Research Article

On Some Relationships of Certain $K$ – Uniformly Analytic Functions Associated with Mittag-Leffler Function

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Received 14 April 2021; Accepted 30 April 2021; Published 28 May 2021

1. Introduction

Let $A$ be the class of analytic functions in the open unit disc $U = \{z : |z| < 1\}$ which in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hfill (1)

For $f(z)$ and $g(z) \in A$, we say that the function $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \text{ or } f(z) \prec g(z).$$  \hfill (2)

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in $U$, then we have the following equivalence relation (cf., e.g., [1, 2]; see also [3]):

$$f(z) \prec g(z) \Leftrightarrow f(0) < g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$  \hfill (3)

Let $f$ be as in (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then Hadamard product (or convolution) of $f(z)$ and $h(z)$ is given by

$$(f * h)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ (z \in U).$$  \hfill (4)

For $\zeta, \eta \in [0, 1]$, we denote by $S^*(\zeta)$, $C(\zeta)$, $K(\zeta, \eta)$, and $K^*(\zeta, \eta)$ the subclasses of $A$ consisting of all analytic functions which are, respectively, starlike of order $\zeta$, convex of order $\zeta$, close-to-convex of order $\zeta$ and type $\eta$, and quasiconvex of order $\zeta$ and type $\eta$ in $U$.

Also, let the subclasses $US(\mu, \zeta)$, $UC(\mu, \zeta)$, $USK(\mu, \zeta, \eta)$, and $UCK(\mu, \zeta, \eta)$ of $A$ $(\eta \in [0, 1]; \mu \geq 0)$ be defined as follows:

$$US(\mu, \zeta) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} - \zeta \right) > \mu \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\},$$

$$UC(\mu, \zeta) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} - \zeta \right) > \mu \left| \frac{zf''(z)}{f'(z)} \right| \right\},$$

$$USK(\mu, \zeta, \eta) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} - \zeta \right) > \mu \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\},$$

$$UCK(\mu, \zeta, \eta) = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} - \zeta \right) > \mu \left| \frac{zf''(z)}{f'(z)} \right| \right\}.$$
USK(μ, μ, η) = \left\{ f \in \mathcal{A} : \exists h \in US(μ, μ) \right. \\
\text{s.t. } \Re \left\{ \frac{zf'(z)}{h(z)} - μ \right\} > μ \left| \frac{zf'(z)}{h(z)} - 1 \right\}

UCK(μ, μ, η) = \left\{ f \in \mathcal{A} : \exists h \in UC(μ, μ) \right. \\
\text{s.t. } \Re \left\{ \frac{(zf'(z))'}{h'(z)} - μ \right\} > μ \left| \frac{(zf'(z))'}{h'(z)} - 1 \right\}

We note that

\begin{align*}
US(0, μ) &= S^μ(μ), \quad UC(0, μ) = C(μ), \\
USK(0, μ, η) &= K(μ, η) \text{ and } UCK(0, μ, η) = K(μ, η) \quad (0 \leq μ; η < 1).
\end{align*}

Moreover, let \( q_{μ, μ}(z) \) be an analytic function which maps \( U \) onto the conic domain \( \Phi_{μ, μ} = \{ u + iv : u > k \sqrt{(u-1)^2 + v^2 + z} \} \) such that \( 1 \in \Phi_{μ, μ} \) defined as follows:

\begin{align*}
q_{μ, μ}(z) &= \begin{cases} \\
1 + (1 - 2μ)z & (μ = 0), \\
1 - μ & (μ = 1), \\
\frac{1 - μ}{2μ - 1} \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 & (μ > 1), \\
\frac{1 - μ}{2μ - 1} \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 & (μ < 1).
\end{cases}
\end{align*}

where \( u(z) = (z - \sqrt{μ})/(1 - \sqrt{μ}z) \) and \( μ(z) \) is such that \( μ = \cosh (nμ'(z)/4c(z)) \). By virtue of properties of the conic domain \( \Phi_{μ, μ} \) (cf., e.g., \([4, 5]\)), we have

\begin{align*}
\Re \left\{ q_{μ, μ}(z) \right\} &\frac{μ + μ}{μ + 1}. \quad (8)
\end{align*}

Making use of the principal of subordination and the definition of \( q_{μ, μ}(z) \), we may rewrite the subclasses \( US(μ, μ) \), \( UCK(μ, μ, η) \), and \( USK(μ, μ, η) \) as follows:

\begin{align*}
US(μ, μ) &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < q_{μ, μ}(z) \right\}, \\
UCK(μ, μ, η) &= \left\{ f \in \mathcal{A} : 1 + \frac{zf'(z)}{f'(z)} < q_{μ, μ}(z) \right\}, \\
USK(μ, μ, η) &= \left\{ f \in \mathcal{A} : \exists h \in US(μ, η) \text{ s.t. } \frac{zf'(z)}{h(z)} < q_{μ, μ}(z) \right\}
\end{align*}

and

\begin{align*}
UCK(μ, μ, η) &= \left\{ f \in \mathcal{A} : \exists h \in UC(μ, η) \right. \\
\text{s.t. } \frac{(zf'(z))'}{h'(z)} < q_{μ, μ}(z) \right\}. \quad (10)
\end{align*}

Attiya \([6]\) introduced the operator \( H_{μ, β}^{γ, k}(f) \), where \( H_{μ, β}^{γ, k}(f) : \mathcal{A} \longrightarrow \mathcal{A} \) is defined by

\begin{align*}
H_{μ, β}^{γ, k}(f) &= μ_{μ, β}^{γ, k} * f(z) (z \in Ω),
\end{align*}

with \( β, γ ∈ \mathbb{C} \), \( Re(α) > max \{ 0, Re(k) - 1 \} \) and \( Re(k) > 0 \). Also, \( Re(α) = 0 \) when \( Re(k) = 1; β ≠ 0 \). Here, \( μ_{μ, β}^{γ, k} \) is the generalized Mittag–Leﬄer function defined by \([7]\), see also \([6]\), and the symbol \( (\ast) \) denotes the Hadamard product.

Due to the importance of the Mittag–Leﬄer function, it is involved in many problems in natural and applied science. A detailed investigation of the Mittag–Leﬄer function has been studied by many authors (see, e.g., \([7–12]\)).
Atiya [6] noted that

$$H_{a,b}^{k}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(y + nk)\Gamma(\alpha + \beta)}{\Gamma(y + k)\Gamma(\beta + an)n!} a_n z^n.$$  \hfill (12)

Also, Atiya [6] showed that

$$z \left( H_{a,b}^{k}(f(z)) \right)' = \left( \frac{y + k}{k} \right) \left( H_{a,b}^{k+1}(f(z)) \right) - \frac{y}{k} \left( H_{a,b}^{k}(f(z)) \right),$$  \hfill (13)

and

$$z \left( H_{a,b}^{k+1}(f(z)) \right)' = \left( \frac{\alpha + \beta}{\alpha} \right) \left( H_{a,b}^{k}(f(z)) \right) - \frac{\beta}{\alpha} \left( H_{a,b}^{k+1}(f(z)) \right).$$  \hfill (14)

Next, by using the operator $H_{a,b}^{k}(f)$, we introduce the following subclasses of analytic functions in $\mathbb{U}$

$$US_{\beta}^{\gamma}(\mu, \zeta) = \left\{ f \in \mathcal{A} : H_{a,b}^{k}(f(z)) \in US(\mu, \zeta) \right\},$$

$$UC_{\beta}^{\gamma}(\mu, \zeta) = \left\{ f \in \mathcal{A} : H_{a,b}^{k}(f(z)) \in UC(\mu, \zeta) \right\},$$

$$USK_{\beta}^{\gamma}(\mu, \zeta, \eta) = \left\{ f \in \mathcal{A} : H_{a,b}^{k}(f(z)) \in USK(\mu, \zeta, \eta) \right\},$$

$$UCK_{\beta}^{\gamma}(\mu, \zeta, \eta) = \left\{ f \in \mathcal{A} : H_{a,b}^{k}(f(z)) \in UCK(\mu, \zeta, \eta) \right\},$$

where $\beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > \max \{0, \Re(k) - 1\}$ and $\Re(k) > 0$. Also, $\Re(\alpha) = 0$ when $\Re(k) = 1; \beta \neq 0$.

Also, we note that

$$f(z) \in UC_{\beta}^{h}(\mu, \zeta) \Rightarrow zf'(z) \in US_{\beta}^{h}(\mu, \zeta),$$  \hfill (16)

$$f(z) \in UCK_{\beta}^{h}(\mu, \zeta, \eta) \Rightarrow zf'(z) \in USK_{\beta}^{h}(\mu, \zeta, \eta).$$  \hfill (17)

In this paper, we introduce several inclusion properties of the classes $US_{\beta}^{h}(\mu, \zeta)$, $UC_{\beta}^{h}(\mu, \zeta)$, $USK_{\beta}^{h}(\mu, \zeta, \eta)$, and $UCK_{\beta}^{h}(\mu, \zeta, \eta)$. Also, integral-preserving properties of these classes associated with generalized Libera integral operator are also obtained.

### 2. Inclusion Properties Associated with $H_{a,b}^{k}(f)$

**Lemma 1** (see [13]). If $h(z)$ is convex univalent in $\mathbb{U}$ with $h(0) = 1$ and $\Re\{\xi h(z) + \zeta\} > 0(\xi \in \mathbb{C})$. Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ which satisfy the following subordination relation

$$p(z) + \frac{zp'(z)}{p(z) - 1} < h(z),$$  \hfill (18)

then

$$p(z) < h(z).$$  \hfill (19)

**Lemma 2** (see [2]). If $h(z)$ is convex univalent in $\mathbb{U}$ and let $w$ be analytic in $\mathbb{U}$ with $\Re\{w(z)\} \geq 0$. Let $p(z)$ be analytic in $\mathbb{U}$ and $p(0) = h(0)$ which satisfy the following subordination relation

$$p(z) + w(z)p'(z) < h(z),$$  \hfill (20)

then

$$p(z) < h(z).$$  \hfill (21)

**Theorem 3.** If $\Re(\gamma/k) > (\mu + \zeta)/(\mu + 1)$, then $US_{\beta}^{\gamma+1}(\mu, \zeta) \subset US_{\beta}^{\gamma}(\mu, \zeta)$.

**Proof.** Let $f(z) \in US_{\beta}^{\gamma+1}(\mu, \zeta)$, put

$$p(z) = \frac{z \left( H_{a,b}^{k}(f(z)) \right)'}{H_{a,b}^{k}(f(z))} \quad (z \in \mathbb{U}),$$  \hfill (22)

we note that $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. From (13) and (12), we have

$$\frac{H_{a,b}^{k+1}(f(z))}{H_{a,b}^{k}(f(z))} = \frac{k}{y + k} \left( p(z) + \frac{y}{k} \right).$$  \hfill (23)

Differentiating (23) with respect to $z$, we obtain

$$\frac{z \left( H_{a,b}^{k+1}(f(z)) \right)'}{H_{a,b}^{k+1}(f(z))} = p(z) + \frac{zp'(z)}{p(z) + (\gamma/k)}. $$  \hfill (24)

From the above relation and using (7), we may write

$$p(z) + \frac{zp'(z)}{p(z) + (\gamma/k)} < q_{\mu, \zeta}(z) \quad (z \in \mathbb{U}).$$  \hfill (25)

Since $\Re\{q_{\mu, \zeta}(z)\} > (\mu + \zeta)/(\mu + 1)$, we see that

$$\Re\{q_{\mu, \zeta}(z) + \frac{\gamma}{k}\} > 0 \quad (z \in \mathbb{U}).$$  \hfill (26)

Applying Lemma 1, it follows that $p(z) < q_{\mu, \zeta}(z)$, that is, $f(z) \in US_{\beta}^{\gamma}(\mu, \zeta)$.

Using the same technique in Theorem 3 with relation (14), we have the following theorem.

**Theorem 4.** If $\Re(\alpha/\beta) > (\mu + \zeta)/(\mu + 1)$, then $US_{\beta}^{\gamma}(\mu, \zeta) \subset US_{\beta+1}^{\gamma}(\mu, \zeta)$.
Theorem 5. If $\Re(y/k) > -(\mu + \zeta)/(\mu + 1)$, then $UC^{\tau+1}_\beta(\mu, \zeta) \subset UC^{\tau+1}_\beta(\mu, \zeta)$.

Proof. Applying Theorem 3 and relation (16), we observe that

$$f(z) \in UC^{\tau+1}_\beta(\mu, \zeta) \Leftrightarrow zf'(z) \in US^{\tau+1}_\beta(\mu, \zeta) \Rightarrow zf'(z) \in US^{\tau+1}_\beta(\mu, \zeta) \Rightarrow f(z) \in UC^{\tau+1}_\beta(\mu, \zeta),$$

(27)

which evidently proves Theorem 5.

Similarly, we can prove the following theorem.

Theorem 6. If $\Re(\alpha/\beta) > -(\mu + \zeta)/(\mu + 1)$, then $UC^{\tau+1}_\beta(\mu, \zeta) \subset UC^{\tau+1}_\beta(\mu, \zeta)$.

Theorem 7. If $\Re(y/k) > -(\mu + \zeta)/(\mu + 1)$, then $USK^{\tau+1}_\beta(\mu, \zeta, \eta) \subset USK^{\tau+1}_\beta(\mu, \zeta, \eta)$.

Proof. Let $f(z) \in USK^{\tau+1}_\beta(\mu, \zeta, \eta)$. Then, there exists a function $r(z) \in US(\mu, \zeta)$ such that

$$z \left( \frac{H^{\tau+1}_{\alpha, \beta}f(z)}{r(z)} \right) < q_{\mu, \zeta}(z).$$

(28)

We can choose the function $h(z)$ such that $H^{\tau+1}_{\alpha, \beta}h(z) = r(z)$. Then, $h(z) \in US^{\tau+1}_\beta(\mu, \zeta)$ and

$$z \left( \frac{H^{\tau+1}_{\alpha, \beta}f(z)}{H^{\tau+1}_{\alpha, \beta}h(z)} \right) < q_{\mu, \zeta}(z).$$

(29)

Now, let

$$p(z) = \frac{z \left( H^{\tau+1}_{\alpha, \beta}f(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)},$$

(30)

where $p(z)$ is analytic in $U$ with $p(0) = 1$. Since $h(z) \in US^{\tau+1}_\beta(\mu, \zeta)$, by Theorem 3, we know that $h(z) \in US^{\tau+1}_\beta(\mu, \zeta)$. Let

$$t(z) = \frac{z \left( H^{\tau+1}_{\alpha, \beta}h(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)} (z \in U),$$

(31)

where $t(z)$ is analytic in $U$ with $\Re \{ t(z) \} > (\mu + \zeta)/(\mu + 1)$.

Also, from (30), we note that

$$z \left( H^{\tau+1}_{\alpha, \beta}f(z) \right)' = H^{\tau+1}_{\alpha, \beta}zf'(z) = \left( H^{\tau+1}_{\alpha, \beta}h(z) \right)p(z).$$

(32)

Differentiating both sides of (32) with respect to $z$, we obtain

$$z \left( \frac{H^{\tau+1}_{\alpha, \beta}f(z)'}{H^{\tau+1}_{\alpha, \beta}h(z)} \right) = \frac{z \left( H^{\tau+1}_{\alpha, \beta}h(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)} p(z) + zp'(z)$$

(33)

$$= t(z)p(z) + zp'(z).$$

Now, using (13) and (33), we obtain

$$z \left( H^{\tau+1}_{\alpha, \beta}f(z) \right)' = H^{\tau+1}_{\alpha, \beta}zf'(z) - \frac{z \left( H^{\tau+1}_{\alpha, \beta}f(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)}$$

$$= \frac{z \left( H^{\tau+1}_{\alpha, \beta}f(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)} + \left( \frac{z \left( H^{\tau+1}_{\alpha, \beta}h(z) \right)'}{H^{\tau+1}_{\alpha, \beta}h(z)} \right) + (y/k)$$

$$= t(z)p(z) + zp'(z) + (y/k)p(z) = p(z) + \frac{zp'(z)}{t(z) + (y/k)}.$$

(34)

Since $\Re(y/k) > -(\mu + \zeta)/(\mu + 1)$, we see that

$$\Re \left\{ t(z) + \frac{y}{K} \right\} > 0 (z \in U).$$

(34)

Hence, applying Lemma 2, we can show that $p(z) < q_{\mu, \zeta}(z)$, so that $f(z) \in USK^{\tau+1}_\beta(\mu, \zeta, \eta)$. This completes the proof of Theorem 7.

Similarly, we can prove the following theorem.

Theorem 8. If $\Re(\alpha/\beta) > -(\mu + \zeta)/(\mu + 1)$, then $USK^{\tau+1}_\beta(\mu, \zeta, \eta) \subset USK^{\tau+1}_\beta(\mu, \zeta, \eta)$.
We can also prove Theorem 9 by using Theorem 7 and relation (17).

**Theorem 9.** If \( \Re (y/k) > -(\mu + \zeta)/(\mu + 1) \), then \( \text{UCH}^{\gamma+1}_\beta(\mu, \zeta, \eta) \subset \text{UCH}^{\gamma}_\beta(\mu, \zeta, \eta) \).

Also, we obtain the following theorem.

**Theorem 10.** If \( \Re (\alpha/\beta) > -(\mu + \zeta)/(\mu + 1) \), then \( \text{UCH}^{\gamma}_\beta(\mu, \zeta, \eta) \subset \text{UCH}^{\gamma+1}_\beta(\mu, \zeta, \eta) \).

Now, we obtain squeeze theorems for inclusion by combining the above theorems as follows:

Combining both theorems 3 and 4, we have the following corollary.

**Corollary 11.** If \( (\mu + \zeta)/(\mu + 1) > -\min \{ \Re (y/k), \Re (\alpha/\beta) \} \), then

\[ \text{US}^{\gamma+1}_\beta(\mu, \zeta) \subset \text{US}^{\gamma}_\beta(\mu, \zeta) \subset \text{US}^{\gamma}_\beta(\mu, \zeta). \quad (36) \]

Combining both theorems 5 and 6, we have the following corollary.

**Corollary 12.** If \( (\mu + \zeta)/(\mu + 1) > -\min \{ \Re (y/k), \Re (\alpha/\beta) \} \), then

\[ \text{UC}^{\gamma+1}_\beta(\mu, \zeta) \subset \text{UC}^{\gamma}_\beta(\mu, \zeta) \subset \text{UC}^{\gamma}_\beta(\mu, \zeta). \quad (37) \]

Combining both theorems 7 and 8, we have the following corollary.

**Corollary 13.** If \( (\mu + \zeta)/(\mu + 1) > -\min \{ \Re (y/k), \Re (\alpha/\beta) \} \), then

\[ \text{USK}^{\gamma+1}_\beta(\mu, \zeta, \eta) \subset \text{USK}^{\gamma}_\beta(\mu, \zeta, \eta) \subset \text{USK}^{\gamma}_\beta(\mu, \zeta, \eta). \quad (38) \]

Combining both theorems 9 and 10, we have the following corollary.

**Corollary 14.** If \( (\mu + \zeta)/(\mu + 1) > -\min \{ \Re (y/k), \Re (\alpha/\beta) \} \), then

\[ \text{UCH}^{\gamma+1}_\beta(\mu, \zeta, \eta) \subset \text{UCH}^{\gamma}_\beta(\mu, \zeta, \eta) \subset \text{UCH}^{\gamma}_\beta(\mu, \zeta, \eta). \quad (39) \]

### 3. Integral Preserving Properties Associated with \( F_\delta \)

The generalized Libera integral operator \( F_\delta \) (see [14–16], also, see related topics [17–19]) is defined by

\[ F_\delta(f)(z) = \frac{\delta + 1}{z^\delta} \int_0^z \frac{\delta f(t)}{z^\delta} \, dt, \quad (40) \]

where \( f(z) \in \mathcal{A} \) and \( \delta > -1 \).

**Theorem 15.** Let \( \delta > -(\mu + \zeta)/(\mu + 1) \). If \( f \in \text{US}^{\gamma}_\beta(\mu, \zeta) \), then \( F_\delta(f) \in \text{US}^{\gamma}_\beta(\mu, \zeta) \).

**Proof.** Let \( f \in \text{US}^{\gamma}_\beta(\mu, \zeta) \) and set

\[ p(z) = \frac{z \left( H_{a,\beta}^{k,1} F_\delta(f)(z) \right)^\prime}{H_{a,\beta}^{k,1} F_\delta(f)(z)} \quad (z \in U), \quad (41) \]

where \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). From definition of \( H_{a,\beta}^{k,1}(f) \) and (40), we have

\[ z \left( H_{a,\beta}^{k,1} F_\delta(f)(z) \right)^\prime = (\delta + 1) H_{a,\beta}^{k,1} f(z) - \delta H_{a,\beta}^{k,1} F_\delta(f)(z). \quad (42) \]

Then, by using (41) and (42), we obtain

\[ (\delta + 1) \frac{H_{a,\beta}^{k,1} f(z)}{H_{a,\beta}^{k,1} F_\delta(f)(z)} = p(z) + \delta. \quad (43) \]

Taking the logarithmic differentiation on both sides of (43) and simple calculations, we have

\[ p(z) + \frac{zp'(z)}{p(z) + \delta} = \frac{z \left( H_{a,\beta}^{k,1} f(z) \right)^\prime}{H_{a,\beta}^{k,1} f(z)} < q_{\mu \zeta}(z). \quad (44) \]

Since \( \Re (q_{\mu \zeta} + \delta) > (\mu + \zeta)/(\mu + 1 + \delta) > 0 \), by virtue of Lemma 1, we conclude that \( p(z) < q_{\mu \zeta}(z) \) in \( U \), which implies that \( F_\delta(f) \in \text{US}^{\gamma}_\beta(\mu, \zeta) \).

**Theorem 16.** Let \( \delta > -(\mu + \zeta)/(\mu + 1) \). If \( f \in \text{UC}^{\gamma}_\beta(\mu, \zeta) \), then \( F_\delta(f) \in \text{UC}^{\gamma}_\beta(\mu, \zeta) \).

**Proof.** By applying Theorem 15, it follows that

\[ f(z) \in \text{UC}^{\gamma}_\beta(\mu, \zeta) \Rightarrow zf'(z) \in \text{US}^{\gamma}_\beta(\mu, \zeta) \]

\[ \Rightarrow F_\delta(zf'(z)) \in \text{US}^{\gamma}_\beta(\mu, \zeta) \]

\[ \Rightarrow z(F_\delta(f(z))^\prime \in \text{US}^{\gamma}_\beta(\mu, \zeta) \]

\[ \Rightarrow F_\delta(f(z)) \in \text{UC}^{\gamma}_\beta(\mu, \zeta), \quad (45) \]

which proves Theorem 16.

**Theorem 17.** Let \( \delta > -(\mu + \zeta)/(\mu + 1) \). If \( f \in \text{USK}^{\gamma}_\beta(\mu, \zeta, \eta) \), then \( F_\delta(f) \in \text{USK}^{\gamma}_\beta(\mu, \zeta, \eta) \).
Proof. Let \( f(z) \in USK^\gamma_{\beta}(\mu, \zeta, \eta) \). Then, there exists a function \( h(z) \in US^\gamma_{\beta}(\mu, \zeta) \) such that

\[
\frac{z\left(\frac{H^{\gamma}_{a,\beta}f(z)}{H^{\gamma}_{a,\beta}h(z)}\right)'}{H^{\gamma}_{a,\beta}h(z)} < q_{\mu,\zeta}(z). \tag{46}
\]

Thus, we set

\[
p(z) = \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)} \quad (z \in \mathbb{U}), \tag{47}
\]

where \( p(z) \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \). Since \( h(z) \in US^\gamma_{\beta}(\mu, \zeta) \), we see from Theorem 15 that \( F_{\delta}(h) \in US^\gamma_{\beta}(\mu, \zeta) \). Let

\[
t(z) = \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}, \tag{48}
\]

where \( t(z) \) is analytic in \( \mathbb{U} \) with \( \Re\{t(z)\} > (\mu + \zeta)/(\mu + 1) \). Using (47), we have

\[
H^{\gamma}_{a,\beta}zF_{\delta}'(f)(z) = \left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)p(z). \tag{49}
\]

Differentiating both sides of (49) with respect to \( z \) and simple calculations, we obtain

\[
z\left(\frac{H^{\gamma}_{a,\beta}zF_{\delta}'(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right) = \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}p(z) + zp'(z) = t(z)p(z) + zp'(z). \tag{50}
\]

Now, using the identity (42) and (50), we obtain

\[
z\left(\frac{H^{\gamma}_{a,\beta}zF_{\delta}'(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)' = \frac{z\left(\frac{H^{\gamma}_{a,\beta}zF_{\delta}'(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)} + \delta \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)} + \delta \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)} + \delta \frac{z\left(\frac{H^{\gamma}_{a,\beta}F_{\delta}(f)(z)}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)}\right)'}{H^{\gamma}_{a,\beta}F_{\delta}(h)(z)} \tag{51}
\]

Since \( \delta > -(\mu + \zeta)/(\mu + 1) \) and \( \Re\{t(z)\} > (\mu + \zeta)/(\mu + 1) \), we see that

\[
\Re\{t(z) + \delta\} > 0 \quad (z \in \mathbb{U}). \tag{52}
\]

Applying Lemma 2 into relation (51), it follows that \( p(z) < q_{\mu,\zeta}(z) \), which is \( F_{\delta}(f) \in USK^\gamma_{\beta}(\mu, \zeta, \eta) \).

We can deduce the integral-preserving property asserted by 18 by using Theorem 17 and relation (17).

**Theorem 18.** Let \( \delta > -(\mu + \zeta)/(\mu + 1) \). If \( f \in UCK^\gamma_{\beta}(\mu, \zeta, \eta) \), then \( F_{\delta}(f) \in UCK^\gamma_{\beta}(\mu, \zeta, \eta) \).

**Data Availability**

All data are available in this paper.

**Conflicts of Interest**

The authors declare no conflict of interest.

**Authors’ Contributions**

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

**Acknowledgments**

This research has been funded by Scientific Research Deanship at the University of Ha’il, Saudi Arabia, through project number RG-20020.

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