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Research Article

Logarithmic Coefficient Bounds and Coefficient Conjectures for Classes Associated with Convex Functions

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It is well-known that the logarithmic coefficients play an important role in the development of the theory of univalent functions. If $\mathcal{S}$ denotes the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the open unit disk $\mathbb{U}$, then the logarithmic coefficients $\gamma_n(f)$ of the function $f \in \mathcal{S}$ are defined by $\log(f(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n$. In the current paper, the bounds for the logarithmic coefficients γ_n for some well-known classes like $\mathcal{C}(1+\alpha z)$ for $\alpha \in (0, 1]$ and $\mathcal{CV}_{\mathrm{hpl}}(1/2)$ were estimated. Further, conjectures for the logarithmic coefficients γ_n for functions f belonging to these classes are stated. For example, it is forecasted that if the function $f \in \mathcal{C}(1+\alpha z)$, then the logarithmic coefficients of f satisfy the inequalities $|\gamma_n| \le \alpha/(2n(n+1)), n \in \mathbb{N}$. Equality is attained for the function $L_{\alpha,n}$, that is, $\log(L_{\alpha,n}(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,n}) z^n = (\alpha/n(n+1)) z^n + \cdots, z \in \mathbb{U}$.

Dedicated to the memory of Professor Gabriela Kohr (1967-2020)

1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the category of analytic functions f in $\mathbb{U}$ for which f has the following representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. Then, the *logarithmic coefficients* γ_n of the function $f \in \mathcal{S}$ are defined with the aid of the following series expansion:

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{U}. \tag{2}$$

These coefficients play an important role for different estimates in the theory of univalent functions, and note that we use γ_n instead of $\gamma_n(f)$. Kayumov [1] solved Brennan's conjecture for conformal mappings with the help of studying the logarithmic coefficients. The significance of the logarithmic coefficients follows from Lebedev-Milin inequalities ([2], chapter 2; see also [3, 4]), where estimates of the logarithmic coefficients were applied to obtain bounds on the coefficients of f. Milin [2] conjectured the inequality

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \le 0, \quad n = 1, 2, 3, \cdots, \tag{3}$$

that implies Robertson's conjecture [5] and hence Bieberbach's conjecture [6], which was the well-known coefficient problem in the theory of univalent functions. De Branges

[7] proved Bieberbach's conjecture by establishing Milin's conjecture.

Recall that we can rewrite (2) in the power series form as follows:

$$2\sum_{n=1}^{\infty}\gamma_n z^n = a_2 z + a_3 z^2 + a_4 z^3 + \cdots - \frac{1}{2}\left(a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)^2 + \frac{1}{3}\left(a_2 z + a_3 z^2 + a_4 z^3 + \cdots\right)^3 + \cdots, \quad z \in \mathbb{U}, \tag{4}$$

and equating the coefficients of z^n for $n = 1, 2, 3$, it follows that

$$\begin{cases} 2\gamma_1 = a_2, \\ 2\gamma_2 = a_3 - \frac{1}{2}a_2^2, \\ 2\gamma_3 = a_4 - a_2 a_3 + \frac{1}{3}a_2^3. \end{cases} \tag{5}$$

If the functions f and g are analytic in $\mathbb{U}$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $|w(z)| < 1, z \in \mathbb{U}$, and $w(0) = 0$, such that $f = g \prec w$. In particular, if g is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{6}$$

Using the principle of subordination, Ma and Minda [8] introduced the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$, where we make here the weaker assumptions that the function φ is analytic in the open unit disk $\mathbb{U}$ and satisfies $\varphi(0) = 1$, such that it has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, z \in \mathbb{U}, \quad \text{with } B_1 \neq 0. \tag{7}$$

They considered the abovementioned classes as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \quad \mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}. \tag{8}$$

Some special subclasses of the class $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ play a significant role in the *Geometric Function Theory* because of their geometric properties.

For example, taking $\varphi(z) = (1 + Az)/(1 + Bz)$ where $A \in \mathbb{C}$, $-1 \leq B \leq 0$, and $A \neq B$, we get the classes $\mathcal{S}^*[A, B]$ and $\mathcal{C}[A, B]$, respectively (see also [9, 10]). The mentioned classes with the restriction $-1 \leq B < A \leq 1$ reduce to the popular *Janowski starlike* and *Janowski convex functions*, respectively. By replacing $A = 1 - 2\alpha$ and $B = -1$, where $0 \leq \alpha < 1$, we obtain the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of the *starlike functions of order* α and *convex functions of order* α, respectively. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the class of starlike functions and of convex functions in the open unit disk $\mathbb{U}$, respectively. Further, by altering $A = \alpha$ and $B = 0$, where $0 \leq \alpha < 1$, we get the classes $\mathcal{S}^*(1 + \alpha z)$ and $\mathcal{C}(1 + \alpha z)$, which are the extensions of the classes $\mathcal{S}^*(1 + z)$ and $\mathcal{C}(1 + z)$, respectively (see [11]), that is,

$$\mathcal{S}^*(1 + \alpha z) := \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \right\}, \quad \mathcal{C}(1 + \alpha z) := \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \right\}, \tag{9}$$

where $0 < \alpha \leq 1$.

Supposing that $\Psi_{\alpha,n} \in \mathcal{S}^*(1 + \alpha z)$ is such that

$$\frac{z\Psi'_{\alpha,n}(z)}{\Psi_{\alpha,n}(z)} = 1 + \alpha z^n, \quad n \in \mathbb{N}, \tag{10}$$

each function $\Psi_{\alpha,n}$ is of the form

$$\Psi_{\alpha,n}(z) = z \exp\left(\int_0^z \frac{1 + \alpha t^n - 1}{t} \, dt \right) = z + \frac{\alpha}{n} z^{n+1} + \cdots, \quad z \in \mathbb{U}, \tag{11}$$

and is the extremal function for various problems in $\mathcal{S}^*(1 + \alpha z)$. Also, suppose that $L_{\alpha,n} \in \mathcal{C}(1 + \alpha z)$ is such that

$$1 + \frac{zL''_{\alpha,n}(z)}{L'_{\alpha,n}(z)} = 1 + \alpha z^n, \quad n \in \mathbb{N}. \tag{12}$$

Then, each function $L_{\alpha,n}$ is of the form

$$L_{\alpha,n}(z) = \int_0^z \exp\left(\int_0^x \frac{1 + \alpha t^n - 1}{t} \, dt \right) dx = z + \frac{\alpha}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{U}, \tag{13}$$

and plays as extremal function for some extremal problems in the set $\mathcal{C}(1 + \alpha z)$.

Lately, Kanas et al. [12] introduced the categories $\mathcal{ST}_{\text{hpl}}(s)$ and $\mathcal{CV}_{\text{hpl}}(s)$ by

$$\mathcal{ST}_{\text{hpl}}(s) =: \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \mathfrak{q}_s(z) := \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\},$$

$$\mathcal{CV}_{\text{hpl}}(s) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \mathfrak{q}_s(z) := \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\}, \tag{14}$$

and obtained some geometric properties in these categories. Further, the functions

$$\Phi_{s,n}(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_s(t^n) - 1}{t}\, \mathrm{d}t\right) = z + \frac{s}{n} z^{n+1} + \cdots, \quad z \in \mathbb{U}, n \in \mathbb{N},$$

$$K_{s,n}(z) = \int_0^z \exp\left(\int_0^x \frac{\mathfrak{q}_s(t^n) - 1}{t}\, \mathrm{d}t\right) \mathrm{d}x = z + \frac{s}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{U}, n \in \mathbb{N}, \tag{15}$$

play as extremal functions for some issues of the families $\mathcal{ST}_{\mathrm{hpl}}(s)$ and $\mathcal{CV}_{\mathrm{hpl}}(s)$, respectively.

Lately, several researchers have subsequently investigated same problems regarding the logarithmic coefficients and the coefficient problems [9, 13–23], to mention a few of them. For instance, the rotation of the *Koebe function* $k(z) = z(1 - e^{i\theta} z)^{-2}$ for each $\theta \in \mathbb{R}$ has the logarithmic coefficients $\gamma_n = e^{i\theta n}/n$, $n \geq 1$. If $f \in \mathcal{S}$, then by applying the Bieberbach inequality for the first relation of (5), it follows that $|\gamma_1| \leq 1$, and using the *Fekete-Szegö inequality* for the second relation of (5) (see [24], Theorem 3.8) leads to

$$|\gamma_2| = \frac{1}{2}\left|a_3 - \frac{1}{2} a_2^2\right| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) \simeq 0.635\cdots. \tag{16}$$

It was established in ([25], Theorem 4) that the logarithmic coefficients γ_n of $f \in \mathcal{S}$ satisfy the inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6}, \tag{17}$$

and the equality is obtained for the Koebe function. For $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \leq 1/n$ holds but is not true for the full class $\mathcal{S}$, even in order of magnitude (see [24], Theorem 8.4). In 2018, some first logarithmic coefficients γ_n were estimated for special subclasses of *close-to-convex functions* in [15, 20]. However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for $n \geq 3$ is presumably still a concern. In [13], the authors obtained the bounds of logarithmic coefficients γ_n, $n \in \mathbb{N}$, for the general class $\mathcal{S}^*(\varphi)$, and the bounds of the logarithmic coefficients γ_n when $n = 1, 2, 3$ for the class $\mathcal{K}(\varphi)$, while the estimated bounds would generalize many of the previous outcomes.

In the present study, which is motivated essentially by the recent works [13, 16], the bounds for the logarithmic coefficients γ_n, $n \in \mathbb{N}$, of the class $\mathcal{C}(1 + \alpha z)$ for $\alpha \in (0, 1]$ and $\mathcal{CV}_{\mathrm{hpl}}(1/2)$ were estimated. Further, conjectures for the logarithmic coefficients γ_n for f belonging to these classes are stated.

2. Main Results

First, we will obtain the bounds for γ_n of the classes $\mathcal{S}^*(1 + \alpha z)$ and $\mathcal{C}(1 + \alpha z)$ for $\alpha \in (0, 1]$. In this regard, the following outcomes will be employed in the key results.

Lemma 1 (see [13], Theorem 1). *Let $f \in \mathcal{S}^*(\varphi)$. If φ is convex univalent, then the logarithmic coefficients of f satisfy the following inequalities:*

$$|\gamma_n| \leq \frac{|B_1|}{2n}, \quad n \in \mathbb{N}, \tag{18}$$

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2}. \tag{19}$$

The inequalities in (18) and (19) are sharp, such that for any $n \in \mathbb{N}$, there exist the function f_n given by $zf_n'(z)/f_n(z) = \varphi(z^n)$ and the function f given by $zf'(z)/f(z) = \varphi(z)$, respectively, for those equalities we obtain.

Lemma 2 (see [13], Theorem 2). *Let $f \in \mathcal{C}(\varphi)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \leq \frac{|B_1|}{4}, \tag{20}$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12}, & \text{if} \quad \left|4B_2 + B_1^2\right| \leq 4|B_1|, \\ \dfrac{\left|4B_2 + B_1^2\right|}{48}, & \text{if} \quad \left|4B_2 + B_1^2\right| > 4|B_1|, \end{cases} \tag{21}$$

and if B_1, B_2, and B_3 are real values, then

$$|\gamma_3| \leq \frac{|B_1|}{24} H(q_1; q_2), \tag{22}$$

where $H(q_1; q_2)$ is given in ([26], Lemma 2) (or [9], Lemma 5), $q_1 = (B_1 + (4B_2/B_1))/2$, and $q_2 = (B_2 + (2B_3/B_1))/2$. The bounds (20) and (21) are sharp.

Lemma 3 (see [18], Theorem 30). *If $f \in \mathcal{CV}_{hpl}(1/2)$, then*

$$|\gamma_1| \leq \frac{1}{8}, |\gamma_2| \leq \frac{1}{24}, |\gamma_3| \leq \frac{1}{48}. \tag{23}$$

The first two bounds are sharp for $f = K_{1/2,1}$ and $f = K_{1/2,2}$, respectively.

If we consider Lemma 1 with the function $\varphi(z) =: 1 + \alpha z$, then we immediately get the next result:

Theorem 4. *If $f \in \mathcal{S}^*(1 + \alpha z)$, then*

$$|\gamma_n| \leq \frac{\alpha}{2n}, \quad n \in \mathbb{N}, \qquad \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\alpha}{4}. \tag{24}$$

These inequalities are sharp for $f = \Psi_{\alpha,n}$ and $f = \Psi_{\alpha,1}$, respectively.

Corollary 5. *Let $f \in \mathscr{C}(1+\alpha z)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_1| \le \frac{\alpha}{4}, \quad |\gamma_2| \le \frac{\alpha}{12}, \quad |\gamma_3| \le \frac{\alpha}{24}. \tag{25}$$

Equalities in these inequalities are attained for the functions $L_{\alpha,n}$ for $n = 1, 2, 3$, respectively.

Proof. For $\varphi(z) = 1 + \alpha z$, where $B_1 = \alpha, B_2 = B_3 = 0$, in Theorem 6, we obtain the required result. Also, since

$$\begin{aligned} \log \frac{L_{\alpha,1}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,1}) z^n = \frac{\alpha}{2} z + \cdots, \quad z \in \mathbb{U}, \\ \log \frac{L_{\alpha,2}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,2}) z^n = \frac{\alpha}{6} z^2 + \cdots, \quad z \in \mathbb{U}, \\ \log \frac{L_{\alpha,3}(z)}{z} &= 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,3}) z^n = \frac{\alpha}{12} z^3 + \cdots, \quad z \in \mathbb{U}, \end{aligned} \tag{26}$$

it follows that these inequalities are attained for the functions $L_{\alpha,n}$ for $n = 1, 2, 3$, respectively. □

Theorem 6. *Let $f \in \mathscr{C}(1+\alpha z)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \le \frac{\alpha}{4n}, \quad n \in \mathbb{N}. \tag{27}$$

This inequality is sharp for $|\gamma_1|$ for the function $L_{\alpha,1}$.

Proof. If $f \in \mathscr{C}(1+\alpha z)$, this is equivalent to $f \in \mathscr{A}$ and

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \alpha z =: \varphi_\alpha(z). \tag{28}$$

If we define $p(z) := zf'(z)/f(z)$, then $p(0) = 1$, and the above subordination relation can be written as

$$p(z) + \frac{zp'(z)}{p(z)} \prec \varphi_\alpha(z). \tag{29}$$

Supposing that the function ψ_α satisfies the differential equation

$$\psi_\alpha(z) + \frac{z\psi_\alpha'(z)}{\psi_\alpha(z)} = \varphi_\alpha(z), \quad \psi_\alpha(0) = 1, \tag{30}$$

we will prove that ψ_α is a convex univalent function in $\mathbb{U}$.

The function φ_α has positive real part in $\mathbb{U}$ whenever $\alpha \in (0, 1]$. Therefore, using ([27], Theorem 1) for $\beta = 1$, $\gamma = 0$, and $c = 1$, it follows that the solution ψ_α of the differential equation (30) is analytic in $\mathbb{U}$, with $\operatorname{Re} \psi_\alpha(z) > 0$ for all $z \in \mathbb{U}$, and

$$\begin{aligned} \psi_\alpha(z) &= H(z)\left(\int_0^z \frac{H(t)}{t}\,\mathrm{d}t\right)^{-1} = \frac{\alpha z \exp(\alpha z)}{\exp(\alpha z) - 1} \\ &= 1 + \frac{\alpha}{2} z + \cdots, \quad z \in \mathbb{U}, \end{aligned} \tag{31}$$

where

$$H(z) = z \exp\left(\int_0^z \frac{\varphi_\alpha(t) - 1}{t}\,\mathrm{d}t\right) = z \exp(\alpha z), \tag{32}$$

and all powers are considered at the principal branch, that is, $\log 1 = 0$.

Since φ_α is convex and ψ_α is analytic with $\operatorname{Re} \psi_\alpha(z) > 0$ for all $z \in \mathbb{U}$, using [28] (Theorem 3.2i) for $n = 1$, we deduce that ψ_α is univalent in $\mathbb{U}$. Moreover, from Figure 1 made with MAPLE software, we get

$$\Psi(z) := \operatorname{Re}\left(1 + \frac{z\psi_\alpha'(z)}{\psi_\alpha'(z)}\right) > 0, \quad z \in \mathbb{U}, \tag{33}$$

and $\psi_\alpha'(0) = \alpha/2 \neq 0$, so ψ_α is a convex function. Hence, it follows that ψ_α is a convex univalent function in $\mathbb{U}$.

Therefore, according to [28] (Theorem 3.2i), the differential subordination (29) implies

$$p(z) \prec \psi_\alpha(z), \tag{34}$$

for all $0 < \alpha \le 1$, and ψ_α is the best dominant. Thus,

$$\frac{zf'(z)}{f(z)} \prec \psi_\alpha(z), \tag{35}$$

for all $0 < \alpha \le 1$. Hence,

$$\mathscr{C}(1+\alpha z) \subset \mathcal{S}^*(\psi_\alpha). \tag{36}$$

From the above relation, we get

$$\sup\{|\gamma_n(f)| : f \in \mathscr{C}(1+\alpha z)\} \le \sup\{|\gamma_n(f)| : f \in \mathcal{S}^*(\psi_\alpha)\}. \tag{37}$$

Hence, from Lemma 1, we obtain

$$\sup\{|\gamma_n(f)| : f \in \mathscr{C}(1+\alpha z)\} \le \frac{\alpha}{4n}. \tag{38}$$

Therefore, for $f \in \mathscr{C}(1+\alpha z)$ and for all $n \in \mathbb{N}$, we conclude that

$$|\gamma_n(f)| \le \frac{\alpha}{4n}. \tag{39}$$

□

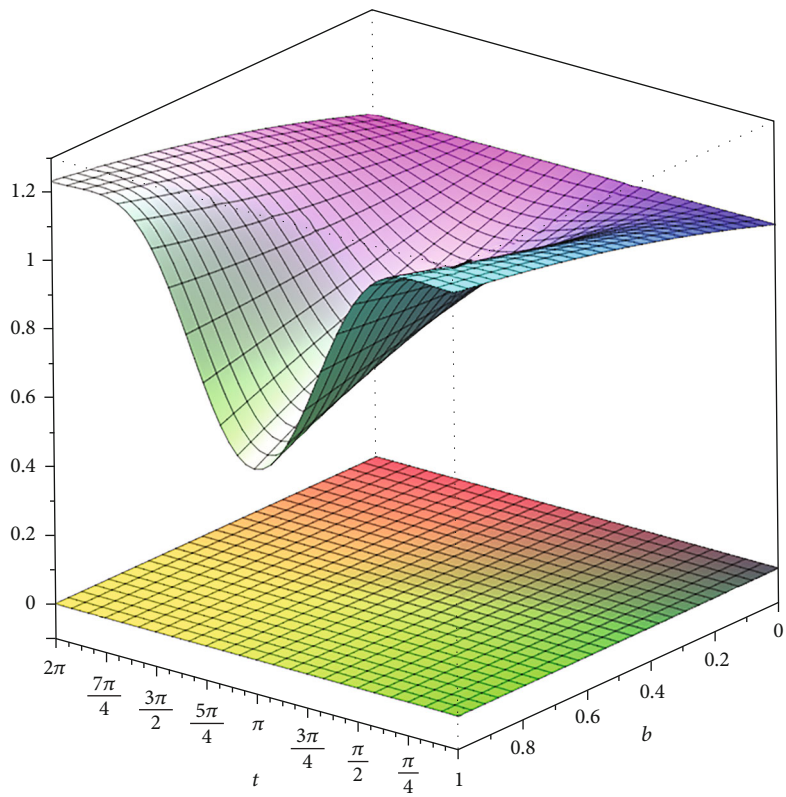

FIGURE 1: The plot of $\Psi(\mathrm{Re}^{it})$ for $t \in [0, 2\pi)$, $R = 1$, and $\alpha \in (10^{-6}, 1]$.

Remark 7. If we compare the results of Corollary 5 with those of Theorem 6, then we conclude that the results of Theorem 6 are not the best possible. We conjecture that if the function $f \in \mathscr{C}(1+\alpha z)$, then the logarithmic coefficients of f satisfy the inequalities

$$|\gamma_n| \le \frac{\alpha}{2n(n+1)}, \quad n \in \mathbb{N}. \tag{40}$$

Equality is attained for the function $L_{\alpha,n}$, that is,

$$\log \frac{L_{\alpha,n}(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(L_{\alpha,n}) z^n = \frac{\alpha}{n(n+1)} z^n + \cdots, \quad z \in \mathbb{U}. \tag{41}$$

Theorem 8. *Let $f \in \mathscr{CV}_{hpl}(1/2)$. Then, the logarithmic coefficients of f satisfy the inequalities*

$$|\gamma_n| \le \frac{1}{8n}, \quad n \in \mathbb{N}. \tag{42}$$

This inequality is sharp for $|\gamma_1|$ for the function $K_{1/2,1}$.

Proof. Letting $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1}{\sqrt{1-z}} =: \mathfrak{q}_{\frac{1}{2}}(z). \tag{43}$$

Suppose that $\mathfrak{p}$ satisfies the differential equation

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\mathfrak{p}(z)} = \frac{1}{\sqrt{1-z}}. \tag{44}$$

If we define $p(z) := zf'(z)/f(z)$, then the subordination (43) can be rewritten as

$$p(z) + \frac{zp'(z)}{p(z)} \prec \mathfrak{q}_{\frac{1}{2}}(z). \tag{45}$$

According to the inequality (20) of [12] (Theorem 2.3), the function $\mathfrak{q}_{1/2}$ is analytic with positive real part in $\mathbb{U}$. Therefore, using [27] (Theorem 1) for $\beta = 1$, $\gamma = 0$, and $c = 1$, it follows that the solution $\mathfrak{p}$ of the differential equation (44) is analytic in $\mathbb{U}$ with $\mathrm{Re}\,\mathfrak{p}(z) > 0$, $z \in \mathbb{U}$, and

$$\begin{aligned}\mathfrak{p}(z) &= H(z)\left(\int_0^z \frac{H(t)}{t}\,\mathrm{d}t\right)^{-1} = \frac{4z}{\left(1+\sqrt{1-z}\right)^2} \\ &\quad \cdot \frac{1}{\left(-8/\left(1+\sqrt{1-z}\right)\right) - 8\ln\left(1+\sqrt{1-z}\right) + 4 + 8\ln 2} \\ &= 1 + \frac{1}{4}z + \cdots, \quad z \in \mathbb{U},\end{aligned} \tag{46}$$

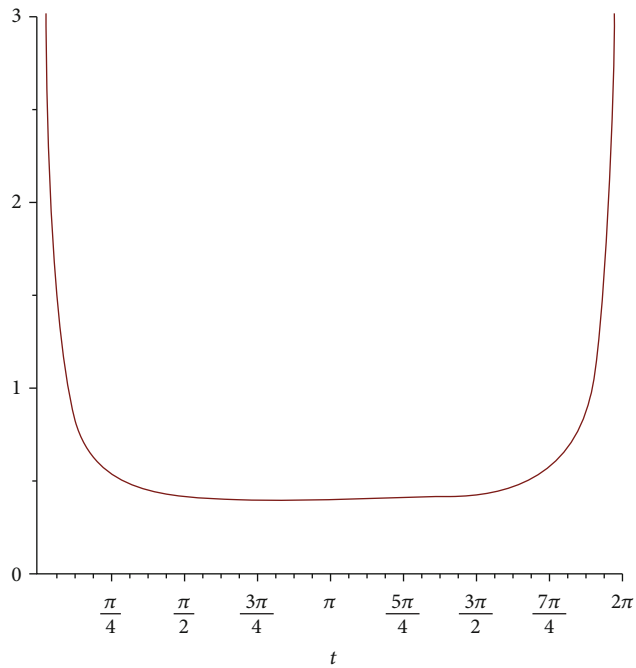

Figure 2: The plot of $\Psi(\mathrm{Re}^{it})$ for $t \in [0, 2\pi)$ and $R = 1$.

where

$$H(z) = z \exp\left(\int_0^z \frac{\mathfrak{q}_{1/2}(t) - 1}{t} \mathrm{d}t\right) = \frac{4z}{\left(1 + \sqrt{1 - z}\right)^2}, \quad (47)$$

and all powers are considered at the principal branch, that is, $\log 1 = 0$. Moreover, from Figure 2 made with MAPLE software, we get

$$\Psi(z) := \mathrm{Re}\left(1 + \frac{z\mathfrak{p}''(z)}{\mathfrak{p}'(z)}\right) > 0, \quad z \in \mathbb{U}, \quad (48)$$

and $\mathfrak{p}'(0) = 1/4 \neq 0$. Hence, ψ is convex in $\mathbb{U}$.

Since p satisfies in the subordination (45), using [28] (Theorem 3.2i), we conclude that $p(z) \prec \psi(z)$, that is,

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{p}(z), \quad (49)$$

and $\mathfrak{p}$ is the best dominant. Thus, $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$ implies $f \in \mathcal{S}^*(\mathfrak{p})$, that is,

$$\mathscr{CV}_{\mathrm{hpl}}(1/2) \subset \mathcal{S}^*(\mathfrak{p}). \quad (50)$$

Therefore, since $\mathfrak{p}$ is convex univalent, from Lemma 1, it follows that

$$\sup\left\{|\gamma_n(f)| : f \in \mathscr{CV}_{\mathrm{hpl}}\left(\frac{1}{2}\right)\right\} \leq \sup\left\{|\gamma_n(f)| : f \in \mathcal{S}^*(\mathfrak{p})\right\} = \frac{1}{8n}, \quad (51)$$

and we obtain the result. This completes the proof.□

Remark 9. If we compare the results of Lemma 1 with those of Theorem 8, then we conclude that the results of Theorem 8 are not the best possible. We conjecture that if the function $f \in \mathscr{CV}_{\mathrm{hpl}}(1/2)$, then the logarithmic coefficients of f satisfy the inequalities

$$|\gamma_n| \leq \frac{1}{4n(n+1)}, \quad n \in \mathbb{N}. \quad (52)$$

Equality is attained for the function $K_{1/2,n}$, that is,

$$\log \frac{K_{1/2,n}(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(K_{1/2,n}) z^n = \frac{1}{2n(n+1)} z^n + \cdots, \quad z \in \mathbb{U}. \quad (53)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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