed disc $\mathcal{D}(u_0, r')$ of $\mathcal{A}$ having a radius greater than r

Proof. Let $u \in \mathcal{D}(u_0, r)$ be any arbitrary point:

$$\begin{aligned} m_v^b(u, \mathcal{A}u) &\leq m_v^b(u, u_0) + m_v^b(u, u) + m_v^b(\mathcal{A}u, u_0) \\ &\quad + m_v^b(\mathcal{A}u, \mathcal{A}u) - 2r - 2m_v^b(u_0, u_0) \\ &\leq r + m_v^b(u_0, u_0) + m_v^b(u, u) + r + m_v^b(u_0, u_0) \\ &\quad - 2r - 2m_v^b(u_0, u_0) = m_v^b(u, u), \end{aligned} \quad (70)$$

$$\text{i.e., } m_v^b(u, \mathcal{A}u) \leq m_v^b(u, u). \quad (71)$$

But by the definition of an M_v^b -metric space, $m_v^b(u, u) \leq m_v^b(u, \mathcal{A}u)$.

So, $m_v^b(u, \mathcal{A}u) = m_v^b(u, u)$. Using (70) and (iii), $m_v^b(u, \mathcal{A}u) \leq m_v^b(\mathcal{A}u, \mathcal{A}u)$. Also, by definition of an M_v^b -metric $m_v^b(\mathcal{A}u, \mathcal{A}u) \leq m_v^b(u, \mathcal{A}u)$. So, $m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u)$, i.e., $m_v^b(u, \mathcal{A}u) = m_v^b(\mathcal{A}u, \mathcal{A}u) = m_v^b(u, u) \implies u$ is a fixed point of $\mathcal{A}, \forall u \in \mathcal{D}(u_0, r)$, i.e., the self-map $\mathcal{A}$ fixes the disc

$\mathcal{D}(\mathbf{u}_0, \mathbf{r})$, i.e., the set of fixed points of a map $\mathcal{A}$ includes a disc.

Let there exist two fixed discs $\mathcal{D}(\mathbf{u}_0, \mathbf{r})$ and $\mathcal{D}(\mathbf{u}_1, \mathbf{r}')$, $\mathbf{r} < \mathbf{r}'$, of $\mathcal{A}$, i.e., $\mathcal{A}$ satisfies the conditions (i) to (iii) for each of the discs $\mathcal{D}(\mathbf{u}_0, \mathbf{r})$ and $\mathcal{D}(\mathbf{u}_1, \mathbf{r}')$.

Let $\mathbf{u} \in \mathcal{D}(\mathbf{u}_0, \mathbf{r})$ and $\mathbf{w} \in \mathcal{D}(\mathbf{u}_1, \mathbf{r}')$ such that $\mathbf{w} \notin \mathcal{D}(\mathbf{u}_0, \mathbf{r})$.

Using (iv), $m_v^b(\mathbf{u}, \mathbf{w}) = m_v^b(\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{w}) \leq \mu m_v^b(\mathbf{u}, \mathbf{w})$, a contradiction. Hence, $\mathcal{D}(\mathbf{u}_0, \mathbf{r}_0)$ is a fixed disc of $\mathcal{A}$ having maximum a radius $\mathbf{r}$.

Remark 43. Following the pattern of Theorem 41, we may prove the existence of a greatest fixed disc of $\mathcal{A}$ having a maximum radius.

Remark 44. For more work on the set of nonunique fixed points forming a circle or a disc or an ellipse, one may refer to [26, 28–31] and references therein.

5. Application to Cantilever Beam

As an application of Theorem 18, we solve fourth-order differential equation arising in two point boundary value problem of a bending of an elastic beam. This problem is employed in the distortion of an elastic beam in equilibrium, of which one end is free whereas another is fixed. Let $\mathcal{J} = [0, 1]$ and $\mathcal{M} = C[\mathcal{J}, \mathbb{R}]$ be the set of all continuous functions on $[0, 1]$. Define m_v^b -metric, $m_v^b : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ as $m_v^b(\mathbf{u}, \mathbf{w}) = (|\mathbf{u}| + |\mathbf{w}|/2)^2$, $\mathbf{u}, \mathbf{w} \in \mathcal{M}$ and $s = 4$.

Theorem 45. We consider the Cantilever beam problem

$$\frac{d^4 u}{dt^4} = \psi\left(t, u, \frac{du}{dt}, \frac{d^2 u}{dt^2}, \frac{d^3 u}{dt^3}\right), \quad (72)$$

$$u(0) = \frac{du}{dt}(0) = \frac{d^2 u}{dt^2}(1) = \frac{d^3 u}{dt^3}(1) = 0, \quad t \in [0, 1],$$

where $\psi : [0, 1] \times \mathbb{R}^4 \rightarrow [0, 1]$ is a continuous function. If

$$\left|\psi\left(\xi, u, \frac{du}{dt}\right)\right| < \sqrt{\mu}|u(\xi)|, \quad \xi \in [0, 1], \mu \in \left(0, \frac{1}{8}\right), \quad (73)$$

then Problem (72) has a unique solution.

Proof. The Cantilever beam problem (72) may be rewritten as

$$\mathbf{u}(t) = \int_0^1 \mathcal{G}(t, \xi) \psi\left(\xi, \mathbf{u}(\xi), \frac{d\mathbf{u}}{dt}(\xi)\right), \quad \mathbf{u} \in \mathcal{M}, \quad (74)$$

where the Green function

$$\mathcal{G}(t, \xi) = \begin{cases} \frac{1}{6} \xi^2 (3t - \xi), & 0 \leq \xi \leq t \leq 1, \\ \frac{1}{6} t^2 (3\xi - t), & 0 \leq t \leq \xi \leq 1. \end{cases} \quad (75)$$

Define a map $\mathcal{A} : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\mathcal{A}\mathbf{u}(t) = \int_0^1 \mathcal{G}(t, \xi) \psi\left(\xi, \mathbf{u}(\xi), \frac{d\mathbf{u}}{dt}(\xi)\right) d\xi. \quad (76)$$

Now, $\mathbf{u} \in \mathcal{M}$ is a solution of (72) iff it is a fixed point of $\mathcal{A}$:

$$\begin{aligned} m_v^b(\mathcal{A}\mathbf{u}(t), \mathcal{A}\mathbf{w}(t)) &= \left(\frac{|\mathcal{A}\mathbf{u}(t)| + |\mathcal{A}\mathbf{w}(t)|}{2}\right)^2 \\ &= \frac{1}{4} \left(\left| \int_0^1 \mathcal{G}(t, \xi) \psi\left(\xi, \mathbf{u}(\xi), \frac{d\mathbf{u}}{dt}(\xi)\right) d\xi \right| \right. \\ &\quad \left. + \left| \int_0^1 \mathcal{G}(t, \xi) \psi\left(\xi, \mathbf{w}(\xi), \frac{d\mathbf{w}}{dt}(\xi)\right) d\xi \right| \right)^2 \\ &\leq \frac{1}{4} \left(\sqrt{\mu}|\mathbf{u}(t)| \left| \int_0^1 \mathcal{G}(t, \xi) d\xi \right| + \sqrt{\mu}|\mathbf{w}(t)| \left| \int_0^1 \mathcal{G}(t, \xi) d\xi \right| \right)^2 \\ &= \frac{\mu}{4} (|\mathbf{u}(t)| + |\mathbf{w}(t)|)^2 \left| \int_0^1 \mathcal{G}(t, \xi) d\xi \right|^2 \\ &= \frac{\mu}{4} (|\mathbf{u}(t)| + |\mathbf{w}(t)|)^2 \left| \int_0^t \frac{1}{6} \xi^2 (3t - \xi) d\xi + \int_t^1 \frac{1}{6} t^2 (3\xi - t) d\xi \right|^2 \\ &= \frac{\mu}{4} (|\mathbf{u}(t)| + |\mathbf{w}(t)|)^2 \left| \frac{1}{8} t^4 + \frac{1}{12} (3t^2 - 2t^3 - t^4) \right|^2 \\ &\leq \frac{\mu}{4} (|\mathbf{u}(t)| + |\mathbf{w}(t)|)^2 \left(\frac{1}{8} + \frac{6}{12} \right)^2 = \frac{25\mu}{256} (|\mathbf{u}(t)| + |\mathbf{w}(t)|)^2, \end{aligned} \quad (77)$$

$$\begin{aligned} m_v^b(\mathbf{u}, \mathcal{A}\mathbf{u}(t)) + m_v^b(\mathbf{w}, \mathcal{A}\mathbf{w}(t)) &= \left(\frac{|\mathbf{u}(t)| + |\mathcal{A}\mathbf{u}(t)|}{2}\right)^2 + \left(\frac{|\mathbf{w}(t)| + |\mathcal{A}\mathbf{w}(t)|}{2}\right)^2 \\ &= \left(\frac{|\mathbf{u}(t)| + \left| \int_0^1 \mathcal{G}(t, \xi) \psi(\xi, \mathbf{u}(\xi), \frac{d\mathbf{u}}{dt}(\xi)) d\xi \right|}{2}\right)^2 \\ &\quad + \left(\frac{|\mathbf{w}(t)| + \left| \int_0^1 \mathcal{G}(t, \xi) \psi(\xi, \mathbf{w}(\xi), \frac{d\mathbf{w}}{dt}(\xi)) d\xi \right|}{2}\right)^2 \\ &= \left(\frac{|\mathbf{u}(t)| + \sqrt{\mu}|\mathbf{u}(t)| \left| \int_0^1 \mathcal{G}(t, \xi) d\xi \right|}{2}\right)^2 \\ &\quad + \left(\frac{|\mathbf{w}(t)| + \mu|\mathbf{w}(t)| \left| \int_0^1 \mathcal{G}(t, \xi) d\xi \right|}{2}\right)^2 \\ &\leq \frac{1}{4} \left[\left(|\mathbf{u}(t)| + \frac{5}{8} \sqrt{\mu}|\mathbf{u}(t)| \right)^2 + \left(|\mathbf{w}(t)| + \frac{5}{8} \sqrt{\mu}|\mathbf{w}(t)| \right)^2 \right] \\ &= \frac{1}{4} \left[\frac{(8 + 5\sqrt{\mu})^2}{64} |\mathbf{u}(t)|^2 + \frac{(8 + 5\sqrt{\mu})^2}{64} |\mathbf{w}(t)|^2 \right] \\ &= \frac{(8 + 5\sqrt{\mu})^2}{256} (|\mathbf{u}(t)|^2 + |\mathbf{w}(t)|^2) < \frac{(8 + 5\sqrt{\mu})^2}{256} (|\mathbf{u}| + |\mathbf{w}(t)|)^2. \end{aligned} \quad (78)$$

Equations (77) and (78) imply that

$$m_v^b(\mathcal{A}\mathbf{u}(t), \mathcal{A}\mathbf{w}(t)) \leq \mu \left[m_v^b(\mathbf{u}(t), \mathcal{A}\mathbf{u}(t)) + m_v^b(\mathbf{w}(t), \mathcal{A}\mathbf{w}(t)) \right], \quad (79)$$

i.e., $\mathcal{A}$ satisfies Theorem 18, and hence, $\mathcal{A}$ has a unique fixed point, i.e., a Cantilever beam problem has a unique solution.

6. Conclusion

We have introduced M_v^b -metric as an improvement and generalization of an M_v -metric which need not be continuous and defined topological notions like open ball, closed ball, convergence of a sequence, Cauchy sequence, and completeness of a space to discuss the topology of M_v^b -metric and to create an environment for the survival of a unique fixed point in an M_v^b -metric space. Further, we have demonstrated that the collection of open ball, which forms a basis on M_v^b -metric space and generates a $\mathcal{T}_0$ topology on it. Also, we have introduced a notion of a fixed circle and a fixed disc to study the geometry of the set of nonunique fixed points of a discontinuous self-map. Our results are sharpened versions of the well-known results, wherein continuity of a self-map is not indispensable for the survival of a unique fixed point. Examples and applications to solve a Cantilever beam problem employed in distortion of an elastic beam in equilibrium substantiate the utility of these improvements and extensions. It is interesting to mention that the Cantilever structures permit overhanging constructions deprived of peripheral bracing.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this research.

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