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Research Article

A Note on the Górnicki-Proinov Type Contraction

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In this paper, we propose a notion of the Górnicki-Proinov type contraction. Then, we prove the uniqueness and existence of the fixed point for such mappings in the framework of the complete metric spaces. Some illustrative examples are also expressed to strengthen the observed results.

1. Introduction and Preliminaries

The history of the fixed point theory goes back about a century. Banach's result initiated the metric fixed point theory in 1922 [1]. The first outstanding extension of this initial theorem was given by Kannan [2] in 1968. In this first generalization, Kannan [2] removed the necessity of the continuity of the contraction mapping. Recently, Górnicki [3] expressed an extension of Kannan type of contraction but the continuity condition was assumed. After then, Bisht [4] refined the result of Górnicki [3] by replacing the continuity condition for the considered mapping with orbitally continuity or p-continuity. Very recently, Górnicki [5] improved these two mentioned results by introducing new contractions, "Geraghty-Kannan type" and "ϕ-Kannan type." He proved the existence of a fixed point for such mappings. On the other hand, Proinov [6] discussed some existing results and noted that these results are particular cases of Skof [7]. He also proposed a very general fixed point theorem that also contains the result of Skof [7].

We first recall the pioneer theorem of Banach [1] and Kannan [2]. On a complete metric space (X, d), a mapping $T : X \to X$ admits a unique fixed point if there exists $0 \leq \mathscr{K} < 1$ such that

$$d(Tu, Tv) \leq k \cdot d(u, v), \tag{1}$$

and

$$d(Tu, Tv) \leq K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{2}$$

for all $u, v \in X$. The inequality (1) belongs to Banach [1] and (2) belongs to Kannan [2]. By using the "asymptotic regularity" concept, Górnicki [3] proved an extension of Kannan Theorem 1.2. Before giving this interesting result, we recollect the interesting concepts:

Let T be a self-mapping on a metric space (X, d) and $\{T^n u\}$ be the Picard iterative sequence, for an initial point $u \in X$.

(o) The set $O(T, u) = \{T^n u : n = 0, 1, 2, \cdots\}$ is called *the orbit of the mapping* T at u.

The mapping T is said to be [3, 5]:

(o-c) *orbitally continuous at a point* $w \in X$ if for any sequence $\{u_n\}$ in $O(T, u)$ for some $u \in X$, $\lim_{n\to\infty} d(u_n, w) = 0$ implies $\lim_{n\to\infty} d(Tu_n, Tw) = 0$.

(p-c) p *-continuous at a point* $w \in X$ $(p = 1, 2, 3, \cdots)$ if for any sequence $\{u_n\}$ in $X \lim_{n\to\infty} d(T^{p-1}u_n, w) = 0$ implies $\lim_{n\to\infty} d(T^p u_n, Tw) = 0$.

(a-r) *asymptotically regular at a point* $u \in X$ if $\lim_{n\to\infty} d(T^n u, T^{n+1}u) = 0$. If T is asymptotically regular at each point of X, we say that it is asymptotically regular.

Remark 1. In [8], it is shown that p-continuity of T and the continuity of Tp are independent conditions for the case $p > 1$.

Theorem 2 (see [3, 5]). *On a complete metric space (X, d), a continuous asymptotically regular mapping $T : X \to X$ admits a unique fixed point if there exist $0 \le k < 1$ and $0 \le K < +\infty$ such that*

$$d(Tu, Tv) \le k \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{3}$$

for all $u, v \in X$.

Later, the assumption of continuity of the mapping T was replaced with weaker notions of continuity.

Theorem 3 (see [4]). *On a complete metric space (X, d) and a mapping $T : X \to X$. Suppose that there exists $0 \le K < 1$ such that*

$$d(Tu, Tv) \le k \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{4}$$

for all $u, v \in X$. Then, T admits a unique fixed point if either T is (o-c) or (p-c) for $p \ge 1$.

In [5], some generalizations of Theorems 2 and 3 are considered, by replacing the constant k with some real-valued functions.

Theorem 4 (see [5]). *Let (X, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that there exist $\psi : [0,\infty) \to [0,\infty)$ and $0 \le K < \infty$ such that*

$$d(Tu, Tv) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{5}$$

for all $u, v \in X$. Suppose that:

(i) $\phi(\theta) < \theta$ for all $\theta > 0$ and ϕ is upper semicontinuous

(ii) either T is (o-c) or T is (p-c) for some $p \ge 1$

Then, T has a unique fixed point $u_* \in X$ and for each $u \in X$, $T^n u \to u_*$ as $n \to \infty$.

Theorem 5 (see [5]). *Let (X, d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that there exist $\varsigma : [0,\infty) \to [0, 1)$ and $0 \le K < \infty$ such that*

$$d(Tu, Tv) \le \varsigma(d(u, v)) \cdot d(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{6}$$

for all $u, v \in X$. Suppose that:

(1) $\varsigma(\theta_n) \to 1 \Rightarrow \theta_n \to 0$;

(2) either T is (o-c) or T is (p-c) for some $p \ge 1$

Then, T has a unique fixed point $u_* \in X$ and for each $u \in X$, $T^n u \to u_*$ as $n \to \infty$.

On the other hand, very recently, Proinov announced some results which unify many known results [6].

Theorem 6 (see [6]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{7}$$

for all $u, v \in X$ with $d(Tu, Tv) > 0$, where the functions ψ, $\phi : (0,\infty) \to \mathbb{R}$ are such that the following conditions are satisfied:

(p_1) $\phi(\theta) < \psi(\theta)$ for any $\theta > 0$;

(p_2) ψ is nondecreasing;

(p_3) $\limsup_{\theta\to e+} \phi(\theta) < \psi(e+)$ for any $e > 0$.

Then, T admits a unique fixed point.

Theorem 7 (see [6]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{8}$$

for all $u, v \in X$ with $d(Tu, Tv) > 0$, where $\psi, \phi : (0,\infty) \to \mathbb{R}$ are two functions such that the following conditions are satisfied:

(p_1) $\phi(\theta) < \psi(\theta)$ for any $\theta > 0$;

(p_4) $\inf_{\theta > e} \psi(\theta) > -\infty$ for any e;

(p_5) $\limsup_{\theta\to e+} \phi(\theta) < \liminf_{\theta\to e} \psi(\theta)$ or $\limsup_{\theta\to e} \phi(\theta) < \liminf_{\theta\to e+} \psi(\theta)$ for any $e > 0$;

(p_6) $\limsup_{\theta\to 0+} \phi(\theta) < \liminf_{\theta\to e} \psi(\theta)$ for any $e > 0$;

(p_7) if the sequences $(\psi(\theta_n))$ and $(\phi(\theta_n))$ are convergent with the same limit and $(\psi(\theta_n))$ is strictly decreasing, then $\theta_n \to 0$ as $n \to \infty$.

Then, T admits a unique fixed point.

Lemma 8 (see [6]). *Let (u_n) be a sequence in a metric space (X, d) such that $d(u_n, u_{n+1}) \to 0$ as $n \to \infty$. If the sequence (u_n) is not Cauchy, then there exist $e > 0$ and two subsequences $\{s_k\}$, $\{r_k\}$ of positive integers such that*

$$\begin{gathered} \lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k+1}\right) = e+, \\ \lim_{k\to\infty} d\left(u_{s_k}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k}, u_{r_k+1}\right) = e \end{gathered} \tag{9}$$

Lemma 9 (see [6]). *Let (u_n) be a sequence in a metric space (X, d) such that $d(u_n, u_{n+1}) \to 0$ as $n \to \infty$. If the sequence (u_n) is not Cauchy, then there exist $e > 0$ and two subsequences $\{s_k\}$, $\{r_k\}$ of positive integers such that*

$$\lim_{k\to\infty} d\left(u_{s_k}, u_{r_k}\right) = e+,$$
$$\lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k+1}\right) = \lim_{k\to\infty} d\left(u_{s_k+1}, u_{r_k}\right) = \lim_{k\to\infty} d\left(u_{s_k}, u_{r_k+1}\right) = e \tag{10}$$

In the end of this section, we recall the notions of α-orbital admissible and triangular α-orbital admissible mappings [9] with mention that these notions were extended in many directions, see, e.g., [10] and it could be potentially extended also to several approaches of recent developments in fixed point theory. See, for instance, [11–21].

On a metric space (X, d), a self-mapping T is called

(i) α-orbital admissible if

$$\alpha(u, Tu) \geq 1 \Rightarrow \alpha\left(Tu, T^2u\right) \geq 1, \tag{11}$$

for any $u, v \in X$, where $\alpha : X \times X \to [0,\infty)$

(ii) triangular α-orbital admissible if it is α-orbital admissible and the following condition is satisfied

$$\alpha(u, v) \geq 1 \text{ and } \alpha(v, Tv) \geq 1 \Rightarrow \alpha(u, Tv) \geq 1, \tag{12}$$

for any $u, v, w \in X$

Lemma 10. *If for an triangular α-orbital admissible mapping $T : X \to X$ there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$, then*

$$\alpha\left(u_n, u_p\right) \geq 1, \text{for all } n, p \in \mathbb{N}, \tag{13}$$

where the sequence $\{u_n\}$ is defined as $u_{n+1} = Tu_n$.

Let (X, d) be a metric space and the function $\alpha : X \times X \to [0,\infty)$. The following conditions will be used further:

$\mathscr{R}$ If for a sequence $\{u_n\}$ in X such that $u_n \to u$ and $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{u_{p_k}\}$ of $\{u_n\}$ such that $\alpha(u_{p_k}, u) \geq 1$.

$(\mathscr{U})$ For all $u, v \in Fix_X T = \{z \in X : Tz = z\}$, we have $\alpha(u, v) \geq 1$.

2. Main Results

Let Λ be the set of all functions $\phi : (0,\infty) \to \mathbb{R}$. For $\phi, \psi \in \Lambda$, we are considering the following conditions:

(a_1) $\phi(\theta) < \psi(\theta)$ for $\theta > 0$

(a_2) $\limsup_{\theta\to e+}\phi(\theta) < \liminf_{\theta\to e}\psi(\theta)$, for any $e > 0$

(a_3) $\limsup_{\theta\to e}\phi(\theta) < \liminf_{\theta\to e+}\psi(\theta)$, for any $e > 0$

(a_4) $\limsup_{\theta\to e+}\phi(\theta) < \psi(e+)$, for any $e > 0$

Definition 11. Let (X, d) be a metric space, the functions ψ, $\phi \in \Lambda$ and $\alpha : X \times X \to [0,\infty)$. An *(a-r)* mapping $T : X \to X$ is said to be (α, ψ, ϕ)-contraction if there exists $0 \leq K < \infty$ such that

$$\alpha(u, v)\psi(d(u, v)) \leq \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{14}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$.

Theorem 12. *On a complete metric space (X, d) an (α, ψ, ϕ)-contraction $T : X \to X$ has a fixed point provided that*

(1) the functions $\psi, \phi \in \Lambda$ satisfy (a_1) and either (a_2) or (a_3)

(2) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$

(3) either T is (o-c) or T is (p-c), for some $p \geq 1$

Moreover, if property (U) is satisfied, then the fixed point of T is unique.

Proof. Let u be any point (but fixed) in X and we build the sequence $\{u_n\}$, where $u_0 = u$ and $u_n = T^n u$ for any $n \in \mathbb{N}$. If there exists $m_0 \in \mathbb{N}$ such that $T^{m_0}u = T^{m_0+1}u = T(T^{m_0}u)$, then $T^{m_0}u$ is a fixed point of T. For this reason, we can suppose that $T^n u \neq T^{n+1}u$, for every $n \in \mathbb{N} \cup \{0\}$ and we claim that $\{u_n\}$ is Cauchy sequence. Assuming the contrary, that the sequence $\{u_n\}$ is not Cauchy, from Lemma 1, it follows that we can find e and two subsequences $\{s_k\}$ and $\{r_k\}$ of positive integers such that (9) holds. Letting $u = u_{s_k}$ and $v = u_{r_k}$ in (14), we have $\alpha(u_{s_k}, u_{r_k}) \geq 1$ (taking into account (1.8)), and then,

$$\begin{aligned}\psi\left(d\left(u_{s_k+1}, u_{r_k+1}\right)\right) &\leq \alpha\left(u_{s_k}, u_{r_k}\right)\psi\left(d\left(u_{s_k+1}, u_{r_k+1}\right)\right)\\ &= \alpha\left(u_{s_k}, u_{r_k}\right)\psi\left(d\left(Tu_{s_k}, Tu_{r_k}\right)\right)\\ &\leq \phi\left(d\left(u_{s_k}, u_{r_k}\right)\right) + K\\ &\quad\cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\},\end{aligned} \tag{15}$$

or denoting $\xi_k = d(u_{s_k+1}, u_{r_k+1})$ and $\zeta_k = d(u_{s_k}, u_{r_k})$

$$\psi(\xi_k) \leq \phi(\zeta_k) + K \cdot \left\{d\left(u_{s_k}, u_{s_k+1}\right) + d\left(u_{r_k}, u_{r_k+1}\right)\right\}. \tag{16}$$

Taking into account the asymptotically regularity of T, from (9), it follows that

$$\xi_k \to e+ \text{ and } \zeta_k \to e. \tag{17}$$

Thus, letting the limit in (16), we have

$$\liminf_{\theta\to e+}\psi(\theta) \leq \liminf_{k\to\infty}\psi(\xi_k) \leq \limsup_{k\to\infty}\phi(\zeta_k) \leq \limsup_{\theta\to e}\phi(\theta). \tag{18}$$

This contradicts the assumption (a_2).

Similarly, if we consider that the functions ψ, ϕ satisfy (a_3), the conclusion follows in the same way, but taking into account Lemma 2.

Therefore, $\{u_n\}$ is a Cauchy sequence, and because the space (X,d) is complete, there exists u_* such that

$$\lim_{n\to\infty} u_n = u_*. \tag{19}$$

We claim that u_* is a fixed point of T.

If T is orbitally continuous, then since $\{u_n\} \in O(T,u)$ and $u_n \to u_*$, we have $u_{n+1} = Tu_n \to Tu_*$ as $n\to\infty$. The uniqueness of the limit gives $Tu_* = u_*$.

If T is p-continuous, for some $p \geq 1$, by (19), we have $\lim_{n\to\infty} T^{p-1}u_n = u^*$ which implies $\lim_{n\to\infty} T^p u_n = Tu^*$ (because T is p-continuous). Therefore, by uniqueness of the limit, we have $Tu_* = u_*$.

Now, supposing that there exists $v_* \in X$ such that $Tv_* = v_* \neq u_* = Tu_*$, from (14) and taking into account the property (U), we have

$$\begin{aligned}\psi(d(u_*,v_*)) &\leq \alpha(u_*,v_*)\psi(d(Tu_*,Tv_*))\\ &\leq \phi(d(u_*,v_*)) + K\cdot\{(d(u_*,Tu_*) + (d(v_*,Tv_*))\}\\ &= \phi(d(u_*,v_*)) < \psi((d(u_*,v_*)),\end{aligned} \tag{20}$$

which is a contradiction. Therefore, $u_* = v_*$.

Letting $\alpha(u,v) = 1$ in Theorem 12, we get the following:

Corollary 13. *Let (X,d) be a complete metric space and an (a-r) mapping $T : X \to X$. Suppose that there exists $0 \leq K < \infty$ such that*

$$\psi(d(u,v)) \leq \phi(d(u,v)) + K\cdot\{d(u,Tu) + d(v,Tv)\}, \tag{21}$$

for each $u,v \in X$ with $d(Tu,Tv) > 0$, where $\psi,\phi \in \Lambda$. Suppose also that:

(1) the functions $\psi,\phi \in \Lambda$ satisfy (a_1) and either (a_2) or (a_3)

(2) either T is (o-r) or T is (p-o), for some $p \geq 1$

Then, T has a unique fixed point.

Corollary 14. *Let (T,d) be a complete metric space and $T : X \to X$ be an (a-r) mapping such that*

$$d(Tu,Tv) \leq \varsigma(d(u,v))d(u,v) + K\cdot\{d(u,Tu) + d(v,Tv)\}, \tag{22}$$

for each $u,v \in X$, where $0 \leq K < \infty$ and the function $\varsigma : (0,\infty) \to (0,1)$ is such that $\limsup_{\theta\to e+}\varsigma(\theta) < 1$ for any $e > 0$. If T is either (o-c) or (p-c) for some $p \geq 1$, then T has a unique fixed point.

Proof. Let $\psi(\theta) = \theta$ in Corollary 1.

Taking $\psi(\theta) = \theta$ and $\phi(\theta) = k\cdot\theta$, with $k \in [0,1)$ Corollary 1 becomes:

Corollary 15. *Let (T,d) be a complete metric space and $T : X \to X$ be an (a-r) mapping. If there exist $k \in [0,1)$ and $0 \leq K < \infty$ such that*

$$d(Tu,Tv) \leq kd(u,v) + K\cdot\{d(u,Tu) + d(v,Tv)\} \tag{23}$$

for each $u,v \in X$, then T admits a unique fixed point provided that T is (o-c) or (p-c) for some $p \geq 1$.

Theorem 16. *Let (X,d) be a complete metric space, $\alpha : X\times X \to [0,\infty)$, $\psi,\varphi \in \Lambda$ such that (a_1) and (a_2) are satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \leq K < \infty$ such that*

$$\psi(\alpha(u,v)d(Tu,Tv)) \leq \varphi(d(u,v)) + K\cdot\{d(u,Tu) + d(v,Tv)\}, \tag{24}$$

for each $u,v \in X$ with $d(Tu,Tv) > 0$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+} < \psi(e+)$ for any $e > 0$

(ii) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$

(iii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a fixed point. Moreover, the fixed point is unique, provided that property (U) is satisfied.

Proof. Let $\{u_n\}$ be the sequence defined as in the previous theorem, as $u_n = T^n u$, where $u \in X$ is arbitrary but fixed. Letting $u = u_{s_k}$ and $v = u_{r_k}$ in (2.7), we have

$$\begin{aligned}&\psi\left(\alpha\left(u_{s_k},u_{r_k}\right)d\left(u_{s_k+1},u_{r_k+1}\right)\right)\\ &\quad\leq \phi\left(d\left(u_{s_k},u_{r_k}\right)\right) + K\cdot\left\{d\left(u_{s_k},u_{s_k+1}\right) + d\left(u_{s_k},u_{s_k+1}\right)\right\},\end{aligned} \tag{25}$$

and taking into account the assumptions (i), (ii), and Lemma 3, we get

$$\begin{aligned}\psi\left(d\left(u_{s_k+1},u_{r_k+1}\right)\right) \leq \phi\left(d\left(u_{s_k},u_{r_k}\right)\right) + K\\ \cdot\left\{d\left(u_{s_k},u_{s_k+1}\right) + d\left(u_{r_k},u_{r_k+1}\right)\right\},\end{aligned} \tag{26}$$

Setting $\xi_k = d(u_{s_k+1},u_{r_k+1})$ and $\zeta_k = d(u_{s_k},u_{r_k})$ and since $\phi(\theta) < \psi(\theta)$, we get

$$\begin{aligned}\psi(\xi_k) &\leq \phi(\zeta_k) + K\cdot\left\{d\left(u_{s_k},u_{s_k+1}\right) + d\left(u_{r_k},u_{r_k+1}\right)\right\}\\ &< \psi(\zeta_k) + K\cdot\left\{d\left(u_{s_k},u_{s_k+1}\right) + d\left(u_{r_k},u_{r_k+1}\right)\right\}.\end{aligned} \tag{27}$$

On the other hand, from 1.5 that $\xi_k \to e+$, $\zeta_k \to e+$ and then, letting the limit as $k\to\infty$ in the above inequality, since T is an (a-r) mapping and taking into account the second part of the assumption (i), we have

$$\psi(e+) = \lim_{k\to\infty}\psi(\xi_k) \le \limsup_{k\to\infty}\phi(\zeta_k) \le \limsup_{\theta\to e+}\phi(\theta) < \psi(e+), \tag{28}$$

which is a contradiction. Thus, the sequence $\{u_n\}$ is Cauchy on a metric space, so there exists u_* such that $u_n \to u_*$ as $n \to \infty$ and following the lines of the previous proof, we get that u_* is the unique fixed point of T.

Again, letting $\alpha(u, v) = 1$ for any $u, v \in X$ we get the following:

Theorem 17. *Let (X, d) be a complete metric space, and two functions $\psi, \varphi \in \Lambda$ such that (a_1) is satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \le K < \infty$ such that*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{29}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$, where $\psi, \phi \in \Lambda$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+} < \psi(e+)$ for any $e > 0$

(ii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a unique fixed point.

Theorem 18. *Let (X, d) be a complete metric space, and two functions $\psi, \varphi \in \Lambda$ such that (a_1) is satisfied. Let $T : X \to X$ be an (a-r) mapping. Suppose that there exists $0 \le K < \infty$ such that*

$$\psi(d(Tu, Tv)) \le \varsigma(d(u, v))\psi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\}, \tag{30}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$, where $\psi \in \Lambda$ and $\varsigma : (0, \infty) \to (0, 1)$. Suppose also that

(i) ψ is nondecreasing and $\limsup_{\theta\to e+}\varsigma(\theta) < 1$ for any $e > 0$

(ii) the mapping T is either (o-c) or (p-c)

Then, the mapping T possesses a unique fixed point.

Proof. Take $\phi(\theta) = \alpha(\theta)\psi(\theta)$, for $\theta > 0$ in Theorem 17.

Next, we consider mappings that satisfy a similar condition as (14), but for which the asymptotic regularity condition is not necessary.

Definition 19. Let (X, d) be a complete metric space, $\alpha : X \times X \to [0,\infty)$ and $\psi, \varphi \in \Lambda$. A mapping $T : X \to X$ is called (α, ψ, ϕ)-contraction of type 2 if there exists $0 \le K < \infty$ such that

$$\begin{aligned}\alpha(u, v)\psi(d(Tu, Tv)) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ \cdot d(u, Tv)d(v, Tu),\end{aligned} \tag{31}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$.

Theorem 20. *On a complete metric space (X, d), an (α, ψ, ϕ) -contraction of type 2, $T : X \to X$ has a fixed point provided that property (R) and the following conditions hold:*

(A) T is triangular α-orbital admissible and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \ge 1$

(B) ψ, ϕ satisfy the assumptions (a_1) and (a_4)

(C) ψ is nondecreasing

(D) $\limsup_{\theta\to 0+}\phi(\theta) < \liminf_{\theta\to e}\psi(\theta)$, for any $e > 0$

Moreover, if the property (U) holds, the fixed point of T is unique.

Proof. Let $\{u\}$ be a sequence in X defined as

$$u_n = T^n u_0, foreveryn \in \mathbb{N}, \tag{32}$$

where u_0 is an arbitrary but fixed point in X. Replacing in (31) and taking into account (11), we have

$$\begin{aligned}\psi(d(u_n, u_{n+1})) &\le \alpha(u_{n-1}, u_n)\psi(d(Tu_{n-1}, Tu_n)) \\ &\le \phi(d(u_{n-1}, u_n)) + +K \cdot \{d(u_{n-1}, Tu_{n-1}) + d(u_n, Tu_n)\} \\ &\quad \cdot d(u_{n-1}, Tu_n)d(u_n, Tu_{n-1}) \\ &= \phi(d(u_{n-1}, u_n)) + K \cdot \{d(u_{n-1}, u_n) + d(u_n, u_{n+1})\} \\ &\quad \cdot d(u_{n-1}, u_{n+1})d(u_n, u_n) = \phi(d(u_{n-1}, u_n)),\end{aligned} \tag{33}$$

or setting $x_n = d(u_{n-1}, u_n)$ (we can suppose that $x_n > 0$) and taking into account the condition (a_1) for any $\theta > 0$, we get

$$\psi(x_n) \le \phi(x_{n-1}) < \psi(x_{n-1}). \tag{34}$$

If the condition (C) holds, from the above inequality, we get $x_n < x_{n-1}$, for every $n \in \mathbb{N}$. Consequently, being positive and strictly decreasing, the sequence $\{x_n\}$ is convergent and there is $x \ge 0 \in X$ such that $x_n \to x$. If we assume that $x > 0$, then from the above inequality, we have

$$\phi(x+) = \lim_{n\to\infty}\psi(x_n) \le \limsup_{n\to\infty}\phi(x_n) \le \limsup_{\theta\to x+}\phi(\theta) < \phi(x+), \tag{35}$$

which is a contradiction. Thus,

$$\lim_{n\to\infty} x_n = x = 0. \tag{36}$$

The aim for the next step is to prove that the sequence $\{u_n\}$ is Cauchy. Supposing by contradiction, the sequence

$\{u_n\}$ is not Cauchy, by (36), and taking into account Lemma 1, we can find $e > 0$ and two subsequences $\{u_{s_k}\}$ and $\{u_{r_k}\}$ of $\{u_n\}$ such that (9) holds. Taking $u = u_{s_k}$ and $v = u_{r_k}$ in (14) and keeping in mind (1.7), we have

$$\begin{aligned}\psi(d(u_{s_k+1}, u_{r_k+1})) &\leq \alpha(u_{s_k}, u_{r_k})\psi(d(Tu_{s_k}, Tu_{r_k})) \\ &\leq \phi(d(u_{s_k}, u_{r_k})) + +K \cdot \{d(u_{s_k}, u_{s_k+1}) + d(u_{r_k}, u_{r_k+1})\} \\ &\quad \cdot d(u_{s_k}, u_{r_k+1})d(u_{r_k}, u_{s_k+1}).\end{aligned} \tag{37}$$

Letting the limit as $k \to \infty$ in the previous inequality (since $d(u_{s_k+1}, u_{r_k+1}) \to e+$ and $d(u_{s_k}, u_{r_k}) \to e$ and using (2.11), we get

$$\begin{aligned}\liminf_{\theta \to e+} \psi(\theta) &\leq \liminf_{k\to\infty} \psi(d(u_{s_k+1}, u_{r_k+1})) \\ &\leq \limsup_{k\to\infty} \phi(d(u_{s_k}, u_{r_k})) \leq \limsup_{\theta\to e} \phi(\theta).\end{aligned} \tag{38}$$

This is a contradiction to (a_4). Thus, $\{u_n\}$ is a Cauchy sequence on a complete metric space, so it is convergent. Let $u_* = \lim_{n\to\infty} u_n$ and we claim that u_* is a fixed point of T. From (31) and (R), for $u = u_n$ and $v = u_*$, we have

$$\begin{aligned}\psi(d(u_{n+1}, Tu_*)) &\leq \alpha(u_n, u_*)\psi(d(Tu_n, Tu_*)) \\ &\leq \phi(d(u_n, u_*)) + +K \cdot \{d(u_n, u_{n+1}) + d(u_*, Tu_*)\} \\ &\quad \cdot d(u_n, Tu_*)d(u_*, u_{n+1}).\end{aligned} \tag{39}$$

Since $\lim_{n\to\infty} d(u_{n+1}, Tu_*) = d(u_*, Tu_*)$ and $\lim_{n\to\infty} d(u_n, u_*) = 0$ if we suppose that $d(u_*, Tu_*) > 0$, the above inequality yields

$$\begin{aligned}\liminf_{\theta \to d(u_*, Tu_*)} \psi(\theta) &\leq \liminf_{n\to\infty} \psi(d(u_{n+1}, Tu_*)) \\ &\leq \limsup_{n\to\infty} \phi(d(u_n, u_*)) \leq \limsup_{\theta\to 0} \phi(\theta),\end{aligned} \tag{40}$$

which is a contradiction to (D). Therefore, $d(u_*, Tu_*) = 0$, that is u_* is a fixed point of T. As in the Theorem 12, adding the condition (U) to the statement of Theorem 20, we are able to prove that the fixed point is unique. Indeed, if we suppose that $v_* \in X$ is such that $Tv_* = v_* \neq u_* = Tu_*$, from (2.10), we have

$$\begin{aligned}\psi(d(u_*, v_*)) &\leq \alpha(u_*, v_*)\psi(d(Tu_*, Tv_*)) \\ &\leq \phi(d(u_*, v_*)) + +K \cdot \{d(u_*, Tu_*) + d(v_*, Tv_*)\} \\ &\quad \cdot (d(u_*, Tv_*) + d(v_*, Tu_*)).\end{aligned} \tag{41}$$

Letting $n \to \infty$ in the above inequality and keeping in mind (a_1), we have

$$\psi(d(u_*, v_*)) \leq \phi(d(u_*, v_*)) < \psi(d(u_*, v_*)), \tag{42}$$

which is a contradiction.

Example 21. Let the set $X = \{A_1, A_2, A_3, A_4, A_5\}$ endowed with the distance $d : X \times X \to [0,\infty)$, where $d(u, u) = 0$, $d(u, v) = d(v, u)$ for any $u, v \in X$ and

$$\begin{gathered}d(A_1, A_2) = d(A_2, A_3) = d(A_3, A_4) = 1, \\ d(A_1, A_3) = d(A_2, A_4) = d(A_4, A_5) = 2, \\ d(A_1, A_4) = d(A_2, A_5) = d(A_3, A_5) = 3, d(A_1, A_5) = 4.\end{gathered} \tag{43}$$

Let the mapping $T : X \to X$ defined by

$$TA_1 = A_1, TA_2 = A_3, TA_3 = A_5, TA_4 = A_2, TA_5 = A_2. \tag{44}$$

Let also the function $\alpha : X \times X \to [0,\infty)$, with

$$\alpha(u, v) = \begin{cases} 2, & \text{if} (u, v) = (A_i, A_1), \text{for } i = 1, 2, 3, 4, 5 \\ 1, & \text{if} (u, v) \in \{(A_3, A_4), (A_4, A_3)\} \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

Then, T does not satisfy Banach, neither Kannan type condition. Indeed, letting for example $u = A_1, v = A_3$,

$$\begin{aligned}d(TA_1, TA_3) &= d(A_1, A_5) = 4 > 2k \\ &= kd(A_1, A_3) for any 0 \leq k < 1,\end{aligned}$$

$$\begin{aligned}d(TA_1, TA_3) &= d(A_1, A_5) = 4 > 3K \\ &= K \cdot \{d(A_1, A_1) + d(A_3, A_5)\} \\ &= K \cdot \{d(A_1, TA_1) + d(A_3, TA_3)\}, \\ &\quad for any 0 \leq K < \frac{1}{2}.\end{aligned} \tag{46}$$

On the other hand, T is not (a-r), so Theorem 3 cannot be applied. Let the functions $\psi, \phi \in \Lambda$, $\phi(\theta) = \theta$, $\psi(\theta) = \theta/2$, for $\theta > 0$ and $K = 8$. For an easier reading, we will set

$$\begin{aligned}A(u, v) &= \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ &\quad \cdot d(u, Tv)d(v, Tu) = \frac{d(u, v)}{2} + 8 \\ &\quad \cdot \{d(u, Tu) + d(v, Tv)\}d(u, Tv)d(v, Tu).\end{aligned} \tag{47}$$

Let us check that the mapping T is an (α, ψ, ϕ)-contraction of type 2. For this purpose, we must consider the following cases:

(i) $u = A_1, v = A_2$,

$$\begin{aligned}\alpha(A_1, A_2)\psi(d(TA_1, TA_2)) = 2d(A_1, A_3) = 4 < \frac{33}{2} \\ = A(A_1, A_2)\end{aligned} \tag{48}$$

(ii) $u = A_1, v = A_3$,

$$\alpha(A_1, A_3)\psi(d(TA_1, TA_3)) = 2d(A_1, A_5) = 8 < 193 = A(A_1, A_3) \tag{49}$$

(iii) $u = A_1, v = A_4$,

$$\alpha(A_1, A_4)\psi(d(TA_1, TA_4)) = 2d(A_1, A_2) = 2 < \frac{99}{2} = A(A_1, A_4) \tag{50}$$

(iv) $u = A_1, v = A_5$,

$$\alpha(A_1, A_5)\psi(d(TA_1, TA_5)) = 2d(A_1, A_2) = 2 < 98 = A(A_1, A_5) \tag{51}$$

(v) $u = A_3, v = A_4$,

$$\alpha(A_3, A_4)\psi(d(TA_3, TA_4)) = d(A_5, A_2) = 3 < \frac{97}{2} = A(A_3, A_4) \tag{52}$$

Moreover, it is easy to see that all the assumptions of Theorem 20 are satisfied, so that T has a unique fixed point.

Example 22. Let the set $X = [0,\infty)$ be endowed with the usual distance d on $\mathbb{R}$. Consider the mapping $T : X \to X$ defined by

$$Tu = \begin{cases} 1 - u, & \text{if} 0 \le u \le 1 \\ \ln (1 + e^u), & if u > 1 \end{cases}. \tag{53}$$

Then,$T$is neither continuous, a contraction, nor (a-r). Define the function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(u, av) = \begin{cases} 2, & \text{if } u \in \left\{\frac{1}{4}, \frac{1}{2}, 1\right\},\ v = \frac{1}{2} \\ 1, & \text{if } u = 2, v = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{54}$$

Consider also, the functions $\psi, \phi \in \Lambda$, where $\psi(\theta) = e^\theta$ and $\phi(\theta) = \theta + 1$, for $\theta > 0$. Let, for example, $K = 64$. Using the same notation as in Example 1, taking into account the definition of the function α, we have the following:

(i) $u = 1/4, v = 1/2$

$$\alpha\left(\frac{1}{4}, \frac{1}{2}\right)\psi\left(d\left(T\frac{1}{4}, T\frac{1}{2}\right)\right) = 2e^{\frac{1}{4}} \le \frac{17}{4} = \phi\left(d\left(\frac{1}{4}, \frac{1}{2}\right)\right) + A\left(\frac{1}{4}, \frac{1}{2}\right) \tag{55}$$

(ii) $u = 1, v = 1/2$

$$\begin{aligned} \alpha\left(1, \frac{1}{2}\right)\psi\left(d\left(T1, T\frac{1}{2}\right)\right) &= 2e^{\frac{1}{2}} \le \frac{35}{2} \\ &= \phi\left(d\left(1, \frac{1}{2}\right)\right) + A\left(1, \frac{1}{2}\right) \end{aligned} \tag{56}$$

(iii) $u = 2, v = 1$

$$\alpha(2, 1)\psi(d(T2, T1)) = e^{\ln (1+e^2)} = 1 + e^2 \le \phi(d(2, 1)) + A(2, 1). \tag{57}$$

Since it easy to check that all the assumptions of Theorem 20 are verified, we can conclude that T has a unique fixed point.

Corollary 23. *Let (X, d) be a complete metric space and a mapping $T : X \to X$ such that for all $u, v \in X$ with $d(Tu, Tv) > 0$,*

$$\begin{aligned} \psi(d(Tu, Tv)) \le \phi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ \cdot d(u, Tv)d(v, Tu), \end{aligned} \tag{58}$$

where $0 \le K < 1$ and the functions $\psi, \phi \in \Lambda$ are such that

(a) ψ, ϕ satisfy (a_1) and (a_4)

(b) ψ is not decreasing

Then, T admits a unique fixed point.

Corollary 24 (Theorem 6). *Let (X, d) be a complete metric space and a mapping $T : X \to X$ such that for all $u, v \in X$ with $d(Tu, Tv) > 0$,*

$$\psi(d(Tu, Tv)) \le \phi(d(u, v)), \tag{59}$$

where the functions $\psi, \phi \in \Lambda$ are such that

(a) ψ, ϕ satisfy the assumptions (a_1) and (a_4)

(b) ψ is not decreasing

Then, T admits a unique fixed point.

Proof. Let $\alpha(u, v) = 0$ and $K = 0$ in Theorem 20.

Corollary 25. *Let (T, d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\begin{aligned} \psi(d(Tu, Tv)) \le \varsigma(d(u, v))\psi(d(u, v)) + K \cdot \{d(u, Tu) + d(v, Tv)\} \\ \cdot d(u, Tv)d(v, Tu), \end{aligned} \tag{60}$$

for each $u, v \in X$ with $d(Tu, Tv) > 0$, where $0 \leq K < \infty$ and the functions $\varsigma : (0,\infty) \to (0, 1)$, $\psi : (0,\infty) \to (0, 1)$ are such that

(i) $\limsup_{\theta \to e+} \varsigma(\theta) < 1$ *for any* $e > 0$

(ii) ψ *is nondecreasing*

Then, T has a unique fixed point.

Proof. Let $\phi(\theta) = \varsigma(d(u, v))\psi(d(u, v))$ in Corollary 4.

Corollary 26. *Let (X, d) be a complete metric space and a mapping $T : X \to X$. Suppose that there exist $0 \leq k < 1$ and $0 \leq K < \infty$ such that for all $u, v \in X$,*

$$d(Tu, Tv) \leq kd(u, v) + K \cdot \{d(u, Tu) + d(v, Tv)\}d(u, Tv)d(v, Tv). \tag{61}$$

Then, T admits a unique fixed point.

Proof. Let $\alpha(u, v) = 0$, $\psi(\theta) = \theta$ and $\phi(\theta) = k \cdot \theta$, with $0 \leq k < 1$ in Theorem 20.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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