Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6683004, 11 pages
https://doi.org/10.1155/2021/6683004

Research Article

On Stancu-Type Generalization of Modified (p, q)-Szász-Mirakjan-Kantorovich Operators

Yong-Mo Hu,[1] **Wen-Tao Cheng**,[2] **Chun-Yan Gui**,[2] **and Wen-Hui Zhang**[3]

[1]*Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Jiangsu, Nanjing 210016, China*
[2]*School of Mathematics and Physics, Anqing Normal University, Anhui, Anqing 246133, China*
[3]*School of Mathematical Sciences and LPMC, Nankai University, Tianjin, Tianjin 300071, China*

Correspondence should be addressed to Wen-Tao Cheng; chengwentao_0517@163.com

Received 24 November 2020; Revised 24 December 2020; Accepted 30 December 2020; Published 9 January 2021

Academic Editor: Tuncer Acar

In the present article, we construct (p, q)-Szász-Mirakjan-Kantorovich-Stancu operators with three parameters λ, α, β. First, the moments and central moments are estimated. Then, local approximation properties of these operators are established via K-functionals and Steklov mean in means of modulus of continuity. Also, a Voronovskaja-type theorem is presented. Finally, the pointwise estimates, rate of convergence, and weighted approximation of these operators are studied.

1. Introduction

During this decades, the applications of (p, q)-calculus transpired as a new area in the field of operator approximation theory. Many researchers constructed and discussed many positive linear operators based on (p, q)-integers, (p, q)-exponential functions, (p, q)-Gamma functions [1], (p, q)-Beta functions, and so on. Since Mursaleen et al. first constructed (p, q)-Bernstein operators [2] and (p, q)-Bernstein-Stancu operators [3], several generalizations of well-known positive linear operators based on (p, q)-calculus have been introduced and studied (see [4–11]). In [12], Acar first proposed (p, q)-Szász-Mirakjan operators defined on $[0, \infty)$. In [13], Kara et al. constructed a modified (p, q)-Szász-Mirakjan as follows:

$$S_n^{p,q}(f;t) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(t) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right), t \in [0,\infty), \quad (1)$$

where $0 < q < p \le 1$, $f \in C[0,\infty)$ and $s_{n,k}^{p,q}(t) = (p^{k(k-n)}/q^{k(k-1)/2})([n]_{p,q}^k t^k/[k]_{p,q}!) e_{p,q}(-[n]_{p,q} p^{k-n+1} q^{-k} t)$. Certain basic notations of (p, q)-calculus are mentioned below (for details see [14]): For each real number λ, (p, q)-analogue of λ named $[\lambda]_{p,q}$ is defined by

$$[\lambda]_{p,q} = \frac{p^\lambda - q^\lambda}{p-q}, p \ne q. \quad (2)$$

And for each nonnegative integer n, the (p, q)-integer $[n]_{p,q}$ and (p, q)-factorial $[n]_{p,q}!$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{n-n}, & p \ne q; \\ np^{n-1}, & p = q; \\ [n]_q, & p = 1; \\ n, & p = q = 1, \end{cases}$$

$$[n]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q}, & n \ge 1; \\ 1, & n = 0. \end{cases} \quad (3)$$

The (p, q)-analogue of the exponential function is defined by

$$e_{p,q}(t) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2} t^n}{[n]_{p,q}!}. \quad (4)$$

Let f be an arbitrary function and $a \in \mathbb{R}$. The (p,q)-Jackson integral [15] was defined by

$$\int_0^a f(u)\mathrm{d}_{p,q}u = (p-q)a\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}\right), 0<q<p\le 1. \quad (5)$$

And the (p,q)-Jackson integral over an interval $[a,b]$ $(a<b)$ can be defined by

$$\int_a^b f(u)\mathrm{d}_{p,q}u = \int_0^b f(u)\mathrm{d}_{p,q}u - \int_0^a f(u)\mathrm{d}_{p,q}u. \quad (6)$$

We easily know that (p,q)-Jackson integral (6) is not positive unless it is assumed that f is a nondecreasing function. To solve this problem, Acar et al. [16] defined the (p,q)-integral of the arbitrary function f on interval $[a,b](a<b)$ as follows:

$$\int_a^b f(u)\mathrm{d}_{p,q}u = (p-q)(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(a+(b-a)\frac{q^n}{p^{n+1}}\right), 0<q<p\le 1. \quad (7)$$

It is obvious that integral (6) and integral (7) of f on $[0,1]$ are equivalence.

The Kantorovich modification of positive linear operators on $[0,\infty)$ is a method to approximate the Riemann integrable functions. The idea behind the Kantorovich modifications mainly depends on replacing the sample value $f(k/n)$ by $n\int_{k+1/n}^{k/n} f(u)\mathrm{d}u$ (see [17, 18]). By definite integral substitution, we have $n\int_{k+1/n}^{k/n} f(u)\mathrm{d}u = \int_0^1 f(k+u/n)\mathrm{d}u$. However, two Kantorovich modifications may be not equivalence or cannot use definite integral substitution in q-calculus and (p,q)-calculus. For the researches about (p,q)-Szász-Mirakjan-Kantorovich-operators, we can see [19–21]. Meantime, the idea behind the Stancu modifications mainly depends on replacing the sample value $f(k/n)$ by $f(k+\alpha/n+\beta)$ with two parameters $0\le\alpha\le\beta$ (see [22]). For the researches about the Stancu modification of (p,q)-operators, we can see [23, 24]. All these achievements motivate us to construct the Stancu and Kantorovich generalizations of (p,q)-Szász-Mirakjan (1) with three parameters λ, α, β as follows:

Definition 1. For $n\in\mathbb{N}$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$ and $f\in C[0,\infty)$, the (p, q)-Szász-Mirakjan-Kantorovich-Stancu operators can be defined by

$$S_{n,\alpha,\beta}^{p,q,\lambda}(f;t) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(t)\int_0^1 f\left(\frac{p^{n-k}[k]_{p,q}+u^\lambda+\alpha}{[n]_{p,q}+\beta}\right)\mathrm{d}_{p,q}u, t\in[0,\infty). \quad (8)$$

2. Auxiliary Results

In order to obtain the approximation properties of the operators $S_{n,\alpha,\beta}^{p,q,\lambda}(f;t)$, we need the following lemmas and corollaries.

Lemma 2. *For $t\in[0,\infty)$, $0<q<p\le 1$, $\lambda>0$, we have $\int_0^1 t^\lambda d_{p,q}t = 1/[\lambda+1]_{p,q}$.*

Proof. Using (7),

$$\int_0^1 t^\lambda \mathrm{d}_{p,q}t = (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{\lambda+1} = \frac{p-q}{p^{\lambda+1}}\sum_{n=0}^{\infty}\left(\frac{q^{\lambda+1}}{p^{\lambda+1}}\right)^n = \frac{1}{[\lambda+1]_{p,q}}. \quad (9)$$

Lemma 3. *([13], Lemma 4) For $0<q<p\le 1$, $n\in\mathbb{N}$, and $t\in[0,\infty)$, we have*

$$S_n^{p,q}(1;t)=1, S_n^{p,q}(u;t)=t, S_n^{p,q}(u^2;t)=t^2+\frac{p^{n-1}}{[n]_{p,q}}t,$$

$$S_n^{p,q}(u^3;t)=t^3+\frac{(2p+q)p^{n-2}}{[n]_{p,q}}t^2+\frac{p^{2n-2}}{[n]_{p,q}^2}t,$$

$$S_n^{p,q}(u^4;t)=t^4+\frac{(3p^2+2qp+q^2)p^{n-3}}{[n]_{p,q}}t^3 + \frac{(3p^2+3qp+q^2)p^{2n-4}}{[n]_{p,q}^2}t^2+\frac{p^{3n-3}}{[n]_{p,q}^3}t. \quad (10)$$

The following lemma will tell us the relation between the moment of the operators $S_n^{p,q}$ and the moment of the operators $S_{n,\alpha,\beta}^{p,q,\lambda}$:

Lemma 4. *For $t\in[0,\infty)$, $n,m\in\mathbb{N}$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$, we have the following recursive relation:*

$$S_{n,\alpha,\beta}^{p,q,\lambda}(u^m;t) = \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}[n]_{p,q}^i S_n^{p,q}(u^i;t)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}}. \quad (11)$$

Proof. By direct computation, we have

$$\begin{aligned} S_{n,\alpha,\beta}^{p,q,\lambda}(u^m;t) &= \sum_{k=0}^{\infty}s_{n,k}^{p,q}(t)\int_0^1\left(\frac{p^{n-k}[k]_{p,q}+u^\lambda+\alpha}{[n]_{p,q}+\beta}\right)^m \mathrm{d}_{p,q}u \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{k=0}^{\infty}\int_0^1\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}\left(p^{n-k}[k]_{p,q}\right)^i u^{\lambda j}\alpha^{m-i-j}\mathrm{d}_{p,q}u \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}[n]_{p,q}^i\left(\sum_{k=0}^{\infty}s_{n,k}^{p,q}(t)\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)^i\right)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}} \\ &= \frac{1}{\left([n]_{p,q}+\beta\right)^m}\sum_{i=0}^{m}\sum_{j=0}^{m-i}\frac{m!}{i!j!(m-i-j)!}[n]_{p,q}^i S_n^{p,q}(u^i;t)\frac{\alpha^{m-i-j}}{[\lambda j+1]_{p,q}}. \end{aligned} \quad (12)$$

Hence, the proof of Lemma 4 is completed.

Then, the following lemma can be obtain immediately:

Lemma 5. *For $t \in [0,\infty)$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$, we have*

$$S_{n,\alpha,\beta}^{p,q,\lambda}(1;t)=1, S_{n,\alpha,\beta}^{p,q,\lambda}(u;t)=\frac{[n]_{p,q}}{[n]_{p,q}+\beta}t+\frac{1}{[n]_{p,q}+\beta}\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha\right),$$

$$S_{n,\alpha,\beta}^{p,q,\lambda}(u^2;t)=\frac{[n]_{p,q}^2}{\left([n]_{p,q}+\beta\right)^2}\left(t^2+\frac{p^{n-1}}{[n]_{p,q}}t\right)+\frac{2[n]_{p,q}}{\left([n]_{p,q}+\beta\right)^2}\cdot\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha\right)t+\frac{1}{\left([n]_{p,q}+\beta\right)^2}\cdot\left(\frac{1}{[2\lambda+1]_{p,q}}+\frac{2\alpha}{[\lambda+1]_{p,q}}+\alpha^2\right). \tag{13}$$

Lemma 6. *Under the condition of Lemma 5, we can easily obtain the following formulas for the first and second central moments:*

$$A_{n,\alpha,\beta}^{p,q,\lambda}(t)\coloneqq S_{n,\alpha,\beta}^{p,q,\lambda}(u-t;t)=\frac{1}{[n]_{p,q}+\beta}\left(\frac{1}{[\lambda+1]_{p,q}}+\alpha-\beta t\right),$$

$$B_{n,\alpha,\beta}^{p,q,\lambda}(t)\coloneqq S_{n,\alpha,\beta}^{p,q,\lambda}((u-t)^2;t)=\frac{p^{n-1}[n]_{p,q}t}{\left([n]_{p,q}+\beta\right)^2}+\frac{1}{\left([n]_{p,q}+\beta\right)^2}\cdot\left(\left(\beta t-\alpha-\frac{1}{[\lambda+1]_{p,q}}\right)^2+\frac{1}{[2\lambda+1]_{p,q}}-\frac{1}{[\lambda+1]_{p,q}^2}\right). \tag{14}$$

Lemma 7. *The sequences (p_n), (q_n) satisfy $0<q_n<p_n\le 1$, such that $q_n\longrightarrow 1$, $p_n^n\longrightarrow\eta\in[0,1]$, $[n]_{p_n,q_n}\longrightarrow\infty$ as $n\longrightarrow\infty$; then for any $t\in[0,\infty)$, $0<q<p\le 1$, $\lambda>0$, $0\le\alpha\le\beta$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)=-\beta t+\alpha+\frac{1}{\lambda+1}, \tag{15}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)=\eta t, \tag{16}$$

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^4;t)=0. \tag{17}$$

Proof. By $[\lambda+1]_{p_n,q_n}=(\lambda+1)\xi_n^\lambda$, $\xi_n\in(q_n,p_n)$, we have $\lim_{n\to\infty}[\lambda+1]_{p_n,q_n}=\lambda+1$. Thus, we easily obtain (15) and (16). As $n\longrightarrow\infty$, we can rewrite

$$S_n^{p_n,q_n}(u^3;t)=t^3+\frac{(2+q_np_n^{-1})p_n^{n-1}}{[n]_{p_n,q_n}}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right),$$

$$S_n^{p_n,q_n}(u^4;t)=t^4+\frac{(3+2q_np_n^{-1}+q_n^2p_n^{-2})p_n^{n-1}}{[n]_{p_n,q_n}}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right). \tag{18}$$

Set $A(n)=(1/[\lambda+1]_{p_n,q_n})+\alpha$. Applying Lemma 4 and $([n]_{p_n,q_n}/[n]_{p_n,q_n}+\beta)^i=1-(i\beta/[n]_{p_n,q_n}+\beta)+o(1/[n]_{p_n,q_n})$, $i=1,2,3,4$, we can also rewrite

$$\begin{aligned}
S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u;t)&=\frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}+\beta}t+\frac{A(n)}{[n]_{p_n,q_n}+\beta}=\left(1-\frac{\beta}{[n]_{p_n,q_n}+\beta}\right)t+\frac{A(n)}{[n]_{p_n,q_n}+\beta}+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\
S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^2;t)&=\frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n}+\beta\right)^2}\left(t^2+\frac{p^{n-1}}{[n]_{p_n,q_n}}t\right)+\frac{2[n]_{p_n,q_n}A(n)}{\left([n]_{p_n,q_n}+\beta\right)^2}t\\
&=\left(1-\frac{2\beta}{[n]_{p_n,q_n}+\beta}\right)t^2+\frac{p^{n-1}}{[n]_{p_n,q_n}}t+\frac{2A(n)}{[n]_{p_n,q_n}+\beta}t+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\
S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^3;t)&=\frac{[n]_{p_n,q_n}^3}{\left([n]_{p_n,q_n}+\beta\right)^3}\left(t^3+\frac{(2+q_np_n^{-1})p_n^{n-1}}{[n]_{p_n,q_n}}t^2\right)+\frac{3[n]_{p_n,q_n}^2A(n)}{\left([n]_{p_n,q_n}+\beta\right)^3}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\\
&=\left(1-\frac{3\beta}{[n]_{p_n,q_n}+\beta}\right)t^3+\frac{(2+q_np_n^{-1})p^{n-1}}{[n]_{p_n,q_n}}t^2+\frac{3A(n)}{[n]_{p_n,q_n}+\beta}t^2+o\left(\frac{1}{[n]_{p_n,q_n}}\right),\\
S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^4;t)&=\frac{[n]_{p_n,q_n}^4}{\left([n]_{p_n,q_n}+\beta\right)^4}\left(t^4+\frac{(3+2q_np_n^{-1}+q_n^2p_n^{-2})p_n^{n-1}}{[n]_{p_n,q_n}}t^3\right)+\frac{4[n]_{p_n,q_n}^3A(n)}{\left([n]_{p_n,q_n}+\beta\right)^4}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right)\\
&=\left(1-\frac{4\beta}{[n]_{p_n,q_n}+\beta}\right)t^4+\frac{(3+2q_np_n^{-1}+q_n^2p_n^{-2})p^{n-1}}{[n]_{p_n,q_n}}t^3+\frac{4\mathrm{A}(n)}{[n]_{p_n,q_n}+\beta}t^3+o\left(\frac{1}{[n]_{p_n,q_n}}\right).
\end{aligned} \tag{19}$$

Combining $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^4;t)=\sum_{m=0}^4\binom{4}{m}(-1)^mS_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^{4-m};t)t^m$, we can obtain

$$[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^4;t)=\left(1-2q_np_n^{-1}+q_n^2p_n^{-2}\right)p_n^{n-1}t^3+o(1)\longrightarrow 0,\text{ as } n\longrightarrow\infty, \tag{20}$$

we obtain the required result.

Lemma 8. *Let $C_B[0,\infty)$ be the set of real-valued continuous bounded functions defined on $[0,\infty)$ endowed with the norm*

$\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. *Under the condition of Lemma 5, for any $f \in C_B[0,\infty)$, we have*

$$\left\| S_{n,\alpha,\beta}^{p,q,\lambda}(f;t) \right\| \le \|f\|. \quad (21)$$

Proof. In view of (8) and Lemma 5, the proof of this lemma can be obtained easily.

3. Local Approximation

In this section, we will establish local approximation theorem for the operators. For any $f \in C_B[0,\infty)$, we consider the following $\mathscr{K}$-functional:

$$\mathscr{K}(f;\delta) = \inf_{h\in \mathbf{W}^2} \left\{ \|f-h\| + \delta \|h''\| \right\}, \quad (22)$$

where $\delta \in (0,\infty)$ and $\mathbf{w}^2 = \{h \in C_B[0,\infty): h', h'' \in C_B[0,\infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of f can be defined as

$$\omega(f;\delta) = \sup_{0<|u|<\delta} \sup_{x\in[0,\infty)} |f(t+u) - f(t)|,$$
$$\omega_2(f;\delta) = \sup_{0<|u|<\delta} \sup_{x\in[0,\infty)} |f(t+2u) - f(t+u) + f(t)|. \quad (23)$$

By ([25], p.177, Theorem 2.4), there exists an absolute positive constant C such that

$$\mathscr{K}(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \delta > 0. \quad (24)$$

In the meantime, for $f \in C_B[0,\infty)$ and $h > 0$, the Steklov mean is defined as

$$f_h(t) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} [2f(t+u+v) - f(t+2(u+v))]\mathrm{d}u\mathrm{d}v. \quad (25)$$

Thus, $f_h \in C_B[0,\infty)$, and we can write

$$f_h(t) - f(t) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2} [2f(t+u+v) - f(t+2(u+v)) - f(t)]\mathrm{d}u\mathrm{d}v. \quad (26)$$

It is obvious that $|f_h(t) - f(t)| \le \omega_2(f;h)$ and $\|f_h - f\| \le \omega_2(f;h)$. If f is continuous, then $f_h', f_h'' \in C_B[0,\infty)$ and

$$f_h'(t) = \frac{4}{h^2}\left[2\int_0^{h/2}\left(f\left(t+u+\frac{h}{2} - f(t+u)\right)\right)\mathrm{d}u - \frac{1}{2}\int_0^{h/2}(f(t+h+2u) - f(t+2u))\mathrm{d}u\right]. \quad (27)$$

Thus, we have $\|f_h'\| \le (5/h)\omega(f;h)$. Similarly, $\|f_h''\| \le (9/h^2)\omega_2(f;h)$.

Theorem 9. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ and $t \in [0,\infty)$, we have*

$$\left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t) \right| \le 2\omega\left(f;\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \quad (28)$$

Proof. For any $\delta > 0$, we have $|f(u) - f(t)| \le \omega(f;|u-t|) \le (1 + (|u-t|/\delta))\omega(f;\delta)$. Applying $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ to both ends and using Lemma 5, we can obtain

$$\left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t) \right| \le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f(u) - f(t)|;t) \le \left(1 + \frac{1}{\delta} S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t)\right)\omega(f;\delta). \quad (29)$$

By using the Chauchy-Schwarz inequality and taking $\delta = \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}$, we have

$$\left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t) \right| \le \left(1 + \frac{1}{\delta}\sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}((u-t)^2;t)}\right)\omega(f;\delta) \le 2\omega\left(f;\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \quad (30)$$

Theorem 9 is proved.

Theorem 10. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ and $t \in [0,\infty)$, there exists an absolute positive constant $C_1 = 4C$ such that*

$$\left| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t) \right| \le C_1\omega_2\left(f;\sqrt{\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right) + \omega\left(f;\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\right). \quad (31)$$

Proof. First, we define the following new positive linear operators as follows:

$$T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) = S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + t\right) + f(t), t \in [0,\infty). \quad (32)$$

It is apparent from Lemma 5, Lemma 6, and Lemma 8 that

$$T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1;t) = 1; T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t;t) = 0, \quad (33)$$

$$\left\| T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) \right\| \le 3\|f\|. \quad (34)$$

Now for any given function $h \in \mathbf{W}^2$ and $u, t \in [0,\infty)$, we write Taylor's expansion formula as follows:

$$h(u) = h(t) + h'(t)(u-t) + \int_t^u h''(v)(u-v)\mathrm{d}v. \tag{35}$$

By applying $T_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ operators to both sides of the above equality, we can obtain

$$\begin{aligned} T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) &= T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(h(t) + h'(t)(u-t) + \int_t^u h''(v)(u-v)\mathrm{d}v;t\right) \\ &= h(t) + T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(h'(t)(u-t);t\right) \\ &\quad + T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\int_t^u h''(v)(u-v)\mathrm{d}v;t\right). \end{aligned} \tag{36}$$

Using (32), (33), and the following inequality,

$$\begin{aligned} \left|\int_t^u h''(v)(u-v)\mathrm{d}v\right| &\le \left|\int_t^u |h''(v)||u-v|\mathrm{d}v\right| \\ &\le \|h''\|\left|\int_t^u |u-v|\mathrm{d}v\right| \\ &\le (u-t)^2\|h''\|, \end{aligned} \tag{37}$$

we can get

$$\begin{aligned} \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| &= \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\int_t^u h''(v)(u-v)\mathrm{d}v;t\right)\right| \\ &\le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\left|\int_t^u h''(v)\,|u-v|\,\mathrm{d}v\right|;t\right) \\ &\quad + \left|\int_t^{A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+t} h''(v)\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+t-v\right)\mathrm{d}v\right| \\ &\le S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)\|h''\| + \left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2\|h''\| \\ &= \left(\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)\|h''\|. \end{aligned} \tag{38}$$

By using (32) and (34), we have

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| &= \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) + f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+t\right) - 2f(t)\right| \\ &\le \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f-h;t) - (f-g)(t)\right| \\ &\quad + \left|T_{n,\alpha,\beta}^{p_n,q_n,\lambda}(h;t) - h(t)\right| \\ &\quad + \left|f\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+t\right) - f(t)\right| \\ &\le 4\|f-h\| + \left(\left(A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^2 + B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)\|h''\| \\ &\quad + \omega\left(f;\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\right). \end{aligned} \tag{39}$$

Taking the infimum on the right-hand side over all $h \in \mathbf{W}^2$ and using (24), we complete the proof of Theorem 10.

Theorem 11. *Under the condition of Lemma 7, then for all $f' \in C_B[0,\infty)$ and $t \in [0,\infty)$, we have*

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le \left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\left|f'(t)\right| + 2\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\,\omega \\ &\quad \cdot \left(f';\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \end{aligned} \tag{40}$$

Proof. Applying $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ to both sides of the equality $f(u) = f(t) + f'(t)(u-t) + f(u) - f(t) - f'(t)(u-t)$, using mean value theorem and the Chauchy-Schwarz inequality and taking $\delta = \sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}$, we can obtain

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le \left|f'(t)\right|\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t;t)\right| + S_{n,\alpha,\beta}^{p_n,q_n,\lambda} \\ &\quad \cdot \left(\left|f(u) - f(t) - f'(t)(u-t)\right|;t\right) \\ &\le |f'(t)|\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u-t;t)\right| + S_{n,\alpha,\beta}^{p_n,q_n,\lambda} \\ &\quad \cdot \left(|u-t|\left(1 + \frac{|u-t|}{\delta}\right)\omega\left(f';\delta\right);t\right) \\ &\le |f'(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right| + \omega\left(f';\delta\right) \\ &\quad \cdot \left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t) + \frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)}{\delta}\right) \\ &\le |f'(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right| + \omega\left(f';\delta\right) \\ &\quad \cdot \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)} \\ &\quad \cdot \left(1 + \frac{\sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)}}{\delta}\right) \\ &\le \left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\left|f'(t)\right| + 2\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\,\omega \\ &\quad \cdot \left(f';\sqrt{B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)}\right). \end{aligned} \tag{41}$$

Theorem 12. *Under the condition of Lemma 7, if $f \in C_B[0,\infty)$, then*

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right| &\le 5\sqrt{[n]_{p_n,q_n}}\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\omega\left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right) \\ &\quad + \left(\frac{9}{2}[n]_{p_n,q_n}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) + 2\right)\omega_2 \\ &\quad \cdot \left(f;\frac{1}{\sqrt{[n]_{p_n,q_n}}}\right). \end{aligned} \tag{42}$$

Proof. For $t \in [0,\infty)$, using the Steklov mean function f_h, we can write

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)-f(t)\right| \leq S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|;t)+\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f_h-f_h(t)\,;t)\right| + |f_h(t)-f(t)|. \tag{43}$$

By Lemma 8 and properties of the Steklov mean, we can obtain

$$S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|;t) \leq \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f-f_h|)\| \leq \|f-f_h\| \leq \omega_2(f\,;h). \tag{44}$$

By Taylor's expansion formula, we have

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f_h-f_h(t)\,;t)\right| &\leq |f_{h'}(t)|\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right| + \frac{1}{2}\|f_h{''}\|B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\\ &\leq \frac{5}{h}\omega(f\,;h)\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\\ &\quad+\frac{9}{2h^2}\omega_2(f\,;h)B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t).\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)-f(t)\right| &\leq \frac{5}{h}\left|A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right|\omega(f\,;h)\\ &\quad+\left(\frac{9}{2h^2}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+2\right)\omega_2(f\,;h).\end{aligned} \tag{46}$$

Setting $h=1/\sqrt{[n]_{p_n,q_n}}$, we can get the desired result.

By the classic Korovkin theorem, we easily get the following corollary:

Corollary 13. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ and any $A>0$, the the sequence $\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)\}$ converges to f uniformly on $[0,A]$.*

4. Voronovskaja-Type Theorem for $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$

In this section, we show a Voronovskaja-type asymptotic formula for the operators $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ by means of the first, second and fourth central moments.

Theorem 14. *Under the condition of Lemma 7, then for all $f \in C_B[0,\infty)$ satisfying $f''(t)$ that exists at a point $t \in [0,\infty)$, we can obtain*

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)-f(t)\right)=\left(\alpha-\beta t+\frac{1}{\lambda+1}\right)f'(t)+\frac{\eta}{2}f''(t)t. \tag{47}$$

Proof. By Taylor's expansion formula for f, we have

$$f(u)=f(t)+f'(t)(u-t)+\frac{1}{2}f''(t)(u-t)^2+\phi(u\,;t)(u-t)^2, \tag{48}$$

where

$$\phi(u\,;t)=\begin{cases}\dfrac{f(u)-f(t)-f'(t)(u-t)-1/2f''(t)(u-t)^2}{(u-t)^2}, & u\neq t;\\ 0, & u=t.\end{cases} \tag{49}$$

Applying L'Hospital's Rule,

$$\lim_{u\to t}\phi(u\,;t)=\frac{1}{2}\lim_{u\to t}\frac{f'(u)-f'(t)}{u-t}-\frac{1}{2}f''(t)=0. \tag{50}$$

Thus, $\phi(.;t) \in C_B[0,\infty)$. Consequently, we can write

$$\begin{aligned}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)-f(x)\right) &= [n]_{p_n,q_n}A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)+\frac{1}{2}[n]_{p_n,q_n}B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\\ &\quad+[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u\,;t)(u-t)^2\,;t\right)\right).\end{aligned} \tag{51}$$

By Schwarz's inequality, we have

$$\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u\,;t)(u-t)^2\,;t\right)\right) \leq \sqrt{\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi^2(u\,;t)\,;t\right)\right)}\cdot\sqrt{\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4\,;t\right)\right)}. \tag{52}$$

We observe that $\phi^2(t\,;t)=0$ and $\phi^2(.;t) \in C_B[0,\infty)$. Then, it follows in Corollary 13 that

$$\lim_{n\to\infty}[n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi^2(u\,;t)\,;t\right)=\phi^2(t\,;t)=0. \tag{53}$$

Hence, from (17), we can obtain

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(\phi(u\,;t)(u-t)^2\,;t\right)\right)=0. \tag{54}$$

Combining, we complete the proof of Theorem 14.

Corollary 15. *Under the condition of Lemma 7, then for all $f', f'' \in C_B[0,\infty)$, we have*

$$\begin{aligned}\lim_{n\to\infty}[n]_{p_n,q_n}\left(S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t)-f(t)\right) &= \left(\alpha-\beta t+\frac{1}{\lambda+1}\right)f'(t)\\ &\quad+\frac{\eta}{2}f''(t)t,\end{aligned} \tag{55}$$

uniformly with respect to any finite interval $I \subset [0,\infty)$.

5. Pointwise Estimates

In this section, we establish two pointwise estimates of the operators $S_{n,\alpha,\beta}^{p,q,\lambda}$. First, we compute the rate of convergence locally by using functions belonging to the Lipschitz class. We denote that $f \in C_B[0,\infty)$ is in $\mathrm{Lip}_M(\gamma, D)$, $\gamma \in (0,1]$, $D \subset [0,\infty)$ if it satisfies the following condition:

$$|f(u)-f(t)| \leq M|u-t|^{\gamma}, u \in D, t \in [0,\infty), \tag{56}$$

where M is a positive constant depending only on γ and f.

Theorem 16. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$, $\gamma \in (0,1]$ and D be any bounded subset on $[0,\infty)$. If $f \in C_B[0,\infty) \cap Lip_M(\gamma, D)$, then for any $t \in [0,\infty)$, we have*

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(x)\right| \leq M\left(\left(B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^{\gamma/2}+2d^{\gamma}(t;D)\right), \tag{57}$$

where $d(t;D)=\inf\{|u-t| : u \in D\}$ denotes the distance between t and D.

Proof. Let $\bar{D}$ be the closure of D. Using the properties of infimum, and there is at least a point $t_0 \in \bar{D}$ such that $\mathrm{d}(t;E) = |t-t_0|$. By the triangle inequality

$$|f(u)-f(t)| \leq |f(u)-f(t_0)| + |f(t)-f(t_0)|. \tag{58}$$

By the monotonicity of $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$, we get

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(x)\right| &\leq S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f(u)-f(t_0)|;t)+S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|f(t)-\mathrm{f}(t_0)|;t)\\ &\leq M\left\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t_0|^{\gamma};t)+S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|t-t_0|^{\gamma};t)\right\}\\ &\leq M\left\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|^{\gamma}+|t-t_0|^{\gamma};t)+|t-t_0|^{\gamma}\right\}\\ &= M\left\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|^{\gamma};t)+2|t-t_0|^{\gamma}\right\}.\end{aligned} \tag{59}$$

Applying the well-known Hölder inequality with $a_1 = 2/\gamma$, $a_2 = 2/2-\gamma$, we obtain

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(x)\right| &\leq M\left\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|^{a_1\gamma};t)^{1/a_1}+2d^{\gamma}(t;D)\right\}\\ &\leq M\left\{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(|u-t|^{2};t\right)^{1/a_1}+2d^{\gamma}(t;D)\right\}\\ &= M\left\{\left(B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^{\gamma/2}+2d^{\gamma}(t;D)\right\}.\end{aligned} \tag{60}$$

Second, we will give a local direct estimation of the operators $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ by using the Lipschitz-type maximal function of the order γ introduced by Lenze [26] as

$$\tilde{\omega}_{\gamma}(f;t) = \sup_{u\neq t, u\in[0,\infty)} \frac{|f(u)-f(t)|}{|u-t|^{\gamma}}, t \in [0,\infty) \text{ and } \gamma \in (0,1]. \tag{61}$$

Theorem 17. *The sequences (p_n), (q_n) satisfy $0 < q_n < p_n \leq 1$ and $\gamma \in (0,1]$. If $f \in C_B[0,\infty)$, then for any $t \in [0,\infty)$, we have*

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| \leq \tilde{\omega}_{\gamma}(f;t)\left(B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^{\gamma/2}. \tag{62}$$

Proof. Using the equality (61), we obtain

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| \leq \tilde{\omega}_{\gamma}(f;t)S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|^{\gamma};t). \tag{63}$$

By the well-known Hölder inequality, we have

$$\begin{aligned}\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| &\leq \tilde{\omega}_{\gamma}(f;t)S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(|u-t|^{2};t\right)^{\gamma/2}\\ &\leq \tilde{\omega}_{\gamma}(f;t)\left(B_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t)\right)^{\gamma/2}.\end{aligned} \tag{64}$$

Thus, the proof of Theorem 17 is completed.

6. Rate of Convergence

Let $B_2[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(t)| \leq C_f(1+t^2)$ with an absolute constant $C_f > 0$ which may depend only on f. $C_2[0,\infty)$ denotes the subspace of all continuous functions $f \in B_2[0,\infty)$ with the norm $\|f\|_2 = \sup_{x\in[0,\infty)} |f(t)|/1+t^2$. By $C_2^0[0,\infty)$, and we denote the subspace of all functions $f \in C_2[0,\infty)$ for which $\lim_{x\to+\infty} |f(t)|/1+t^2$ is finite. Meantime, we denote the modulus of continuity of f on the interval $[0,a]$, $a > 0$ by

$$\omega_a(f;\delta) = \sup_{|u-t|\leq\delta} \sup_{u,t\in[0,a]} |f(u)-f(t)|. \tag{65}$$

Theorem 18. *Let $f \in C_2[0,\infty)$, $0 < q < p \leq 1$, and $a > 0$. Then, for all $t \in [0,a]$, we have*

$$\left|S_{n,\alpha,\beta}^{p,q,\lambda}(f;t)-f(t)\right| \leq C_f\left(4+3a^2\right)B_{n,\alpha,\beta}^{p,q,\lambda}(t)+2\omega_{a+1}\left(f;\sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}\right). \tag{66}$$

Proof. For any $t \in [0,a]$ and $u > a+1$, we easily have $1 \leq (u-a)^2 \leq (u-t)^2$; thus

$$\begin{aligned}|f(u)-f(t)| &\leq |f(u)|+|f(t)| \leq C_f\left(2+u^2+t^2\right)\\ &= C_f\left(2+t^2+(u-t+t)^2\right) \leq C_f\left(2+3t^2+2(u-t)^2\right)\\ &\leq C_f\left(4+3t^2\right)(u-t)^2 \leq M_f\left(4+3a^2\right)(u-t)^2,\end{aligned} \tag{67}$$

and for any $t \in [0, a]$, $u \in [0, a+1]$) and $\delta > 0$, we have

$$|f(u)-f(t)| \le \omega_{a+1}(|u-t|;t) \le \left(1+\frac{|u-t|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{68}$$

For (67) and (68), we can get

$$|f(u)-f(t)| \le C_f\left(4+3a^2\right)(u-t)^2 + \left(1+\frac{|u-t|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{69}$$

Applying the Cauchy-Schwarz inequality and choosing $\delta = \sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}$, we have

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p,q,\lambda}(f;t)-f(t)\right| &\le S_{n,\alpha,\beta}^{p,q,\lambda}(|f(u)-f(t)|;t) \\ &\le C_f\left(4+3a^2\right)S_{n,\alpha,\beta}^{p,q,\lambda}\left((u-t)^2;t\right) \\ &\quad + S_{n,\alpha,\beta}^{p,q,\lambda}\left(\left(1+\frac{|u-t|}{\delta}\right);t\right)\omega_{a+1}(f;\delta) \\ &\le C_f\left(4+3a^2\right)B_{n,\alpha,\beta}^{p,q,\lambda}(t)+\omega_{a+1}(f;\delta) \\ &\quad \cdot \left(1+\frac{\sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}}{\delta}\right) = C_f\left(4+3a^2\right)B_{n,\alpha,\beta}^{p,q,\lambda}(t) \\ &\quad + 2\omega_{a+1}\left(f;\sqrt{B_{n,\alpha,\beta}^{p,q,\lambda}(t)}\right). \end{aligned} \tag{70}$$

This completes the proof of Theorem 18.

7. Weighted Approximation

As is known, if $f \in C[0,\infty)$ is not uniform, the limit $\lim_{\delta\to 0^+}\omega(f;\delta)=0$ may be not true. In [27], Ispir defined the following weighted modulus of continuity:

$$\Omega(f;\delta) = \sup_{t\in[0,\infty),0<h\le\delta}\frac{|f(t+h)-f(t)|}{\left(1+t^2\right)\left(1+h^2\right)} \text{ for } f \in C_2^0[0,\infty), \tag{71}$$

and proved the properties of monotone increasing about $\Omega(f;\delta)$ as $\delta>0$, $\lim_{\delta\to 0^+}\Omega(f;\delta)=0$ and the inequality

$$\Omega(f;\tau\delta) \le 2(1+\tau)\left(1+\delta^2\right)\Omega(f;\delta), \tau>0. \tag{72}$$

Theorem 19. *Under the condition of Lemma 7, $f \in C_2^0[0,\infty)$, then for sufficiently large n, the inequality*

$$\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| \le K\left(1+t^2\right)^{2+\theta}\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right) \tag{73}$$

holds, where $\theta \ge 1/2$ and K is a positive constant depending only on f and n.

Proof. Applying (71) and (72), we can obtain

$$\begin{aligned} |f(u)-f(t)| &\le \left(1+(u-t)^2\right)\left(1+t^2\right)\Omega(f;|u-t|) \\ &\le 2\left(1+\frac{|u-t|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(u-t)^2\right)\left(1+t^2\right) \\ &\le \begin{cases} 4\left(1+\delta^2\right)^2\left(1+t^2\right)\Omega(f;\delta), & |u-t|\le\delta, \\ 4\left(1+\delta^2\right)\left(1+t^2\right)\Omega(f;\delta)\dfrac{|u-t|+|u-t|^3}{\delta}, & |u-t|>\delta. \end{cases} \end{aligned} \tag{74}$$

Thus, for any $\delta \in (0, 1/2)$ and $u, t \in [0,\infty)$, the above inequality can be rewritten

$$|f(u)-f(t)| \le 5\left(1+t^2\right)\Omega(f;\delta)\left(\frac{5}{4}+\frac{|u-t|+|u-t|^3}{\delta}\right). \tag{75}$$

Applying (16) and (17), there exists sufficiently large n such that

$$\begin{aligned} [n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right) &\le K_1^2\left(1+t^2\right), \\ [n]_{p_n,q_n}S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right) &\le K_2^2\left(1+t^2\right)^2. \end{aligned} \tag{76}$$

By Schwarz's inequality, we can obtain

$$\begin{aligned} S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(|u-t|;t) &\le \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)} \\ &\le \frac{K_1}{[n]_{p_n,q_n}}\sqrt{1+t^2}, \\ S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(|u-t|^3;t\right) &\le \sqrt{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^2;t\right)S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left((u-t)^4;t\right)} \\ &\le \frac{K_2}{[n]_{p_n,q_n}}\sqrt{\left(1+t^2\right)^3}. \end{aligned} \tag{77}$$

Using $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}$ as linear and positive and choosing $\delta = 1/[n]_{p_n,q_n}$, we can obtain

$$\begin{aligned} \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t)-f(t)\right| &\le 5\left(1+t^2\right)\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right) \\ &\quad \cdot\left(\frac{5}{4}+K_1\left(1+t^2\right)+K_2\sqrt{\left(1+t^2\right)^3}\right) \\ &\le K\left(1+t^2\right)^{5/2}\Omega\left(f;\frac{1}{[n]_{p_n,q_n}}\right), \end{aligned} \tag{78}$$

for sufficiently large n and $t \in [0,\infty)$, where $K := 5\max\{5/4, K_1, K_2\}$.

Theorem 20. *Under the condition of Lemma 7, then for any $f \in C_2^0[0,\infty)$, we have*

$$\lim_{n\to\infty} \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f\|_2 = 0. \tag{79}$$

Proof. Applying the Korovkin theorem [28], we only see that it is sufficient to prove the following three conditions:

$$\lim_{n\to\infty} \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^m;t) - t^m\|_2 = 0, m = 0, 1, 2. \tag{80}$$

Since $S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1;t) = 1$, the condition holds for $m = 0$. By Lemma 6, we can obtain

$$\begin{aligned}\lim_{n\to\infty} \left\| S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u;t) - t \right\|_2 &= \lim_{n\to\infty} \left\| A_{n,\alpha,\beta}^{p_n,q_n,\lambda}(t) \right\|_2 \\ &\le \frac{1}{[n]_{p_n,q_n} + \beta} \left(\left(\frac{1}{[\lambda+1]_{p_n,q_n}} + \alpha \right) \sup_{t\in[0,\infty)} \frac{1}{1+t^2} \right. \\ &\quad \left. + \beta \sup_{t\in[0,\infty)} \frac{t}{1+t^2} \right) \longrightarrow 0, \text{as} n \longrightarrow \infty.\end{aligned} \tag{81}$$

Hence, (80) holds for $m = 1$. Similarly, by Lemma 5, we can write for $m = 2$,

$$\begin{aligned}\lim_{n\to\infty} \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(u^2;t) - t^2\|_2 &\le \left| \frac{[n]_{p_n,q_n}^2}{\left([n]_{p_n,q_n} + \beta\right)^2} - 1 \right| \sup_{t\in[0,\infty)} \frac{t^2}{1+t^2} + \frac{[n]_{p_n,q_n}}{\left([n]_{p_n,q_n} + \beta\right)^2} \\ &\quad \cdot \left(p_n^{n-1} + \frac{2}{[\lambda+1]_{p_n,q_n}} + 2\alpha \right) \sup_{t\in[0,\infty)} \frac{t}{1+t^2} + \frac{1}{\left([n]_{p_n,q_n} + \beta\right)^2} \\ &\quad \cdot \left(\frac{1}{[2\lambda+1]_{p_n,q_n}} + \frac{2\alpha}{[\lambda+1]_{p_n,q_n}} + \alpha^2 \right) \sup_{t\in[0,\infty)} \frac{1}{1+t^2} \longrightarrow 0, \text{as} n \longrightarrow \infty.\end{aligned} \tag{82}$$

Thus, (80) holds for $m = 2$. Hence, the proof is completed.

Theorem 21. *Under the condition of Lemma 7, then for any $f \in C_2^0[0,\infty)$ and $\kappa > 0$, we have*

$$\lim_{n\to\infty} \sup_{t\in[0,\infty)} \frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)}{(1+t^2)^{1+\kappa}} = 0. \tag{83}$$

Proof. Let $t_0 \in (0,\infty)$ be arbitrary but fixed.

$$\begin{aligned}\sup_{t\in[0,\infty)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right|}{(1+t^2)^{1+\kappa}} &\le \sup_{t\in[0,t_0)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right|}{(1+t^2)^{1+\kappa}} + \sup_{t\in[t_0,\infty)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f(t)\right|}{(1+t^2)^{1+\kappa}} \\ &\le \|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f;t) - f\|_{[0,t_0)} + \|f\|_2 \sup_{t\in[t_0,\infty)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1+u^2;t)\right|}{(1+t^2)^{1+\kappa}} \\ &\quad + \sup_{t\in[t_0,\infty)} \frac{|f(t)|}{(1+t^2)^{1+\kappa}} \coloneqq I_1 + I_2 + I_3.\end{aligned} \tag{84}$$

Applying $|f(t)| \le \|f\|_2(1+t^2)$, we have

$$I_3 = \sup_{t\in[t_0,\infty)} \frac{|f(t)|}{(1+t^2)^{1+\kappa}} \le \sup_{t\in[t_0,\infty)} \frac{\|f\|_2(1+t^2)}{(1+t^2)^{1+\kappa}} \le \frac{\|f\|_2}{(1+t_0^2)^\kappa}. \tag{85}$$

Let $\varepsilon > 0$. By Lemma 5, there exists $N_1 \in \mathbb{N}$, such that for all $n > N_1$:

$$\begin{aligned}\frac{\|f\|_2 \left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(1+u^2;t)\right|}{(1+t^2)^{1+\kappa}} &\le \frac{\|f\|_2}{(1+t^2)^{1+\kappa}} \left((1+t^2) + \frac{\varepsilon}{3\|f\|_2} \right) \\ &\le \frac{\|f\|_2}{(1+t^2)^\kappa} + \frac{\varepsilon}{3}.\end{aligned} \tag{86}$$

Hence

$$\|f\|_2 \sup_{t\in[t_0,\infty)} \frac{\left|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}\left(1+u^2\,;t\right)\right|}{\left(1+t^2\right)^{1+\kappa}} \le \frac{\|f\|_2}{\left(1+t_0^2\right)^{\kappa}} + \frac{\varepsilon}{3}, \forall n \ge N_1. \tag{87}$$

Thus

$$I_2 + I_3 < \frac{2\|f\|_2}{\left(1+t_0^2\right)^{\kappa}} + \frac{\varepsilon}{3}, \forall n \ge N_1. \tag{88}$$

Next, for sufficiently large t_0 such that $\|f\|_2/(1+t_0^2)^{\kappa} < \varepsilon/6$. Then, $I_2 + I_3 < 2\varepsilon/3, \forall n \ge N_1$. Applying Corollary 13, there exists $N_2 \in \mathbb{N}$, such that for all $n > N_2$,

$$\left\|S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f\right\|_{[0,t_0)} < \frac{\varepsilon}{3}. \tag{89}$$

Let $N = \max\{N_1, N_2\}$. Combining (86), (88), and (89), we have

$$\sup_{t\in[0,\infty)} \frac{S_{n,\alpha,\beta}^{p_n,q_n,\lambda}(f\,;t) - f(t)}{\left(1+t^2\right)^{1+\kappa}} < \varepsilon, \forall n \ge N. \tag{90}$$

Hence, the proof of Theorem 21 is completed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11626031), the Key Natural Science Research Project in Universities of Anhui Province (Grant No. KJ2019A0572 and KJ2020A0503), the Philosophy and Social Sciences General Planning Project of Anhui Province of China (Grant No. AHSKYG2017D153), and the Natural Science Foundation of Anhui Province of China (Grant No. 1908085QA29).

References

[1] U. Kadak and S. A. Mohiuddine, “Generalized statistically almost convergence based on the difference operator which includes the (p,q)-gamma function and related approximation theorems,” *Results in Mathematics*, vol. 73, no. 1, article 9, 2018.

[2] M. Mursaleen, K. J. Ansari, and A. Khan, “On (p,q)-analogue of Bernstein operators,” *Applied Mathematics and Computation*, vol. 266, pp. 874–882, 2015.

[3] M. Mursaleen, K. J. Ansari, and A. Khan, “Some approximation results by (p,q)-analogue of Bernstein-Stancu operators,” *Applied Mathematics and Computation*, vol. 264, pp. 392–402, 2015.

[4] T. Acar, P. N. Agrawal, and A. S. Kumar, “On a modification of (p,q)-Szász–Mirakyan operators,” *Complex Analysis and Operator Theory*, vol. 12, no. 1, pp. 155–167, 2018.

[5] T. Acar, A. Aral, and S. A. Mohiuddine, “Approximation by bivariate (p,q)-Bernstein–Kantorovich operators,” *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 2, pp. 655–662, 2018.

[6] T. Acar, A. Aral, and S. A. Mohiuddine, “On Kantorovich modification of (p,q)-Bernstein operators,” *Iranian Journal of Science and Technology, Transactions A: Science*, vol. 42, no. 3, pp. 1459–1464, 2018.

[7] T. Acar, A. Aral, and M. Mursaleen, “Approximation by Baskakov-Durrmeyer operators based on (p,q)-integers,” *Mathematica Slovaca*, vol. 68, no. 4, pp. 897–906, 2018.

[8] T. Acar, M. Mursaleen, and S. A. Mohiuddine, “Stancu type (p,q)-Szász-Mirakyan-Baskakov operators,” *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, vol. 67, no. 1, pp. 116–128, 2018.

[9] H. G. I. Ilarslan and T. Acar, “Approximation by bivariate (p,q)-Baskakov–Kantorovich operators,” *Georgian Mathematical Journal*, vol. 25, no. 3, pp. 397–407, 2018.

[10] R. Maurya, H. Sharma, and C. Gupta, “Approximation properties of Kantorovich type modifications of (p,q)-Meyer-König-Zeller operators,” *Constructive Mathematical Analysis*, vol. 1, no. 1, pp. 58–72, 2018.

[11] S. A. Mohiuddine, A. Alotaibi, and T. Acar, “Durrmeyer type (p,q)-Baskakov operators preserving linear functions,” *Journal of Mathematical Inequalities*, vol. 12, no. 4, pp. 961–973, 2007.

[12] T. Acar, “(p,q)-Generalization of Szász-Mirakyan operators,” *Mathematical Methods in the Applied Sciences*, vol. 39, no. 10, pp. 2685–2695, 2016.

[13] M. Kara and N. I. Mahmudov, “(p,q)-Generalization of Szász-Mirakjan operators and their approximation properties,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, Article ID 116, 2020.

[14] V. Gupta, T. M. Rassias, P. N. Agrawal, and A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Springer, New York, 2018.

[15] P. N. Sadiang, “On the Fundamental theorem of (p,q)-calculus and (p,q)-Taylor formulas,” *Results Math*, vol. 73, no. 1, article 39, 2018.

[16] T. Acar, A. Aral, and S. A. Mohiuddine, “On Kantorovich modification of (p,q)-Baskakov operators,” *Journal of Inequalities and Applications*, vol. 2016, no. 1, Article ID 98, 2016.

[17] S. A. Mohiuddine, T. Acar, and A. Alotaibi, “Construction of a new family of Bernstein-Kantorovich operators,” *Mathematical Methods in the Applied Sciences*, vol. 40, no. 18, pp. 7749–7759, 2017.

[18] S. A. Mohiuddine and F. Özger, “Approximation of functions by Stancu variant of Bernstein–Kantorovich operators based on shape parameter α,” *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, no. 2, article 70, 2020.

[19] M. Mursaleen, A. Alotaibi, and K. J. Ansari, “On a Kantorovich variant of (p,q)-Szász-Mirakjan operators,” *Journal of Function Spaces*, vol. 2016, Article ID 1035253, 9 pages, 2016.

[20] M. Mursaleen, A. Naaz, and A. Khan, “Improved approximation and error estimations by King type (p,q)-Szász-Mirakjan Kantorovich operators,” *Applied Mathematics and Computation*, vol. 348, pp. 175–185, 2019.

[21] Z. B. Zheng, J. W. Fang, W. T. Cheng, Z. D. Guo, and X. L. Zhou, "Approximation properties of modified (p,q)-Szász-Mirakyan-Kantorovich operators," *AIMS Mathematics*, vol. 5, no. 5, pp. 4959–4973, 2020.

[22] D. D. Stancu, "Approximation of functions by a new class of linear polynomials operators," *Revue Roumaine des Mathematiques Pures et Appliquees*, vol. 13, pp. 1173–1194, 1968.

[23] T. Acar, S. A. Mohiuddine, and M. Mursaleen, "Approximation by (p,q)-Baskakov–Durrmeyer–Stancu operators," *Complex Analysis and Operator Theory*, vol. 12, no. 6, pp. 1453–1468, 2018.

[24] Q. B. Cai and G. R. Zhou, "On (p,q)-analogue of Kantorovich type Bernstein-Stancu-Schurer operators," *Applied Mathematics and Computation*, vol. 276, pp. 12–20, 2016.

[25] R. A. Devore and G. G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[26] B. Lenze, "On Lipschitz-type maximal functions and their smoothness spaces," *Indagationes Mathematicae*, vol. 91, no. 1, pp. 53–63, 1988.

[27] N. Ispir, "On modified Baskakov operators on weighted spaces," *Turkish Journal of Mathematics*, vol. 25, pp. 355–365, 2001.

[28] A. D. Gadijev and P. P. On, "Korovkin type theorems," *Math Zametki*, vol. 20, no. 5, pp. 781–786, 1976.