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Research Article

Global Existence for Two Singular One-Dimensional Nonlinear Viscoelastic Equations with respect to Distributed Delay Term

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In this current work, we are interested in a system of two singular one-dimensional nonlinear equations with a viscoelastic, general source and distributed delay terms. The existence of a global solution is established by the theory of potential well, and by using the energy method with the function of Lyapunov, we prove the general decay result of our system.

1. Introduction

We are interested in the following system:

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s)\frac{1}{x}(xu_x(x,s))_x ds + \mu_1 u_t \\ +\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| u_t(x, t-\rho) d\rho = f_1(u, v), \text{ in } Q, \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s)\frac{1}{x}(xv_x(x,s))_x ds + \mu_3 v_t \\ +\int_{\tau_1}^{\tau_2} |\mu_4(\rho)| v_t(x, t-\rho) d\rho = f_2(u, v), \text{ in } Q, \end{cases} \quad (1)$$

with

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in (0, L), \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in (0, L), \\ u_t(x,-t) = f_0(x,t), v_t(x,-t) = g_0(x,t), t \in (0, \tau_2), \\ u(L,t) = v(L,t) = 0, \int_0^L xu(x,t)dx = \int_0^L xv(x,t)dx = 0, \end{cases} \quad (2)$$

where $Q = (0, L) \times (0, T)$, $L < \infty$, $T < \infty$, $g_1(.)$, $g_2(.)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $\mu_1, \mu_3 > 0$, the second integral represents the distributed delay and $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are bounded functions, where τ_1, τ_2 are two real numbers satisfying $0 \le \tau_1 < \tau_2$, and $f_1(.,.)$, $f_2(.,.)$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined functions later.

Three decades ago, these problems that arise in one-dimensional elasticity have been studied and developed with regard to viscosity with long-term memory. And it has been studied in many fields of science, engineering, medical sciences, and chemistry, as well as population and other matters; see, for example, [1–24]. Recently, in the absence of delay ($\mu_i = 0, i = 1..4$), problem (1) was studied in [25], and also later in [26], the authors considered problem (1) with localized frictional damping term. We also know that delay, especially distributed delay, is a phenomenon in our life and is almost found in various fields, and its inclusion in any problem makes it more important. The distributed delay in many works has been studied and many authors have taken care of it, for example, [5, 9, 27, 28]. Based on all this and the results of the research papers [14, 15, 17, 28–30, 31], the introduction of the term distributed delay as a damping mechanism in problem (1) makes it a new problem from what has been previously studied.

And we have divided this paper into the following. We present in the second section the definitions, basics, and theories of function spaces that are required throughout the rest of the paper. In Section 3, we present the energy function while proving to be decreasing. And in the final section, the general decay is obtained by applying the energy method and the function of Lyapunov.

2. Preliminaries

Let $L_x^p = L_x^p((0, L))$ be the weighted Banach space equipped with the norm

$$\|u\|_{L_x^p} = \left(\int_0^L x|u|^p dx \right)^{1/p}, \quad (3)$$

$H = L_x^2((0, L))$ be the Hilbert space of square integral functions having the finite norm

$$\|u\|_H = \left(\int_0^L xu^2 dx \right)^{1/2}, \quad (4)$$

and $K = L_x^2((0, L) \times (0, 1) \times (\tau_1, \tau_2))$ be the Hilbert space equipped with the norm

$$\|z\|_{K,\mu_2} = \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| \|z\|_H d\rho d\rho. \quad (5)$$

$V = V_x^1$ is the Hilbert space equipped with the norm

$$\|u\|_V = \left(\|u\|_H^2 + \|u_x\|_H^2 \right)^{1/2}, \quad (6)$$

$$V_0 = \{u \in V \text{ such that } u(L) = 0\}. \quad (7)$$

Theorem 1 [27]. *For $2 < p < 4$ and $\forall v$ in V_0, we have*

$$\int_0^L x|v|^p dx \leq C_* \|v_x\|_{H=L_x^2(0,L)}^p, \quad (8)$$

where C_* is a constant depending on L and p only.

As in [18], introducing the new variables

$$\begin{cases} z(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho), \\ y(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{cases} \quad (9)$$

yields

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = u_t(x, t), \end{cases}$$
$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = v_t(x, t). \end{cases} \quad (10)$$

Problem (1) arrives at

$$\begin{cases} u_{tt} - \frac{1}{x}(xu_x)_x + \int_0^t g_1(t-s) \frac{1}{x}(xu_x(x, s))_x ds + \mu_1 u_t \\ + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x, t - \varrho) d\varrho = f_1(u, v), \\ v_{tt} - \frac{1}{x}(xv_x)_x + \int_0^t g_2(t-s) \frac{1}{x}(xv_x(x, s))_x ds + \mu_3 v_t \\ + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x, t - \varrho) d\varrho = f_2(u, v), \\ \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \end{cases} \quad (11)$$

where

$$(x, \rho, s, t) \in (0, L) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty). \quad (12)$$

With the initial data and boundary conditions

$$\begin{cases} (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \text{ in } (0, L), \\ (u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), \text{ in } (0, L), \\ (u_t(x, -t), v_t(x, -t)) = (f_0(x, t), g_0(x, t)), \text{ in } (0, L) \times (0, \tau_2), \\ u(0, t) = u(L, t) = v(0, t) = v(L, t) = 0, \\ z(x, \rho, \varrho, 0) = f_0(x, \rho\varrho), \text{ in } (0, L) \times (0, 1) \times (0, \tau_2), \\ y(x, \rho, \varrho, 0) = g_0(x, \rho\varrho), \\ u(L, t) = v(L, t) = 0, \int_0^L xu(x, t) dx = \int_0^L xv(x, t) dx = 0. \end{cases} \quad (13)$$

We have the following assumptions:

(G1) $g_i(t)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are C^1, nonincreasing functions satisfying

$$\begin{cases} g_i(s) \geq 0, \quad g_i'(s) \leq 0, \\ g_i(0) > 0, 1 - \int_0^\infty g_i(s) ds = l_i > 0, i = 1, 2, \end{cases} \quad (14)$$

(G2) $\exists \xi(t) > 0$ a differentiable function, such that

$$g_i'(t) \leq -\xi(t) g_i^\sigma(t), i = 1, 2, t \geq 0, 1 \leq \sigma < \frac{3}{2}, \quad (15)$$

and $\xi(t)$ satisfies for some $l < 1$

$$\xi'(t) \leq 0, \left| \frac{\xi'(t)}{\xi(t)} \right| \leq l, \int_0^\infty \xi(s) ds = +\infty, \forall t > 0. \quad (16)$$

And also, where $1 < \sigma < 3/2$, $\forall t_0 > 0$ fixed, $\exists C_\sigma(\sigma) > 0$, such that

$$\frac{t}{\left(1+\int_{t_0}^{t}\xi(s)ds\right)^{1/(2(\sigma-1))}} \le C_\sigma, \quad \forall t \ge t_0. \tag{17}$$

(G3) we take

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2}, \\ f_2(u,v) &= a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2}, \end{aligned} \tag{18}$$

where $a, b > 0$ and $r > -1$.

We have

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v), \forall (u,v) \in \mathbb{R}^2, \tag{19}$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \tag{20}$$

(G4) $\mu_2, \mu_4 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|d\varrho < \mu_1, \tag{21}$$

$$\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|d\varrho < \mu_3. \tag{22}$$

Theorem 2. *Assume (14) and $p < 3$. Then, $\forall (u_0, v_0) \in V_0^2$, $(v_1, v_2) \in H^2$ and $(f_0, g_0) \in K^2$ problem (1) has a unique local solution*

$$(u,v,z,y) \in C\left(0,t_*; V_0^2 \times K^2\right) \cap C^1\left(0,t_*; H^2 \times K^2\right), \tag{23}$$

for $t_* > 0$ small enough.

Lemma 3. *For $r > -1$, $\exists \eta > 0$ such that $\forall u, v \in V \cap V_0(0,L)$, we have*

$$\|u+v\|_{L_x^{2(r+2)}}^{2(r+2)} + 2\|uv\|_{L_x^{(r+2)}}^{(r+2)} \le \eta\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right)^{r+2}. \tag{24}$$

Proof. It is clear that by using the Minkowski inequality we get

$$\|u+v\|_{L_x^{2(r+2)}}^2 \le 2\left(\|u\|_{L_x^{2(r+2)}}^2 + \|v\|_{L_x^{2(r+2)}}^2\right). \tag{25}$$

Also, Hölder's and Young's inequalities give us

$$\|uv\|_{L_x^{(r+2)}}^{(r+2)} \le \|u\|_{L_x^{2(r+2)}}\|v\|_{L_x^{2(r+2)}} \tag{26}$$

$$\le c\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right). \tag{27}$$

By applying the embedding $V \cap V_0(0,L) \hookrightarrow L_x^{2(r+2)}(0,L)$ and (25), (27) gives (15).

Lemma 4. *$\exists \Lambda_1, \Lambda_2 > 0$ such that*

$$\int_0^L x|f_i(u,v)|^2 dx \le \Lambda_i\left(l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx\right)^{2r+3}, \forall x \in (0,L), i = 1,2. \tag{28}$$

Proof. We prove inequality for f_1 and the same result also holds for f_2.

It is clear that

$$\begin{aligned} |f_1(u,v)| &\le C\left(|u+v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right) \\ &\le C\left[|u|^{2r+3} + |v|^{2r+3} + |u|^{r+1}|v|^{r+2}\right]. \end{aligned} \tag{29}$$

By Young's inequality, with

$$\begin{aligned} q &= \frac{2r+3}{r+1}, \\ q' &= \frac{2r+3}{r+2}, \end{aligned} \tag{30}$$

we get

$$|u|^{r+1}|v|^{r+2} \le c_1|u|^{2r+3} + c_2|v|^{2r+3}. \tag{31}$$

Therefore,

$$|f_1(u,v)| \le C\left[|u|^{2r+3} + |v|^{2r+3}\right]. \tag{32}$$

Hence, by Poincaré's inequality and (11), we obtain

$$\begin{aligned} \int_0^L x|f_i(u,v)|^2 dx &\le C\left(\|u_x\|_H^{2(2r+3)} + \|v_x\|_H^{2(2r+3)}\right) \\ &\le \Lambda_1\left(l_1\|u_x\|_H^2 + l_2\|v_x\|_H^2\right)^{(2r+3)}. \end{aligned} \tag{33}$$

The proof of lemma is complete.

The energy function (see, e.g., [8, 19] and reference therein) is defined by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\int_0^L xu_t^2 dx + \frac{1}{2}\int_0^L xv_t^2 dx + \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right) \\ & \cdot \int_0^L xu_x^2 dx + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right) \\ & \cdot \int_0^L xv_x^2(x,t)dx + \frac{1}{2}K(z,y) \\ & + \frac{1}{2}(g_1 \circ u_x)(t) + \frac{1}{2}(g_2 \circ v_x)(t) - \int_0^L F(u,v)dx, \end{aligned} \tag{34}$$

where

$$K(z,y)=\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)|z^2(x,\rho,\varrho,t)+|\mu_4(\varrho)|y^2(x,\rho,\varrho,t)\right)d\varrho d\rho dx,$$
$$(g\circ u_x)(t)=\int_0^L\int_0^t xg(t-s)|u_x(x,t)-u_x(x,s)|^2dsdx. \tag{35}$$

Lemma 5. *Let (u,v,z,y) be the solution of system (11); then, $E(t)$ is a nonincreasing function, that is, $\forall t\geq 0$*

$$E'(t)\leq -d_1\int_0^L xu_t^2dx-d_2\int_0^L xv_t^2dx+\frac{1}{2}\left(g_1'\circ u_x\right)(t)+\frac{1}{2}\left(g_2'\circ v_x\right)(t)-\frac{1}{2}g_1(t)\int_0^L xu_x^2dx-\frac{1}{2}g_2(t)\int_0^L xv_x^2dx\leq 0, \tag{36}$$

where

$$d_1=\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)>0,$$
$$d_2=\mu_3-\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)>0. \tag{37}$$

Proof. Multiplying equation $(11)_{1,2}$ by xu_t, xv_t, and integrating over $(0,L)$, we find

$$\begin{aligned}&\int_0^L xu_{tt}u_tdx-\int_0^L (xu_x)_xu_tdx+\mu_1\int_0^L xu_t^2dx\\&+\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\rho dx\\&+\int_0^L\int_0^t g_1(t-s)(xu_x(x,s))_xdsu_tdx\\&+\int_0^L xv_{tt}v_tdx-\int_0^L (xv_x)_xv_tdx+\mu_3\int_0^L xv_t^2dx\\&+\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx\\&+\int_0^L\int_0^t g_2(t-s)(xv_x(x,s))_xdsv_tdx\\&=\int_0^L\left[a|u+v|^{2(r+1)}(u+v)+b|u|^ru|v|^{r+2}\right]xu_tdx\\&+\int_0^L\left[a|u+v|^{2(r+1)}(u+v)+b|v|^rv|u|^{r+2}\right]xv_tdx.\end{aligned} \tag{38}$$

Using integration by parts, we get

$$\int_0^L xu_{tt}u_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_t^2dx\right], \tag{39}$$

$$\int_0^L xv_{tt}v_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_t^2dx\right], \tag{40}$$

$$-\int_0^L (xu_x)_xu_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xu_x^2dx\right], \tag{41}$$

$$-\int_0^L (xv_x)_xv_tdx=\frac{1}{2}\frac{d}{dt}\left[\int_0^L xv_x^2dx\right], \tag{42}$$

$$\begin{aligned}&\frac{1}{2(r+2)}\int_0^L xf_1(u,v)uu_tdx+\frac{1}{2(r+2)}\int_0^L xf_2(u,v)vv_tdx\\&=\frac{1}{2(r+2)}\frac{d}{dt}\int_0^L\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]xdx,\end{aligned} \tag{43}$$

$$\begin{aligned}&\int_0^L\int_0^t g_1(t-s)(xu_x(s))_xdsu_t(t)dx\\&=\frac{1}{2}\frac{d}{dt}\left[(g_1\circ u_x)(t)-\int_0^t g_1(s)ds\int_0^L xu_x^2dx\right]\end{aligned} \tag{44}$$

$$-\frac{1}{2}\left(g_1'\circ u_x\right)(t)+\frac{1}{2}g_1(t)\int_0^L xu_x^2dx, \tag{45}$$

$$\begin{aligned}&\int_0^L\int_0^t g_2(t-s)(xv_x(s))_xdsv_t(t)dx\\&=\frac{1}{2}\frac{d}{dt}\left[(g_2\circ v_x)(t)-\int_0^t g_2(t)ds\int_0^L xv_x^2dx\right]\\&-\frac{1}{2}\left(g_2'\circ v_x\right)(t)+\frac{1}{2}g_2(t)\int_0^L xv_x^2dx.\end{aligned} \tag{46}$$

Now, multiplying equation $(11)_3$ by $xz|\mu_2(\varrho)|$ and integrating over $(0,L)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|xz^2d\varrho d\rho dx\\&=-\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|xzz_\rho d\varrho d\rho dx\\&=-\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|\frac{d}{d\rho}z^2d\varrho d\rho dx\\&=\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|\left((z(x,0,\varrho,t))^2-(z(x,1,\varrho,t))^2\right)d\varrho dx\\&=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\int_0^L|xu_t|^2dx\\&-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}x|\mu_2(\varrho)|(z(x,1,\varrho,t))^2d\varrho dx.\end{aligned} \tag{47}$$

Similarly, by multiplying equation $(11)_4$ by $xy|\mu_4(\varrho)|$

and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, we get

$$\frac{d}{dt}\frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_4(\varrho)|xy^2 d\varrho d\rho dx = \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\int_0^L xv_t^2 dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx. \quad (48)$$

Using Young's and Cauchy-Schwartz inequalities, we have

$$-\int_0^L xu_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z(x,1,\varrho,t)d\varrho dx \le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} \mid \mu_2(\varrho) \mid d\varrho\right)\int_0^L xu_t^2 dx + \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|xz^2(x,1,\varrho,t)d\varrho dx. \quad (49)$$

Similarly, we get

$$-\int_0^L xv_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y(x,1,\varrho,t)d\varrho dx \le \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\int_0^L xv_t^2 dx + \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|xy^2(x,1,\varrho,t)d\varrho dx. \quad (50)$$

By combining (39), (40), (41), (42), (43), (45), (46), (47), (48), (49), and (50) in (38), we get (34) and (36).

3. Global Existence

In this section, we showed the global existence of the solutions of the system (11).

First, introducing the following notation

$$I(t) := I(u(t), v(t)) = \left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx + (g_1 \circ u_x)(t) + \left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx + (g_2 \circ v_x)(t) + K(z,y) - 2(r+2)\int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx, \quad (51)$$

$$J(t) := J(u(t), v(t)) = \frac{1}{2}\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx + \frac{1}{2}(g_1 \circ u_x)(t) + \frac{1}{2}\left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx + \frac{1}{2}(g_2 \circ v_x)(t) + \frac{1}{2}K(z,y) - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx, \quad (52)$$

note that

$$E(t) = J(t) + \frac{1}{2}\int_0^L xu_t^2 dx + \frac{1}{2}\int_0^L xv_t^2 dx. \quad (53)$$

Lemma 6. *Assume that (24), (14), (15), (16), (17), and (22) hold, and* $\forall (u_0, v_0) \in V_0^2$, $(u_1, v_1) \in H^2$ *and* $(f_0, g_0) \in L_x^2((0,L),(0,1),(\tau_1,\tau_2))$ *satisfying*

$$I(0) > 0, \beta := \eta\left(\frac{2(r+2)}{(r+1)}E(0)\right)^{r+1} < 1. \quad (54)$$

Then, $\exists t_* > 0$ *such that*

$$I(t) > 0, \forall t \in [0, t_*), \quad (55)$$

where

$$E(0) = J(0) + \frac{1}{2}\int_0^L xu_1^2 dx + \frac{1}{2}\int_0^L xv_1^2 dx. \quad (56)$$

Proof. As $I(0) > 0$, then by continuity of $I(t)$, $\exists T_m \le t_*$ such that $I(t) \ge 0$, $\forall t \in [0, T_m)$; this implies that we have a maximum time value noting T_m such that

$$\{I(T_m) = 0 \text{ and } I(t) > 0, \quad \text{for all } 0 \le t < T_m\}. \quad (57)$$

This, with (51), (52), and (14), we have

$$J(t) = \frac{r+1}{2(r+2)}\left[\left(1 - \int_0^t g_1(s)ds\right)\int_0^L xu_x^2 dx + \left(1 - \int_0^t g_2(s)ds\right)\int_0^L xv_x^2 dx\right] + \frac{r+1}{2(r+2)}[(g_1 \circ u_x)(t) + (g_2 \circ v_x)(t) + K(z,y)] + \frac{1}{2(r+2)}I(t) \ge \frac{r+1}{2(r+2)}\left[\left(l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx\right) + (g_1 \circ u_x)(t) + (g_2 \circ v_x)(t) + K(z,y)\right]. \quad (58)$$

Hence,

$$l_1\int_0^L xu_x^2 dx + l_2\int_0^L xv_x^2 dx \le \frac{2(r+2)}{r+1}J(t) \le \frac{2(r+2)}{r+1}E(t) \le \frac{2(r+2)}{r+1}E(0), \quad \forall t \in [0, T_m). \quad (59)$$

By (24) and (54), we get

$$\begin{aligned}2(r+2)\int_0^L F(u(T_m),v(T_m))dx &\le \eta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)^{r+2}\\ &\le \eta\left(\frac{2(r+2)}{r+1}E(0)\right)^{r+1}\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)\\ &= \beta\left(l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx\right)\\ &< \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx\\ &\quad + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx\\ &\quad + (g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y).\end{aligned} \tag{60}$$

Hence,

$$\begin{aligned}&\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx + \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx\\ &\quad + (g_1\circ u_x)(t) + (g_2\circ v_x)(t) + K(z,y)\\ &\quad - 2(r+2)\int_0^L xF(u,v)dx > 0.\end{aligned} \tag{61}$$

This proves that $I(t) > 0, \forall t \in [0, T_m)$. By repeating the procedure, T_m is extended to t_*.

Theorem 7. *Let (14), (15), (16), (17), (22), and (24) hold. Then, $\forall (u_0, v_0) \in V_0^2$, $(u_1, v_1) \in H^2$, and $(f_0, g_0) \in L_x^2((0,L),(0,1),(\tau_1,\tau_2))$ satisfying (54) the solution of system (11) is bounded and global.*

Proof. To prove that $\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \|y\|_{K,\mu_4}^2$ is bounded independently of t, using (36) yields

$$E(0) \ge E(t). \tag{62}$$

Using (52), we find

$$\begin{aligned}&-2(r+2)\int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx\\ &= I(t) - \left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx\\ &\quad - \left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2dx\\ &\quad - (g_1\circ u_x)(t) - (g_2\circ v_x)(t) - K(z,y).\end{aligned} \tag{63}$$

By using (62) in (63), we get

$$\begin{aligned}E(0) \ge E(t) &= \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx\\ &\quad + \frac{1}{2}\left(1-\int_0^t g_1(s)ds\right)\int_0^L xu_x^2dx\\ &\quad + \frac{1}{2}\left(1-\int_0^t g_2(s)ds\right)\int_0^L xv_x^2(x,t)dx\\ &\quad + \frac{1}{2}(g_1\circ u_x)(t) + \frac{1}{2}(g_2\circ v_x)(t)\\ &\quad + \frac{1}{2}K(z,y) + I(t),\end{aligned} \tag{64}$$

and using (14), (15), and (54) in (64), we get

$$\begin{aligned}E(0) \ge E(t) &\ge \frac{1}{2}\int_0^L xu_t^2dx + \frac{1}{2}\int_0^L xv_t^2dx\\ &\quad + \left(\frac{r+1}{2(r+2)}\right)\left\{l_1\int_0^L xu_x^2dx + l_2\int_0^L xv_x^2dx + K(z,y)\right\}\\ &\ge \mu_0\left(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx\right.\\ &\quad \left. + \int_0^L xv_x^2dx + K(z,y)\right).\end{aligned} \tag{65}$$

So

$$\|u_x\|_H^2 + \|v_x\|_H^2 + \|u_t\|_H^2 + \|v_t\|_H^2 + \|z\|_{K,\mu_2}^2 + \|y\|_{K,\mu_4}^2 \le \mu E(0)/\mu := \frac{1}{\mu_0}, \tag{66}$$

where

$$\mu_0 := \min\left\{\frac{1}{2}, \frac{(r+1)}{2(r+2)}l_1, \frac{(r+1)}{2(r+2)}l_2, \frac{(r+1)}{2(r+2)}\right\}. \tag{67}$$

Hence, the solution of system (11) is bounded and global.

4. Decay of Solutions

In this section, the decay result is showed by using several lemmas.

As, we let

$$F(t) := E(t) + \varepsilon_1\Phi(t) + \varepsilon_2\chi(t) + \varepsilon_3\Psi(t), \tag{68}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, and

$$\Phi(t) := \xi(t)\int_0^L xu_tu\,dx + \xi(t)\int_0^L xv_tv\,dx, \tag{69}$$

$$\chi(t) := -\xi(t)\int_0^L xu_t \int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ -\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx, \quad (70)$$

$$\Psi(z,y) := \xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\rho\varrho}\left(|\mu_2(\varrho)|z^2 + |\mu_4(\varrho)|y^2\right)d\varrho d\rho dx. \quad (71)$$

Lemma 8. *There exist $\alpha_1, \alpha_2 > 0$, such that*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t), \quad (72)$$

for ε_1, ε_2, and ε_3 small enough.

Proof. Using the inequality of Young and the Poincaré-type inequality and $0 < \xi(t) \le \xi(0)$, we find

$$\varepsilon_1\xi(t)\int_0^L xu_t u dx \le \frac{\varepsilon_1}{2}\xi(0)\int_0^L xu_t^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx, \quad (73)$$

$$\varepsilon_1\xi(t)\int_0^L xv_t v dx \le \frac{\varepsilon_1}{2}\xi(0)\int_0^L xv_t^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx, \quad (74)$$

$$-\varepsilon_2\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ \le \frac{\varepsilon_2}{2}\xi(0)\int_0^L xu_t^2dx + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t), \quad (75)$$

$$-\varepsilon_2\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ \le \frac{\varepsilon_2}{2}\xi(0)\int_0^L xv_t^2dx + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t), \quad (76)$$

$$\Psi(z,y) \le \xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2 + |\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx, \quad (77)$$

where $C_p > 0$.

A combination of (73), (74), (75), (76), and (77) in (68) gives

$$F(t) \le E(t) + \left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L xu_t^2dx + \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L xv_t^2dx \\ + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx + \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx \\ + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t) \\ + \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t) \\ + \varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)\mid z^2 + |\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx. \quad (78)$$

Then, $\exists\alpha_1 > 0$, for ε_1, ε_2, and ε_3 small enough, such that

$$F(t) \le \frac{1}{\alpha_1}E(t). \quad (79)$$

Similarly, thanks to the inequalities of Young and Poincaré-type and using $0 < \xi(t) \le \xi(0)$ gives

$$\varepsilon_1\xi(t)\int_0^L xu_t u dx \ge \frac{-\varepsilon_1}{2}\xi(0)\int_0^L xu_t^2dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2dx, \quad (80)$$

$$\varepsilon_1\xi(t)\int_0^L xv_t v dx \ge \frac{-\varepsilon_1}{2}\xi(0)\int_0^L xv_t^2dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2dx, \quad (81)$$

$$-\varepsilon_2\xi(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ \ge \frac{-\varepsilon_2}{2}\xi(0)\int_0^L xu_t^2dx - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t), \quad (82)$$

$$-\varepsilon_2\xi(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ \ge \frac{-\varepsilon_2}{2}\xi(0)\int_0^L xv_t^2dx - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t), \quad (83)$$

and

$$-\varepsilon_3\Psi(z,y) \ge -\varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2 + |\mu_4(\rho\varrho)\mid y^2\right)d\varrho d\rho dx. \quad (84)$$

By combining (80), (81), (82), (83), and (84) in (68), we find

$$F(t) \geq E(t) - \left(\frac{\varepsilon_{1+}\varepsilon_2}{2}\right)\xi(0)\int_0^L xu_t^2 dx - \left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\xi(0)\int_0^L xv_t^2 dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xu_x^2 dx - \frac{\varepsilon_1}{2}C_p\xi(0)\int_0^L xv_x^2 dx - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_1)(g_1\circ u_x)(t) - \frac{\varepsilon_2}{2}C_p\xi(0)(1-l_2)(g_2\circ v_x)(t) - \varepsilon_3\xi(0)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx. \tag{85}$$

Then, $\exists\alpha_2>0$, for ε_1, ε_2, and ε_3 small enough, such that

$$F(t) \geq \frac{1}{\alpha_2}E(t). \tag{86}$$

This completes the proof.

Lemma 9. *For $\sigma>1$ and $0<\theta<1$, we have*

$$\int_0^t g(t-s)\|w(s)\|^2 ds \leq \left(\int_0^t g^{1-\theta}(t-s)\|w(s)\|^2 ds\right)^{1/\sigma} \times \left(\int_0^t g^{(\sigma-1+\theta)/\sigma-1}(t-s)\|w(s)\|^2 ds\right)^{(\sigma-1)/\sigma}, \tag{87}$$

$\forall w \in H$.

Proof. It suffices to note that

$$\int_0^t g(t-s)\|w(s)\|^2 ds = \int_0^t g^{(1-\theta)/r}(t-s)\|w(s)\|^{2/r} g^{(\sigma-1+\theta)/\sigma}\cdot(t-s)\|w(s)\|^{(2(\sigma-1))/\sigma}ds, \tag{88}$$

using Hölder's inequality for

$$p=\sigma, \quad q=\frac{\sigma}{\sigma-1}, \quad r>1. \tag{89}$$

This completes the proof.

Lemma 10. *Let $v\in L^\infty((0,T);H)$ be such that $v_x\in L^\infty((0,t);H)$ and g be a continuous function on $[0,T]$ and suppose that $0<\theta<1$ and $\rho>1$. Then, $\exists C>0$ so that*

$$\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq C\left(\sup_{0<s<T}\|v(.,s)\|_H^2\int_0^t g^{1-\theta}(s)ds\right)^{(\rho-1)/(\rho-1+\theta)} \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{\theta/(\rho-1+\theta)}. \tag{90}$$

Proof. By applying Lemma 8 with $\sigma=(\rho-1+\theta)/(\rho-1)$ gives

$$\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq \left(\int_0^t g^{1-\theta}(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/(\rho-1+\theta)} \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2\right)^{\theta/(\rho-1+\theta)}. \tag{91}$$

We also have

$$\int_0^t g^{1-\theta}(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq C\sup_{0<s<T}\|v_x(.,s)\|_H^2\int_0^t g^{1-\theta}(s)ds, \tag{92}$$

by combining (82) and (83). This completes the proof.

Lemma 11. *Suppose that $v\in L^\infty((0,T);H)$ be such that $v_x\in L^\infty((0,T);H)$ and g be a continuous function on $[0,T]$ and assume $\rho>1$. Then, $\exists C>0$ so that*

$$\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq c\left(t\|v_x(.,t)\|_H^2+\int_0^t\|v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/\rho} \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{1/\rho}. \tag{93}$$

Proof. By using (82) for $\theta=1$ gives

$$\int_0^t g(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds \leq \left(\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{(\rho-1)/\rho} \times \left(\int_0^t g^\rho(t-s)\|v_x(.,t)-v_x(.,s)\|_H^2 ds\right)^{1/\rho}, \tag{94}$$

where

$$\int_0^t \|v_x(.,t)-v_x(.,s)\|_H^2 ds \le 2t\|v_x(.,t)\|_H^2+2\int_0^t \|v_x(.,s)\|_H^2 ds, \quad (95)$$

to obtain (93). Hence, this ends the proof.

Lemma 12. *Suppose that r satisfies (15) and (52) hold. Then, the functional $\Phi(t)$, given by (69), satisfies*

$$\begin{aligned}\Phi'(t) \le{}& \left(1+\frac{l}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L xu_t^2 dx \\ &+ \left(1+\frac{l}{2\delta}+\frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L xv_t^2 dx - \xi(t) \\ &\cdot \left(\frac{l_1 - C_p(\delta l - 2\delta_1\mu_1)}{2}\right)\int_0^L xu_x^2 dx - \xi(t) \\ &\cdot \left(\frac{l_2 - C_p(\delta l - 2\delta_2\mu_3)}{2}\right)\int_0^L xv_x^2 dx \\ &+ \frac{\xi(t)}{2l_1}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) + \frac{\xi(t)}{2l_2} \\ &\cdot \left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t) \\ &+ \frac{\xi(t)}{2\delta_1}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\ &+ \frac{\xi(t)}{2\delta_2}\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &+ \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \quad (96)$$

For any $\delta, \delta_1, \delta_2 > 0$.

Proof. The derivation of (11) gives

$$\begin{aligned}\Phi'(t) ={}& \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx + \xi(t)\int_0^L xu_{tt} u dx \\ &+ \xi'(t)\int_0^L xv_t v dx + \xi(t)\int_0^L xv_t^2 dx + \xi(t)\int_0^L xv_{tt} v dx \\ ={}& \xi'(t)\int_0^L xu_t u dx + \xi(t)\int_0^L xu_t^2 dx - \xi(t)\int_0^L xu_x^2 dx \\ &- \xi(t)\mu_1\int_0^L xuu_t dx - \xi(t)\int_0^L xu\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\ &+ \xi(t)\int_0^L xu_x\int_0^t g_1(t-s)u_x(s)ds dx + \xi'(t)\int_0^L xv_t v dx \\ &+ \xi(t)\int_0^L xv_t^2 dx - \xi(t)\int_0^L xv_x^2 dx - \xi(t)\mu_3\int_0^L xvv_t dx \\ &- \xi(t)\int_0^L xv\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &+ \xi(t)\int_0^L xv_x\int_0^t g_2(t-s)v_x(s)ds dx \\ &+ \frac{\xi(t)}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \quad (97)$$

By Young's and Poincaré inequalities and (14) and (15), we find

$$\begin{aligned}&\xi(t)\int_0^L xu_x(t)\left(\int_0^t g_1(t-s)u_x(s)ds\right)dx \\ &\le \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}\int_0^L x\left(\int_0^t g_1(t-s)(|u_x(s) - u_x(t)| + |u_x(t)|)ds\right)^2 dx \\ &\le \frac{\xi(t)}{2}\int_0^L xu_x^2 dx + \frac{\xi(t)}{2}(1+\eta_1)(1-l_1)^2\int_0^L xu_x^2(t)dx \\ &\quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\ &= \xi(t)\left(\frac{1+(1+\eta_1)(1-l_1)^2}{2}\right)\int_0^L xu_x^2 dx \\ &\quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma \circ u_x)(t) \\ &\quad + \frac{\xi(t)}{r+2}\int_0^L\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx.\end{aligned} \quad (98)$$

Similarly, we get

$$\begin{aligned}&\int_0^L xv_x(t)\left(\int_0^t g_1(t-s)v_x(s)ds\right)dx \\ &\le \xi(t)\left(\frac{1+(1+\eta_2)(1-l_2)^2}{2}\right)\int_0^L xv_x^2 dx \\ &\quad + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma \circ v_x)(t).\end{aligned} \quad (99)$$

$\forall \eta_1, \eta_2 > 0$. As we have

$$\begin{aligned}\xi'(t)\int_0^L xu_t u dx &\le \frac{\xi(t)}{2}\left|\frac{\xi'(t)}{\xi(t)}\right|\left(C_p\delta\int_0^L xu_x^2 dx + \frac{1}{\delta}\int_0^L xu_t^2 dx\right) \\ &\le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xu_x^2 dx + \frac{l}{\delta}\int_0^L xu_t^2 dx\right), \forall \delta > 0,\end{aligned} \quad (100)$$

and similarly, we find

$$\xi'(t)\int_0^L xv_t v dx \le \frac{\xi(t)}{2}\left(C_p l\delta\int_0^L xv_x^2 dx + \frac{l}{\delta}\int_0^L xv_t^2 dx\right). \quad (101)$$

By using Young's and Poincaré's inequalities and (22) gives

$$-\xi(t)\int_0^L xu\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \le \frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{1}{\delta_1}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\right), \quad (102)$$

$$-\xi(t)\int_0^L xv\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \le \frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{1}{\delta_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\right). \quad (103)$$

Similarly, we get

$$\xi(t)\int_0^L xu_tudx \le \frac{\xi(t)}{2}\left(C_p\delta_1\mu_1\int_0^L xu_x^2dx+\frac{\mu_1}{\delta_1}\int_0^L xu_t^2dx\right)\xi(t)\int_0^L xv_tvdx \le \frac{\xi(t)}{2}\left(C_p\delta_2\mu_3\int_0^L xv_x^2dx+\frac{\mu_3}{\delta_2}\int_0^L xv_t^2dx\right). \quad (104)$$

In a combination of (98), (99), (100), (101), (102), (103), and (104) in (97), we obtain

$$\begin{aligned}\Phi'(t) \le{}& \left(1+\frac{l}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\xi(t)\int_0^L xu_t^2dx + \left(1+\frac{l}{2\delta}+\frac{\mu_3}{2\delta_2}\right)\xi(t)\int_0^L xv_t^2dx \\ &- \frac{\xi(t)}{2}\left[1-(1+\eta_1)(1-l_1)^2-\delta C_pl-2\delta_1C_p\mu_1\right]\cdot\int_0^L xu_x^2dx - \frac{\xi(t)}{2}\left[1-(1+\eta_2)(1-l_2)^2-\delta C_pl-2\delta_2C_p\mu_3\right] \\ &\cdot\int_0^L xv_x^2dx + \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_1}\right)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\ &+ \frac{\xi(t)}{2}\left(1+\frac{1}{\eta_2}\right)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) \\ &+ \frac{\xi(t)}{2}\int_0^L\int_{\tau_1}^{\tau_2}\left(\frac{1}{\delta_1}|\mu_2(\varrho)|z^2(x,1,\varrho,t) + \frac{1}{\delta_2}|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx \\ &+ \frac{\xi(t)}{2(r+2)}\int_0^L\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx,\end{aligned} \quad (105)$$

by choosing η_1, η_1, so that $\eta_1 = l_1/(1-l_1)$; hence, $(1/2)(-1+(1+\eta_1)(1-l_1)^2) = -l_1/2$ and $(1+(1/\eta_1)) = 1/l_1$,and $\eta_2 = l_2/(1-l_2)$; therefore, $(1/2)(-1+(1+\eta_2)(1-l_2)^2) = -l_2/2$ and $(1+(1/\eta_2)) = 1/l_2$.

Then, (96) is proved.

Lemma 13. *Assuming that r satisfies (15), (14), and (15) and (22) and (52) hold. Then, the functional $\chi(t)$ given by (70) satisfies along the solution of (11)*

$$\begin{aligned}\chi'(t) \le{}& \xi(t)\theta\left[1+c_1+c_1'+2(1-l_1)^2\right]\left(\int_0^L xu_x^2dx\right) \\ &+ \xi(t)\theta\left[1+c_2+c_2'+2(1-l_2)^2\right]\left(\int_0^L xv_x^2dx\right) \\ &+ \xi(t)\left[\theta-\left(\int_0^t g_1(s)ds\right)+\theta l+\theta_1\mu_1\right]\left(\int_0^L xu_t^2dx\right) \\ &+ \xi(t)\left[\theta-\left(\int_0^t g_2(s)ds\right)+\theta l+\theta_2\mu_3\right]\left(\int_0^L xv_t^2dx\right) \\ &+ \left[\frac{1}{2\theta}+2\theta+\frac{\mu_1C_p}{2\theta_1}+\frac{C_p(1+l)}{4\theta}\right] \times\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\ &+ \left[\frac{1}{2\theta}+2\theta+\frac{\mu_3C_p}{2\theta_2}+\frac{C_p(1+l)}{4\theta}\right] \times\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) \\ &- \frac{C_p}{4\theta}\xi(t)g_1(0)\left(g_1'\circ u_x\right)(t) - \frac{C_p}{4\theta}\xi(t)g_2(0)\left(g_2'\circ v_x\right)(t) \\ &+ \xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x\big(\theta_1|\mu_2(\varrho)|z^2(x,1,\varrho,t) + \theta_2|\mu_4(\varrho)|y^2(x,1,\varrho,t)\big)d\varrho dx,\end{aligned} \quad (106)$$

for any $\theta, \theta_1, \theta_2 > 0$.

Proof. Direct calculation gives

$$\begin{aligned}\chi'(t) ={}& -\xi'(t)\int_0^L xu_t\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &- \xi(t)\int_0^L xu_{tt}\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\ &- \xi(t)\int_0^L xu_t\frac{d}{dt}\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\ &- \xi'(t)\int_0^L xv_t\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ &- \xi(t)\int_0^L xv_{tt}\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\ &- \xi(t)\int_0^L xv_t\frac{d}{dt}\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx,\end{aligned} \quad (107)$$

by using

$$\frac{d}{dt}\left(\int_{\alpha(t)}^{\beta(t)} f(t,s)ds\right) = \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(t,s)}{\partial t} ds + \frac{\partial \beta(t)}{\partial t} f(t,\beta(t)) - \frac{\partial \alpha(t)}{\partial t} f(t,\alpha(t)). \quad (108)$$

As we have (u, v, z, y) the solution of (11), we find

$$\begin{aligned}
\chi'(t) = &-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&+\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right) \\
&\cdot\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&-\xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| z^2(x,1,\rho,t)d\rho\right) \\
&\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx-\xi(t)\int_0^L xf_1(u,v) \\
&\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx-\xi(t)\int_0^L xu_t \\
&\cdot\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx-\xi(t) \\
&\cdot\left(\int_0^t g_1(s)ds\right)\int_0^L xu_t^2dx-\xi(t)\left(\int_0^t g_2(s)ds\right)\int_0^L xv_t^2dx \\
&-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&+\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\
&-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right) \\
&\cdot\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx-\xi(t)\mu_3\int_0^L xv_t \\
&\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx-\xi(t)\int_0^L x \\
&\cdot\left(\int_{\tau_1}^{\tau_2} |\mu_4(\rho)| y^2(x,1,\rho,t)d\rho\right) \\
&\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&-\xi(t)\int_0^L xf_2(u,v)\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx.
\end{aligned} \quad (109)$$

By Young's inequality and (14) and (15), we arrive to

$$\begin{aligned}
&-\xi'(t)\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\leq \xi(t)\left|\frac{\xi'(t)}{\xi(t)}\right|\left[\theta\int_0^L xu_t^2dx+\frac{C_p}{4\theta}\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\right] \\
&\leq \theta l\xi(t)\int_0^L xu_t^2dx+\frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),
\end{aligned} \quad (110)$$

$$\begin{aligned}
&\xi(t)\int_0^L xu_x\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&\leq \theta\xi(t)\int_0^L xu_x^2dx+\frac{1}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t).
\end{aligned} \quad (111)$$

Similarly, we get

$$\begin{aligned}
&\xi(t)\mu_1\int_0^L xu_t\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\leq \theta_1\mu_1\xi(t)\int_0^L xu_t^2dx+\frac{1}{4\theta_1}C_p\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t),
\end{aligned} \quad (112)$$

$$\begin{aligned}
&\xi(t)\mu_3\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\
&\leq \theta_2\mu_3\xi(t)\int_0^L xv_t^2dx+\frac{1}{4\theta_2}C_p\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t),
\end{aligned} \quad (113)$$

with

$$\begin{aligned}
&-\xi(t)\int_0^L x\left(\int_0^t g_1(t-s)u_x(s)ds\right)\left(\int_0^t g_1(t-s)(u_x(t)-u_x(s))ds\right)dx \\
&\leq 2\theta(1-l_1)^2\xi(t)\int_0^L xu_x^2dx+\left(2\theta+\frac{1}{4\theta}\right)\xi(t) \\
&\quad\cdot\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t).
\end{aligned} \quad (114)$$

So

$$\begin{aligned}
&-\xi(t)\int_0^L xf_1(u,v)\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx \\
&\leq \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t) \\
&\quad+c_1\theta\xi(t)\int_0^L xu_x^2dx+c_2\theta\xi(t)\int_0^L xv_x^2dx,
\end{aligned} \quad (115)$$

where

$$\begin{cases} c_1 := \Lambda_1\left(\dfrac{2(r+2)}{(r+1)}E(0)\right)^{2(r+1)}, \\ c_2 := \Lambda_2\left(\dfrac{2(r+2)}{(r+1)}E(0)\right)^{2(r+1)}, \end{cases} \tag{116}$$

$$\begin{aligned} &-\xi(t)\int_0^L xu_t\left(\int_0^t g_1'(t-s)(u(t)-u(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xu_t^2dx - \frac{g_1(0)}{4\theta}C_p\xi(t)\left(g_1'\circ u_x\right)(t). \end{aligned} \tag{117}$$

Then,

$$\begin{aligned} &-\xi'(t)\int_0^L xv_t\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ &\quad\le \theta l\xi(t)\int_0^L xv_t^2dx + \frac{C_pl}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{118}$$

$$\begin{aligned} &\xi(t)\int_0^L xv_x\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xv_x^2dx + \frac{1}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t). \end{aligned} \tag{119}$$

Thus,

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_0^t g_2(t-s)v_x(s)ds\right)\left(\int_0^t g_2(t-s)(v_x(t)-v_x(s))ds\right)dx \\ &\le 2\theta(1-l_2)^2\xi(t)\int_0^L xv_x^2dx + \left(2\theta+\frac{1}{4\theta}\right)\xi(t) \\ &\quad\cdot\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t), \end{aligned} \tag{120}$$

$$\begin{aligned} &-\frac{\xi(t)}{2(r+2)}\int_0^L xf_2(u,v)\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx \\ &\le \frac{C_p}{4\theta}\xi(t)\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t) + c_1'\theta\xi(t) \\ &\quad\cdot\int_0^L xu_x^2dx + c_2'\theta\xi(t)\int_0^L xv_x^2dx, \end{aligned} \tag{121}$$

where

$$\begin{cases} c_1' := \Lambda_1'\left(\dfrac{2(r+2)}{r+1}E(0)\right)^{2(r+1)}, \\ c_2' := \Lambda_2'\left(\dfrac{2(r+2)}{r+1}E(0)\right)^{2(r+1)}, \end{cases} \tag{122}$$

$$\begin{aligned} &-\xi(t)\int_0^L xv_t\left(\int_0^t g_2'(t-s)(v(t)-v(s))ds\right)dx \\ &\quad\le \theta\xi(t)\int_0^L xv_t^2dx - \frac{g_2(0)}{4\theta}C_p\xi(t)\left(g_2'\circ v_x\right)(t). \end{aligned} \tag{123}$$

Similarly, we have

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} \mid\mu_2(\rho)\mid z^2(x,1,\rho,t)d\rho\right) \\ &\quad\cdot\left(\int_0^t g_1(t-s)(u(t)-u(s))ds\right)dx\theta_1\xi(t) \\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\rho)|z^2(x,1,\rho,t)d\rho dx \\ &\quad+\frac{1}{4\theta_1}\mu_1C_p\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t), \end{aligned} \tag{124}$$

$$\begin{aligned} &-\xi(t)\int_0^L x\left(\int_{\tau_1}^{\tau_2} \mid\mu_4(\varrho)\mid y^2(x,1,\varrho,t)d\varrho\right) \\ &\quad\cdot\left(\int_0^t g_2(t-s)(v(t)-v(s))ds\right)dx\theta_2\xi(t) \\ &\quad\cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ &\quad+\frac{1}{4\theta_2}\mu_3C_p\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t). \end{aligned} \tag{125}$$

A combination of (110), (111), (112), (113), (114), (115), (117), (118), (119), (120), (121), (123), (124), and (125) into (109) gives (106).

Lemma 14. *Let (u,v,z,y) be the solution of (11). Then, for $\eta_3>0$, the functional $\Psi(t)$ satisfies*

$$\begin{aligned} \Psi'(t) \le& -\xi(t)\eta_4\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)\mid z^2+|\mu_4(\varrho)\mid y^2\right)d\varrho d\rho dx \\ &+\xi(t)\mu_1\int_0^L xu_t^2dx + \xi(t)\mu_3\int_0^L xv_t^2dx \\ &-\xi(t)\eta_3\int_0^L\int_{\tau_1}^{\tau_2} x\left(|\mu_2(\varrho)\mid z^2(x,1,\varrho,t)\right. \\ &\left.+|\mu_4(\varrho)\mid y^2(x,1,\varrho,t)\right)d\varrho dx, \end{aligned} \tag{126}$$

where $\eta_3>0$ and $\eta_4=\eta_3(1-l)>0>0$.

Proof. By differentiating $\Psi(t)$ and using equations $(11)_3$ and $(11)_4$, we get

$$\begin{aligned}\Psi'(t) &= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad -2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_2(\varrho)|zz_\rho(x,\rho,\varrho,t)d\varrho d\rho dx\\ &\quad -2\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} e^{-\varrho\rho}|\mu_4(\varrho)|yy_\rho(x,\rho,\varrho,t)d\varrho d\rho dx\\ &= \xi'(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_2(\varrho)|z^2 d\varrho d\rho dx\\ &\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|\left[e^{-\varrho}z^2(x,1,\varrho,t)-z^2(x,0,\varrho,t)\right]d\varrho dx\\ &\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}|\mu_4(p)|y^2 d\rho\varrho\rho dx\\ &\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|\left[e^{-\varrho}y^2(x,1,\varrho,t)-y^2(x,0,\varrho,t)\right]d\varrho dx.\end{aligned} \tag{127}$$

Using the equality $z(x,0,\varrho,t)=u_t(x,t)$, $y(x,0,\varrho,t)=v_t(x,t)$, and $e^{-\varrho}\le e^{-\rho\varrho}\le 1$, for any $0<\rho<1$, we find

$$\begin{aligned}\Psi'(t) &\le \xi(t)l\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|z^2\right)d\varrho d\rho dx\\ &\quad -\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\varrho e^{-\varrho\rho}\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx\\ &\quad -\xi(t)\int_0^L\int_{\tau_1}^{\tau_2} xe^{-\varrho}\left(|\mu_2(\varrho)|z^2(x,1,\varrho,t)+|\mu_4(\varrho)|y^2(x,1,\varrho,t)\right)d\varrho dx\\ &\quad +\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\xi(t)\int_0^L xu_t^2dx\\ &\quad +\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\right)\xi(t)\int_0^L xv_t^2dx.\end{aligned} \tag{128}$$

As $-e^{-\varrho}$ is an increasing function, we have $-e^{-\varrho}\le -e^{-\tau_2}$, for any $\rho\in[\tau_1,\tau_2]$.

Then, setting $\eta_3=e^{-\tau_2}$ and (22), we obtain (126).

Theorem 15. *Let $(u_0,v_0)\in V_0^2$, $(u_1,v_1)\in H^2$, and $(f_0,g_0)\in L_x^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$ be defined and satisfy (163). Assume that r satisfies (24), (14), (15), (16), (17), and (22) hold. Then, for each $t_0>0$, $\exists K$ and k such that the solution of (11) satisfies $\forall t\ge t_0$, we have the following inequality for the energy function*

$$E(t)\le\begin{cases}Ke^{-k\int_{t_0}^t\xi(s)ds}, & \sigma=1,\\ K\left(1+\int_{t_0}^t\xi(s)ds\right)^{-(1/(\sigma-1))}, & 1<\sigma<\dfrac{3}{2}.\end{cases} \tag{129}$$

Proof. As g_1,g_2 is continuous and $g_1(0),g_2(0)>0$, hence $\forall t_0>0$; we have

$$\begin{cases}\int_0^t g_1(s)ds\ge\int_{t_0}^t g_1(s)ds=g_{1,0}>0, & \forall t\ge t_0,\\ \int_0^t g_2(s)ds\ge\int_{t_0}^t g_2(s)ds=g_{2,0}>0, & \forall t\ge t_0.\end{cases} \tag{130}$$

By using (36), (96), (106), (126), and (130) and $0<\xi(t)\le\xi(0)$) (hence $(\xi(t)/\xi(0))<1$), we get

$$\begin{aligned}F'(t) &= E'(t)+\varepsilon_1\Phi'(t)+\varepsilon_2\chi'(t)+\varepsilon_3\Psi'(t)\\ &\le -\Bigg[d_1-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_1}{2\delta_1}\right)\\ &\qquad +\varepsilon_2\left(g_{1,0}-\theta-\theta l-\mu_1\theta_1\right)-\varepsilon_3\mu_1\Bigg]\xi(t)\left(\int_0^L xu_t^2dx\right)\\ &\quad -\Bigg[d_2-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_3}{2\delta_2}\right)+\varepsilon_2\left(g_{2,0}-\theta-\theta l-\mu_1\theta_1\right)\\ &\qquad -\varepsilon_3\mu_3\Bigg]\xi(t)\left(\int_0^L xv_t^2dx\right)+2\varepsilon_1\xi(t)\\ &\quad \cdot\int_0^L x\left[a|u+v|^{2(r+2)}+2b|uv|^{r+2}\right]dx\\ &\quad +\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_1(0)\right)\left(g_1'\circ u_x\right)(t)\\ &\quad +\left(\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}C_pg_2(0)\right)\left(g_2'\circ v_x\right)(t)\\ &\quad -\Bigg[\frac{\varepsilon_1}{2}\left(l_1-\delta C_pl-2\delta_1\mu_1C_p\right)-\varepsilon_2\theta\\ &\qquad\cdot\left(1+c_1+c_1'+2(1-l_1)^2\right)\Bigg]\xi(t)\\ &\quad \cdot\left(\int_0^L xu_x^2dx\right)-\Bigg[\frac{\varepsilon_1}{2}\left(l_2-\delta C_pl-2\delta_2\mu_3C_p\right)\\ &\qquad -\varepsilon_2\theta\left(1+c_2+c_2'+2(1-l_2)^2\right)\Bigg]\xi(t)\left(\int_0^L xv_x^2dx\right)\\ &\quad +\left[\frac{\varepsilon_1}{2l_1}+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1C_p}{2\theta_1}+\frac{C_p+lC_p}{4\theta}\right)\right]\xi(t)\\ &\quad \cdot\left(\int_0^t g_1^{2-\sigma}(s)ds\right)(g_1^\sigma\circ u_x)(t)\\ &\quad +\left[\frac{\varepsilon_1}{2l_2}+\varepsilon_2\left(\frac{1}{2\theta}+2\theta+\frac{\mu_1C_p}{2\theta_1}+\frac{C_p+lC_p}{4\theta}\right)\right]\xi(t)\\ &\quad \cdot\left(\int_0^t g_2^{2-\sigma}(s)ds\right)(g_2^\sigma\circ v_x)(t)\\ &\quad -\left[\varepsilon_3\eta_3-\varepsilon_1\frac{1}{2\delta_1}-\varepsilon_2\theta_1\right]\xi(t)\\ &\quad \cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_2(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ &\quad -\left[\varepsilon_3\eta_3-\varepsilon_1\frac{1}{2\delta_2}-\varepsilon_2\theta_2\right]\xi(t)\\ &\quad \cdot\int_0^L\int_{\tau_1}^{\tau_2} x|\mu_4(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &\quad -\varepsilon_3\eta_4\xi(t)\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} x\rho\left(|\mu_2(\varrho)|z^2+|\mu_4(\varrho)|y^2\right)d\varrho d\rho dx.\end{aligned} \tag{131}$$

By choosing δ, δ_1, and δ_2 so small that

$$\begin{cases} \left(l_1 - \delta C_p l - 2\mu_1\delta_1 C_p\right) > \dfrac{l_1}{2}, \\ \left(l_2 - \delta C_p l - 2\mu_3\delta_2 C_p\right) > \dfrac{l_2}{2}. \end{cases} \tag{132}$$

Then,

$$\begin{aligned} \delta &< \frac{1}{4C_p l} \min\{l_1, l_2\}, \\ \delta_1 &< \frac{1}{8\mu_1 C_p} \min\{l_1, l_2\}, \\ \delta_2 &< \frac{1}{8\mu_3 C_p} \min\{l_1, l_2\}. \end{aligned} \tag{133}$$

At this point, we choose θ small enough, such that

$$\begin{aligned} k_3 &:= \frac{\varepsilon_1 l_1}{4} - \varepsilon_2\theta\left(1 + c_1 + c_1' + 2(1 - l_1)^2\right) > 0, \\ k_4 &:= \frac{\varepsilon_1 l_2}{4} - \varepsilon_2\theta\left(1 + c_2 + c_2' + 2(1 - l_2)^2\right) > 0. \end{aligned} \tag{134}$$

Then,

$$\theta < \min\left\{\frac{\varepsilon_1 l_1}{4\varepsilon_2\left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}, \frac{\varepsilon_1 l_2}{4\varepsilon_2\left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)}\right\}. \tag{135}$$

Now, δ, δ_1, δ_2, and θ are fixed. Then, we select ε_1, ε_2, ε_3, θ_1, and θ_2 so small that (72) and (162) remain correct and

$$\begin{aligned} k_1 &:= \left[d_1 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\left(g_{1,0} - \mu_1\theta_1 - (1 + l)\theta\right) - \varepsilon_3\mu_1\right] > 0, \\ k_2 &:= \left[d_2 - \varepsilon_1\left(1 + \frac{1}{2\delta} + \frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\left(g_{2,0} - \mu_3\theta_2 - (1 + l)\theta\right) - \varepsilon_3\mu_3\right] > 0, \\ k_5 &:= \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta} C_p g_1(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)ds\right)\right\} > 0, \\ k_6 &:= \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta} C_p g_2(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_3 C_p}{2\theta_2} + \frac{C_p + lC_p}{4\theta}\right)\right]\left(\int_0^t g_2^{2-\sigma}(s)ds\right)\right\} > 0, \\ k_7 &:= \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1 > 0, \\ k_8 &:= \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2 > 0. \end{aligned} \tag{136}$$

Hence, by using (15) gives, for some $\sigma > 0$,

$$\begin{aligned} F'(t) \le -\sigma\xi(t)\Bigg[&\int_0^L xu_t^2 dx + \int_0^L xv_t^2 dx \\ &- \int_0^L x\left[a|u + v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &+ \int_0^L xu_x^2 dx + \int_0^L xv_x^2 dx + (g_1^\sigma \circ u_x)(t) \\ &+ (g_2^\sigma \circ v_x)(t) + K(z, y)\Bigg]. \end{aligned} \tag{137}$$

We choose θ, θ_1, and θ_2 so small that

$$\begin{aligned} \left(g_{1,0} - \mu_1\theta_1 - (1 + l)\theta\right) &> \frac{1}{2} g_{1,0}, \\ \left(g_{2,0} - \mu_3\theta_2 - (1 + l)\theta\right) &> \frac{1}{2} g_{2,0}. \end{aligned} \tag{138}$$

By (134), we get

$$\begin{aligned} &\theta < \min\left\{\frac{\varepsilon_1 l_1}{4\varepsilon_2\left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}, \frac{\varepsilon_1 l_2}{4\varepsilon_2\left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)}, \frac{1}{4(1 + l)} g_{1,0}, \frac{1}{4(1 + l)} g_{2,0}\right\}, \\ &\theta_1 < \frac{1}{4\mu_1(1 + l)} g_{1,0}, \theta_2 < \frac{1}{4\mu_3(1 + l)} g_{2,0}, \\ &\frac{4\theta\left(1 + c_1 + c_1' + 2(1 - l_1)^2\right)}{l_1} < \frac{g_{1,0}}{2 + (l/\delta) + (\mu_1/\delta_1)}, \\ &\frac{4\theta\left(1 + c_2 + c_2' + 2(1 - l_2)^2\right)}{l_2} < \frac{g_{2,0}}{2 + (l/\delta) + (\mu_3/\delta_2)}. \end{aligned} \tag{139}$$

With $\theta, \theta_1, \theta_2$, and α' fixed, we pick $\varepsilon_1, \varepsilon_2$, and ε_3 such that

$$\max\left\{\frac{4\theta\left(1+c_1+c_1'+2(1-l_1)^2\right)}{l_1}, \frac{4\theta\left(1+c_2+c_2'+2(1-l_2)^2\right)}{l_2}\right\}\varepsilon_2 < \varepsilon_1 < \frac{1}{2+(l/\delta)+\min\left((\mu_1/\delta_1),(\mu_3/\delta_2)\right)}(\min(d_1,d_2) + \varepsilon_2 \min\left\{g_{1,0},_{2,0}\right\} + \varepsilon_3 \min(\mu_1+\mu_3)). \tag{140}$$

We will make

$$\begin{cases} k_1 := \left[d_1 - \varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_1}{2\delta_1}\right) + \varepsilon_2\left(g_{1,0} - \mu_1\theta_1 - (1+l)\theta\right) - \varepsilon_3\mu_1\right] > 0, \\ k_2 := \left[-\varepsilon_1\left(1+\frac{1}{2\delta}+\frac{\mu_3}{2\delta_2}\right) + \varepsilon_2\left(g_{2,0} - \mu_3\theta_2 - \theta - \theta l\right) - \varepsilon_3\mu_3\right] > 0, \\ k_3 := \frac{\varepsilon_1}{2}\left(l_1 - 2\mu_1\delta_1 C_p - \delta C_p l\right) - \varepsilon_2\theta\left(1+c_1+c_1'+2(1-l_1)^2\right) > 0, \\ k_4 := \frac{\varepsilon_1}{2}\left(l_2 - 2\mu_3\delta_2 C_p - \delta C_p l\right) - \varepsilon_2\theta\left(1+c_2+c_2'+2(1-l_2)^2\right) > 0, \\ k_7 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_1} - \varepsilon_2\theta_1 > 0, \\ k_8 := \varepsilon_3\eta_3 - \varepsilon_1\frac{1}{2\delta_2} - \varepsilon_2\theta_2 > 0. \end{cases} \tag{141}$$

Then, we select $\varepsilon_1, \varepsilon_2$, and ε_3 so small that (72) and (137) remain correct and

$$\begin{aligned} k_5 &:= \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_1(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_1} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_1 C_p}{2\theta_1} + \frac{C_p + lC_p}{4\theta}\right)\right]\left(\int_0^t g_1^{2-\sigma}(s)ds\right)\right\} > 0, \\ k_6 &:= \left(\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}C_p g_2(0)\right) - \left\{\left[\frac{\varepsilon_1}{2l_2} + \varepsilon_2\left(\frac{1}{2\theta} + 2\theta + \frac{\mu_3 C_p}{2\theta_2} + \frac{C_p + lC_p}{4\theta}\right)\right]\left(\int_0^t g_2^{2-\sigma}(s)ds\right)\right\} > 0. \end{aligned} \tag{142}$$

Next, as (137) is showed, according to the different ranges of r, we give the following two cases.

Case 1. $\sigma = 1$.

By choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3, \theta_1, \theta_2$, and θ, (137) gives, for $\gamma > 0$ is constant so that,

$$F'(t) \le -\gamma\xi(t)E(t), \quad \forall t \ge t_0. \tag{143}$$

Therefore, with the help of the LHS of (72) and (143), we obtain

$$F'(t) \le -\gamma\alpha_1\xi(t)F(t), \forall t \ge t_0. \tag{144}$$

By integration of (144) over (t_0, t) gives

$$F'(t) \le F(t_0)e^{(-\gamma\alpha_1)\int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0. \tag{145}$$

Therefore, $(129)_1$ is proved by (72) as well.

Case 2. $1 < \sigma < 3/2$.

We use (11), which gives

$$\begin{aligned} g_1(t)^{1-\sigma} &\ge (\sigma - 1)\left(\int_{t_0}^t \xi(s)ds\right) + g_1(t_0)^{1-\sigma}, \\ g_2(t)^{1-\sigma} &\ge (\sigma - 1)\left(\int_{t_0}^t \xi(s)ds\right) + g_2(t_0)^{1-\sigma}. \end{aligned} \tag{146}$$

We have, for $0 < \tau < 1$,

$$\begin{aligned} \int_0^\infty g_1^{1-\tau}(s)ds &\le \int_0^\infty \frac{1}{\left[(\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_1(t_0)^{1-\sigma}\right]^{(1-\tau)/(\sigma-1)}}ds, \\ \int_0^\infty g_2^{1-\tau}(s)ds &\le \int_0^\infty \frac{1}{\left[(\sigma-1)\left(\int_{t_0}^t \xi(s)ds\right) + g_2(t_0)^{1-\sigma}\right]^{(1-\tau)/(\sigma-1)}}ds. \end{aligned} \tag{147}$$

For $0 < \tau < 2 - \sigma < 1$, we have $(1-\tau)/(\sigma-1) > 1$ and (15), we find

$$\begin{aligned} \int_0^\infty g_1^{1-\tau}(s)ds &< \infty, \quad \forall 0 < \tau < 2 - \sigma, \\ \int_0^\infty g_2^{1-\tau}(s)ds &< \infty, \quad \forall 0 < \tau < 2 - \sigma. \end{aligned} \tag{148}$$

From (72) (for $\theta = \tau$ and $\rho = \sigma$) and (55) gives

$$\begin{aligned} (g_1 \circ u_x)(t) &\le C_1\left(E(0)\int_0^\infty g_1^{1-\tau}(s)ds\right)^{(\sigma-1)/(\sigma-1+\tau)} \cdot \left(\left(g_1^\sigma \circ u_x\right)(t)\right)^{\tau/(\sigma-1+\tau)} \\ &\le C_1'\left(\left(g_1^\sigma \circ v_x\right)(t)\right)^{\tau/(\sigma-1+\tau)}. \end{aligned} \tag{149}$$

Similarly, we have

$$(g_2 \circ v_x)(t) \le C_2'\left(\left(g_2^\sigma \circ v_x\right)(t)\right)^{\tau/(\sigma-1+\tau)}, \tag{150}$$

for some $C_1', C_2' > 0$. Hence, $\forall \sigma_1 > 1$, we find

$$\begin{aligned} E^{\sigma_1}(t) &\le C''E^{\sigma_1-1}(0)\Big(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \\ &+ \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + K(z,y)\Big) \\ &+ C_1'((g_1\circ u_x)(t))^{\sigma_1} + C_2'((g_2\circ v_x)(t))^{\sigma_1} \\ &\le C''E^{\sigma_1-1}(0)\Big(\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \\ &+ \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx + K(z,y)\Big) \\ &+ C_1'((g_1^\sigma\circ u_x)(t))^{\tau\sigma_1/(\sigma-1+\tau)} + C_2'((g_2^\sigma\circ v_x)(t))^{\tau\sigma_1/(\sigma-1+\tau)}. \end{aligned} \tag{151}$$

We choose $\tau = 1/2$ and $\sigma_1 = 2\sigma - 1$ (therefore, $\tau\sigma_1/(\sigma-1+\tau) = 1$) and (144); we get, for some $\Gamma > 0$,

$$\begin{aligned} E^{\sigma_1}(t) \le \Gamma\Big[&\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx + \int_0^L xv_x^2dx \\ &+ K(z,y) - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &+ (g_1^\sigma\circ u_x)(t) + (g_2^\sigma\circ v_x)(t)\Big]. \end{aligned} \tag{152}$$

By combining (72), (137), and (151), we find

$$F'(t) \le -\frac{\sigma}{\Gamma}\xi(t)E^{\sigma_1}(t) \le -\frac{\sigma}{\Gamma}\alpha_1^{\sigma_1}F^{\sigma_1}(t)\xi(t), \forall t \ge t_0. \tag{153}$$

By integrating (153) gives

$$F(t) \le C_1^*\left(1 + \int_{t_0}^t \xi(s)ds\right)^{-(1/(\sigma_1-1))}, \quad \forall t \ge t_0. \tag{154}$$

Hence,

$$\int_{t_0}^\infty F(t)dt \le C_1^*\int_{t_0}^\infty \frac{1}{\left(1 + \int_{t_0}^t \xi(s)ds\right)^{1/(\sigma_1-1)}}dt. \tag{155}$$

From $(1/(\sigma_1-1)) > 0$ and $(1 + \int_{t_0}^t \xi(s)ds) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we find

$$\int_{t_0}^\infty F(t)dt < \infty. \tag{156}$$

Also, we use (24), and we get

$$tF(t) \le \frac{C_1^* t}{\left(1 + \int_{t_0}^t \xi(s)ds\right)^{1/(\sigma_1-1)}} \le C_\sigma. \tag{157}$$

Hence, we find

$$\sup_{t\ge t_0} tF(t) < \infty. \tag{158}$$

From $E(t)$ which is bounded, using (72), (156), and (158) to get

$$\int_{t_0}^\infty F(t)dt + \sup_{t\ge 0}(tF(t)) < \infty. \tag{159}$$

Therefore, using (55) and Lemma 10 (for $\rho = \sigma$) gives

$$\begin{aligned} (g_1\circ u_x)(t) &\le C_2^*\left(t\|u_x(x,t)\|_H^2 + \int_0^t \|u_x(x,s)\|_H^2ds\right)^{(\sigma-1)/\sigma} \\ &\times\left(\int_0^t g^\sigma(t-s)\|u_x(x,t) - u_x(x,s)\|_H^2ds\right) \\ &\le C_2^*\left(tF(t) + \int_{t_0}^t F(s)ds\right)^{(\sigma-1)/\sigma}((g_1^\sigma\circ u_x)(t))^{1/\sigma} \\ &\le C_3^*((g_1^\sigma\circ u_x)(t))^{1/\sigma}. \end{aligned} \tag{160}$$

This means

$$(g_1^\sigma\circ u_x)(t) \ge C_4((g_1\circ u_x)(t))^\sigma, \tag{161}$$

$$(g_2^\sigma\circ v_x)(t) \ge C_5((g_2\circ v_x)(t))^\sigma, \tag{162}$$

for some $C_4, C_5 > 0$.
Then, combining (137), (161), and (162) yields

$$\begin{aligned} F'(t) \le -C_6\xi(t)\Big\{&\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \\ &+ \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &+ K(z,y) + ((g_1\circ u_x)(t))^\sigma + ((g_2\circ v_x)(t))^\sigma\Big\}, \end{aligned} \tag{163}$$

for some $C_6 > 0$.
As in [1], we obtain

$$\begin{aligned} E^\sigma(t) \le C_7\xi(t)\Big\{&\int_0^L xu_t^2dx + \int_0^L xv_t^2dx + \int_0^L xu_x^2dx \\ &+ \int_0^L xv_x^2dx - \int_0^L x\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]dx \\ &+ K(z,y) + ((g_1\circ u_xt)(t))^\sigma + ((g_2\circ v_x)(t))^\sigma\Big\}, \end{aligned} \tag{164}$$

$\forall t \ge 0$ and some $C_7 > 0$.

Combining (163), (164), and (72), we find

$$F'(t) \le -C_8 \xi(t) F^{\sigma}(t), \forall t \ge t_0, \quad (165)$$

for some $C_8 > 0$.

By integrating (163) over (t_0, t), we get

$$F(t) \le C_9 \left(1 + \int_{t_0}^{t} \xi(s) ds\right)^{-(1/(\sigma-1))}, \quad \forall t \ge t_0. \quad (166)$$

Hence, $(129)_2$ is showed by (72) as well.

Data Availability

No data were used to support this study.

Conflicts of Interest

This work does not have any conflicts of interest.

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References

[1] B. Cahlon, D. M. Kulkarni, and P. Shi, "Stepwise stability for the heat equation with a nonlocal constraint," *SIAM Journal on Numerical Analysis*, vol. 32, no. 2, pp. 571–593, 1995.

[2] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, no. 2, pp. 155–160, 1963.

[3] V. Capasso and K. Kunisch, "A reaction-diffusion system arising in modelling man-environment diseases," *Quarterly of Applied Mathematics*, vol. 46, no. 3, pp. 431–450, 1988.

[4] Y. S. Choi and K. Y. Chan, "A parabolic equation with nonlocal boundary conditions arising from electrochemistry," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 18, no. 4, pp. 317–331, 1992.

[5] A. Choucha, D. Ouchenane, K. Zennir, and F. Feng, "Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term," *Mathematical Methods in the Applied Sciences*, vol. 42, pp. 1–26, 2020.

[6] C. Mu and J. Ma, "On a system of nonlinear wave equations with Balakrishnan-Taylor damping," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 65, no. 1, pp. 91–113, 2014.

[7] R. E. Ewing and T. Lin, "A class of parameter estimation techniques for fluid flow in porous media," *Advances in Water Resources*, vol. 14, no. 2, pp. 89–97, 1991.

[8] M. A. Goodrich and M. A. Ragusa, "Hölder continuity of weak solutions of p-Laplacian PDEs with VMO coefficients," *Nonlinear Analysis-Theory Methods and Applications*, vol. 185, pp. 336–355, 2019.

[9] J. Hao and F. Wang, "Energy decay in a Timoshenko-type system for thermoelasticity of type III with distributed delay and past history," *Electronic Journal of Differential Equations*, vol. 2018, pp. 1–27, 2018.

[10] E. Pişkin and F. Ekinci, "General decay and blowup of solutions for coupled viscoelastic equation of Kirchhoff type with degenerate damping terms," *Mathematical Methods in the Applied Sciences*, vol. 46, pp. 5468–5488, 2019.

[11] N. I. Ionkin and E. I. Moiseev, "A problem for the heat conduction equation with two-point boundary condition," *Differentsial'nye Uravneniya*, vol. 15, pp. 1284–1295, 1979.

[12] N. I. Ionkin, "Solution of boundary value problem in heat conduction theory with nonclassical boundary conditions," *Differentsial'nye Uravneniya*, vol. 13, pp. 1177–1182, 1977.

[13] A. V. Kartynnik, "Three-point boundary value problem with an integral space-variable condition for a second order parabolic equation," *Differential Equations*, vol. 26, pp. 1160–1162, 1990.

[14] L. I. Kamynin, "A boundary-value problem in the theory of heat conduction with non-classical boundary conditions," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, pp. 1006–1024, 1964.

[15] M. R. Li and L. Y. Tsai, "Existence and nonexistence of global solutions of some system of semilinear wave equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 54, no. 8, pp. 1397–1415, 2003.

[16] L. S. Pulkina, "A nonlocal problem with integral conditions for hyperbolic equations," *Electronic Journal of Differential Equations*, vol. 45, pp. 1–6, 1999.

[17] L. S. Pul'kina, "TheL^2 solvability of a nonlocal problem with integral conditions for a hyperbolic equation," *Differential Equations*, vol. 36, no. 2, pp. 316–318, 2000.

[18] A. S. Nicaise and C. Pignotti, "Stabilization of the wave equation with boundary or internal distributed delay," *Differential and Integral Equations*, vol. 21, no. 9-10, pp. 935–958, 2008.

[19] M. A. Ragusa and A. Tachikawa, "Partial regularity of the minimizers of quadratic functionals with *VMO* coefficients," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 609–620, 2005.

[20] M. A. Ragusa and A. Tachikawa, "Regularity for minimizers for functionals of double phase with variable exponents," *Adv. Nonlinear Anal.*, vol. 9, no. 1, pp. 710–728, 2019.

[21] P. Shi and M. Shillor, "On design of contact patterns in one dimensional thermoelasticity," in *Theoretical Aspects of Industrial Design*, SIAM, Philadelphia, PA, USA, 1992.

[22] P. Shi, "Weak solution to an evolution problem with a nonlocal constraint," *SIAM Journal on Mathematical Analysis*, vol. 24, no. 1, pp. 46–58, 1993.

[23] W. Shuntang, "Blow-up of solutions for a singular nonlocal viscoelastic equation," *Journal of Partial Differential Equations*, vol. 24, pp. 140–149, 2018.

[24] N. I. Yurchuk, "Mixed problem with an integral condition for certain parabolic equations," *Differentsial'nye Uravneniya*, vol. 22, pp. 2117–2126, 1986.

[25] S. Boulaaras, R. Guefaifia, and N. Mezouar, "Global existence and decay for a system of two singular one-dimensional nonlinear viscoelastic equations with general source terms," *Applicable Analysis*, pp. 1–25, 2020.

[26] S. Boulaaras and N. Mezouar, "Global existence and decay of solutions of a singular nonlocal viscoelastic system with a nonlinear source term, nonlocal boundary condition, and localized damping term," *Mathematical Methods in the Applied Sciences*, vol. 43, no. 10, pp. 6140–6164, 2020.

[27] T. A. Apalara, "Uniform decay in weakly dissipative Timoshenko system with internal distributed delay feedbacks," *Acta Mathematica Scientia*, vol. 36, no. 3, pp. 815–830, 2016.

[28] S. Boulaaras, A. Choucha, D. Ouchenane, and B. Cherif, "Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms," *Adv. Difference Equ.*, vol. 2020, no. 1, article 310, 2020.

[29] W. Liu, Y. Sun, and G. Li, "On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term," *Topological Methods in Nonlinear Analysis*, vol. 49, pp. 299–323, 2017.

[30] A. Draifia, A. Zarai, and S. Boulaaras, "Global existence and decay of solutions of a singular nonlocal viscoelastic system," *Rendiconti del Circolo Matematico di Palermo Series 2*, vol. 69, no. 1, pp. 125–149, 2020.

[31] N. Mezouar and S. Boulaaras, "Global existence and decay of solutions of a singular nonlocal viscoelastic system with damping terms," *Topological Methods in Nonlinear Analysis*, vol. 56, no. 1, pp. 283–312, 2020.