Research Article
Approximation Properties of New Modified Gamma Operators

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This paper is aimed at constructing new modified Gamma operators using the second central moment of the classic Gamma operators. And we will compute the first, second, fourth, and sixth order central moments by the moment computation formulas, and their quantitative properties are researched. Then, the global results are established in certain weighted spaces and the direct results including the Voronovskaya-type asymptotic formula, and point-wise estimates are investigated. Also, weighted approximation of these operators is discussed. Finally, the quantitative Voronovskaya-type asymptotic formula and Grüss Voronovskaya-type approximation are presented.

1. Introduction

Recently, Karsli et al. [1] constructed and estimated the rate of convergence for functions with derivatives of bounded variation on \( \mathbb{R}_+ := (0, \infty) \) of new Gamma type operators preserving \( z^2 \) as (see also [2])

\[
(\Phi_1 \lambda(t))(z) = \frac{(2l+3)!z^{l+3}}{l!(l+2)!} \int_0^\infty \frac{t^l}{(z+t)^{2l+4}} \lambda(t)dt, \quad z \in \mathbb{R}_+.
\]

(1)

In [3], Karsli et al. used analysis methods to obtain the rate of point-wise convergence for the operators (1). In [4], Karsli and Ozarslan obtained some direct local and global approximation results for the operators (1). In [5], Izgi studied some direct results in asymptotic approximation about the operators (1). In [6], Krech gave a note about the results of Izgi in [5] and obtained an error estimate for the operators (1). In [7], Krech gave direct approximation theorems for the operators (1) in certain weighted spaces. In [8], Cai and Zeng constructed \( q \)-Gamma operators and gave their approximation properties. In [9], Zhao et al. extended the works of Cai and Zeng and considered the stancu generalization of \( q \)-Gamma operators. Recently, Cheng et al. constructed \( (p, q) \)-Gamma operators using \( (p, q) \)-Beta function of the second kind and discussed their approximation properties in [10]. In [11], Zhou et al. extended the works of Cheng et al. in [10] and constructed \( (p, q) \)-Gamma-Stancu operators. There are many papers about the research and application of other Gamma-type operators, and we mention some of them [12–17].

In this paper, we construct new modified Gamma operators using the second central moment of the operators (1) as follows:

**Definition 1.** For \( l = 1, 2, \cdots \) and \( \lambda : \mathbb{R}_+ \rightarrow \mathbb{R} \), we construct new modified Gamma operators by

\[
(Y_l \lambda(t))(z) = \int_0^\infty K_l(t, z) \lambda(t)dt, \quad z \in \mathbb{R}_+,
\]

(2)

where

\[
K_l(t, z) = \frac{2^l(2l+3)!(z+t)^{l+1}}{l!} \frac{t^l(z-t)^2}{(z+t)^{2l+4}}, \quad t, z \in \mathbb{R}_+.
\]

(3)

The paper is organized as follows: In Section 1, we introduce the history of Gamma operators and construct new modified Gamma operators using the second central moment. In Section 2, we obtain the basic results by the moment computation formulas. And the first, second,
2. Basic Results

In this section, we present certain auxiliary results which will be used to prove our main theorems for the operators (2).

**Lemma 2** (see [1]). For any \( l \in \mathbb{N}, p = 0, 1, 2, \ldots, l + 2, \) we have

\[
\phi_l(p) := (\Phi_l p^p)(z) = \frac{\Gamma(l + p + 1)}{\Gamma(l + 2)} z^p, z \in \mathbb{R}.
\]  

(4)

**Lemma 3.** If we define \( \phi_l(p) := (Y_l p^p)(z) \), then there holds the following relation

\[
\frac{2z^2}{l + 2} \phi_l(p) = \phi_l(p + 2) - 2\phi_l(p + 1)z + \phi_l(p)z^2,
\]  

(5)

where \( p = 0, 1, \ldots, l, z \in \mathbb{R}. \)

Then, the following lemma can be obtained immediately:

**Lemma 4.** For any \( l \in \mathbb{N}, z \in \mathbb{R}, \) we have

\[
\phi_l(0) = 1; \phi_l(1) = \frac{l + 3}{l + 2} z; \phi_l(2) = \frac{(l + 2)(l + 9)}{(l + 1)^2} z^2, \text{ for } l > 1;
\]  

(6)

\[
\phi_l(3) = \frac{(l + 2)(l + 3)(l + 19)}{(l + 1)(l - 2)} z^3, \text{ for } l > 2;
\]  

(7)

\[
\phi_l(4) = \frac{(l + 2)(l + 3)(l + 4)(l + 33)}{(l - 1)(l - 2)(l - 3)} z^4, \text{ for } l > 3;
\]  

(8)

\[
\phi_l(p) = \frac{(l - p)!(l + p)!(l + 2p^2 + 1)}{(l + 1)!} z^p, \text{ for } l \geq p;
\]  

(9)

\[
A_l(z) = (Y_l(t - z)(t - z))^3 = 3 z^3;
\]  

(10)

\[
B_l(z) = (Y_l(t - z)^2(z)) = \frac{6l + 24}{l(l - 1)} z^2, \text{ for } l > 1;
\]  

(11)

\[
(Y_l(t - z)^3(z)) = \frac{90l + 230}{l(l - 1)(l - 2)} z^3, \text{ for } l > 2;
\]  

(12)

\[
(Y_l(t - z)^4(z)) = \frac{60l^2 + 23l + 48}{l(l - 1)(l - 2)(l - 3)} z^4, \text{ for } l > 3;
\]  

(13)

\[
(Y_l(t - z)^5(z)) = 840l^3 + 69l^2 + 506l + 768 z^5, \text{ for } l > 5;
\]  

(14)

\[
\lim_{l \to \infty} A_l(z) = 3 z;
\]  

(15)

\[
\lim_{l \to \infty} B_l(z) = 6z^2;
\]  

(16)

\[
\lim_{l \to \infty} \frac{1}{l^2} (Y_l(t - z)^4(z)) = 60z^4;
\]  

(17)

\[
\lim_{l \to \infty} \frac{1}{l} (Y_l(t - z)^5(z)) = 840z^5.
\]  

(18)

By the classical Korovkin theorem, we easily obtain the following lemma:

**Lemma 5.** For all \( \lambda \in C_0(\mathbb{R}_+) \) and any finite interval \( I \subset \mathbb{R}_+ \),
then the sequence \( \{Y_l(t)\} \) converges to \( \lambda \) uniformly on \( I \), where \( C_0(\mathbb{R}_+) \) denotes the set of all real-valued bounded and continuous functions defined on \( \mathbb{R}_+ \), endowed with the norm \( ||\lambda|| = \sup_{z \in \mathbb{R}_+} |\lambda(z)| \).

3. Global Results

In this section, we establish some global results by using certain Lipschitz classes. We first recall some basic definitions. Let \( r \in \mathbb{N} := \{0, 1, 2, \ldots\} \) and define the weighted function \( w_r \) as follows:

\[
w_0(z) = 1 \text{ and } w_r(z) = \frac{1}{1 + z^r} \text{ for } z \in \mathbb{R} \text{ and } r \in \mathbb{N} \setminus \{0\}.
\]  

(19)

Meantime, we consider the following subspace \( S_r(\mathbb{R}_+) \) of \( C(\mathbb{R}_+) \) generated by \( w_r \):

\[
S_r(\mathbb{R}_+) := \{\lambda \in C(\mathbb{R}_+): w_r \lambda \text{ is uniformly continuous and bounded on } \mathbb{R}_+\}
\]  

(20)

endowed with the norm \( ||\lambda||_r := \sup_{z \in \mathbb{R}_+} |w_r(z)\lambda(z)| \) for \( \lambda \in S_r(\mathbb{R}_+) \). For every \( \lambda \in S_r(\mathbb{R}_+), \delta > 0, \) and \( \alpha \in (0, 2] \), the usual weighted modulus of continuity, the second-order weighted modulus of smoothness, and the corresponding Lipschitz classes are, respectively, defined as

\[
\omega^1_r(\lambda; \delta) := \sup_{y \in [0, \delta]} \{w_r(y)\lambda(y) - \lambda(z): |y - z| \leq \delta, y, z \in \mathbb{R}_+\};
\]  

(21)

\[
\omega^2_r(\lambda; \delta) := \sup_{z \in [0, \delta]} ||\lambda(z + 2t) - 2\lambda(z + t) + \lambda(z)||_r;
\]  

\[
\operatorname{Lip}_r^2 \alpha := \{\lambda \in S_r(\mathbb{R}_+): w_r^2(\lambda; \delta) = O(\delta^\alpha) \text{ as } \delta \rightarrow 0^+ \}.
\]  

(22)
Proof. Inequality (22) is obvious for \( r = 0 \). Assume that \( l \geq r \geq 1 \), using (6), we have

\[
\phi_r(z)(Y_l(z)) \left( \frac{1}{e} \right)(z) = \phi_r(z)(Y_l(1 + t^r))(z) = \phi_r(z)(Y_l(1)(z) + \phi_r(z)(Y_l(t^r))(z)) = \phi_r(z) + \phi_r(z) (l - r)!l + r! \left( 1 + z^2 + 1 \right) l(l + 1) \ni = C_r \phi_r(z)(1 + z^2) = C_r,
\]

where \( C_r = \max \{ 1, \sup_{l \geq r} ((l - r)!l + r!l + 2r^2 + 1)l/l(l + 1)! \} \), and then we obtain (22). Moreover, for every \( \lambda \in S_r(\mathbb{R}_+) \) and \( z \in \mathbb{R}_+ \), we have

\[
\phi_r(z)(Y_l(\lambda(t))(z)) = \phi_r(z)(Y_l(t))(\nu(t)(z) + \nu(t)(z)(Y_l(t^r))(z)) \leq \phi_r(z)(1 + z^2) = C_r \lambda \nu(t)(z) \ni = C_r, \nu(t)(z) \ni.
\]

Taking the supremum over \( z \in \mathbb{R}_+ \), we obtain (23).

Theorem 7. For any fixed \( r \in \{ 0, 1, \ldots, l - 2 \} \), \( l \geq 2 \), there exists a positive constant \( C_r \) such that

\[
\phi_r(z)(Y_l(z)) \left( \frac{1}{e} \right)(z) \leq C_r z^2 T. \tag{26}
\]

Proof. The formula (11) implies (26) for \( r = 0 \). If \( r = 1 \), then we obtain

\[
(Y_l(t))(z) = (Y_l((t-z)^r))(z) + (Y_l((t-z)^2))(z) = (Y_l((t-z)^r))(z) + (1 + z)(Y_l((t-z)^2))(z),
\]

which by (11) and (12) yield (26) for \( r = 1 \). Assuming \( l - 2 \geq r \geq 2 \) and using (11) and (6), we obtain

\[
(Y_l(t-z))(z) = (Y_l((t-z)^2))(z) + (Y_l((t-z)^r))(z)
\]

Furthermore, for all \( \lambda \in S_r(\mathbb{R}_+) \), we have

\[
\phi_r(z)(Y_l(\lambda(t))(z)) \leq C_r \phi_r(z)(\lambda(t))(z) \ni \leq C_r \lambda(t)(z) \ni.
\]

Thus, \( Y_l(z) \) is a linear positive operator from \( S_r(\mathbb{R}_+) \) to \( S_r(\mathbb{R}_+) \) for any \( r \in \{ 0, 1, \ldots, l \} \).

Now, for \( r \in \{ 0, 1, \ldots, l \} \), we consider the two spaces \( S_l^r(\mathbb{R}_+) = \{ \lambda \in S_r(\mathbb{R}_+) : \lambda' \in S_r(\mathbb{R}_+) \} \) and \( S_l^2(\mathbb{R}_+) = \{ \lambda \in S_r(\mathbb{R}_+) : \lambda' \in S_r(\mathbb{R}_+) \} \), and we have the three following theorems:

Theorem 8. For any fixed \( r \), if \( \lambda \in S_l^r(\mathbb{R}_+) \), there exists a positive constant \( C_r \) such that

\[
\phi_r(z)(Y_l(\lambda(t))(z) - \lambda(z)) \leq C_r \phi_r(z)(\lambda(t))(z) \ni \leq C_r \phi_r(z)(\lambda(t))(z) \ni.
\]

for all \( z \in \mathbb{R}_+ \) and \( l \geq 2 \).
Applying the well-known Cauchy-Schwarz inequality, we can obtain
\[
(Y_l | t - z |)(z) \leq \sqrt{(Y_l(t - z)^2)(z)},
\]
\[
(Y_l \left( \frac{|t - z|}{w_r(t)} \right))(z) \leq \sqrt{\left( Y_l \left( \frac{1}{w_r(t)} \right) \right)(z) \left( Y_l \left( \frac{(t - z)^2}{w_r(t)} \right) \right)(z)}.
\]
(33)

Combining (22) and (26), we can get the required result.

**Theorem 9.** For any fixed \( r \), if \( \lambda \in S^r_l(R_+) \), then there exists a positive constant \( C \), such that
\[
w_r(z)(Y_l \lambda(t))(z) - \lambda(z) \leq C \omega^l_1(\lambda; z)
\]
for all \( z \in R_+ \) and \( l \geq r + 2 \).

**Proof.** Let \( z \in R_+ \). We denote the Steklov means of \( \lambda \) by \( \lambda_s \), \( s \in R_+ \):
\[
\lambda_s(z) = \frac{1}{s} \int_0^s \lambda(\nu + z)du, z, s \in R_+.
\]
(35)

It is obvious that
\[
\lambda_s(z) - \lambda(z) = \frac{1}{s} \int_0^s (\lambda(\nu + z) - \lambda(z))du,
\]
\[
\lambda'_s(z) = \frac{1}{s} \left( \lambda(z + s) - \lambda(z) \right)
\]
for \( z, s \in R_+ \). Hence, if \( \lambda \in S^r_l(R_+) \), then \( \lambda_s \in S^r_l(R_+) \) for every fixed \( s \in R_+ \). Furthermore, we have
\[
\| \lambda_s - \lambda \| \leq \omega^l_1(\lambda; s), \| \lambda'_s \| \leq \frac{1}{s} \omega^l_1(\lambda; s).
\]
(37)

By
\[
w_r(z)(Y_l \lambda(t))(z) - \lambda(z) \leq w_r(z)(Y_l(\lambda(t) - \lambda_s(t))(z)) \nonumber\] \[+ w_r(z)(Y_l \lambda_s(t))(z) - \lambda_s(z) + w_r(z) \lambda(z) - \lambda_s(z),
\]
(38)

Using (23) and (37), we have
\[
w_r(z)(Y_l(\lambda(t) - \lambda_s(t))(z)) \leq C_r \| \lambda - \lambda_s \| \leq C_r \omega^l_1(\lambda; s)
\]
(39)

for any \( z, s \in R_+ \). From (29) and (37), we have
\[
w_r(z)(Y_l \lambda_s(t))(z) - \lambda_s(z) \leq C_r \omega^l_1(\lambda; s) \frac{z}{\sqrt{l}} \leq C_r \frac{1}{s} \omega^l_1(\lambda; s) \frac{z}{\sqrt{l}},
\]
(40)

By (37), we have
\[
w_r(z)(\lambda(t) - \lambda_s(z)) \leq \| \lambda - \lambda_s \| \leq \omega^l_1(\lambda; s)
\]
for any \( z, s \in R_+ \). Finally, we have
\[
w_r(z)(Y_l \lambda(t))(z) - \lambda(z) \leq \omega^l_1(\lambda; z) \left( C_r + \frac{1}{s} C_r \frac{z}{\sqrt{l}} + 1 \right)
\]
(42)

for any \( z, s \in R_+ \). Choosing \( s = z/\sqrt{l} \), the proof is proved.

**Theorem 10.** Defining a new operator,
\[
(Y_l^* \lambda(t))(z) = (Y_l \lambda(t))(z) - \lambda(z + A_l(z)) + \lambda(z).
\]
(43)

For any fixed \( r \), if \( \lambda \in S^r_l(R_+) \), then there exists a positive constant \( C \), such that
\[
w_r(z)(Y_l^* \lambda(t))(z) - \lambda(z) \leq C \| \lambda'' \| \frac{z^2}{T},
\]
(44)

for all \( z \in R_+ \) and \( l \geq r + 2 \).

**Proof.** Using Taylor’s expansion, we have
\[
\lambda(t) - \lambda(z) = (t - z)\lambda'(z) + \int_z^t (t - u)\lambda''(u)du, z, t \in R_+.
\]
(45)

By \( Y_l^* (t - z))(z) = 0 \) and \( Y_l^* (1)(z) = 1 \), we have
\[
|(Y_l^* \lambda(t))(z) - \lambda(z)| \leq (Y_l^* (\lambda(t) - \lambda(z)))(z) \nonumber\] \[\leq \left( Y_l^* \left( \int_z^t (t - u)\lambda''(u)du \right) \right)(z) \left( Y_l^* \left( \int_z^t (t - u)\lambda''(u)du \right) \right)(z) \nonumber\] \[\leq \left( \int_z^{z + A_l(z)} (z + A_l(z) - u)\lambda''(u)du \right) \]
(46)

Since
\[
\int_z^t (t - u)\lambda''(u)du \leq \frac{\| \lambda'' \|}{2} \left( \frac{(t - z)^2}{2w_r(z)} + \frac{1}{w_r(t)} \right),
\]
\[
\int_z^{z + A_l(z)} (z + A_l(z) - u)\lambda''(u)du \leq \frac{\| \lambda'' \|}{2w_r(z)} (A_l(z))^2,
\]
(47)
Combining (23) and (44), we have
\[
\omega_r(z)\|Y_1^r\lambda(t)(z) - \lambda(z)\| \leq \left(C_r + 3\right)\sqrt{\frac{|\lambda''(z)|}{T}} + C_r\|\lambda''(z)\| T^{n/2} w_r(z)|\lambda(z + A_{r}(z)) - \lambda(z)| - \lambda(z))
\]
\[
\leq C_r\omega_r^r(\lambda; s) \left(1 + \frac{1}{2} z^2 + 2\right) + \omega_r^r(\lambda; A_{r}(z))
\]
\]
(55)

Hence, choosing \( s = z/\sqrt{T} \), the first part of the proof is proved. The second part of the proof can be directly observed from the definition of the space \( \text{Lip}_r^\alpha \).

4. Direct Results

4.1. Voronovskaya-Type Theorem

Theorem 12. If \( \lambda \in C_B(\mathbb{R}_+) \) and \( \lambda'' \) exists at a point \( z \in \mathbb{R}_+ \), then
\[
\lim_{t \to z^-} l((Y_1^r\lambda(t)(z) - \lambda(z)) = 3z\left(\lambda'(z) + z\lambda''(z)\right).
\]
(56)

Proof. By the Taylor’s expansion formula for \( \lambda \), we have
\[
\lambda(t) = \lambda(z) + \lambda'(z)(t-z) + 1/2 \lambda''(z)(t-z)^2 + R(t; z)(t-z)^2,
\]
(57)
where
\[
R(t; z) = \left\{
\begin{array}{ll}
\lambda(t) - \lambda(z) - \lambda'(z)(t-z) - 1/2\lambda''(z)(t-z)^2, & t \neq z; \\
0, & t = z.
\end{array}
\right.
\]
(58)

Applying the L’Hospital’s Rule,
\[
\lim_{t \to z^-} R(t; z) = \frac{1}{2} \lim_{t \to z^-} \frac{\lambda'(t) - \lambda'(z)}{t - z} - \frac{1}{2} \lambda''(z) = 0.
\]
(59)

Thus, \( R(t; z) \in C_B(\mathbb{R}_+) \). Consequently, we can write
\[
(Y_1^r\lambda(t))(z) - \lambda(z) = A_r(z)\lambda'(z) + \frac{1}{2} B(z)\lambda''(z)
\]
\[
+ (Y_1^r\lambda(z))(z) + \lambda'(z) - \lambda'(z)(t-z)^2) + R(t; z)(t-z)^2
\]
(60)

By the Cauchy-Schwarz inequality, we have
\[
l((Y_1^r\lambda(t))(z) - \lambda(z))^2) \leq \sqrt{(Y_1^r\lambda^2(t; z))(z)\sqrt{\lambda^2(t; z)^2)}(z).}
\]
(61)

We observe that \( R^2(z; z) = 0 \) and \( R^2(t; z) \in C_B(\mathbb{R}_+) \).

Then, it follows in Lemma 5 that
Proof. Let \( \lambda \in C_b(\mathbb{R}_+ \cap \text{Lip}_M(y, D)) \) be a positive constant depending only on \( y \) and \( \lambda \).

Theorem 14. If \( \lambda \in C_b(\mathbb{R}_+) \), then for any \( z \in \mathbb{R}_+ \), we have

\[
|Y_1(\lambda(t)) - \lambda| \leq M |t - z|^\gamma, \quad t \in D, z \in \mathbb{R}_+,
\]

where \( M \) is a positive constant depending only on \( \gamma \) and \( \lambda \).

Proof. Let \( D \) be the closure of the set \( \{ d(z; D) < \infty \} \). Using the properties of infimum, there is at least a point \( t_0 \in D \) such that \( d(z; D) = |z - t_0| \). By the triangle inequality

\[
|\lambda(t) - \lambda(z)| \leq |\lambda(t) - \lambda(t_0)| + |\lambda(t_0) - \lambda(z)|,
\]

we have

\[
|Y_1(\lambda(t)) - \lambda(z)| \leq (|Y_1(\lambda(t)) - \lambda(t_0)|) + (|Y_1(\lambda(z) - \lambda(t_0))|)\]

\[
\leq M(|(t - t_0)|^\gamma + |z - t_0|^\gamma)
\]

\[
= M(|(t - t_0)|^\gamma + |z - t_0|^\gamma)
\]

\[
\leq M(|Y_1(t - t_0|) + |z - t_0|)|^\gamma
\]

Choosing \( p = 2/\gamma \) and \( q = 2/2 - \gamma \) and using the well-known H"older inequality, we have

\[
|Y_1(\lambda(t)) - \lambda(z)|
\]

\[
\leq M \left\{ (|Y_1(t - t_0|)^p)(z) + |z - t_0|)^q \right\}^\gamma
\]

\[
\leq M \left\{ (|Y_1(t - t_0|)^p) + 2d(z; D) \right\}
\]

Finally, we establish point-wise estimate of the operators (2) in the following Lipschitz-type space (see [26]) with two distinct parameters \( \mu_1, \mu_2 \in \mathbb{R}_+ \):

\[
\text{Lip}_{\mu_1, \mu_2}(y) = \left\{ \lambda \in C(\mathbb{R}_+) : |\lambda(t) - \lambda(z)| \leq M |t - z|^\gamma \right\},
\]

where \( y \in (0, 1] \), \( M \) is a positive constant depending only on \( y, \mu_1, \mu_2 \) and \( \lambda \).

Theorem 16. If \( \lambda \in \text{Lip}_{\mu_1, \mu_2}(y) \), then for any \( z \in \mathbb{R}_+ \), we have

\[
|Y_1(\lambda(t)) - \lambda(z)| \leq M \left( \frac{B(z)}{\mu_1 |z|^\gamma + \mu_2} \right)^\gamma
\]

Proof. Applying the well-known H"older inequality with \( p = 2/\gamma \) and \( q = 2/2 - \gamma \), we have

\[
\left| \frac{Y_1(\lambda(t)) - \lambda(z)}{Y_1(\lambda(t)) - \lambda(z)} \right| \leq M \left( \frac{B(z)}{\mu_1 |z|^\gamma + \mu_2} \right)^\gamma
\]
Thus, the proof is completed.

5. Weighted Approximation

Let $B_2(\mathbb{R}_+)$ be the set of all functions $\lambda$ defined on $\mathbb{R}_+$ satisfying the condition $|\lambda(z)| \leq M_1(1 + z^2)$ with an absolute constant $M_1 > 0$ which depends only on $\lambda$. $C^0_2(\mathbb{R}_+)$ denotes the subspace of all continuous functions $\lambda \in B_2(\mathbb{R}_+)$ with the norm $\|\lambda\|_2 = \sup((|\lambda(z)|/1 + z^2))$. By $C_2^0(\mathbb{R}_+)$, we denote the subspace of all functions $f \in C_2(\mathbb{R}_+)$ for which $\lim_{z \rightarrow +\infty} |\lambda(z)|/1 + z^2$ is finite.

**Theorem 17.** If $\lambda \in C^0_2(\mathbb{R}_+)$ and $\kappa > 0$, we have

$$
\lim_{l \rightarrow \infty} \sup_{z \in \mathbb{R}_+} \frac{|(Y_1\lambda(t))(z) - \lambda(z)|}{(1 + z^2)^{1+\kappa}} = 0.
$$

**Proof.** Let $z_0 \in \mathbb{R}_+$ be arbitrary but fixed.

$$
\sup_{z \in \mathbb{R}_+} \frac{|(Y_1\lambda(t))(z) - \lambda(z)|}{(1 + z^2)^{1+\kappa}} \leq \sup_{z \in (0, z_0)} \frac{|(Y_1\lambda(t))(z) - \lambda(z)|}{(1 + z^2)^{1+\kappa}} + \sup_{z \in [z_0, \infty)} \frac{|(Y_1\lambda(t))(z) - \lambda(z)|}{(1 + z^2)^{1+\kappa}}
$$

$$
\leq ||(Y_1\lambda(t))(z) - \lambda||_{(0, z_0)} + ||\lambda||_2 \sup_{z \in [z_0, \infty)} \frac{|(Y_1(t))(z)|}{(1 + z^2)^{1+\kappa}} + \sup_{z \in [z_0, \infty)} \frac{|\lambda(z)|}{(1 + z^2)^{1+\kappa}} \leq I_1 + I_2 + I_3.
$$

(78)

Applying $|\lambda(z)| \leq ||\lambda||_2(1 + z^2)$, we have

$$
I_3 = \sup_{z \in [z_0, \infty)} \frac{|\lambda(z)|}{(1 + z^2)^{1+\kappa}} \leq \sup_{z \in [z_0, \infty)} \frac{||\lambda||_2(1 + z^2)}{(1 + z^2)^{1+\kappa}} \leq \frac{||\lambda||_2}{(1 + z^2)^{1+\kappa}}.
$$

(79)

Let $\varepsilon > 0$. Since $\lim_{l \rightarrow \infty} \sup_{z \in [z_0, \infty)} |(Y_1(t))(z)|/(1 + z^2) = 1$, there exists $L_1 \in \mathbb{N}$, such that for all $l > L_1$,

$$
||(Y_1(t))(z) - z^2||_2 = \sup_{z \in \mathbb{R}_+} \frac{|(l + z^2)|}{(l - 1)} \leq \frac{12l + 19}{l(l - 1)}.
$$

(88)

Hence, the proof of Theorem 17 is completed.

**Theorem 18.** If $\lambda \in C^0_2(\mathbb{R}_+)$, then we have

$$
\lim_{l \rightarrow \infty} ||(Y_1\lambda(t))(z) - \lambda||_2 = 0.
$$

(85)

**Proof.** Applying the Korovkin theorem [27], it is sufficient to show the following three conditions:

$$
\lim_{l \rightarrow \infty} ||(Y_1(t))(z) - z^2||_2 = 0, p = 0, 1, 2.
$$

(86)

Since $(Y_1(t))(z) = 1$, the condition (86) holds for $p = 0$. From Lemma (11), we have

$$
||(Y_1(t))(z) - z||_2 = \sup_{z \in \mathbb{R}_+} \frac{1}{1 + z^2} \frac{|l + z|}{l - 1} \leq \frac{3}{l}.
$$

(87)

Thus, $\lim_{l \rightarrow \infty} ||(Y_1(t))(z) - z||_2 = 0$. Finally, we have

$$
||(Y_1(t))(z) - z^2||_2 = \sup_{z \in \mathbb{R}_+} \frac{1}{1 + z^2} \frac{(l + 2)(l + 1)}{l(l - 1)} \leq \frac{12l + 19}{l(l - 1)}.
$$

(88)
6. Some Voronovskaya-Type Approximation Theorem

As is known, if \( \lambda \in C(\mathbb{R}_+) \) is not uniform, the limit \( \lim_{\delta \to 0^+} \omega(\lambda; \delta) = 0 \) may not be true. In [28], Yuksel and Ispir defined the following weighted modulus of continuity:

\[ \Omega(\lambda; \delta) = \sup_{z \in \mathbb{R}_+, |h| < \delta} \left| \frac{\lambda(z + h) - \lambda(z)}{(1 + z^2)(1 + h^2)} \right| \text{ for } \lambda \in C^2(\mathbb{R}_+) \quad (89) \]

and proved the properties of monotone increasing about \( \Omega(\lambda; \delta) \) as \( \delta > 0 \), \( \lim_{\delta \to 0^+} \Omega(\lambda; \delta) = 0 \), and the inequality

\[ \Omega(\lambda; \tau \delta) \leq 2(1 + \tau)(1 + \delta^2)\Omega(\lambda; \delta), \quad \tau > 0. \quad (90) \]

For any \( \lambda \in C^2(\mathbb{R}_+) \), it follows from (89) and (90) that

\[ |\lambda(t) - \lambda(z)| \leq (1 + (t - z)^2)\Omega(\lambda; |t - z|) \leq 2 \left( 1 + \frac{|t - z|}{\delta} \right)(1 + \delta^2)\Omega(\lambda; \delta)(1 + (t - z)^2)(1 + z^2). \quad (91) \]

In the next theorem, we obtain the degree of approximation of \( \lambda \) by the operators (2) in the weighted space of continuous functions \( C^2(\mathbb{R}_+) \) in terms of the weighted modulus of smoothness \( \Omega(\lambda; \delta), \delta > 0 \).

6.1. Quantitative Voronovskaya-Type Theorem

**Theorem 19.** If \( \lambda \in C^2(\mathbb{R}_+) \) satisfies \( \lambda', \lambda'' \in C^2(\mathbb{R}_+) \), then for sufficiently large \( I \) and any \( z \in \mathbb{R}_+ \),

\[ l|y, \lambda(t)(t)| - |y, \lambda(z) - \lambda'(z)A_1(z)| \leq m \| \lambda''(z) \| R_1(t, z) \leq O(1)\Omega(\lambda''; \frac{1}{\sqrt{I}}). \quad (92) \]

**Proof.** By Taylor's expansion formula for \( \lambda \), we have

\[ \lambda(t) = \lambda(z) + \lambda'(z)(t - z) + \frac{\lambda''(y)}{2!}(t - z)^2 \]

\[ = \lambda(z) + \lambda'(z)(t - z) + \frac{\lambda''(z)}{2!}(t - z)^2 + R_1(t, z), \quad (93) \]

where \( |y - z| \leq |t - z| \) and hence

\[ R_1(t, z) = \frac{\lambda''(y) - \lambda''(z)}{2!}(t - z)^2. \quad (94) \]

Applying the inequality (91) of the weighted modulus of continuity, we have

\[ |\lambda''(y) - \lambda''(z)| \leq (1 + (y - z)^2)(1 + z^2)\Omega(\lambda''; |y - z|) \leq (1 + (t - z)^2)(1 + \delta^2)\Omega(\lambda''; |t - z|) \leq 2 \left( 1 + \frac{|t - z|}{\delta} \right)(1 + \delta^2)\Omega(\lambda''; \delta)(1 + (t - z)^2)(1 + z^2). \]

Combining (94) and (95) and choosing \( \delta = (0, 1) \), we have

\[ |R_1(t, z)| \leq 2(1 + \delta^2)(1 + z^2)\Omega(\lambda''; \delta) \left( 1 + \frac{(t - z)^4}{\delta} \right)(t - z)^2. \quad (96) \]

Using the operator (2) and Lemma 4 on both sides of (94), we have

\[ l|y, \lambda(t)(t)| - |y, \lambda(z) - \lambda'(z)A_1(z)| \leq \frac{\lambda''(z)}{2!}B_1(z) \leq (Y_1|R_1(t, z)|)(z). \quad (97) \]

Applying (16), (18), and (96), we have

\[ \left| (Y_1|R_1(t, z)|)(z) \right| \leq 2 \left( 1 + \delta^2 \right)^2(1 + z^2)\Omega(\lambda''; \delta) \left( 1 + \frac{(t - z)^6}{\delta^4} \right) \left( (t - z)^2 \right) \leq 2 \left( 1 + \delta^2 \right)^2(1 + z^2)\Omega(\lambda''; \delta) \left( 1 + \frac{(t - z)^6}{\delta^4} \right) \left( (t - z)^2 \right). \quad (98) \]

Choosing \( \delta = 1/\sqrt{I} \), we have

\[ l|y, \lambda(t)(t)| - |y, \lambda(z) - \lambda'(z)A_1(z)| \leq O(1)\Omega(\lambda''; \frac{1}{\sqrt{I}}). \quad (99) \]

Combining (97)-(99), we complete the proof of Theorem 19.

6.2. Grüss Voronovskaya-Type Theorem
Theorem 20. If \( \lambda, \mu \in C^2_\infty(\mathbb{R}_+) \) satisfy \( \lambda \mu, \lambda', \mu', (\lambda \mu) , \lambda'', \mu'' \) and \( (\lambda \mu)'' \in C^2_\infty(\mathbb{R}_+) \). Then, for any \( z \in \mathbb{R}_+ \),

\[
\lim_{t \to \infty} l((Y(t \lambda \mu)(t))(z) - (Y(t \lambda)(t))(z) - (Y(t \mu)(t))(z)) = 6 \lambda'(z) \mu'(z) z^2.
\]

(100)

Proof. Using the equalities

\[
(\lambda \cdot \mu)(z) = \lambda(z) \cdot \mu(z), (\lambda \cdot \mu)'(z) = \lambda'(z) \cdot \mu(z) + \lambda(z) \cdot \mu'(z),
\]

(101)

by simple computations, for any \( z \in \mathbb{R}_+ \), we have

\[
(Y(t \lambda \mu)(t))(z) - (Y(t \lambda)(t))(z) - (Y(t \mu)(t))(z) = \left\{ Y(t \lambda \mu)(t))(z) - (\lambda \cdot \mu)(z) - \lambda'(z) \cdot \mu(z) - \frac{\mu'(z)}{2} B_2(z) \right\}
\]

\[
- \mu(z) \left\{ Y(t \lambda)(t))(z) - \lambda(z) - \lambda'(z) A_2(z) - \frac{\lambda''(z)}{2} B_2(z) \right\}
\]

\[
- (Y(t \mu)(t))(z) - \mu(z) - \mu'(z) A_2(z) - \frac{\mu''(z)}{2} B_2(z)
\]

\[
+ \frac{1}{2} \frac{B_2(z)}{2} \left\{ \lambda(z) - \mu'(z) + 2 \lambda'(z) \mu'(z) - \mu''(z) \cdot (Y(t \lambda)(t))(z) \right\}
\]

\[
+ A_2(z) \left\{ \lambda(z) \cdot \mu'(z) + \mu'(z) - [Y(t \lambda)(t))(z)] \right\}.
\]

(102)

By using (16), Lemma 5, and Theorem 19, we have

\[
\lim_{t \to \infty} l((Y(t \lambda \mu)(t))(z) - (Y(t \lambda)(t))(z) - (Y(t \mu)(t))(z)) = 6 \lambda'(z) \mu'(z) z^2,
\]

(103)

which proves our theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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