Research Article

Logarithmic Coefficient Bounds and Coefficient Conjectures for Classes Associated with Convex Functions

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It is well-known that the logarithmic coefficients play an important role in the development of the theory of univalent functions. If \( \mathcal{A} \) denotes the class of functions \( f(z) = z + \sum_{n=1}^{\infty} a_n z^n \) analytic and univalent in the open unit disk \( \mathbb{U} \), then the logarithmic coefficients \( \gamma_n(f) \) of the function \( f \in \mathcal{A} \) are defined by \( \log \left( \frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n \). In the current paper, the bounds for the logarithmic coefficients \( \gamma_n \) for some well-known classes like \( \mathcal{B}(1+az) \) for \( a \in (0, 1) \) and \( \mathcal{B}_{\mathcal{V}}(1/2) \) were estimated. Further, conjectures for the logarithmic coefficients \( \gamma_n \) for functions \( f \) belonging to these classes are stated. For example, it is forecasted that if the function \( f \in \mathcal{B}(1+az) \), then the logarithmic coefficients of \( f \) satisfy the inequalities \( |\gamma_n| \leq a/(2n(n+1)) \), \( n \in \mathbb{N} \). Equality is attained for the function \( L_{a,n} \), that is, \( \log \left( L_{a,n}(z)/z \right) = 2 \sum_{n=1}^{\infty} \gamma_n(L_{a,n}) z^n = (a/n(n+1)) z^n + \cdots, z \in \mathbb{U} \).

Dedicated to the memory of Professor Gabriela Kohr (1967-2020)

1. Introduction

Let \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk in the complex plane \( \mathbb{C} \). Let \( \mathcal{A} \) be the category of analytic functions \( f \) in \( \mathbb{U} \) for which \( f \) has the following representation:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}.
\]

These coefficients play an important role for different estimates in the theory of univalent functions, and note that we use \( \gamma_n \) instead of \( \gamma_n(f) \). Kaukumov [1] solved Brennan’s conjecture for conformal mappings with the help of studying the logarithmic coefficients. The significance of the logarithmic coefficients follows from Lebedev-Milin inequalities ([2], chapter 2; see also [3, 4]), where estimates of the logarithmic coefficients were applied to obtain bounds on the coefficients of \( f \). Milin [2] conjectured the inequality

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \left( k |\gamma_k| - \frac{1}{k} \right) \leq 0, \quad n = 1, 2, 3, \ldots,
\]

that implies Robertson’s conjecture [5] and hence Bieberbach’s conjecture [6], which was the well-known coefficient problem in the theory of univalent functions. De Branges

Recall that we can rewrite (2) in the power series form as follows:

\[
2 \sum_{n=1}^{\infty} \gamma_n z^n = a_2 z + a_3 z^2 + a_4 z^3 + \cdots \frac{1}{2} (a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^2 \\
+ \frac{1}{3} (a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^3 + \cdots, \quad z \in \mathbb{U}, 
\]

and equating the coefficients of \(z^n\) for \(n = 1, 2, 3\), it follows that

\[
\begin{aligned}
2 \gamma_1 &= a_2, \\
2 \gamma_2 &= a_3 - \frac{1}{2} a_2^2, \\
2 \gamma_3 &= a_4 - a_2 a_3 + \frac{1}{3} a_2^3.
\end{aligned} 
\]

If the functions \(f\) and \(g\) are analytic in \(\mathbb{U}\), the function \(f\) is called to be \textit{subordinate} to the function \(g\), written \(f(z) < g(z)\), if there exists a function \(w\) analytic in \(\mathbb{U}\) with \(|w(z)| < 1, z \in \mathbb{U}\), and \(w(0) = 0\), such that \(f = g \circ w\). In particular, if \(g\) is univalent in \(\mathbb{U}\), then the following equivalence relationship holds true:

\[
f(z) \prec g(z) \iff f(0) = g(0), \\
f(\mathbb{U}) \subset g(\mathbb{U}).
\]

Using the principle of subordination, Ma and Minda [8] introduced the classes \(\Delta^*(\varphi)\) and \(\mathcal{C}(\varphi)\), where we make here the weaker assumptions that the function \(\varphi\) is analytic in the open unit disk \(\mathbb{U}\) and satisfies \(\varphi(0) = 1\), such that it has a series expansion of the form

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad z \in \mathbb{U}, \quad \text{with } B_1 \neq 0.
\]

They considered the abovementioned classes as follows:

\[
\Delta^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\},
\]

\[
\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \right\}.
\]

Some special subclasses of the class \(\Delta^*(\varphi)\) and \(\mathcal{C}(\varphi)\) play a significant role in the \textit{Geometric Function Theory} because of their geometric properties. 

For example, taking \(\varphi(z) = (1 + Az)/(1 + Bz)\) where \(A \in \mathbb{C}, \quad -1 \leq B \leq 0\), and \(A \neq B\), we get the classes \(\Delta^*[A, B]\) and \(\mathcal{C}[A, B]\), respectively (see also [9, 10]). The mentioned classes with the restriction \(-1 \leq B < A \leq 1\) reduce to the popular \textit{Janowski starlike} and \textit{Janowski convex functions}, respectively. By replacing \(A = 1 - 2\alpha\) and \(B = -1\), where \(0 \leq \alpha < 1\), we obtain the classes \(\Delta^*(\alpha)\) and \(\mathcal{C}(\alpha)\) of the \textit{starlike functions of order} \(\alpha\) and \textit{convex functions of order} \(\alpha\), respectively. In particular, \(\Delta^* = \Delta^*(0)\) and \(\mathcal{C} = \mathcal{C}(0)\) are the classes of starlike functions and of convex functions in the open unit disk \(\mathbb{U}\), respectively. Further, by altering \(A = \alpha\) and \(B = 0\), where \(0 \leq \alpha < 1\), we get the classes \(\Delta^*(1 + \alpha z)\) and \(\mathcal{C}(1 + \alpha z)\), which are the extensions of the classes \(\Delta^*(1 + z)\) and \(\mathcal{C}(1 + z)\), respectively (see [11]), that is,

\[
\Delta^*(1 + \alpha z) = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \right\},
\]

\[
\mathcal{C}(1 + \alpha z) = \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \right\},
\]

where \(0 < \alpha \leq 1\).

Supposing that \(\Psi_{\alpha, n} \in \Delta^*(1 + \alpha z)\) is such that

\[
\frac{z\Psi_{\alpha, n}'}{\Psi_{\alpha, n}}(z) = 1 + \alpha z^n, \quad n \in \mathbb{N},
\]

each function \(\Psi_{\alpha, n}\) is of the form

\[
\Psi_{\alpha, n}(z) = z \exp \left( \int_0^z \frac{1 + \alpha t^{n-1} - 1}{t} \, dt \right) = z + \frac{\alpha}{n} z^{n+1} + \cdots, \quad z \in \mathbb{U},
\]

and is the extremal function for various problems in \(\Delta^*(1 + \alpha z)\). Also, suppose that \(L_{\alpha, n} \in \mathcal{C}(1 + \alpha z)\) is such that

\[
1 + \frac{zL_{\alpha, n}'}{L_{\alpha, n}} = 1 + \alpha z^n, \quad n \in \mathbb{N}.
\]

Then, each function \(L_{\alpha, n}\) is of the form

\[
L_{\alpha, n}(z) = \int_0^z \exp \left( \int_0^t \frac{1 + \alpha u^{n-1} - 1}{u} \, du \right) \, dx = z + \frac{\alpha}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{U},
\]

and plays as extremal function for some extremal problems in the set \(\mathcal{C}(1 + \alpha z)\).

Lately, Kanas et al. [12] introduced the categories \(\mathfrak{J}_{hp}(s)\) and \(\mathcal{C}(s)\) by

\[
\mathfrak{J}_{hp}(s) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < q_s(z) = \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\},
\]

\[
\mathcal{C}(s) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < q_s(z) = \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\},
\]

and obtained some geometric properties in these categories. Further, the functions
\[ \Phi_{r,n}(z) = z \exp \left( \int_0^r \frac{q_n(t^n) - 1}{t} \, dt \right) = z + \frac{s}{n} z^{n+1} + \cdots, \quad z \in U, \ n \in \mathbb{N}, \]

\[-\left(1 - e^{\theta^2} z^2 \right)^{-2} \quad \text{for each} \ \theta \in \mathbb{R} \]

play as extremal functions for some issues of the families \( \mathcal{S}_{\text{hyp}}(s) \) and \( \mathcal{S}_{\text{hyp}}^-(s) \), respectively.

Lately, several researchers have subsequently investigated some problems regarding the logarithmic coefficients and the coefficients problems [9, 13–23], to mention a few of them. For instance, the rotation of the Koebe function \( k(z) = (1-e^{\theta^2} z^2)^{-2} \) for each \( \theta \in \mathbb{R} \) has the logarithmic coefficients \( \gamma_n = e^{\theta n}/n, n \geq 1 \). If \( f \in \mathcal{S}' \), then applying the Bieberbach inequality for the first relation of (5), it follows that \( |\gamma_1| \leq 1 \), and using the Fekete-Szeg"o inequality for the second relation of (5) (see [24], Theorem 3.8) leads to

\[ |\gamma_2| = \frac{1}{2} \left| a_3 - \frac{1}{2} a_2^2 \right| \leq \frac{1}{2} \left( 1 + 2e^{-2} \right) = 0.635 \cdots. \] (16)

It was established in ([25], Theorem 4) that the logarithmic coefficients \( \gamma_n \) of \( f \in \mathcal{S}' \) satisfy the inequality

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6}, \] (17)

and the equality is obtained for the Koebe function. For \( f \in \mathcal{S}' \), the inequality \( |\gamma_n| \leq 1/n \) holds but is not true for the full class \( \mathcal{S}' \), even in order of magnitude (see [24], Theorem 8.4).

In 2018, some first logarithmic coefficients \( \gamma_n \) were estimated for special subclasses of close-to-convex functions in [15, 20]. However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for \( n \geq 3 \) is presumably still a concern. In [13], the authors obtained the bounds of logarithmic coefficients \( \gamma_n, n \in \mathbb{N} \), for the general class \( \mathcal{S}'^{\phi}(\mathbb{R}) \), and the bounds of the logarithmic coefficients \( \gamma_n \) when \( n = 1, 2, 3 \) for the class \( \mathcal{S}(\mathbb{R}) \), while the estimated bounds would generalize many of the previous outcomes.

In the present study, which is motivated essentially by the recent works [13, 16], the bounds for the logarithmic coefficients \( \gamma_n, n \in \mathbb{N} \), of the class \( \mathcal{C}(1 + az) \) for \( a \in (0, 1] \) and \( \mathcal{C}_{\text{hyp}}(1/2) \) were estimated. Further, conjectures for the logarithmic coefficients \( \gamma_n \) for \( f \) belonging to these classes are stated.

2. **Main Results**

First, we will obtain the bounds for \( \gamma_n \) of the classes \( \mathcal{S}'^{\phi}(1 + az) \) and \( \mathcal{C}(1 + az) \) for \( a \in (0, 1] \). In this regard, the following outcomes will be employed in the key results.

**Lemma 1** (see [13], Theorem 1). Let \( f \in \mathcal{S}'^{\phi}(\mathbb{R}) \). If \( \phi \) is convex univalent, then the logarithmic coefficients \( f \) satisfy the following inequalities:

\[ |\gamma_n| \leq \frac{|B_1|}{2n}, \quad n \in \mathbb{N}, \] (18)

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2}. \] (19)

The inequalities in (18) and (19) are sharp, such that for any \( n \in \mathbb{N} \), there exist the function \( f_n \) given by \( z f_n(z)/f_n(z) = \phi(z^n) \) and the function \( f \) given by \( z f(z)/f(z) = \phi(z) \), respectively, for those equalities we obtain.

**Lemma 2** (see [13], Theorem 2). Let \( f \in \mathcal{C}_{\text{hyp}}(\mathbb{R}) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities

\[ |\gamma_1| \leq \frac{|B_1|}{4}, \] (20)

\[ \begin{cases} |\gamma_2| \leq \frac{|B_1|}{12}, & \text{if} \quad |4B_2 + B_1^2| \leq 4|B_1|, \\ |\gamma_2| \leq \frac{|4B_2 + B_1^2|}{48}, & \text{if} \quad |4B_2 + B_1^2| > 4|B_1|, \end{cases} \] (21)

and if \( B_1, B_2, \) and \( B_3 \) are real values, then

\[ |\gamma_3| \leq \frac{|B_1|}{24} H(q_1, q_2). \] (22)

where \( H(q_1, q_2) \) is given in ([26], Lemma 2) (or [9], Lemma 5), \( q_1 = (B_1 + (4B_2/B_1))/2, \) and \( q_2 = (B_2 + (2B_3/B_1))/2 \). The bounds (20) and (21) are sharp.

**Lemma 3** (see [18], Theorem 30). If \( f \in \mathcal{C}_{\text{hyp}}(1/2) \), then

\[ |\gamma_1| \leq \frac{1}{8}, \quad |\gamma_2| \leq \frac{1}{24}, \quad |\gamma_3| \leq \frac{1}{48}. \] (23)

The first two bounds are sharp for \( f = K_{1/2,1} \) and \( f = K_{1/2,2} \), respectively.

If we consider Lemma 1 with the function \( \phi(z) = 1 + az \), then we immediately get the next result:

**Theorem 4.** If \( f \in \mathcal{S}'^{\phi}(1 + az) \), then

\[ |\gamma_n| \leq \frac{a}{2n}, \quad n \in \mathbb{N}, \] (24)

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{a}{4}. \]

These inequalities are sharp for \( f = \Psi_{a,n} \) and \( f = \Psi_{a,1} \), respectively.
Corollary 5. Let \( f \in \mathcal{C}(1 + az) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities
\[
|y_1| \leq \frac{\alpha}{4}, \\
|y_2| \leq \frac{\alpha}{12}, \\
|y_3| \leq \frac{\alpha}{24}.
\]  
(Eq. 25)

Equalities in these inequalities are attained for the functions \( L_{a,n} \) for \( n = 1, 2, 3 \), respectively.

Proof. For \( \varphi(z) = 1 + az \), where \( B_1 = a, B_2 = B_3 = 0 \), in Theorem 6, we obtain the required result. Also, since
\[
\log \frac{L_{a,1}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,1}) z^n = \frac{\alpha}{2} z + \cdots, \quad z \in \mathbb{U},
\]
\[
\log \frac{L_{a,2}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,2}) z^n = \frac{\alpha}{6} z^2 + \cdots, \quad z \in \mathbb{U},
\]
\[
\log \frac{L_{a,3}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,3}) z^n = \frac{\alpha}{12} z^3 + \cdots, \quad z \in \mathbb{U},
\]

it follows that these inequalities are attained for the functions \( L_{a,n} \) for \( n = 1, 2, 3 \), respectively. \( \square \)

Theorem 6. Let \( f \in \mathcal{C}(1 + az) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities
\[
|y_n| \leq \frac{\alpha}{4n}, \quad n \in \mathbb{N}.
\]  
(Eq. 27)

This inequality is sharp for \( |y_1| \) for the function \( L_{a,1} \).

Proof. If \( f \in \mathcal{C}(1 + az) \), this is equivalent to \( f \in \mathcal{A} \) and
\[
1 + \frac{zf^{(n)}(z)}{f(z)} < 1 + az = \varphi_a(z).
\]  
(Eq. 28)

If we define \( p(z) = zf^{(n)}(z)/f(z) \), then \( p(0) = 1 \), and the above subordination relation can be written as
\[
p(z) + \frac{zp^{(n)}(z)}{p(z)} < \varphi_a(z).
\]  
(Eq. 29)

Supposing that the function \( \psi_a \) satisfies the differential equation
\[
\psi_a(z) + \frac{z\psi_a(z)'}{\psi_a(z)} = \varphi_a(z), \quad \psi_a(0) = 1,
\]  
(Eq. 30)

we will prove that \( \psi_a \) is a convex univalent function in \( \mathbb{U} \).

The function \( \varphi_a \) has positive real part in \( \mathbb{U} \) whenever \( \alpha \in (0, 1] \). Therefore, using ([27], Theorem 1) for \( \beta = 1, \gamma = 0, \) and \( c = 1 \), it follows that the solution \( \psi_a \) of the differential equation (30) is analytic in \( \mathbb{U} \), with \( \Re \psi_a(z) > 0 \) for all \( z \in \mathbb{U} \), and
\[
\psi_a(z) = H(z) \left( \int_0^z \frac{H(t)}{t} \; dt \right)^{-1} = \frac{az \exp(az)}{\exp(az) - 1}
\]  
(Eq. 31)

where
\[
H(z) = z \exp \left( \int_0^z \frac{\varphi_a(t) - 1}{t} \; dt \right) = z \exp(az),
\]  
(Eq. 32)

and all powers are considered at the principal branch, that is, \( \log 1 = 0 \).

Since \( \varphi_a \) is convex and \( \psi_a \) is analytic with \( \Re \psi_a(z) > 0 \) for all \( z \in \mathbb{U} \), using [28] (Theorem 3.2i) for \( n = 1 \), we deduce that \( \psi_a \) is univalent in \( \mathbb{U} \). Moreover, from Figure 1 made with MAPLE software, we get
\[
\Psi(z) = \Re \left( 1 + \frac{z\psi_a(z)}{\psi_a(z)} \right) > 0, \quad z \in \mathbb{U},
\]  
(Eq. 33)

and \( \psi_a(0) = a/2 \neq 0 \), so \( \psi_a \) is a convex function. Hence, it follows that \( \psi_a \) is a convex univalent function in \( \mathbb{U} \).

Therefore, according to [28] (Theorem 3.2i), the differential subordination (29) implies
\[
p(z) < \psi_a(z),
\]  
(Eq. 34)

for all \( 0 < \alpha \leq 1 \), and \( \psi_a \) is the best dominant. Thus,
\[
\frac{zf^{(n)}(z)}{f(z)} < \psi_a(z),
\]  
(Eq. 35)

for all \( 0 < \alpha \leq 1 \). Hence,
\[
\mathcal{C}(1 + az) \subset \mathcal{D}^*(\psi_a).
\]  
(Eq. 36)

From the above relation, we get
\[
\sup \{ |y_n(f)| : f \in \mathcal{C}(1 + az) \} \leq \sup \{ |y_n(f)| : f \in \mathcal{D}^*(\psi_a) \}.
\]  
(Eq. 37)

Hence, from Lemma 1, we obtain
\[
\sup \{ |y_n(f)| : f \in \mathcal{C}(1 + az) \} \leq \frac{\alpha}{4n},
\]  
(Eq. 38)

Therefore, for \( f \in \mathcal{C}(1 + az) \) and for all \( n \in \mathbb{N} \), we conclude that
\[
|y_n(f)| \leq \frac{\alpha}{4n},
\]  
(Eq. 39)

\( \square \)
Remark 7. If we compare the results of Corollary 5 with those of Theorem 6, then we conclude that the results of Theorem 6 are not the best possible. We conjecture that if the function \( f \in \mathcal{C}(1 + \alpha z) \), then the logarithmic coefficients of \( f \) satisfy the inequalities

\[
|y_n| \leq \frac{\alpha}{2n(n + 1)}, \quad n \in \mathbb{N}.
\] (40)

Equality is attained for the function \( L_{\alpha,n} \), that is,

\[
\log \frac{L_{\alpha,n}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{\alpha,n}) z^n = \frac{\alpha}{n(n + 1)} z^n + \cdots, \quad z \in \mathbb{U}.
\] (41)

Theorem 8. Let \( f \in \mathcal{C}^{\mu}(1/2) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities

\[
|y_n| \leq \frac{1}{8n}, \quad n \in \mathbb{N}.
\] (42)

This inequality is sharp for \( |y_n| \) for the function \( K_{1/2,1} \).

Proof. Letting \( f \in \mathcal{C}^{\mu}(1/2) \), it follows that

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{1}{\sqrt{1 - z}} = q_1(z).
\] (43)

Suppose that \( p \) satisfies the differential equation

\[
p(z) + \frac{zp'(z)}{p(z)} = \frac{1}{\sqrt{1 - z}}.
\] (44)

If we define \( p(z) = zq_1(z)/f(z) \), then the subordination (43) can be rewritten as

\[
p(z) + \frac{zp'(z)}{p(z)} < q_1(z).
\] (45)

According to the inequality (20) of [12] (Theorem 2.3), the function \( q_{1/3} \) is analytic with positive real part in \( \mathbb{U} \). Therefore, using [27] (Theorem 1) for \( \beta = 1, \gamma = 0, \) and \( c = 1 \), it follows that the solution \( p \) of the differential equation (44) is analytic in \( \mathbb{U} \) with \( \text{Re} \ p(z) > 0, z \in \mathbb{U} \), and

\[
p(z) = H(z) \left( \int_0^z \frac{H(t)}{t} \, dt \right)^{-1} = \frac{4z}{\left(1 + \sqrt{1 - z} \right)^2} - \frac{1}{8} \frac{1}{\left(1 + \sqrt{1 - z} \right)} - 8 \ln \left(1 + \sqrt{1 - z} \right) + 4 + 8 \ln 2
\]

\[
= 1 + \frac{1}{4} z + \cdots, \quad z \in \mathbb{U},
\] (46)
and we obtain the result. This completes the proof. \(\square\)

**Remark 9.** If we compare the results of Lemma 1 with those of Theorem 8, then we conclude that the results of Theorem 8 are not the best possible. We conjecture that if the function \(f \in \mathcal{C} \mathcal{Y} \mathcal{Y} \mathcal{Y}_{hpl}(1/2)\), then the logarithmic coefficients of \(f\) satisfy the inequalities

\[
|\gamma_n| \leq \frac{1}{4n(n+1)}, \quad n \in \mathbb{N}.
\]  

Equality is attained for the function \(K_{1/2,n}\), that is,

\[
\log \frac{K_{1/2,n}(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(K_{1/2,n})z^n = \frac{1}{2n(n+1)}z^n + \cdots, \quad z \in \mathbb{U}.
\]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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