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Research Article

On L^2-Boundedness of h-Pseudodifferential Operators

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Let T_a^h be the h-pseudodifferential operators with symbol a. When $a \in S^m_{\rho,1}$ and $m = n(\rho-1)/2$, it is well known that T_a^h is not always bounded in $L^2(\mathbb{R}^n)$. In this paper, under the condition $a(x,\xi) \in L^\infty S_\rho^{n(\rho-1)/2}(\omega)$, we show that T_a^h is bounded on L^2.

1. Introduction and Main Results

Let $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $S^m_{\rho,\delta}$ be the sets of all the symbol $a(x,\xi)$ satisfying, for $m \in \mathbb{R}$, $0 \le \rho, \delta \le 1$, and $\alpha, \beta \in \mathbb{N}^n$,

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \lesssim \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{1}$$

The classical pseudodifferential operator T_a associated with the symbol $a(x,\xi) \in S^m_{\rho,\delta}$ is defined by

$$T_a f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi) \hat{f}(\xi) d\xi, \tag{2}$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

The generalized form of pseudodifferential operator is h-pseudodifferential operator, which is introduced by Helffer in [1]. Helffer studied this operator to discuss the operator $-h^2\Delta + V(x)$ associated with the parameter h, where Δ is the Laplace operator and $V(x)$ denotes some potential function for any $x \in \mathbb{R}^n$. Moveover, the h-pseudodifferential operator also provides a rigorous way to establish relationship each other for quantum physics and classical mechanics; see, for example, [2, 3]. Furthermore, by using h-pseudodifferential operator, the Cauchy problem of semiclassical elliptic partial differential equations is studied in [3–5]. Because of these operators importance, many scholars have studied one; see, for example, [6–12] and the references given there.

Now, we consider the h-pseudodifferential operator as follows:

$$\begin{aligned} T_a^h f &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x, h\xi) e^{i(x-y)\xi} f(y) dy \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi) e^{(i/h)(x-y)\xi} f(y) dy \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)x\xi} a(x,\xi) \hat{f}(\xi) d\xi, \end{aligned} \tag{3}$$

where $h \in (0, h_0]$ and

$$\hat{f}(\xi, h) = \int_{\mathbb{R}^n} e^{-(i/h)y\xi} f(y) dy. \tag{4}$$

First of all, we review L^2 regularity theory for the classical pseudodifferential operators T_a. In [13], Hörmander shown that T_a is bounded in $L^2(\mathbb{R}^n)$, when $a \in S^m_{\rho,\delta}$, $\delta < 1$ and $m \le n(\rho-\delta)/2$. For $a \in S^0_{1,1}$, Ching [14] proved that T_a is not bounded in $L^2(\mathbb{R}^n)$. Moveover, for $a \in S^m_{\rho,1}$, Rodino [15] shown that T_a is bounded in $L^2(\mathbb{R}^n)$ if and only if $m < n(\rho-1)/2$. However, the operator T_a is not always L^2-boundedness for $a \in S^{n(\rho-1)/2}_{\rho,1}$; see, for example, [13–15]. The necessary and sufficient conditions of L^2-boundedness of T_a are obtained by Higuchi [16] as $m = n(\rho-1)/2$. Recently, Aitemrar and Senoussaoui [6] studied boundedness and compactness on L^2 for h-Fourier integral operators, where the symbol $a \in \Gamma_\rho^\mu$. Moveover, Aitemrar and Senoussaoui [9] also discussed regularity on Bessel potential spaces. Furthermore, Aitemrar and Senoussaoui [7] considered the symbol for h-pseudodifferential operator, which belongs to the class $L^\infty S_\rho^m$, namely,

$$\left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}. \tag{5}$$

By using this condition and the compact set $\mathrm{Supp}_\xi a(x,\xi)$, they obtained the global L^p-boundedness for h-Fourier integral operators. The noncompact case for $\mathrm{Supp}_\xi a(x,\xi)$, they also proved L^p-boundedness for $m < n(\rho-1)/2$. Motivated by this noncompact case, for $\delta = 1$, $m = n(\rho-1)/2$, we consider the L^2-boundedness of h-pseudodifferential operators. In what follows, we mainly concentrate on the L^2-boundedness for h-pseudodifferential operator T_a^h with the specific symbol a. Our main result could be stated as follows:

Theorem 1. *Let T_a^h be the h-pseudodifferential operators given by (3) with symbol $a(x,\xi) \in L^\infty S_\rho^{n(\rho-1)/2}(\omega)$, where $L^\infty S_\rho^{n(\rho-1)/2}(\omega)$ is as in Definition 3. Then, for $0 \le \rho \le 1$, there exists a constant $C > 0$ such that*

$$\left\|T_a^h u\right\|_{L^2} \le C\|u\|_{L^2}. \tag{6}$$

Remark 2. For the classical pseudodifferential operator, in [16, 17], they have constructed an example that the operator is not bounded in $L^2(\mathbb{R}^n)$ as $a \in S_{\rho,1}^{n(\rho-1)/2}$. Moveover, here, we remark that the symbol $L^\infty S_\rho^{n(\rho-1)/2}(\omega)$ is different from the symbol $a \in \Gamma_\rho^\mu$ in [6].

2. Definitions, Notations, and Preliminaries

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$, $\mathbb{N} = \{1,2,\cdots\}$, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, and $\mathbb{Z}_+^n = (\mathbb{Z}_+)^n$. For any multi-index $\alpha = (\alpha_1,\cdots,\alpha_n)$ and $\beta = (\beta_1,\cdots,\beta_n) \in \mathbb{Z}_+^n$, we let

$$|\alpha| = \sum_{j=1}^n \alpha_j, \alpha+\beta = (\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n), \partial_x^\alpha = \frac{\partial^\alpha}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}}, \tag{7}$$

and $\nabla_\xi = (\partial_{\xi_1},\cdots,\partial_{\xi_n})$. Also, we also define $|\xi|$ and $\langle\xi\rangle$ as follows:

$$|\xi| = \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}, \quad \langle\xi\rangle = \left(1+|\xi|^2\right)^{1/2}. \tag{8}$$

Throughout this article, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We sometimes write $A \lesssim B$ as shorthand for $A \le CB$.

We give the class $L^\infty S_\rho^m(\omega)$ defined below, which will play significant role in our investigations.

Definition 3. Let m be a real number. A function $a(x,\xi)$ which is smooth in the frequency variable ξ and bounded measurable in the spatial variable x, belongs to the symbol class $L^\infty S_\rho^m(\omega)$, if it satisfies, for all multi-indices α,

$$\left\|\partial_\xi^\alpha a(x,\xi)\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}\omega(\langle\xi\rangle), \tag{9}$$

where $\omega(t)$ is a nonnegative and decreasing function on $[1,\infty)$ and satisfies

$$\int_1^\infty \frac{\omega(t)^2}{t}dt < \infty. \tag{10}$$

Remark 4. If $\omega(t)$ satisfies (10), then $\sum_{j=0}^\infty \omega^2(2^j) < \infty$.

To prove the main theorem, we need the following lemma.

Lemma 5. *Let T_a^h be the h-pseudodifferential operators given by (3) with symbol $a(x,\xi) \in L^\infty S_\rho^{n(\rho-1)/2}$ and $\mathrm{Supp}_\xi a(x,\xi) \subset \{\xi : |\xi| \le R\}$. Then, for $0 \le \rho \le 1$, there exists a constant $C > 0$ such that*

$$\left\|T_a^h u\right\|_{L^2} \le C\|u\|_{L^2}. \tag{11}$$

The proof method of lemma is similar to Corollary 3.2 in [18], the details being omitted.

3. Proof of Main Result

In this section, we shall prove the main result Theorem 1.

First we need a dyadic partition of unity, let A be the annulus $A = \{\xi \in \mathbb{R}^n ; 1/2 \le |\xi| \le 2\}$,

$$\chi_0(\xi) + \sum_{j=1}^\infty \chi_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n, \tag{12}$$

where $\chi_0(\xi) \in C_0^\infty(B(0,2))$ and $\chi_j(\xi) = \chi(2^{-j}\xi)$. When $j \ge 1$ with $\chi(\xi) \in C_0^\infty(A)$, we decompose the operator T_a^h as

$$T_a^h = T_{\chi_0}(D) + \sum_{j=1}^\infty T_{\chi_j}(D) = T_0(D) + \sum_{j=1}^\infty T_j(D). \tag{13}$$

The first term in (13) is bounded on $L^2(\mathbb{R}^n)$ from Lemma

5. After a change of variables, we have

$$\begin{aligned} T_j(D) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)x\xi} \chi_j(\xi) a(x,\xi) \hat{u}(\xi) d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)\cdot 2^{j\rho} x\xi} \chi_j(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) \hat{u}(2^{j\rho}\xi) d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)\cdot 2^{j\rho} x\xi} \chi_j(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) \int_{\mathbb{R}^n} e^{-(i/h)2^{j\rho}\xi\cdot y} u(y) dy d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{(i/h)\cdot 2^{j\rho}(x\xi - y\cdot\xi)} \chi_j(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) u(y) d\xi dy. \end{aligned} \tag{14}$$

The kernel of the operator $T_j(D)$ is given by

$$T_j(x,y) = \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)\cdot 2^{j\rho}(x\xi - y\cdot\xi)} \chi_j(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) d\xi. \tag{15}$$

Let

$$a_j(x,\xi) = \chi\left(2^{j(\rho-1)}\xi\right) a\left(x, 2^{j\rho}\xi\right). \tag{16}$$

Then

$$A_j = \operatorname*{Supp}_{\xi} a_j \subset \left\{\xi\,; 2^{-1} 2^{j(1-\rho)} < |\xi| < 2\cdot 2^{j(1-\rho)}\right\}, \tag{17}$$

and satisfying

$$\left|\partial_\xi^\alpha a_j(x,\xi)\right| \le C_\alpha 2^{jn(\rho-1)/2}. \tag{18}$$

Now, we deal with the high frequency component T_j of T_φ^a. Let $S_j = T_j T_j^*$. Then

$$\begin{aligned} S_j u(x) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{(i/h)(x\xi - y\xi)} \chi_j^2(\xi) a(x,\xi) a(\bar{y},\xi) u(y) dy d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{(i/h)2^{j\rho}(x\xi - y\xi)} \chi_j^2(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi) u(y) d\xi dy. \end{aligned} \tag{19}$$

The kernel of the operator $S_j = T_j T_j^*$ reads

$$S_j(x,y) = \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)2^{j\rho}(x\xi - y\xi)} \chi_j^2(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi) d\xi. \tag{20}$$

We define

$$b_j(x,y,\xi) = \chi_j^2(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi). \tag{21}$$

By this, we have

$$\operatorname{Supp} b_j \subset \left\{\xi : \frac{2^{j(1-\rho)}}{2} < |\xi| < 2\cdot 2^{j(1-\rho)}\right\}. \tag{22}$$

Now, we claim that

$$\left|\partial_\xi^\alpha b_j(x,y,\xi)\right| \le C_\alpha 2^{jn(\rho-1)} \omega^2(2^j). \tag{23}$$

In fact, by using (18), we have

$$\begin{aligned} \left|\partial_\xi^\alpha b_j(x,y,\xi)\right| &= \left|\partial_\xi^\alpha\left[\chi_j^2(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi)\right]\right| \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \left|\partial_\xi^{\alpha_1}\left[a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi)\right]\right| \left|\partial_\xi^{\alpha_2} \chi^2\left(2^{-j(1-\rho)}\xi\right)\right| \\ &\lesssim \sum_{\alpha_1+\alpha_2=\alpha} \left(2^{j\rho}\right)^{|\alpha_1|} \left|\left(\partial_\xi^{\alpha_1}(a\cdot\bar{a})\right)(x, 2^{j\rho}\xi)\right| \omega^2(2^j) \cdot 2^{-j(1-\rho)|\alpha_2|} \left|\left(\partial_\xi^{\alpha_2}\chi\right)\left(2^{-j(1-\rho)}\xi\right)\right| \\ &\lesssim \sum_{\alpha_1+\alpha_2=\alpha} 2^{j\rho|\alpha_1|} \langle 2^{j\rho}\xi \rangle^{n(\rho-1)-\rho|\alpha_1|} 2^{-j(1-\rho)|\alpha_2|} \omega^2(2^j) \\ &\lesssim \sum_{\alpha_1+\alpha_2=\alpha} 2^{j\rho|\alpha_1|} 2^{j(n(\rho-1)-\rho|\alpha_1|)} 2^{-j(1-\rho)|\alpha_2|} \omega^2(2^j) \\ &= \sum_{\alpha_1+\alpha_2=\alpha} 2^{jn(\rho-1)-j(1-\rho)|\alpha_2|} \omega^2(2^j) = 2^{jn(\rho-1)} \sum_{\alpha_2} 2^{-j(1-\rho)|\alpha_2|} \omega^2(2^j) \lesssim 2^{jn(\rho-1)} \omega^2(2^j). \end{aligned} \tag{24}$$

Next, we consider the following differential operators, for $j \in \mathbb{N}$,

$$L_j(x,y,D) = \frac{h\nabla_\xi}{i 2^{j\rho}(x-y)}. \tag{25}$$

So

$$L_j^N(x,y,D) e^{(i/h)2^{j\rho}(x-y)\xi} = e^{(i/h)2^{j\rho}(x-y)\xi}, \tag{26}$$

$$L_j^*(x,y,D) = -\frac{h\nabla_\xi}{i 2^{j\rho}(x-y)}. \tag{27}$$

From this and (25), it follows that

$$\begin{aligned} S_j(x,y) &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)2^{j\rho}(x\xi - y\xi)} \chi_j^2(2^{j\rho}\xi) a(x, 2^{j\rho}\xi) a(y, \bar{2}^{j\rho}\xi) d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} \left(L_j^N e^{(i/h)2^{j\rho}(x-y)\xi}\right) b_j(x, y.\xi) d\xi \\ &= \frac{2^{j\rho n}}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{(i/h)2^{j\rho}(x-y)\xi} \left(L_j^*\right)^N b_j d\xi. \end{aligned} \tag{28}$$

Moreover, by (27) and (24), we further obtain

$$\begin{aligned} \left|\left(L_j^*\right)^N b_j\right| &\lesssim \frac{1}{2^{j\rho N}} \cdot \frac{h^N}{|x-y|^N} 2^{jn(\rho-1)} \omega^2(2^j) \\ &= \frac{h^N}{\left[2^{j\rho}|x-y|\right]^N} 2^{jn(\rho-1)} \omega^2(2^j). \end{aligned} \tag{29}$$

Integration by parts yields

$$S_j(x,y) = \frac{2^{j\rho n}}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{(i/h)2^{j\rho}(x-y)\xi}\left(L_j^*\right)^N b_j d\xi \lesssim \frac{h^N 2^{j\rho n}}{\left[2^{j\rho}\mid x-y\mid\right]^N} 2^{jn(\rho-1)}\omega^2\left(2^j\right)2^{j(1-\rho)n} \lesssim \frac{2^{j\rho n}}{\left[2^{j\rho}\mid x-y\mid\right]^N}\omega^2\left(2^j\right). \tag{30}$$

Thus,

$$\left|S_j(x,y)\right|\sum_{l=0}^{N}\left(2^{j\rho}|x-y|\right)^l \lesssim 2^{j\rho n}\omega^2\left(2^j\right), \tag{31}$$

which implies that

$$\left|S_j(x,y)\right| \lesssim \frac{2^{j\rho n}}{\left(1+2^{j\rho}|x-y|\right)^N}\omega^2\left(2^j\right), \tag{32}$$

and hence

$$\sup_x \int_{\mathbb{R}^n}\left|S_j(x,y)\right|dy \lesssim 2^{j\rho n}\omega^2\left(2^j\right)\int_{\mathbb{R}^n}\frac{1}{\left(1+2^{j\rho}|x-y|\right)^N}dy \lesssim \omega^2\left(2^j\right)\int_{\mathbb{R}^n}\frac{1}{(1+|z|)^N}dz \lesssim \omega^2\left(2^j\right). \tag{33}$$

By Young's inequality, we obtain

$$\left\|S_j u(x)\right\|_{L^2} \lesssim \omega^2\left(2^j\right)\|u(x)\|_{L^2}. \tag{34}$$

Therefore, we have

$$\left\|T_j^* u\right\|_{L^2}^2 = \left\langle T_j^* u, T_j^* u\right\rangle = \left\langle u, T_j T_j^* u\right\rangle \le \|u\|_{L^2}\left\|S_j u\right\|_{L^2} \le C\omega^2\left(2^j\right)\|u\|_{L^2}^2. \tag{35}$$

Namely

$$\left\|T_j u\right\|_{L^2} \le C\omega\left(2^j\right)\|u\|_{L^2}. \tag{36}$$

Next, we need the Littlewood-Paley decomposition. Let $\psi_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth radial function which equals to one on the unit ball centric at the origin and supported on its concentric double. Set $\psi(\xi) = \psi_0(\xi) - \psi_0(2\xi)$ and $\psi_k(\xi) = \psi(2^{-k}\xi)$. Then, for any $\xi \in \mathbb{R}^n$,

$$\psi_0(\xi) + \sum_{k=1}^{\infty}\psi_k(\xi) = 1, \tag{37}$$

and supp $\psi_k(\xi) \subset \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$. Thus

$$\widehat{u}(\xi) = \sum_{k=0}^{\infty}\widehat{u}(\xi)\psi_k(\xi). \tag{38}$$

Furthermore, we have

$$T_a^h u = \sum_{j=0}^{\infty} T_j u = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} e^{(i/h)x\xi} a_j(x,\xi)\widehat{u}_k(\xi)d\xi, \tag{39}$$

where $|j-k| \le 2$; in order to simplify, let

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} e^{(i/h)x\xi} a_j(x,\xi)\widehat{u}_k(\xi)d\xi = \sum_{j=1}^{\infty} T_j u_j. \tag{40}$$

And hence

$$\left\|T_a^h u\right\|_{L^2} = \left\|\sum_{j=0}^{\infty} T_j u_j\right\|_{L^2} \lesssim \sum_{j=0}^{\infty}\omega\left(2^j\right)\left\|u_j\right\|_{L^2} \lesssim \left(\sum_{j=0}^{\infty}\omega^2\left(2^j\right)\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}\|u_j\|_{L^2}^2\right)^{\frac{1}{2}} \lesssim \|u\|_{L^2}. \tag{41}$$

We use Remark 4 here. This finishes the proof of Theorem 1.

Data Availability

The data is available at https://journals.tubitak.gov.tr/math/issues/mat-18-42-4/mat-42-4-14-1610-104.pdf.

Conflicts of Interest

The author declares that he/she has no conflicts of interest.

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