

Research Article

Some Integral Inequalities for n -Polynomial ζ -Preinvex Functions

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In this paper, we study the properties of n -polynomial ζ -preinvex functions and establish some integral inequalities of Hermite-Hadamard type via this class of convex functions. Moreover, we discuss some special cases which provide a significant complement to the integral estimations of preinvex functions. Applications of the obtained results to the inequalities for special means are also considered.

1. Introduction and Preliminaries

The geometric inequalities involving volume, surface area, mean width, etc. in the Orlicz space have attracted considerable attention of researchers, and the convexity properties of functions have been a powerful tool for dealing with various problems of convex geometry (see [1, 2]). This suggests that it is a significant work to develop new inequalities for generalized convex functions. For this purpose, let us start with recalling some concepts and notations on the convexity of functions.

A set $\mathcal{C} \subset \mathbb{R}$ is said to be convex if

$$(1-t)x + ty \in \mathcal{C}, \quad (1)$$

for any $x, y \in \mathcal{C}$ and $t \in [0, 1]$.

A function $\mathcal{F} : \mathcal{C} \rightarrow \mathbb{R}$ is said to be convex if the inequality

$$\mathcal{F}((1-t)x + ty) \leq (1-t)\mathcal{F}(x) + t\mathcal{F}(y) \quad (2)$$

holds for any $x, y \in \mathcal{C}$ and $t \in [0, 1]$.

In recent years, the classical concept of convexity has been extended and generalized in different directions. Mititelu [3] introduced the notion of invex set, as follows.

Definition 1 [3]. Let $\mathcal{X} \subset \mathbb{R}$ be a nonempty set and $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function. A set \mathcal{X} is said to be invex with respect to η if

$$x + t\eta(y, x) \in \mathcal{X}, \quad (3)$$

for all $x, y \in \mathcal{X}$ and $t \in [0, 1]$.

The invexity would reduce to the classical convexity if $\eta(y, x) = y - x$. Weir and Mond [4] defined the class of preinvex functions as follows.

Definition 2 [4]. Let $\mathcal{X} \subset \mathbb{R}$ be a nonempty invex set with respect to $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. A function $\mathcal{F} : \mathcal{X} \rightarrow \mathbb{R}$ is said to be preinvex with respect to η if the inequality

$$\mathcal{F}(x + t\eta(y, x)) \leq (1-t)\mathcal{F}(x) + t\mathcal{F}(y) \quad (4)$$

holds for all $x, y \in \mathcal{X}$ and $t \in [0, 1]$.

As a generalization of convex functions, Gordji et al. [5] introduced the notion of ζ -convex function.

Definition 3 [5]. A function $\mathcal{F} : \mathcal{F} \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be ζ -convex function with respect to $\zeta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if the inequality

$$\mathcal{F}(tx + (1-t)y) \leq \mathcal{F}(y) + t\zeta(\mathcal{F}(x), \mathcal{F}(y)) \quad (5)$$

holds for all $x, y \in \mathcal{I}$ and $t \in [0, 1]$.

The properties of convexity have numerous applications in different fields of pure and applied mathematics; especially, the concept of convexity has close relation with the theory of inequalities. Many inequalities are direct consequences of the applications of classical convexity. As is known to us, the Hermite-Hadamard inequality is one of the most significant result associated with convex functions, it reads as follows.

Let $\mathcal{F} : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, then

$$\mathcal{F}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \mathcal{F}(x) dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2}. \quad (6)$$

Noor [6] obtained a generalization of classical Hermite-Hadamard's inequality using the class of preinvex functions, as follows.

Let $\mathcal{F} : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a preinvex function, then

$$\mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \leq \frac{\mathcal{F}(a) + \mathcal{F}(b)}{2}. \quad (7)$$

The result of Noor has inspired a lot of investigators to deal with new generalizations and refinements of Hermite-Hadamard's inequality via preinvexity. For example, Barani et al. [7] obtained the generalizations of Hermite-Hadamard's inequality for functions whose derivative absolute values are preinvex. Du et al. [8] and Noor et al. [9] obtained several generalizations of Hermite-Hadamard's inequality via (s, m) -preinvex functions and h -preinvex functions, respectively. Park [10, 11] derived several variations of Hermite-Hadamard's inequality from differentiable preinvex functions. Sarikaya et al. [12] and Wu et al. [13] established the Hermite-Hadamard-like type inequalities via log-preinvex functions and harmonically (p, h, m) -preinvex functions, respectively. Wang and Liu [14] and Li [15] obtained different refinements of Hermite-Hadamard's inequality using s -preinvex functions. Deng et al. [16, 17] and Wu et al. [18] deduced some quantum Hermite-Hadamard-type inequalities by using generalized (s, m) -preinvex functions and strongly preinvex functions, respectively.

Recently, Toplu et al. [19] proposed the concept of n -polynomial convex functions and investigated their properties.

In this paper, we shall introduce a new class of n -polynomial convex functions based on a different form of inequality in the definition compared with [19], which is convenient to the generalizations and applications of n -polynomial convexity. More specifically, we will define a class of convex functions called as n -polynomial ζ -preinvex functions. We then show that this class of convex functions contains a number of other classes of convex functions. Furthermore, we establish some new integral inequalities of Hermite-Hadamard type for n -polynomial ζ -preinvex func-

tions. Finally, we apply the obtained inequalities to establish two inequalities for special means.

Firstly, we introduce the notion of n -polynomial ζ -preinvex functions.

Definition 4. Let $n \in \mathbb{N}$. A nonnegative function $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}$ is said to be n -polynomial ζ -preinvex with respect to bifunctions $\eta, \zeta : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ if the inequality

$$\mathcal{F}(a + t\eta(b, a)) \leq \mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \zeta(\mathcal{F}(b), \mathcal{F}(a)) \quad (8)$$

holds for all $a, b \in \mathcal{X}$ and $t \in [0, 1]$.

Note that if we take $n = 1$, then we have 1-polynomial ζ -preinvexity, which is just the ζ -preinvex functions defined by the inequality

$$\mathcal{F}(a + t\eta(b, a)) \leq \mathcal{F}(a) + t\zeta(\mathcal{F}(b), \mathcal{F}(a)), \quad \forall a, b \in \mathcal{X}, t \in [0, 1]. \quad (9)$$

If we take $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$, then we obtain the class of n -polynomial preinvex functions, which is defined by the inequality

$$\begin{aligned} \mathcal{F}(a + t\eta(b, a)) &\leq \frac{1}{n} \sum_{s=1}^n (1-t)^s \mathcal{F}(a) \\ &\quad + \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \mathcal{F}(b), \quad \forall a, b \in \mathcal{X}, t \in [0, 1]. \end{aligned} \quad (10)$$

If we take $\eta(b, a) = b - a$, then we get the class of n -polynomial ζ -convex functions, which is defined by the inequality

$$\begin{aligned} \mathcal{F}(a + t(b-a)) &\leq \mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \zeta(\mathcal{F}(b), \mathcal{F}(a)), \\ &\quad \forall a, b \in \mathcal{X}, t \in [0, 1]. \end{aligned} \quad (11)$$

If we take $n = 1$ in inequality (11), then we have the class of ζ -convex functions. Furthermore, we obtain the classical convex functions by setting $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$.

If we take $n = 2$ in Definition 4, then we have the class of 2-polynomial ζ -preinvex functions, which is defined by the following inequality:

$$\mathcal{F}(a + t\eta(b, a)) \leq \mathcal{F}(a) + \frac{3t - t^2}{2} \zeta(\mathcal{F}(b), \mathcal{F}(a)), \quad \forall a, b \in \mathcal{X}, t \in [0, 1]. \quad (12)$$

Note that $0 \leq t \leq 3t - t^2/2$, this shows that, for every nonnegative bifunction ζ , the ζ -preinvex function is also the 2-polynomial ζ -preinvex functions. More generally, we have the following result.

Proposition 5. For every nonnegative bifunction ζ and $n \geq 2$, if $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}$ is a $(n - 1)$ -polynomial ζ -preinvex function, then \mathcal{F} is a n -polynomial ζ -preinvex function.

To verify the validity of Proposition 5, it is enough to show that

$$\frac{1}{n-1} \sum_{s=1}^{n-1} [1 - (1-t)^s] \leq \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s], \quad (13)$$

for any $n \geq 2$ and $t \in [0, 1]$.
Direct computation gives

$$\begin{aligned} & \frac{1}{n-1} \sum_{s=1}^{n-1} [1 - (1-t)^s] - \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s] \\ &= \left(\frac{1-t}{t}\right) \left[\frac{1 - (1-t)^n}{n} - \frac{1 - (1-t)^{n-1}}{n-1} \right] \\ &= \left(\frac{1-t}{t}\right) \left[\frac{(1-t)^n + nt(1-t)^{n-1} - 1}{n(n-1)} \right] \\ &= \left(\frac{1-t}{t}\right) \left[\frac{(1-t)^n + nt(1-t)^{n-1} - ((1-t) + t)^n}{n(n-1)} \right] \\ &= -\left(\frac{1-t}{t}\right) \left[\frac{C_n^2 t^2 (1-t)^{n-2} + C_n^3 t^3 (1-t)^{n-3} + \dots + C_n^n t^n}{n(n-1)} \right] \\ &\leq 0, \end{aligned} \quad (14)$$

which implies the required inequality (13).

As a consequence, we obtain the following.

Proposition 6. For every nonnegative bifunction ζ , if $\mathcal{F} : \mathcal{X} \mapsto \mathbb{R}$ is a ζ -preinvex function, then \mathcal{F} is a n -polynomial ζ -preinvex function.

Choosing $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$ in Proposition 6 gives the following.

Proposition 7. If $\mathcal{F} : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ is a preinvex function with $\mathcal{F}(b) - \mathcal{F}(a) \geq 0$, then \mathcal{F} is a n -polynomial preinvex function.

2. Main Results

In this section, we establish some new Hermite-Hadamard-type inequalities using the class of n -polynomial ζ -preinvex functions. We first need to introduce the notation called Condition C, which was presented by Mohan and Neogy in [20].

Condition C. Let $\mathcal{X} \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot, \cdot)$, we say that the bifunction $\eta(\cdot, \cdot)$ satisfies the Condition C, if for any $x, y \in \mathcal{X}$ and $t \in [0, 1]$, we have

$$\begin{aligned} \eta(x, x + t\eta(y, x)) &= -t\eta(y, x), \\ \eta(y, x + t\eta(y, x)) &= (1-t)\eta(y, x). \end{aligned} \quad (15)$$

Note that for any $x, y \in \mathcal{X}$, $t_1, t_2 \in [0, 1]$ and from Condition C, we can deduce

$$\eta(x + t_2\eta(y, x), x + t_1\eta(y, x)) = (t_2 - t_1)\eta(y, x). \quad (16)$$

Throughout the paper we assume that Condition C is satisfied for the domain with respect to bifunction $\eta(\cdot, \cdot)$ as a precondition.

Theorem 8. Let $\mathcal{F} : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a n -polynomial ζ -preinvex function. If $\eta(b, a) > 0$ and $\mathcal{F} \in L[a, a + \eta(b, a)]$, then we have

$$\begin{aligned} & \mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{n + 2^n - 1}{n} M_\zeta \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} \mathcal{F}(x) dx \\ & \leq \mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n \frac{s}{s+1} \zeta(\mathcal{F}(b), \mathcal{F}(a)), \end{aligned} \quad (17)$$

where M_ζ is the upper bound of bifunction ζ .

Proof. Using the definition of n -polynomial ζ -preinvex function and Condition C, we have

$$\begin{aligned} \mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) &= \mathcal{F}\left(a + (1-t)\eta(b, a) + \frac{1}{2}(t - (1-t))\eta(b, a)\right) \\ &= \mathcal{F}\left(a + (1-t)\eta(b, a) + \frac{1}{2}\eta(a + t\eta(b, a), a) + (1-t)\eta(b, a)\right) \\ &\leq \mathcal{F}(a + (1-t)\eta(b, a)) \\ &\quad + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{1}{2}\right)^s\right) \zeta(\mathcal{F}(a + t\eta(b, a)), \mathcal{F}(a + (1-t)\eta(b, a))) \\ &\leq \mathcal{F}(a + (1-t)\eta(b, a)) \\ &\quad + \frac{n + 2^n - 1}{n} M_\zeta. \end{aligned} \quad (18)$$

Hence, we obtain

$$\mathcal{F}(a + (1-t)\eta(b, a)) \geq \mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{n + 2^n - 1}{n} M_\zeta. \quad (19)$$

Integrating both sides of the above inequality with respect to t on $[0, 1]$, it follows that

$$\int_0^1 \mathcal{F}(a + (1-t)\eta(b, a)) dt \geq \mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{n + 2^n - 1}{n} M_\zeta, \quad (20)$$

that is,

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \geq \mathcal{F}\left(\frac{2a + \eta(b, a)}{2}\right) - \frac{n + 2^{-n} - 1}{n} M_\zeta. \tag{21}$$

The left-hand side inequality of (17) is proved.

On the other hand, from the definition of n -polynomial ζ -preinvex function, one has

$$\mathcal{F}(a + t\eta(b, a)) \leq \mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n (1 - (1-t)^s) \zeta(\mathcal{F}(b), \mathcal{F}(a)). \tag{22}$$

Integrating both sides of the above inequality with respect to t on $[0, 1]$, we obtain

$$\begin{aligned} \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx &= \int_0^1 \mathcal{F}(a + t\eta(b, a)) dt \\ &\leq \int_0^1 \left(\mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n (1 - (1-t)^s) \zeta(\mathcal{F}(b), \mathcal{F}(a)) \right) dt \\ &= \mathcal{F}(a) + \frac{1}{n} \sum_{s=1}^n \frac{s}{s+1} \zeta(\mathcal{F}(b), \mathcal{F}(a)). \end{aligned} \tag{23}$$

This proves the right-hand side inequality of (17). The proof of Theorem 8 is complete.

Before we put forward another kind of integral inequality of Hermite-Hadamard type, we need to prove an auxiliary result, which will play a key role in deducing subsequent results. For the sake of simplicity, we let $\mathcal{J} = [a, a + \eta(b, a)]$ and let \mathcal{J}° be the interior of \mathcal{J} .

Lemma 9. *Let $\mathcal{F} : \mathcal{J} \rightarrow \mathbb{R}$ be a differentiable function on \mathcal{J}° with $\eta(b, a) > 0$, $\min\{\lambda, \mu\} \geq t > 0$. If $\mathcal{F}' \in L[\mathcal{J}]$, then*

$$\begin{aligned} &\frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \\ &= \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu (-t) \mathcal{F}'\left(a + \frac{\mu-t}{\lambda + \mu} \eta(b, a)\right) dt \right. \\ &\quad \left. + \int_0^\lambda t \mathcal{F}'\left(a + \frac{\mu+t}{\lambda + \mu} \eta(b, a)\right) dt \right]. \end{aligned} \tag{24}$$

Proof. Let

$$\begin{aligned} I &= \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu (-t) \mathcal{F}'\left(a + \frac{\mu-t}{\lambda + \mu} \eta(b, a)\right) dt \right. \\ &\quad \left. + \int_0^\lambda t \mathcal{F}'\left(a + \frac{\mu+t}{\lambda + \mu} \eta(b, a)\right) dt \right] = I_1 + I_2. \end{aligned} \tag{25}$$

Integrating by parts yields

$$\begin{aligned} I_1 &= \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu (-t) \mathcal{F}'\left(a + \frac{\mu-t}{\lambda + \mu} \eta(b, a)\right) dt \right] \\ &= \frac{1}{\lambda + \mu} \left(\mu \mathcal{F}(a) - \int_0^\mu \mathcal{F}\left(a + \frac{\mu-t}{\lambda + \mu} \eta(b, a)\right) dt \right) \\ &= \frac{\mu}{\lambda + \mu} \mathcal{F}(a) - \frac{1}{\eta(b, a)} \int_a^{a+\frac{\mu}{\lambda + \mu} \eta(b, a)} \mathcal{F}(x) dx. \end{aligned} \tag{26}$$

Similarly,

$$I_2 = \frac{\lambda}{\lambda + \mu} \mathcal{F}(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_{a+\frac{\mu}{\lambda + \mu} \eta(b, a)}^{a+\eta(b, a)} \mathcal{F}(x) dx. \tag{27}$$

Substituting the formulations of I_1 and I_2 in (25) leads to the desired identity (24). The proof of Lemma 9 is complete.

We shall now give some estimations of bounds for Hermite-Hadamard-type inequalities.

Theorem 10. *Let $\mathcal{F} : \mathcal{J} \rightarrow \mathbb{R}$ be a differentiable function on \mathcal{J}° with $\eta(b, a) > 0$, $\lambda > 0$, $\mu > 0$, and let $\mathcal{F}' \in L[\mathcal{J}]$. If $|\mathcal{F}'|$ is n -polynomial ζ -preinvex function, then*

$$\begin{aligned} &\left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ &\leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{\lambda^2 + \mu^2}{2} |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n K_1 \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right. \\ &\quad \left. + \frac{1}{n} \sum_{s=1}^n K_2 \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right], \end{aligned} \tag{28}$$

where

$$\begin{aligned} K_1 &= \frac{\mu^2}{2} - \frac{\mu(s+2)(\lambda + \mu)^{s+1} - (\lambda + \mu)^{s+2} + \lambda^{s+2}}{(s+1)(s+2)(\lambda + \mu)^s}, \\ K_2 &= \frac{\lambda^2}{2} - \frac{\lambda^{s+2}}{(s+1)(s+2)(\lambda + \mu)^s}. \end{aligned} \tag{29}$$

Proof. Using Lemma 9 and the assumption that $|\mathcal{F}'|$ is n -polynomial ζ -preinvex function, we have

$$\begin{aligned}
 & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu |t| \left| \mathcal{F}' \left(a + \frac{\mu - t}{\mu + \lambda} \eta(b, a) \right) \right| dt + \int_0^\lambda |t| \left| \mathcal{F}' \left(a + \frac{\mu + t}{\lambda + \mu} \eta(b, a) \right) \right| dt \right] \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu t \left\{ |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \frac{\mu - t}{\mu + \lambda} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right\} dt \right. \\
 & \quad \left. + \int_0^\lambda t \left\{ |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(1 - \frac{\mu + t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right\} dt \right] \\
 & = \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu t |\mathcal{F}'(a)| dt + \frac{1}{n} \sum_{s=1}^n \int_0^\mu t \left(1 - \left(\frac{\lambda + t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) dt \right. \\
 & \quad \left. + \int_0^\lambda t |\mathcal{F}'(a)| dt + \frac{1}{n} \sum_{s=1}^n \int_0^\lambda t \left(1 - \left(\frac{\lambda - t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) dt \right] \\
 & = \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{\lambda^2 + \mu^2}{2} |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n \int_0^\mu t \left(1 - \left(\frac{\lambda + t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) dt \right. \\
 & \quad \left. + \frac{1}{n} \sum_{s=1}^n \int_0^\lambda t \left(1 - \left(\frac{\lambda - t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) dt \right], \tag{30}
 \end{aligned}$$

which implies the desired inequality (28) since

$$\int_0^\mu t \left(1 - \left(\frac{\lambda + t}{\lambda + \mu} \right)^s \right) dt = \frac{\mu^2}{2} - \frac{\mu(s+2)(\lambda + \mu)^{s+1} - (\lambda + \mu)^{s+2} + \lambda^{s+2}}{(s+1)(s+2)(\lambda + \mu)^s} = K_1,$$

$$\int_0^\lambda t \left(1 - \left(\frac{\lambda - t}{\lambda + \mu} \right)^s \right) dt = \frac{\lambda^2}{2} - \frac{\lambda^{s+2}}{(s+1)(s+2)(\lambda + \mu)^s} = K_2. \tag{31}$$

This completes the proof of Theorem 10.

Next, we discuss some special cases of Theorem 10.

(I) If we consider $\lambda = \mu = 1$ in Theorem 10, then we have

$$\begin{aligned}
 & \left| \frac{\mathcal{F}(a) + \mathcal{F}(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
 & \leq \frac{\eta(b, a)}{4} \left[|\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n \frac{2^s(s+1)(s+2) - 2(1 + 2^{s+1}s)}{2^{s+1}(s+1)(s+2)} \zeta \right. \\
 & \quad \left. \cdot (|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) + \frac{1}{n} \sum_{s=1}^n \frac{2^s(s+1)(s+2) - 2}{(s+1)(s+2)2^{s+1}} \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right]. \tag{32}
 \end{aligned}$$

(II) If we take $\eta(b, a) = b - a$ in Theorem 10, then we get

$$\begin{aligned}
 & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(b)}{\lambda + \mu} - \frac{1}{b - a} \int_a^b \mathcal{F}(x) dx \right| \\
 & \leq \frac{b - a}{(\lambda + \mu)^2} \left[\frac{\lambda^2 + \mu^2}{2} |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n K_1 \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right. \\
 & \quad \left. + \frac{1}{n} \sum_{s=1}^n K_2 \zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) \right]. \tag{33}
 \end{aligned}$$

(III) If we choose $\zeta(|\mathcal{F}'(b)|, |\mathcal{F}'(a)|) = |\mathcal{F}'(b)| - |\mathcal{F}'(a)|$ in Theorem 10, then we obtain

$$\begin{aligned}
 & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{\lambda^2 + \mu^2}{2} |\mathcal{F}'(a)| + \frac{1}{n} \sum_{s=1}^n (K_1 + K_2) (|\mathcal{F}'(b)| - |\mathcal{F}'(a)|) \right]. \tag{34}
 \end{aligned}$$

Theorem 11. Let $\mathcal{F} : \mathcal{I} \rightarrow \mathbb{R}$ be a differentiable function on \mathcal{I}° with $\eta(b, a) > 0$, $\lambda > 0$, $\mu > 0$, and let $\mathcal{F}' \in L[\mathcal{I}]$, $(1/p) + (1/q) = 1$, $p > 1$, $q > 1$. If $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, then

$$\begin{aligned}
 & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\left(\frac{\mu^{p+1}}{p+1} \right)^{1/p} \left(\mu |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_3 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\
 & \quad \left. + \left(\frac{\lambda^{p+1}}{p+1} \right)^{1/p} \left(\lambda |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_4 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right], \tag{35}
 \end{aligned}$$

where

$$\begin{aligned}
 K_3 &= \mu - \frac{(\lambda + \mu)^{s+1} - \lambda^{s+1}}{(s+1)(\lambda + \mu)^s}, \\
 K_4 &= \lambda - \frac{\lambda^{s+1}}{(s+1)(\lambda + \mu)^s}. \tag{36}
 \end{aligned}$$

Proof. Using Lemma 9, Hölder's inequality, and the fact that $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, it follows that

$$\begin{aligned}
 & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\int_0^\mu |t| \left| \mathcal{F}' \left(a + \frac{\mu - t}{\mu + \lambda} \eta(b, a) \right) \right| dt \right. \\
 & \quad \left. + \int_0^\lambda |t| \left| \mathcal{F}' \left(a + \frac{\mu + t}{\lambda + \mu} \eta(b, a) \right) \right| dt \right] \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\left(\int_0^\mu |t|^p dt \right)^{1/p} \left(\int_0^\mu \left| \mathcal{F}' \left(a + \frac{\mu - t}{\lambda + \mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right. \\
 & \quad \left. + \left(\int_0^\lambda |t|^p dt \right)^{1/p} \left(\int_0^\lambda \left| \mathcal{F}' \left(a + \frac{\mu + t}{\lambda + \mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right] \\
 & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\left(\frac{\mu^{p+1}}{p+1} \right)^{1/p} \left(\int_0^\mu (|\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{\lambda + t}{\lambda + \mu} \right)^s \right) \zeta \right. \right. \\
 & \quad \left. \left. \cdot (|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) dt \right)^{1/q} + \left(\frac{\lambda^{p+1}}{p+1} \right)^{1/p} \left(\int_0^\lambda (|\mathcal{F}'(a)|^q \right. \right. \\
 & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{\lambda - t}{\lambda + \mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) dt \right)^{1/q} \right] \\
 & = \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\left(\frac{\mu^{p+1}}{p+1} \right)^{1/p} \left(\mu |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_3 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\
 & \quad \left. + \left(\frac{\lambda^{p+1}}{p+1} \right)^{1/p} \left(\lambda |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_4 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right], \tag{37}
 \end{aligned}$$

where

$$\begin{aligned} K_3 &= \int_0^\mu \left(1 - \left(\frac{\lambda+t}{\lambda+\mu}\right)^s\right) dt = \mu - \frac{(\lambda+\mu)^{s+1} - \lambda^{s+1}}{(s+1)(\lambda+\mu)^s}, \\ K_4 &= \int_0^\lambda \left(1 - \left(\frac{\lambda-t}{\lambda+\mu}\right)^s\right) dt = \lambda - \frac{\lambda^{s+1}}{(s+1)(\lambda+\mu)^s}. \end{aligned} \quad (38)$$

The proof of Theorem 11 is complete.

We now discuss some special cases of Theorem 11.

(I) If we choose $\lambda = \mu = 1$ in Theorem 11, then

$$\begin{aligned} & \left| \frac{\mathcal{F}(a) + \mathcal{F}(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{4(p+1)^{1/p}} \left[\left\{ |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \frac{2^s(s+1) - (2^{s+1} - 1)}{2^s(s+1)} \right. \right. \\ & \quad \cdot \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \left. \left. \right\}^{1/q} + \left\{ |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \frac{2^s(s+1) - 1}{2^s(s+1)} \right. \right. \\ & \quad \cdot \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \left. \left. \right\}^{1/q} \right]. \end{aligned} \quad (39)$$

(II) If we take $\eta(b, a) = b - a$ in Theorem 11, then

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b \mathcal{F}(x) dx \right| \\ & \leq \frac{b-a}{(\lambda+\mu)^2} \left[\left(\frac{\mu^{p+1}}{p+1} \right)^{1/p} \left(\mu |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_3 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{\lambda^{p+1}}{p+1} \right)^{1/p} \left(\lambda |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_4 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right]. \end{aligned} \quad (40)$$

(III) If we put $\zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) = |\mathcal{F}'(b)|^q - |\mathcal{F}'(a)|^q$ in Theorem 11, then

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\left(\frac{\mu^{p+1}}{p+1} \right)^{1/p} \left(\left(\mu - \frac{1}{n} \sum_{s=1}^n K_3 \right) |\mathcal{F}'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_3 |\mathcal{F}'(b)|^q \right)^{1/q} + \left(\frac{\lambda^{p+1}}{p+1} \right)^{1/p} \left(\left(\lambda - \frac{1}{n} \sum_{s=1}^n K_4 \right) |\mathcal{F}'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_4 |\mathcal{F}'(b)|^q \right)^{1/q} \right]. \end{aligned} \quad (41)$$

Theorem 12. Let $\mathcal{F} : \mathcal{I} \rightarrow \mathbb{R}$ be a differentiable function on \mathcal{I}° with $\eta(b, a) > 0$, $\lambda > 0$, $\mu > 0$, and let $\mathcal{F}' \in L[\mathcal{I}]$, $q \geq 1$. If $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, then

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\left(\frac{\mu^2}{2} \right)^{1-(1/q)} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_1 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{\lambda^2}{2} \right)^{1-(1/q)} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_2 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right], \end{aligned} \quad (42)$$

where K_1 and K_2 are the expressions as described in Theorem 10.

Proof. Note that $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, by using the power mean inequality, we have

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\int_0^\mu |1-t| \left| \mathcal{F}' \left(a + \frac{\mu-t}{\mu+\lambda} \eta(b, a) \right) \right| dt \right. \\ & \quad \left. + \int_0^\lambda |t| \left| \mathcal{F}' \left(a + \frac{\mu+t}{\lambda+\mu} \eta(b, a) \right) \right| dt \right] \\ & \leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\left(\int_0^\mu |1-t| dt \right)^{1-(1/q)} \left(\int_0^\mu \left| \mathcal{F}' \left(a + \frac{\mu-t}{\mu+\lambda} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right. \\ & \quad \left. + \left(\int_0^\lambda |t| dt \right)^{1-(1/q)} \left(\int_0^\lambda \left| \mathcal{F}' \left(a + \frac{\mu+t}{\lambda+\mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right] \\ & \leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\left(\frac{\mu^2}{2} \right)^{1-(1/q)} \left(\int_0^\mu t |\mathcal{F}'(a)|^q dt + \frac{1}{n} \sum_{s=1}^n \int_0^\mu t \left(1 - \left(\frac{\lambda+t}{\lambda+\mu} \right)^s \right) \zeta \right. \right. \\ & \quad \left. \left. \cdot \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) dt \right)^{1/q} + \left(\frac{\lambda^2}{2} \right)^{1-(1/q)} \right. \\ & \quad \left. \cdot \left(\int_0^\lambda t |\mathcal{F}'(a)|^q dt + \frac{1}{n} \sum_{s=1}^n \int_0^\lambda t \left(1 - \left(\frac{\lambda-t}{\lambda+\mu} \right)^s \right) \zeta \right. \right. \\ & \quad \left. \left. \cdot \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) dt \right)^{1/q} \right] \\ & = \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\left(\frac{\mu^2}{2} \right)^{1-(1/q)} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_1 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{\lambda^2}{2} \right)^{1-(1/q)} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_2 \zeta \left(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q \right) \right)^{1/q} \right]. \end{aligned} \quad (43)$$

Here, K_1 and K_2 are formulated as that of Theorem 10. This completes the proof of Theorem 12.

We now discuss some special cases of Theorem 12.

(I) Choosing $\lambda = \mu = 1$ in Theorem 12, we get

$$\begin{aligned} & \left| \frac{\mathcal{F}(a) + \mathcal{F}(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{2^{3-1/q}} \left[\left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \left(\frac{(s+1)(s+2)2^s - 2^{s+2}s - 2}{2^{s+1}(s+1)(s+2)} \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \frac{(s+1)(s+2)2^s - 2}{2^{s+1}(s+1)(s+2)} \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right]. \end{aligned} \tag{44}$$

(II) Choosing $\eta(b, a) = b - a$ in Theorem 12, we have

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b \mathcal{F}(x) dx \right| \\ & \leq \frac{b-a}{(\lambda + \mu)^2} \left[\left(\frac{\mu^2}{2} \right)^{1-(1/q)} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_1 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{\lambda^2}{2} \right)^{1-(1/q)} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_2 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right]. \end{aligned} \tag{45}$$

(III) Taking $\zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) = |\mathcal{F}'(b)|^q - |\mathcal{F}'(a)|^q$ in Theorem 12, we obtain

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\left(\frac{\mu^2}{2} \right)^{1-(1/q)} \left(\left(\frac{\mu^2}{2} - \frac{1}{n} \sum_{s=1}^n K_1 \right) |\mathcal{F}'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_1 |\mathcal{F}'(b)|^q \right)^{1/q} + \left(\frac{\lambda^2}{2} \right)^{1-(1/q)} \left(\left(\frac{\lambda^2}{2} - \frac{1}{n} \sum_{s=1}^n K_2 \right) |\mathcal{F}'(a)|^q \right. \right. \\ & \quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_2 |\mathcal{F}'(b)|^q \right)^{1/q} \right]. \end{aligned} \tag{46}$$

Theorem 13. Let $\mathcal{F} : \mathcal{J} \rightarrow \mathbb{R}$ be a differentiable function on \mathcal{J}° with $\eta(b, a)\lambda > 0$, $\lambda > 0$, $\mu > 0$, and let $\mathcal{F}' \in L[\mathcal{J}]$, $1/p + 1/q = 1$, $p > 1$, $q > 1$. If $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, then

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\ & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{1}{\mu} \left\{ \left(\frac{\mu^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_5 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\frac{\mu^{p+2}}{p+2} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_6 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right. \\ & \quad \left. + \frac{1}{\lambda} \left\{ \left(\frac{\lambda^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_7 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\frac{\lambda^{p+2}}{p+2} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_8 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right], \end{aligned} \tag{47}$$

where

$$\begin{aligned} K_5 &= \frac{\mu^2}{2} - \frac{\mu^{s+2}}{(s+2)(\lambda + \mu)^s}, \\ K_6 &= \frac{\mu^2}{2} - \frac{\mu^{s+2}}{(s+1)(s+2)(\lambda + \mu)^s}, \\ K_7 &= \frac{\lambda^2}{2} - \frac{(\mu + \lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)\mu^{s+1}}{(s+1)(s+2)(\lambda + \mu)^s}, \\ K_8 &= \frac{\lambda^2}{2} - \frac{(\mu + \lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)(\lambda + \mu)^{s+1}}{(s+1)(s+2)(\lambda + \mu)^s}. \end{aligned} \tag{48}$$

Proof. Note that $|\mathcal{F}'|^q$ is n -polynomial ζ -preinvex function, by using the refined Hölder inequality (see [19]), we obtain

$$\begin{aligned} & \left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \\ & \quad \cdot \left[\int_0^\mu |t| \left| \mathcal{F}' \left(a + \frac{\mu-t}{\mu+\lambda} \eta(b, a) \right) \right| dt + \int_0^\lambda |t| \left| \mathcal{F}' \left(a + \frac{\mu+t}{\lambda+\mu} \eta(b, a) \right) \right| dt \right] \\ & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{1}{\mu} \left\{ \left(\int_0^\mu (\mu-t)|t|^p dt \right)^{1/p} \left(\int_0^\mu (\mu-t) \left| \mathcal{F}' \left(a + \frac{\mu-t}{\lambda+\mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\int_0^\mu t|t|^p dt \right)^{1/p} \left(\int_0^\mu t \left| \mathcal{F}' \left(a + \frac{\mu-t}{\mu+\lambda} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right\} \right. \\ & \quad \left. + \frac{1}{\lambda} \left\{ \left(\int_0^\lambda (\lambda-t)|t|^p dt \right)^{1/p} \left(\int_0^\lambda (\lambda-t) \left| \mathcal{F}' \left(a + \frac{\mu-t}{\lambda+\mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\int_0^\lambda t|t|^p dt \right)^{1/p} \left(\int_0^\lambda t \left| \mathcal{F}' \left(a + \frac{\mu+t}{\lambda+\mu} \eta(b, a) \right) \right|^q dt \right)^{1/q} \right\} \right] \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \\ & \quad \cdot \left[\frac{1}{\mu} \left\{ \left(\frac{\mu^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\int_0^\mu \mu - t \bar{n} \left(|\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \right. \right. \right. \right. \\ & \quad \left. \left. \left. \cdot \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right) dt \right)^{1/q} + \left(\frac{\mu^{p+2}}{p+2} \right)^{1/q} \right. \right. \\ & \quad \left. \left. \cdot \left(\int_0^\mu \left(|\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right) dt \right)^{1/q} \right\} \right. \\ & \quad \left. + \frac{1}{\lambda} \left\{ \left(\frac{\lambda^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\int_0^\lambda \lambda - t \bar{n} \left(|\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) \right. \right. \right. \right. \right. \\ & \quad \left. \left. \left. \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right) dt \right)^{1/q} + \left(\frac{\lambda^{p+2}}{p+2} \right)^{1/q} \right. \right. \\ & \quad \left. \left. \cdot \left(\int_0^\lambda \left(|\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right) dt \right)^{1/q} \right\} \right] \\ & = \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[\frac{1}{\mu} \left\{ \left(\frac{\mu^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_5 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\frac{\mu^{p+2}}{p+2} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_6 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right. \\ & \quad \left. + \frac{1}{\lambda} \left\{ \left(\frac{\lambda^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_7 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\ & \quad \left. \left. + \left(\frac{\lambda^{p+2}}{p+2} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_8 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right] \end{aligned} \tag{49}$$

A direct computation gives

$$K_5 = \int_0^\mu (\mu-t) \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) dt = \frac{\mu^2}{2} - \frac{\mu^{s+2}}{(s+2)(\lambda + \mu)^s},$$

$$\begin{aligned}
K_6 &= \int_0^\mu t \left(1 - \left(\frac{\mu-t}{\lambda+\mu} \right)^s \right) dt = \frac{\mu^2}{2} - \frac{\mu^{s+2}}{(s+1)(s+2)(\lambda+\mu)^s}, \\
K_7 &= \int_0^\lambda (\lambda-t) \left(1 - \left(\frac{\mu+t}{\lambda+\mu} \right)^s \right) dt \\
&= \frac{\lambda^2}{2} - \frac{(\mu+\lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)\mu^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}, \\
K_8 &= \int_0^\lambda t \left(1 - \left(\frac{\mu+t}{\lambda+\mu} \right)^s \right) dt \\
&= \frac{\lambda^2}{2} - \frac{(\mu+\lambda)^{s+2} - \mu^{s+2} - \lambda(s+2)(\lambda+\mu)^{s+1}}{(s+1)(s+2)(\lambda+\mu)^s}. \tag{50}
\end{aligned}$$

This completes the proof of Theorem 13.

Let us now discuss some special cases of Theorem 13.

(I) If we take $\lambda = \mu = 1$ in Theorem 13, we get

$$\begin{aligned}
&\left| \frac{\mathcal{F}(a) + \mathcal{F}(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
&\leq \frac{\eta(b, a)}{4} \left[\left(\frac{1}{(p+1)(p+2)} \right)^{1/p} \right. \\
&\quad \times \left\{ \left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \left(\frac{1}{2} - \frac{1}{(s+2)2^s} \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\
&\quad \left. + \left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \left(\frac{1}{2} - \frac{1}{(s+1)(s+2)2^s} \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \\
&\quad + \left(\frac{1}{p+2} \right)^{1/p} \left\{ \left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \left(\frac{1}{2} - \frac{2^{s+2} - s - 3}{(s+1)(s+2)2^s} \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \\
&\quad \left. + \left(\frac{|\mathcal{F}'(a)|^q}{2} + \frac{1}{n} \sum_{s=1}^n \left(\frac{1}{2} + \frac{1 + s2^{s+1}}{(s+1)(s+2)2^s} \right) \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\}. \tag{51}
\end{aligned}$$

(II) If we put $\eta(b, a) = b - a$ in Theorem 13, then

$$\begin{aligned}
&\left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(b)}{\lambda + \mu} - \frac{1}{b-a} \int_a^b \mathcal{F}(x) dx \right| \\
&\leq \frac{b-a}{(\lambda+\mu)^2} \left[\frac{1}{\mu} \left\{ \left(\frac{\mu^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_5 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\
&\quad \left. \left. + \left(\frac{\mu^{p+2}}{p+2} \right)^{1/p} \left(\frac{\mu^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_6 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right. \\
&\quad \left. + \frac{1}{\lambda} \left\{ \left(\frac{\lambda^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_7 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right. \right. \\
&\quad \left. \left. + \left(\frac{\lambda^{p+2}}{p+2} \right)^{1/p} \left(\frac{\lambda^2}{2} |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_8 \zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) \right)^{1/q} \right\} \right]. \tag{52}
\end{aligned}$$

(III) If we choose $\zeta(|\mathcal{F}'(b)|^q, |\mathcal{F}'(a)|^q) = |\mathcal{F}'(b)|^q - |\mathcal{F}'(a)|^q$ in Theorem 13, then

$$\begin{aligned}
&\left| \frac{\mu \mathcal{F}(a) + \lambda \mathcal{F}(a + \eta(b, a))}{\lambda + \mu} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \mathcal{F}(x) dx \right| \\
&\leq \frac{\eta(b, a)}{(\lambda+\mu)^2} \left[\frac{1}{\mu} \left\{ \left(\frac{\mu^{p+2}}{(p+1)(p+2)} \right)^{1/p} \left(\left(\frac{\mu^2}{2} - \frac{1}{n} \sum_{s=1}^n K_5 \right) |\mathcal{F}'(a)|^q \right. \right. \right. \\
&\quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_5 |\mathcal{F}'(b)|^q \right)^{1/q} + \left(\frac{\mu^{p+2}}{p+2} \right)^{1/p} \left(\left(\frac{\mu^2}{2} - \frac{1}{n} \sum_{s=1}^n K_6 \right) |\mathcal{F}'(a)|^q \right. \right. \\
&\quad \left. \left. + \frac{1}{n} \sum_{s=1}^n K_6 |\mathcal{F}'(b)|^q \right)^{1/q} \right\} + \frac{1}{\lambda} \left\{ \left(\frac{\lambda^{p+2}}{(p+1)(p+2)} \right)^{1/p} \right. \\
&\quad \cdot \left(\left(\frac{\lambda^2}{2} - \frac{1}{n} \sum_{s=1}^n K_7 \right) |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_7 |\mathcal{F}'(b)|^q \right)^{1/q} \\
&\quad \left. \left. + \left(\frac{\lambda^{p+2}}{p+2} \right)^{1/p} \left(\left(\frac{\lambda^2}{2} - \frac{1}{n} \sum_{s=1}^n K_8 \right) |\mathcal{F}'(a)|^q + \frac{1}{n} \sum_{s=1}^n K_8 |\mathcal{F}'(b)|^q \right)^{1/q} \right\} \right]. \tag{53}
\end{aligned}$$

3. Application to Special Means

Let us recall the definitions of the arithmetic mean, weighted arithmetic mean, and the mean for functions, as follows:

(1) The arithmetic mean

$$\mathcal{A}(a_1, a_2, \dots, a_n) = \frac{a_1 + a_2 + \dots + a_n}{n}. \tag{54}$$

(2) The weighted arithmetic mean

$$\mathcal{A}(a_1, a_2, \dots, a_n; p_1, p_2, \dots, p_n) = \frac{p_1 a_1 + p_2 a_2 + \dots + p_n a_n}{p_1 + p_2 + \dots + p_n}. \tag{55}$$

(3) The mean of the function Φ on $[a, b]$

$$\mathcal{A}_\Phi(a, b) = \frac{1}{b-a} \int_a^b \Phi(x) dx. \tag{56}$$

We establish the following inequalities for special means.

Proposition 14. Let $\Phi : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a preinvex function with $\Phi(b) - \Phi(a) \geq 0$. If $\eta(b, a) > 0$ and $\Phi \in L[a, a + \eta(b, a)]$, then we have the following inequality

$$\mathcal{A}_\Phi(a, a + \eta(b, a)) \leq \Phi(a) + (\Phi(b) - \Phi(a)) \mathcal{A} \left(\frac{1}{2}, \frac{2}{3}, \dots, \frac{n}{n+1} \right). \tag{57}$$

Proof. Taking $\mathcal{F}(x) = \Phi(x)$, $x \in [a, a + \eta(b, a)]$ and $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$. Since $\Phi(x)$ is a preinvex function with $\Phi(b) - \Phi(a) \geq 0$, we deduce from Proposition 7 that $\Phi(x)$ is a n -polynomial preinvex function. Using Theorem 8, we obtain the desired inequality (57).

Proposition 15. Let Φ be a differentiable function, and let $|\Phi'|: [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a preinvex function with $|\Phi'(b)| - |\Phi'(a)| \geq 0$. If $\eta(b, a) > 0$, $\lambda > 0$, $\mu > 0$, and $|\Phi'| \in L[a, a + \eta(b, a)]$, then we have the following inequality

$$\begin{aligned} & |\mathcal{A}(\Phi(a), \Phi(a + \eta(b, a)); \mu, \lambda) - \mathcal{A}_\Phi(a, a + \eta(b, a))| \\ & \leq \frac{\eta(b, a)}{(\lambda + \mu)^2} \left[|\Phi'(a)| \mathcal{A}(\lambda^2, \mu^2) + \frac{2}{n} \left(|\Phi'(b)| - |\Phi'(a)| \right) \sum_{s=1}^n \mathcal{A}(K_1, K_2) \right], \end{aligned} \quad (58)$$

where K_1 and K_2 are the expressions as that described in Theorem 10.

Proof. Choosing $\mathcal{F}(x) = \Phi(x)$, $x \in [a, a + \eta(b, a)]$ and $\zeta(\mathcal{F}(b), \mathcal{F}(a)) = \mathcal{F}(b) - \mathcal{F}(a)$. Since $|\Phi'(x)|$ is a preinvex function with $|\Phi'(b)| - |\Phi'(a)| \geq 0$, it follows from Proposition 7 that $|\Phi'(x)|$ is a n -polynomial preinvex function. Using Theorem 10 leads to the desired inequality (58).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

S.W. and M.U.A. finished the proofs of the main results and the writing work. M.U.U., S.T., and A.K. gave lots of advice on the proofs of the main results and the writing work. All authors read and approved the final manuscript.

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