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Research Article

On Some Relationships of Certain $K-$ Uniformly Analytic Functions Associated with Mittag-Leffler Function

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In this paper, we introduce and investigate several inclusion relationships of new k -uniformly classes of analytic functions defined by the Mittag-Leffler function. Also, integral-preserving properties of these classes associated with the certain integral operator are also obtained.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ which in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

For $f(z)$ and $g(z) \in \mathcal{A}$, we say that the function $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \text{ or } f(z) \prec g(z), \quad (2)$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [1, 2]; see also [3]):

$$f(z) \prec g(z) \Leftrightarrow f(0) \prec g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \quad (3)$$

Let f be as in (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then Hadamard product (or convolution) of $f(z)$ and $h(z)$ is given by

$$(f * h)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \, (z \in U). \quad (4)$$

For $\zeta, \eta \in [0, 1)$, we denote by $S^*(\zeta)$, $C(\zeta)$, $K(\zeta, \eta)$, and $K^*(\zeta, \eta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order ζ, convex of order ζ, close-to-convex of order ζ and type η, and quasiconvex of order ζ and type η in $\mathbb{U}$.

Also, let the subclasses $US(\mu, \zeta)$, $UC(\mu, \zeta)$, $USK(\mu, \zeta, \eta)$, and $UCK(\mu, \zeta, \eta)$ of $\mathcal{A}$ $(\eta \in 0, 1) < 1\,;\, \mu \ge 0)$ be defined as follows:

$$US(\mu, \zeta) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)} - \zeta \right) > \mu \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\},$$

$$UC(\mu, \zeta) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \zeta \right) > \mu \left| \frac{zf''(z)}{f'(z)} \right| \right\},$$

$$USK(\mu,\zeta,\eta)=\left\{f\in\mathcal{A}:\exists h\in US(\mu,\zeta)\right.$$
$$\left. s.t.\,\mathfrak{R}\left(\frac{zf'(z)}{h(z)}-\zeta\right)>\mu\left|\frac{zf'(z)}{h(z)}-1\right|\right\}$$

$$UCK(\mu,\zeta,\eta)=\left\{f\in\mathcal{A}:\exists h\in UC(\mu,\zeta)\right.$$
$$\left. s.t.\,\mathfrak{R}\left(\frac{\left(zf'(z)\right)'}{h'(z)}-\zeta\right)>\mu\left|\frac{\left(zf'(z)\right)'}{h'(z)}-1\right|\right\}. \tag{5}$$

We note that

$$US(0,\zeta)=S^*(\zeta),\ UC(0,\zeta)=C(\zeta),$$
$$USK(0,\zeta,\eta)=K(\zeta,\eta)\text{ and }UCK(0,\zeta,\eta)$$
$$=K^*(\zeta,\eta)\ (0\le\zeta\,;\eta<1). \tag{6}$$

Moreover, let $q_{\mu,\zeta}(z)$ be an analytic function which maps $\mathbb{U}$ onto the conic domain $\Phi_{\mu,\zeta}=\{u+iv:u>k\sqrt{(u-1)^2+v^2}+\zeta\}$ such that $1\in\Phi_{\mu,\zeta}$ defined as follows:

$$q_{\mu,\zeta}(z)=\begin{cases}\dfrac{1+(1-2\zeta)z}{1-z}\ (\mu=0),\\[2mm] \dfrac{1-\zeta}{1-\mu^2}\cos\left\{\dfrac{2}{\pi}\left(\cos^{-1}\mu\right)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\dfrac{\mu^2-\zeta}{1-\mu^2}\ (0<\mu<1),\\[2mm] 1+\dfrac{2(1-\zeta)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\ (\mu=1),\\[2mm] \dfrac{1-\zeta}{\mu^2-1}\sin\left\{\dfrac{\pi}{2\varsigma(\mu)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\mu}}}\dfrac{dt}{\sqrt{1-t^2\sqrt{1-\mu^2t^2}}}\right\}+\dfrac{\mu^2-\zeta}{\mu^2-1}\ (\mu>1),\end{cases} \tag{7}$$

where $u(z)=(z-\sqrt{\mu})/(1-\sqrt{\mu}z)$ and $\varsigma(\mu)$ is such that $\mu=\cosh\left(\pi\varsigma'(z)/4\varsigma(z)\right)$. By virtue of properties of the conic domain $\Phi_{\mu,\zeta}$ (cf., e.g., [4, 5]), we have

$$\mathfrak{R}\left\{q_{\mu,\zeta}(z)\right\}>\frac{\mu+\zeta}{\mu+1}. \tag{8}$$

Making use of the principal of subordination and the definition of $q_{\mu,\zeta}(z)$, we may rewrite the subclasses $US(\mu,\zeta)$, $UC(\mu,\zeta)$, $USK(\mu,\zeta,\eta)$, and $UCK(\mu,\zeta,\eta)$ as follows:

$$US(\mu,\zeta)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec q_{\mu,\zeta}(z)\right\},$$
$$UC(\mu,\zeta)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec q_{\mu,\zeta}(z)\right\},$$
$$USK(\mu,\zeta,\eta)=\left\{f\in\mathcal{A}:\exists h\in US(\mu,\eta)\ s.t.\frac{zf'(z)}{h(z)}\prec q_{\mu,\zeta}(z)\right\} \tag{9}$$

and

$$UCK(\mu,\zeta,\eta)=\left\{f\in\mathcal{A}:\exists h\in UC(\mu,\zeta)\right.$$
$$\left. s.t.\frac{\left(zf'(z)\right)'}{h'(z)}\prec q_{\mu,\zeta}(z)\right\}. \tag{10}$$

Attiya [6] introduced the operator $H^{\gamma,k}_{\alpha,\beta}(f)$, where $H^{\gamma,k}_{\alpha,\beta}(f)$: $\mathcal{A}\longrightarrow\mathcal{A}$ is defined by

$$H^{\gamma,k}_{\alpha,\beta}(f)=\mu^{\gamma,k}_{\alpha,\beta}*f(z)\ (z\in\mathbb{U}), \tag{11}$$

with $\beta,\gamma\in\mathbb{C}$, $\mathrm{Re}\,(\alpha)>\max\{0,\mathrm{Re}\,(k)-1\}$ and $\mathrm{Re}\,(k)>0$. Also, $\mathrm{Re}\,(\alpha)=0$ when $\mathrm{Re}\,(k)=1$; $\beta\neq0$. Here, $\mu^{\gamma,k}_{\alpha,\beta}$ is the generalized Mittag–Leffler function defined by [7], see also [6], and the symbol $(*)$ denotes the Hadamard product.

Due to the importance of the Mittag–Leffler function, it is involved in many problems in natural and applied science. A detailed investigation of the Mittag–Leffler function has been studied by many authors (see, e.g., [7–12]).

Attiya [6] noted that

$$H_{\alpha,\beta}^{\gamma,k}(f)(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!}a_nz^n. \quad (12)$$

Also, Attiya [6] showed that

$$z\left(H_{\alpha,\beta}^{\gamma,k}(f(z))\right)'=\left(\frac{\gamma+k}{k}\right)\left(H_{\alpha,\beta}^{\gamma+1,k}f(z)\right)-\frac{\gamma}{k}\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right), \quad (13)$$

and

$$z\left(H_{\alpha,\beta+1}^{\gamma,k}(f(z))\right)'=\left(\frac{\alpha+\beta}{\alpha}\right)\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right)-\frac{\beta}{\alpha}\left(H_{\alpha,\beta+1}^{\gamma,k}f(z)\right). \quad (14)$$

Next, by using the operator $H_{\alpha,\beta}^{\gamma,k}(f)$, we introduce the following subclasses of analytic functions in $\mathbb{U}$

$$\begin{aligned}US_{\beta}^{\gamma}(\mu,\zeta)&=\left\{f\in\mathscr{A}:H_{\alpha,\beta}^{\gamma,k}f(z)\in US(\mu,\zeta)\right\},\\UC_{\beta}^{\gamma}(\mu,\zeta)&=\left\{f\in\mathscr{A}:H_{\alpha,\beta}^{\gamma,k}f(z)\in UC(\mu,\zeta)\right\},\\USK_{\beta}^{\gamma}(\mu,\zeta,\eta)&=\left\{f\in\mathscr{A}:H_{\alpha,\beta}^{\gamma,k}f(z)\in USK(\mu,\zeta,\eta)\right\},\\UCK_{\beta}^{\gamma}(\mu,\zeta,\eta)&=\left\{f\in\mathscr{A}:H_{\alpha,\beta}^{\gamma,k}f(z)\in UCK(\mu,\zeta,\eta)\right\},\end{aligned} \quad (15)$$

where $\beta,\gamma\in\mathbb{C}$, $\Re(\alpha)>\max\{0,\Re(k)-1\}$ and $\Re(k)>0$. Also, $\Re(\alpha)=0$ when $\Re(k)=1$; $\beta\neq0$.

Also, we note that

$$f(z)\in UC_{\beta}^{\gamma}(\mu,\zeta)\Leftrightarrow zf'(z)\in US_{\beta}^{\gamma}(\mu,\zeta), \quad (16)$$

$$f(z)\in UCK_{\beta}^{\gamma}(\mu,\zeta,\eta)\Leftrightarrow zf'(z)\in USK_{\beta}^{\gamma}(\mu,\zeta,\eta). \quad (17)$$

In this paper, we introduce several inclusion properties of the classes $US_{\beta}^{\gamma}(\mu,\zeta)$, $UC_{\beta}^{\gamma}(\mu,\zeta)$, $USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$, and $UCK_{\beta}^{\gamma}(\mu,\zeta,\eta)$. Also, integral-preserving properties of these classes associated with generalized Libera integral operator are also obtained.

2. Inclusion Properties Associated with $H_{\alpha,\beta}^{\gamma,k}f(z)$

Lemma 1 (see [13]). *If $h(z)$ is convex univalent in $\mathbb{U}$ with $h(0)=1$ and $\Re\{\xi h(z)+\zeta\}>0(\zeta\in\mathbb{C})$. Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0)=1$ which satisfy the following subordination relation*

$$p(z)+\frac{zp'(z)}{\xi p(z)+\zeta}\prec h(z), \quad (18)$$

then

$$p(z)\prec h(z). \quad (19)$$

Lemma 2 (see [2]). *If $h(z)$ is convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $\Re\{w(z)\}\geq0$. Let $p(z)$ be analytic in $\mathbb{U}$ and $p(0)=h(0)$ which satisfy the following subordination relation*

$$p(z)+w(z)zp'(z)\prec h(z), \quad (20)$$

then

$$p(z)\prec h(z). \quad (21)$$

Theorem 3. *If $\Re(\gamma/k)>-(\mu+\zeta)/(\mu+1)$, then $US_{\beta}^{\gamma+1}(\mu,\zeta)\subset US_{\beta}^{\gamma}(\mu,\zeta)$.*

Proof. Let $f(z)\in US_{\beta}^{\gamma+1}(\mu,\zeta)$, put

$$p(z)=\frac{z\left(H_{\alpha,\beta}^{\gamma,k}f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k}f(z)}\ (z\in\mathbb{U}), \quad (22)$$

we note that $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. From (13) and (22), we have

$$\frac{H_{\alpha,\beta}^{\gamma+1,k}f(z)}{H_{\alpha,\beta}^{\gamma,k}f(z)}=\frac{k}{\gamma+k}\left(p(z)+\frac{\gamma}{k}\right). \quad (23)$$

Differentiating (23) with respect to z, we obtain

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{H_{\alpha,\beta}^{\gamma+1,k}f(z)}=p(z)+\frac{zp'(z)}{p(z)+(\gamma/k)}. \quad (24)$$

From the above relation and using (7), we may write

$$p(z)+\frac{zp'(z)}{p(z)+(\gamma/k)}\prec q_{\mu,\zeta}(z)\ (z\in\mathbb{U}). \quad (25)$$

Since $\Re\{q_{\mu,\zeta}(z)\}>(\mu+\zeta)/(\mu+1)$, we see that

$$\Re\left(q_{\mu,\zeta}(z)+\frac{\gamma}{k}\right)>0\ (z\in\mathbb{U}). \quad (26)$$

Applying Lemma 1, it follows that $p(z)\prec q_{\mu,\zeta}(z)$, that is, $f(z)\in US_{\beta}^{\gamma}(\mu,\zeta)$.

Using the same technique in Theorem 3 with relation (14), we have the following theorem.

Theorem 4. *If $\Re(\alpha/\beta)>-(\mu+\zeta)/(\mu+1)$, then $US_{\beta}^{\gamma}(\mu,\zeta)\subset US_{\beta+1}^{\gamma}(\mu,\zeta)$.*

Theorem 5. *If* $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, *then* $UC_{\beta}^{\gamma+1}(\mu,\zeta) \subset UC_{\beta}^{\gamma}(\mu,\zeta)$.

Proof. Applying Theorem 3 and relation (16), we observe that

$$\begin{aligned} f(z) \in UC_{\beta}^{\gamma+1}(\mu,\zeta) &\Leftrightarrow zf'(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta) \\ &\Rightarrow zf'(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \Leftrightarrow f(z) \in UC_{\beta}^{\gamma}(\mu,\zeta), \end{aligned} \tag{27}$$

which evidently proves Theorem 5.

Similarly, we can prove the following theorem.

Theorem 6. *If* $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, *then* $UC_{\beta}^{\gamma}(\mu,\zeta) \subset UC_{\beta+1}^{\gamma}(\mu,\zeta)$.

Theorem 7. *If* $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, *then* $USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.

Proof. Let $f(z) \in USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta)$. Then, there exists a function $r(z) \in US(\mu,\zeta)$ such that

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{r(z)} \prec q_{\mu,\zeta}(z). \tag{28}$$

We can choose the function $h(z)$ such that $H_{\alpha,\beta}^{\gamma+1,k} h(z) = r(z)$. Then, $h(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta)$ and

$$\frac{z\left(H_{\alpha,\beta}^{\gamma+1,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma+1,k} h(z)} \prec q_{\mu,\zeta}(z). \tag{29}$$

Now, let

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)}, \tag{30}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=1$. Since $h(z) \in US_{\beta}^{\gamma+1}(\mu,\zeta)$, by Theorem 3, we know that $h(z) \in US_{\beta}^{\gamma}(\mu,\zeta)$. Let

$$t(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} \quad (z \in \mathbb{U}), \tag{31}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > (\mu+\zeta)/(\mu+1)$. Also, from(30), we note that

$$z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)' = H_{\alpha,\beta}^{\gamma,k} zf'(z) = \left(H_{\alpha,\beta}^{\gamma,k} h(z)\right) p(z). \tag{32}$$

Differentiating both sides of (32) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} p(z) + zp'(z) \\ &= t(z)p(z) + zp'(z). \end{aligned} \tag{33}$$

Now, using (13) and (33), we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} = \frac{H_{\alpha,\beta}^{\gamma+1,k} zf'(z)}{H_{\alpha,\beta}^{\gamma+1,k} h(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)' + (\gamma/k) H_{\alpha,\beta}^{\gamma,k} zf'(z)}{z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)' + (\gamma/k) H_{\alpha,\beta}^{\gamma,k} h(z)} \\ &= \frac{\left(z\left(H_{\alpha,\beta}^{\gamma,k} zf'(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right) + (\gamma/k)\left(z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right)}{\left(z\left(H_{\alpha,\beta}^{\gamma,k} h(z)\right)'/H_{\alpha,\beta}^{\gamma,k} h(z)\right) + (\gamma/k)} \\ &= \frac{t(z)p(z) + zp'(z) + (\gamma/k)p(z)}{t(z) + (\gamma/k)} = p(z) + \frac{zp'(z)}{t(z) + (\gamma/k)}. \end{aligned} \tag{34}$$

Since $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, we see that

$$\Re\left\{t(z) + \frac{\gamma}{k}\right\} > 0 \; (z \in \mathbb{U}). \tag{34}$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{\mu,\zeta}(z)$, so that $f(z) \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$. This completes the proof of Theorem 7.

Similarly, we can prove the following theorem.

Theorem 8. *If* $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, *then* $USK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset USK_{\beta+1}^{\gamma}(\mu,\zeta,\eta)$.

We can also prove Theorem 9 by using Theorem 7 and relation (17).

Theorem 9. *If* $\Re(\gamma/k) > -(\mu+\zeta)/(\mu+1)$, *then* $UCK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset UCK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.

Also, we obtain the following theorem.

Theorem 10. *If* $\Re(\alpha/\beta) > -(\mu+\zeta)/(\mu+1)$, *then* $UCK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset UCK_{\beta+1}^{\gamma}(\mu,\zeta,\eta)$.

Now, we obtain squeeze theorems for inclusion by combining the above theorems as follows:

Combining both theorems 3 and 4, we have the following corollary.

Corollary 11. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k), \Re(\alpha/\beta)\}$, *then*

$$US_{\beta}^{\gamma+1}(\mu,\zeta) \subset US_{\beta}^{\gamma}(\mu,\zeta) \subset US_{\beta+1}^{\gamma}(\mu,\zeta). \quad (36)$$

Combining both theorems 5 and 6, we have the following corollary.

Corollary 12. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k), \Re(\alpha/\beta)\}$, *then*

$$UC_{\beta}^{\gamma+1}(\mu,\zeta) \subset UC_{\beta}^{\gamma}(\mu,\zeta) \subset UC_{\beta+1}^{\gamma}(\mu,\zeta). \quad (37)$$

Combining both theorems 7 and 8, we have the following corollary.

Corollary 13. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k), \Re(\alpha/\beta)\}$, *then*

$$USK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset USK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset USK_{\beta+1}^{\gamma}(\mu,\zeta,\eta). \quad (38)$$

Combining both theorems 9 and 10, we have the following corollary.

Corollary 14. *If* $(\mu+\zeta)/(\mu+1) > -\min\{\Re(\gamma/k), \Re(\alpha/\beta)\}$, *then*

$$UCK_{\beta}^{\gamma+1}(\mu,\zeta,\eta) \subset UCK_{\beta}^{\gamma}(\mu,\zeta,\eta) \subset UCK_{\beta+1}^{\gamma}(\mu,\zeta,\eta). \quad (39)$$

3. Integral Preserving Properties Associated with F_δ

The generalized Libera integral operator F_δ (see [14–16], also, see related topics [17–19]) is defined by

$$F_\delta(f)(z) = \frac{\delta+1}{z^\delta}\int_0^z t^{\delta-1} f(t)\,dt, \quad (40)$$

where $f(z) \in \mathscr{A}$ and $\delta > -1$.

Theorem 15. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$. *If* $f \in US_{\beta}^{\gamma}(\mu,\zeta)$, *then* $F_\delta(f) \in US_{\beta}^{\gamma}(\mu,\zeta)$.

Proof. Let $f \in US_{\beta}^{\gamma}(\mu,\zeta)$ and set

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)} \quad (z \in \mathbb{U}), \quad (41)$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=1$. From definition of $H_{\alpha,\beta}^{\gamma,k}(f)$ and (40), we have

$$z\left(H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)\right)' = (\delta+1) H_{\alpha,\beta}^{\gamma,k} f(z) - \delta H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z). \quad (42)$$

Then, by using (41) and (42), we obtain

$$(\delta+1)\frac{H_{\alpha,\beta}^{\gamma,k} f(z)}{H_{\alpha,\beta}^{\gamma,k} F_\delta(f)(z)} = p(z) + \delta. \quad (43)$$

Taking the logarithmic differentiation on both sides of (43) and simple calculations, we have

$$p(z) + \frac{zp'(z)}{p(z)+\delta} = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} f(z)} \prec q_{\mu,\zeta}(z). \quad (44)$$

Since $\Re(q_{\mu,\zeta}+\delta) > ((\mu+\zeta)/(\mu+1)+\delta) > 0$, by virtue of Lemma 1, we conclude that $p(z) \prec q_{\mu,\zeta}(z)$ in $\mathbb{U}$, which implies that $F_\delta(f) \in US_{\beta}^{\gamma}(\mu,\zeta)$.

Theorem 16. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$. *If* $f \in UC_{\beta}^{\gamma}(\mu,\zeta)$, *then* $F_\delta(f) \in UC_{\beta}^{\gamma}(\mu,\zeta)$.

Proof. By applying Theorem 15, it follows that

$$\begin{aligned} f(z) \in UC_{\beta}^{\gamma}(\mu,\zeta) &\Leftrightarrow zf'(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Rightarrow F_\delta\left(zf'\right)(z) \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Leftrightarrow z(F_\delta(f)(z))' \in US_{\beta}^{\gamma}(\mu,\zeta) \\ &\Leftrightarrow F_\delta(f)(z) \in UC_{\beta}^{\gamma}(\mu,\zeta), \end{aligned} \quad (45)$$

which proves Theorem 16.

Theorem 17. *Let* $\delta > -(\mu+\zeta)/(\mu+1)$. *If* $f \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$, *then* $F_\delta(f) \in USK_{\beta}^{\gamma}(\mu,\zeta,\eta)$.

Proof. Let $f(z) \in USK_{\beta}^{\gamma}(\mu, \zeta, \eta)$. Then, there exists a function $h(z) \in US_{\beta}^{\gamma}(\mu, \zeta)$ such that

$$\frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} \prec q_{\mu,\zeta}(z). \quad (46)$$

Thus, we set

$$p(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} \ (z \in \mathbb{U}), \quad (47)$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Since $h(z) \in US_{\beta}^{\gamma}(\mu, \zeta)$, we see from Theorem 15 that $F_{\delta}(h) \in US_{\beta}^{\gamma}(\mu, \zeta)$. Let

$$t(z) = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)}, \quad (48)$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > (\mu + \zeta)/(\mu + 1)$. Using (47), we have

$$H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z) = \left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) p(z). \quad (49)$$

Differentiating both sides of (49) with respect to z and simple calculations, we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} &= \frac{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} p(z) + zp'(z) \\ &= t(z)p(z) + zp'(z). \end{aligned} \quad (50)$$

Now, using the identity (42) and (50), we obtain

$$\begin{aligned} \frac{z\left(H_{\alpha,\beta}^{\gamma,k} f(z)\right)'}{H_{\alpha,\beta}^{\gamma,k} h(z)} &= \frac{H_{\alpha,\beta}^{\gamma,k} z f'(z)}{H_{\alpha,\beta}^{\gamma,k} h(z)} = \frac{z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)' + \delta H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)}{z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)' + \delta H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)} \\ &= \frac{\left(z\left(H_{\alpha,\beta}^{\gamma,k} z F_{\delta}{}'(f)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) + \delta\left(z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(f)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)}{\left(z\left(H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right)' / H_{\alpha,\beta}^{\gamma,k} F_{\delta}(h)(z)\right) + \delta} \\ &= \frac{t(z)p(z) + zp'(z) + \delta p(z)}{t(z) + \delta} = p(z) + \frac{zp'(z)}{t(z) + \delta}. \end{aligned} \quad (51)$$

Since $\delta > -(\mu + \zeta)/(\mu + 1)$ and $\Re\{t(z)\} > (\mu + \zeta)/(\mu + 1)$, we see that

$$\Re\{t(z) + \delta\} > 0 \ (z \in \mathbb{U}). \quad (52)$$

Applying Lemma 2 into relation (51), it follows that $p(z) \prec q_{\mu,\zeta}(z)$, which is $F_{\delta}(f) \in USK_{\beta}^{\gamma}(\mu, \zeta, \eta)$.

We can deduce the integral-preserving property asserted by 18 by using Theorem 17 and relation (17).

Theorem 18. *Let $\delta > (-\mu + \zeta)/(\mu + 1)$. If $f \in UCK_{\beta}^{\gamma}(\mu, \zeta, \eta)$, then $F_{\delta}(f) \in UCK_{\beta}^{\gamma}(\mu, \zeta, \eta)$.*

Data Availability

All data are available in this paper.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to the writing of this paper. All authors approved the final version of the manuscript.

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