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Research Article

Higher-Order Riesz Transforms in the Inverse Gaussian Setting and UMD Banach Spaces

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In this paper, we study higher-order Riesz transforms associated with the inverse Gaussian measure given by $\pi^{n/2}e^{|x|^2}dx$ on $\mathbb{R}^n$. We establish $L^p(\mathbb{R}^n, e^{|x|^2}dx)$-boundedness properties and obtain representations as principal values singular integrals for the higher-order Riesz transforms. New characterizations of the Banach spaces having the UMD property by means of the Riesz transforms and imaginary powers of the operator involved in the inverse Gaussian setting are given.

1. Introduction

Our setting is $\mathbb{R}^n$ endowed with the measure γ_{-1} whose density with respect to the Lebesgue measure is $\pi^{n/2}e^{|x|^2}$, $x \in \mathbb{R}^n$. The measure γ_{-1} is called the inverse Gaussian measure. The study of harmonic analysis operators in $(\mathbb{R}^n, \gamma_{-1})$ was began by Salogni [1]. The principal motivation for Salogni's studies was the connection with the Gaussian setting. However, as Bruno and Sjögren [2] pointed out, $(\mathbb{R}^n, \gamma_{-1})$ can be seen as a model of a variety of settings where a theory of singular integrals has not been developed. Also, the natural Laplacian on $(\mathbb{R}^n, \gamma_{-1})$, that we will denote by $\mathscr{A}$, can be interpreted as a restriction of the Laplace-Beltrami operator associated with a warped-product manifold whose Ricci tensor is unbounded from below. A complete exposition of the theory of this kind of manifolds can be found in [3].

The aim of this paper is to study $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness properties of higher-order Riesz transforms in the inverse Gaussian setting. Also, we characterize the UMD Banach spaces by using these Riesz transforms.

We consider the second-order differential operator $\mathscr{A}_0$ defined by

$$\mathscr{A}_0 f(x) = -\frac{1}{2}\Delta f(x) - x \cdot \nabla f(x), x \in \mathbb{R}^n, \quad (1)$$

where $f \in C_c^\infty(\mathbb{R}^n)$, the space of the smooth functions with compact support in $\mathbb{R}^n$. Here, Δ and ∇ denote the usual Euclidean Laplacian and gradient, respectively.

$\mathscr{A}_0$ is essentially selfadjoint in $L^2(\mathbb{R}^n, \gamma_{-1})$. $\mathscr{A}$ denotes the closure of $\mathscr{A}_0$ in $L^2(\mathbb{R}^n, \gamma_{-1})$.

For every $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$ by H_k, we represent the k-th Hermite polynomial given by $H_k(x) = \prod_{i=1}^n H_{k_i}(x_i)$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, where for every $m \in \mathbb{N}$,

$$H_m(z) = (-1)^m e^{z^2} \frac{d^m}{dz^m} e^{-z^2}, z \in \mathbb{R}. \quad (2)$$

We have that, for every $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$,

$$\mathscr{A}\tilde{H}_k = (|k| + n)\tilde{H}_k, \quad (3)$$

where $|k| = k_1 + k_2 + \cdots + k_n$ and $\tilde{H}_k(x) = e^{-|x|^2}H_k(x)$, $x \in \mathbb{R}^n$. The spectrum of $\mathscr{A}$ in $L^2(\mathbb{R}^n, \gamma_{-1})$ is the discrete set $\{n + m\}_{m\in\mathbb{N}}$.

The operator $-\mathscr{A}$ generates a diffusion semigroup (in the Stein sense [4]) $\{T_t^{\mathscr{A}}\}_{t>0}$ in $(\mathbb{R}^n, \gamma_{-1})$ where for every $t > 0$, we have that

$$T_t^{\mathcal{A}}(f)(x) = \int_{\mathbb{R}^n} T_t^{\mathcal{A}}(x,y) f(y) dy, x \in \mathbb{R}^n, \quad (4)$$

for every $f \in L^p(\mathbb{R}^n, \gamma_{-1})$, $1 \le p < \infty$, and being

$$T_t^{\mathcal{A}}(x,y) = \frac{e^{-nt}}{\pi^{n/2}(1-e^{-2t})^{n/2}} \exp\left(-\frac{|x-e^{-t}y|^2}{1-e^{-2t}}\right), x, y \in \mathbb{R}^n, t > 0. \quad (5)$$

The maximal operator $T_*^{\mathcal{A}}$ defined by

$$T_*^{\mathcal{A}} f = \sup_{t>0} \left| T_t^{\mathcal{A}} f \right|, \quad (6)$$

was studied by Salogni ([1]). She proved that $T_*^{\mathcal{A}}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. From the general results in [4], it can be deduced that $T_*^{\mathcal{A}}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 < p < \infty$. Recently, Betancor et al. [5] have characterized the Köethe function spaces with the Hardy-Littlewood property by using the maximal operators

$$T_{*,k}^{\mathcal{A}} f = \sup_{t>0} \left| t^k \partial_t^k T_t f \right|, k \in \mathbb{N}. \quad (7)$$

In [1], $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness properties with $1 < p < \infty$ for some spectral multipliers associated with the operator $\mathcal{A}$ were proved. The imaginary power $\mathcal{A}^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, of $\mathcal{A}$ is a special case of the multipliers studied in [1]. Bruno ([6]) established endpoints results for $\mathcal{A}^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, proving that $\mathcal{A}^{i\sigma}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. Also, he showed that, for $\lambda \ge 1$, the shifted first-order Riesz transform $\nabla(\mathcal{A} + \lambda I)^{-1/2}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. These operators are studied on new Hardy type H^1-spaces.

Higher-order Riesz transforms associated with the operator $\mathcal{A}$ were studied by Bruno and Sjögren [2]. For every $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \setminus \{0\}$, the α-th Riesz transform is defined by $R_\alpha = \partial^\alpha \mathcal{A}^{-|\alpha|/2}$, where $\partial^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. In ([2], Theorem 1.1), it was established that R_α is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$ if and only if $|\alpha| \le 2$.

In ([6], Remark 2.6), Bruno proved that, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = 1$, R_α is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 < p < \infty$. In [2], Bruno and Sjögren say that they do not know whether R_α is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself for every $1 < p < \infty$ and $\alpha \in \mathbb{N}^n$, $|\alpha| > 1$, though they expect so. In our first result, we prove that, as they expected, R_α is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself when $1 < p < \infty$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$. We also obtain a representation of R_α as a principal value singular integral. In order to prove our result, we need to use some properties of the negative power $\mathcal{A}^{-\beta}$, $\beta > 0$, of $\mathcal{A}$. In Section 2, we analyze $\mathcal{A}^{-\beta}$, $\beta > 0$. We obtain that, for every $\beta > 0$, the operator $\mathcal{A}^{-\beta}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. This result contrasts with the one in ([7], Proposition 6.2) where it is proved that $\mathcal{L}^{-\beta}$, $\beta > 0$, is not bounded from $L^1(\mathbb{R}^n, \gamma)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma)$, where $\mathcal{L}$ represents the Ornstein-Uhlenbeck operator and γ denotes the Gaussian measure ($d\gamma(x) = \pi^{-n/2} e^{-|x|^2} dx$) on $\mathbb{R}^n$.

Theorem 1. *Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \setminus \{0\}$. For every $f \in C_c^\infty(\mathbb{R}^n)$, the derivative $\partial^\alpha \mathcal{A}^{-|\alpha|/2}(f)(x)$ exists for almost all $x \in \mathbb{R}^n$ and there exists $c_\alpha \in \mathbb{R}$ such that*

$$\partial_x^\alpha \mathcal{A}^{-|\alpha|/2}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_\alpha(x,y) f(y) dy + c_\alpha f(x), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \quad (8)$$

where $c_\alpha = 0$ if α_i is odd for some $i = 1, \cdots, n$.

Furthermore, when $n = 1$ and α is even, the last integral is actually absolutely convergent for every $x \in \mathbb{R}$, and in this case, no principal value is needed. Here,

$$R_\alpha(x,y) = \frac{1}{\Gamma(|\alpha|/2)} \int_0^\infty \partial_x^\alpha T_t^{\mathcal{A}}(x,y) t^{|\alpha|/2-1} dt, x, y \in \mathbb{R}^n, x \ne y. \quad (9)$$

Let $\alpha \in \mathbb{N}^n$. Since, for every $\ell \in \mathbb{N}$,

$$\frac{d}{dz} \tilde{H}_\ell(z) = -\tilde{H}_{\ell+1}(z), z \in \mathbb{R}, \quad (10)$$

we have that, for every $k \in \mathbb{N}^n$,

$$\partial_x^\alpha \mathcal{A}^{-|\alpha|/2} \left(\tilde{H}_k\right)(x) = \frac{(-1)^{|\alpha|}}{(|k|+n)^{|\alpha|/2}} \tilde{H}_{k+\alpha}(x), x \in \mathbb{R}^n. \quad (11)$$

Let $f \in L^2(\mathbb{R}^n, \gamma_{-1})$. We can write $f = \sum_{k \in \mathbb{N}^n} c_k(f) \tilde{H}_k$ in $L^2(\mathbb{R}^n, \gamma_{-1})$, where for every $k \in \mathbb{N}^n$,

$$c_k(f) = \frac{\pi^{n/2}}{\left\|\tilde{H}_k\right\|_{L^2(\mathbb{R}^n, \gamma_{-1})}^2} \int_{\mathbb{R}^n} f(y) \tilde{H}_k(y) e^{|y|^2} dy. \quad (12)$$

We define

$$R_\alpha f = (-1)^{|\alpha|} \sum_{k \in \mathbb{N}^n} \frac{c_k(f)}{(|k|+n)^{|\alpha|/2}} \tilde{H}_{k+\alpha}. \quad (13)$$

For every $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$,

$$\left\|\tilde{H}_{k+\alpha}\right\|_{L^2(\mathbb{R}^n, \gamma_{-1})}^2 = \int_{\mathbb{R}^n} (H_{k+\alpha}(y))^2 e^{-|y|^2} dy = \pi^n 2^{|\alpha|+|k|} \prod_{i=1}^n \Gamma(k_i + \alpha_i + 1). \quad (14)$$

Then

$$\|R_\alpha f\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})} = \sum_{k\in\mathbb{N}^n} \frac{|c_k(f)|^2 \|\tilde{H}_{k+\alpha}\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})}}{(|k|+n)^{|\alpha|}} = \sum_{k\in\mathbb{N}^n} \left(c_k(f)\|\tilde{H}_k\|_{L^2(\mathbb{R}^n,\gamma_{-1})}\right)^2 \frac{\|\tilde{H}_{k+\alpha}\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})}}{(|k|+n)^{|\alpha|}\|\tilde{H}_k\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})}} \le C\sum_{k\in\mathbb{N}^n}\left(c_k(f)\|\tilde{H}_k\|_{L^2(\mathbb{R}^n,\gamma_{-1})}\right)^2 = C\|f\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})}, \tag{15}$$

where the constant C depends on n and α. Hence, R_α is bounded from $L^2(\mathbb{R}^n,\gamma_{-1})$ into itself.

If $f\in C_c^\infty(\mathbb{R}^n)$, then $R_\alpha(f)(x) = \partial_x^\alpha \mathscr{A}^{-|\alpha|/2}(f)(x)$, $x\in\mathbb{R}^n$.

Theorem 2. *Let $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\setminus\{0\}$ and $1<p<\infty$. The Riesz transform R_α can be extended from $L^2(\mathbb{R}^n,\gamma_{-1})\cap L^p(\mathbb{R}^n,\gamma_{-1})$ to $L^p(\mathbb{R}^n,\gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself. By denoting again R_α to this extension, we have that, there exists $c_\alpha\in\mathbb{R}$ such that, for every $f\in L^p(\mathbb{R}^n,\gamma_{-1})$,*

$$R_\alpha(f)(x) = \lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_\alpha(x,y)f(y)dy + c_\alpha f(x), \quad \text{for almost all} \quad x\in\mathbb{R}^n, \tag{16}$$

where $c_\alpha=0$ if α_i is odd for some $i=1,\cdots,n$.

When $n=1$ and $\alpha\in\mathbb{N}$ is even the integral defining R_α is absolutely convergent.

As it was mentioned, Bruno and Sjögren ([2], Theorem 1.1]) proved that R_α is bounded from $L^1(\mathbb{R}^n,\gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n,\gamma_{-1})$ if and only if $1\le|\alpha|\le 2$. This property also holds in the Gaussian setting (see [8, 9]). Aimar et al. ([10]) introduced Riesz type operators $\mathfrak{R}_\alpha$, $\alpha\in\mathbb{N}^n\setminus\{0\}$, related to the Ornstein-Uhlenbeck operator. $\mathfrak{R}_\alpha$ is bounded from $L^1(\mathbb{R}^n,\gamma)$ into $L^{1,\infty}(\mathbb{R}^n,\gamma)$ for every $\alpha\in\mathbb{N}^n\setminus\{0\}$. Motivated by the results in [10], we define Riesz transform in the inverse Gaussian setting whose behavior in $L^1(\mathbb{R}^n,\gamma_{-1})$ is different from the one for R_α.

We can write $\mathscr{A}_0=\sum_{i=1}^n \delta_i\partial_{x_i}$, where for every $i=1,\cdots,n$, $\delta_i=-2^{-1}e^{-x_i^2}\partial_{x_i}e^{x_i^2}$. We consider the operator $\bar{\mathscr{A}}_0=\sum_{i=1}^n\partial_{x_i}\delta_i$. We have that $\bar{\mathscr{A}}_0=-n+\mathscr{A}_0$. Let $\bar{\mathscr{A}}$ the closure of $\bar{\mathscr{A}}_0$ in $L^2(\mathbb{R}^n,\gamma_{-1})$. For every $k\in\mathbb{N}^n$, $\bar{\mathscr{A}}\tilde{H}_k=|k|\tilde{H}_k$, and the spectrum of $\bar{\mathscr{A}}$ in $L^2(\mathbb{R}^n,\gamma_{-1})$ is the set of nonnegative integers. For every $\ell,m\in\mathbb{N}$, we have that $\delta_u^m\tilde{H}_\ell(u)=(-1)^m(\Gamma(\ell+1)/\Gamma(\ell-m+1))\tilde{H}_{\ell-m}(u)$, $u\in\mathbb{R}$. Here, we understand $\tilde{H}_\ell=0$ when $\ell<0$. Then, for every $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n$, by denoting $\delta^\alpha=\prod_{i=1}^n\delta_i^{\alpha_i}$, we get

$$\delta^\alpha\tilde{H}_k = (-1)^{|\alpha|}\prod_{i=1}^n\frac{\Gamma(k_i+1)}{\Gamma(k_i-\alpha_i+1)}\tilde{H}_{k-\alpha}, k = (k_1,\cdots,k_n)\in\mathbb{N}^n, k_r\ge\alpha_r, r=1,\cdots,n. \tag{17}$$

If $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\setminus\{0\}$ and $k=(k_1,\cdots,k_n)\in\mathbb{N}^n\setminus\{0\}$, with $k_r\ge\alpha_r$, $r=1,\cdots,n$,

$$\delta^\alpha\bar{\mathscr{A}}^{-|\alpha|/2}(\tilde{H}_k) = \frac{(-1)^{|\alpha|}}{|k|^{|\alpha|/2}}\prod_{i=1}^n\frac{\Gamma(k_i+1)}{\Gamma(k_i-\alpha_i+1)}\tilde{H}_{k-\alpha}. \tag{18}$$

In other cases, $\delta^\alpha\bar{\mathscr{A}}^{-|\alpha|/2}(\tilde{H}_k)=0$ (see Section 4 for details).

Let $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\setminus\{0\}$. We define the Riesz transform $\bar{R}_\alpha$ on $L^2(\mathbb{R}^n,\gamma_{-1})$ as follows

$$\bar{R}_\alpha(f) = (-1)^{|\alpha|}\sum_{\substack{k=(k_1,\cdots,k_n)\in\mathbb{N}^n\\ k_r\ge\alpha_r, r=1,\cdots,n}}\frac{1}{|k|^{|\alpha|/2}}\prod_{i=1}^n\frac{\Gamma(k_i+1)}{\Gamma(k_i-\alpha_i+1)}c_k(f)\tilde{H}_{k-\alpha}. \tag{19}$$

Thus, $\bar{R}_\alpha$ is bounded from $L^2(\mathbb{R}^n,\gamma_{-1})$ into itself. If $f\in C_c^\infty(\mathbb{R}^n)$ and $c_0(f)=0$ then $\bar{R}_\alpha f=\delta^\alpha\bar{\mathscr{A}}^{-|\alpha|/2}f$.

Theorem 3. *Let $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\setminus\{0\}$. The Riesz transform $\bar{R}_\alpha$ can be extended from $L^2(\mathbb{R}^n,\gamma_{-1})\cap L^p(\mathbb{R}^n,\gamma_{-1})$ to $L^p(\mathbb{R}^n,\gamma_{-1})$ as a bounded operator from*

(i) $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself, for every $1<p<\infty$

(ii) $L^1(\mathbb{R}^n,\gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n,\gamma_{-1})$, provided that $n=1$ or $|\alpha|>n$, when $n>1$

By denoting again $\bar{R}_\alpha$ to the extension we have that, for every $f\in L^p(\mathbb{R}^n,\gamma_{-1})$, $1\le p<\infty$,

$$\bar{R}_\alpha(f)(x) = \lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon}\bar{R}_\alpha(x,y)f(y)dy + c_\alpha f(x), \quad \text{for almost all} \quad x\in\mathbb{R}^n, \tag{20}$$

where $c_\alpha=0$ when α_i is odd for some $i=1,\cdots,n$. Here

$$\bar{R}_\alpha(x,y) = \frac{1}{\Gamma(|\alpha|/2)}\int_0^\infty \delta_x^\alpha T_t^{\bar{\mathscr{A}}}(x,y)t^{|\alpha|/2-1}dt, x,y\in\mathbb{R}^n, x\ne y. \tag{21}$$

Let X be a Banach space. Suppose that $\{M_r\}_{r=1}^m$ is a X-valued martingale. The sequence $\{d_r=M_r-M_{r-1}\}_{r=1}^m$, where M_0 is understood as 0, is called the martingale difference associated with $\{M_r\}_{r=1}^m$. We say that $\{d_r\}_{r=1}^m$ is a L^p-martingale difference sequence when it is the difference sequence associated with a L^p-martingale. If $1<p<\infty$, X is said to be a UMD_p-space when there exists $\beta>0$ such that for all X-valued L^p-martingale difference sequence $\{d_r\}_{r=1}^m$ and for all $(\varepsilon_r)_{r=1}^m\in\{-1,1\}^m$,

$$\mathbb{E}\left\|\sum_{r=1}^m\varepsilon_r d_r\right\|^p \le \beta\mathbb{E}\left\|\sum_{r=1}^m d_r\right\|^p. \tag{22}$$

UMD is an abbreviation of unconditional martingale difference. If X is UMD_p for some $1<p<\infty$, then X is UMD_p

for every $1 < p < \infty$. This fact justifies to call UMD to the property without any reference to p. Burkholder [11] and Bourgain [12] proved that the UMD property of X is necessary and sufficient for the boundedness of the Hilbert transform in $L^p(\mathbb{R}, X)$, $1 < p < \infty$. The UMD property is a central notion in the development of the harmonic analysis when the functions are taking values in infinite-dimensional spaces. UMD Banach spaces have been characterized by using other singular integrals that can be seen as Riesz transforms associated to orthogonal systems (see [13–16], for instance). In the following result, we characterize the Banach spaces with the UMD property by using Riesz transforms in the inverse Gaussian setting. For every $i = 1, \cdots, n$, we define $e^i = (e^i_1, \cdots, e^i_n)$ where $e^i_j = 0$, $i \neq j$, and $e^i_i = 1$.

Theorem 4. *Let X be a Banach space. The following assertions are equivalent.*

(i) *X is UMD*

(ii) *For every $i = 1, \cdots, n$, R_{e^i} can be extended from $L^p(\mathbb{R}^n, \gamma_{-1}) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself, for every $1 < p < \infty$*

(iii) *For every $i = 1, \cdots, n$, R_{e^i} can be extended from $L^p(\mathbb{R}^n, \gamma_{-1}) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself, for some $1 < p < \infty$*

(iv) *For every $i = 1, \cdots, n$, R_{e^i} can be extended from $L^1(\mathbb{R}^n, \gamma_{-1}) \otimes X$ to $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1}, X)$*

Also, the equivalences hold when in the properties (ii), (iii), and (iv), we replace R_{e^i} by the maximal operator $R^*_{e^i}$ defined by

$$R^*_{e^i}(f)(x) = \sup_{\varepsilon > 0} \left\| \int_{|x-y|>\varepsilon} R_{e^i}(x, y) f(y) dy \right\|, x \in \mathbb{R}^n \quad \text{and} \quad i = 1, \cdots, n. \tag{23}$$

Theorem 5. *Let X be a Banach space. The following assertions are equivalent.*

(i) *X is UMD*

(ii) *For every $1 \leq p < \infty$ there exists, for each $i = 1, \cdots, n$, the limit*

$$\lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} R_{e^i}(x, y) f(y) dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{24}$$

for every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$.

(iii) *For some $1 \leq p < \infty$, there exists, for every $i = 1, \cdots, n$, the limit*

$$\lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} R_{e^i}(x, y) f(y) dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{25}$$

for each $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$.

(iv) *For every $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$ and $i = 1, \cdots, n$, $R^*_{e^i}(f)(x) < \infty$, for almost all $x \in \mathbb{R}^n$*

(v) *For some $1 \leq p < \infty$ and for every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$ and $i = 1, \cdots, n$, $R^*_{e^i}(f)(x) < \infty$, for almost all $x \in \mathbb{R}^n$*

The properties stated in Theorems 4 and 5 can also be established when we replace R-Riesz transforms by $\bar{R}$-Riesz transforms.

Next aim is to state characterizations of the Banach spaces with the UMD property by using imaginary powers $\mathscr{A}^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, of $\mathscr{A}$. Salogni ([1], Theorem 3.4.3) proved that, for every $1 < p < \infty$ and $\sigma \in \mathbb{R} \setminus \{0\}$,

$$\left\| \mathscr{A}^{i\sigma} \right\|_{L^p(\mathbb{R}^n,\gamma_{-1}) \longrightarrow L^p(\mathbb{R}^n,\gamma_{-1})} \sim e^{\phi^*_p |\sigma|}, \tag{26}$$

as $\sigma \longrightarrow \infty$, when $\phi_p = \arcsin |2/p - 1|$. Actually, $\mathscr{A}^{i\sigma}$ is a Laplace transform type multiplier associated with $\mathscr{A}$ defined by the function $\phi_\sigma(t) = t^{-i\sigma}/\Gamma(1 - i\sigma)$, $t > 0$, for every $\sigma \in \mathbb{R} \setminus \{0\}$. Then, since $\{T^{\mathscr{A}}_t\}_{t>0}$ is a Stein diffusion semigroup, the $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness of $\mathscr{A}^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, follows from the general results established in ([4], Chapter III). Recently, Bruno ([6], Theorem 4.1) proved that $\mathscr{A}^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

Let $\sigma \in \mathbb{R} \setminus \{0\}$. We have that

$$\mathscr{A}^{i\sigma} f = \sum_{k \in \mathbb{N}^n} (|k| + n)^{i\sigma} c_k(f) \tilde{H}_k, f \in L^2(\mathbb{R}^n, \gamma_{-1}). \tag{27}$$

It is immediate to see that $\mathscr{A}^{i\sigma}$ is bounded from $L^2(\mathbb{R}^n, \gamma_{-1})$ into itself. For every $f \in L^2(\mathbb{R}^n, \gamma_{-1}) \otimes X$, we define $\mathscr{A}^{i\sigma}$ in the obvious way when X is a Banach space. In order to the operator $\mathscr{A}^{i\sigma}$ is bounded from $L^2(\mathbb{R}^n, \gamma_{-1}) \otimes X$ into itself as subspace of $L^2(\mathbb{R}^n, \gamma_{-1}, X)$, we need to impose some additional property to the Banach space X. For instance, if X is isomorphic to a Hilbert space, then $\mathscr{A}^{i\sigma}$ can be extended from $L^2(\mathbb{R}^n, \gamma_{-1}) \otimes X$ to $L^2(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^2(\mathbb{R}^n, \gamma_{-1}, X)$ into itself. We are going to characterize the UMD Banach spaces as those Banach spaces for which $\mathscr{A}^{i\sigma}$ can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1}, X)$. Our result is motivated by the one in ([17], p. 402) where UMD Banach spaces are characterized by the $L^p(\mathbb{R}, dx)$-boundedness properties of the imaginary power $(-d^2/dx^2)^{i\sigma}$, $\sigma \in \mathbb{R} \setminus \{0\}$, of $-d^2/dx^2$. Guerre-Delabriére's result was extended to higher dimensions by considering imaginary powers of the Laplacian in ([18], Proposition 1). In [19], this kind of characterization for UMD Banach spaces is obtained in Hermite and Laguerre settings. As far as we know, this property has not been proved for the Ornstein-Uhlenbeck operator in the Gaussian framework.

Theorem 6. *Let $\sigma \in \mathbb{R} \setminus \{0\}$. For every $f \in L^2(\mathbb{R}^n, \gamma_{-1})$, we have that*

$$\mathscr{A}^{i\sigma}(f)(x) = \lim_{\varepsilon \longrightarrow 0^+} \left(\int_{|x-y|>\varepsilon} K_\sigma^{\mathscr{A}}(x, y) f(y) dy + \alpha(\varepsilon) f(x) \right), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{28}$$

where

$$K_\sigma^{\mathscr{A}}(x, y) = -\int_0^\infty \phi_\sigma(t) \partial_t T_t^{\mathscr{A}}(x, y) dt, x, y \in \mathbb{R}^n, x \neq y, \tag{29}$$

and

$$\alpha(\varepsilon) = \frac{1}{\Gamma(n/2)} \int_0^\infty \phi_\sigma\left(\frac{\varepsilon^2}{4u}\right) e^{-u} u^{n/2-1} du, \varepsilon \in (0, \infty), \tag{30}$$

being $\phi_\sigma(t) = t^{-i\sigma}/\Gamma(1 - i\sigma)$, $t \in (0, \infty)$.

Let X be a Banach space. The following assertions are equivalent.

(i) X is UMD

(ii) For every $1 < p < \infty$, $\mathscr{A}^{i\sigma}$ can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself

(iii) For some $1 < p < \infty$, $\mathscr{A}^{i\sigma}$ can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself

(iv) $\mathscr{A}^{i\sigma}$ can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^1(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1}, X)$

We define the maximal operator $\mathscr{A}_*^{i\sigma}$ by

$$\mathscr{A}_*^{i\sigma}(f)(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} K_\sigma^{\mathscr{A}}(x, y) f(y) dy \right\|, f \in \bigcup_{p \geq 1} L^p(\mathbb{R}^n, \gamma_{-1}, X). \tag{31}$$

The following assertions are equivalent to (i).

(v) For every $1 < p < \infty$, $\mathscr{A}_*^{i\sigma}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself

(vi) For some $1 < p < \infty$, $\mathscr{A}_*^{i\sigma}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself

(vii) $\mathscr{A}_*^{i\sigma}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1}, X)$

(viii) For every $1 \leq p < \infty$ and every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$ there exists the limit

$$\lim_{\varepsilon \longrightarrow 0^+} \left(\int_{|x-y|>\varepsilon} K_\sigma^{\mathscr{A}}(x, y) f(y) dy + \alpha(\varepsilon) f(x) \right), \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{32}$$

(ix) For some $1 \leq p < \infty$ and every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$, there exists the limit

$$\lim_{\varepsilon \longrightarrow 0^+} \left(\int_{|x-y|>\varepsilon} K_\sigma^{\mathscr{A}}(x, y) f(y) dy + \alpha(\varepsilon) f(x) \right), \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{33}$$

(x) For every $1 \leq p < \infty$ and every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$, $\mathscr{A}_*^{i\sigma}(f)(x) < \infty$, for almost all $x \in \mathbb{R}^n$

(xi) For some $1 \leq p < \infty$ and every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$, $\mathscr{A}_*^{i\sigma}(f)(x) < \infty$, for almost all $x \in \mathbb{R}^n$

This paper is organized as follows. In Section 2 we study the negative power $\mathscr{A}^{-\beta}$, $\beta > 0$, of $\mathscr{A}$. Higher order Riesz transforms in the inverse Gauss setting are considered in Section 3 where we prove Theorems 1 and 2. Theorem 3 is established in Section 4 and Theorems 4 and 5 in Section 5. Finally, Section 6 is devoted to show the proof of Theorem 6.

Throughout this paper, C and c denote positive constants that can change in each occurrence.

2. Negative Powers of $\mathscr{A}$

In this section, we prove $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness properties of the negative powers $\mathscr{A}^{-\beta}$, $\beta > 0$, of $\mathscr{A}$. These properties are different than the ones of the negative powers of the Ornstein-Uhlenbeck operator $\mathscr{L} = -\Delta/2 + x\nabla$. We prove that, for every $\beta > 0$, $\mathscr{A}^{-\beta}$ defines a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. However, in ([7], Proposition 6.2), it was proved that if $\beta > 0$, $\mathscr{L}^{-\beta}$ is not bounded from $L^1(\mathbb{R}^n, \gamma)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma)$.

Let $\beta > 0$. We define

$$\mathscr{A}^{-\beta}(f) = \sum_{k \in \mathbb{N}^n} \frac{c_k(f)}{(|k| + n)^\beta} \tilde{H}_k, f \in L^2(\mathbb{R}^n, \gamma_{-1}). \tag{34}$$

$\mathscr{A}^{-\beta}$ is bounded from $L^2(\mathbb{R}^n, \gamma_{-1})$ into itself. Moreover, when $\beta > 1$, the series also converges pointwisely in $\mathbb{R}^n$. Indeed, let $f \in L^2(\mathbb{R}^n, \gamma_{-1})$. We have that

$$|c_k(f)| \leq \pi^{n/2} \|f\|_{L^2(\mathbb{R}^n, \gamma_{-1})} \left\|\tilde{H}_k\right\|_{L^2(\mathbb{R}^n, \gamma_{-1})}^{-1}, k \in \mathbb{N}^n. \tag{35}$$

Also, for every $k \in \mathbb{N}^n$, (see (14)),

$$\|\tilde{H}_k\|_{L^2(\mathbb{R}^n,\gamma_{-1})} = \pi^{n/2} 2^{|k|/2} \left(\prod_{j=1}^{n} \Gamma(k_j + 1) \right)^{1/2}, \tag{36}$$

and according to ([20], p. 324),

$$|H_j(z)| \leq 2\sqrt{\Gamma(j+1)} 2^{j/2} e^{z^2/2}, z \in \mathbb{R} \quad \text{and} \quad j \in \mathbb{N}. \tag{37}$$

Then,

$$|c_k(f)\tilde{H}_k(x)| \leq 2^n e^{-|x|^2/2} \|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}, x \in \mathbb{R}^n, k \in \mathbb{N}^n, \tag{38}$$

and if $\beta > 1$ it follows that

$$\sum_{k\in\mathbb{N}^n} \frac{|c_k(f)|}{(|k| + n)^\beta} |\tilde{H}_k(x)| \leq C e^{-|x|^2/2} \|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}, x \in \mathbb{R}^n. \tag{39}$$

The series in (34) converges pointwise absolutely for each $\beta > 0$ when $f \in C_c^\infty(\mathbb{R}^n)$. Indeed, let $f \in C_c^\infty(\mathbb{R}^n)$. Partial integration allows us to see that, for every $r \in \mathbb{N}$, there exists $C = C(f, r) > 0$ such that

$$|c_k(f)| \leq \frac{C}{(|k| + n)^r \|\tilde{H}_k\|_{L^2(\mathbb{R}^n,\gamma_{-1})}}, k \in \mathbb{N}^n. \tag{40}$$

Then,

$$\sum_{k\in\mathbb{N}^n} \frac{|c_k(f)|}{(|k| + n)^\beta} |\tilde{H}_k(x)| \leq C e^{-|x|^2/2} \sum_{k\in\mathbb{N}^n} \frac{1}{(|k| + n)^{\beta+r}} < \infty. \tag{41}$$

We also consider the operator defined, for every $f \in L^2(\mathbb{R}^n, \gamma_{-1})$, by

$$S_\beta(f) = \frac{1}{\Gamma(\beta)} \int_0^\infty T_t^{\mathcal{A}}(f) t^{\beta-1} dt, \tag{42}$$

where the integral is understood in the $L^2(\mathbb{R}^n, \gamma_{-1})$-Bochner sense.

For each $t > 0$, we can write

$$T_t^{\mathcal{A}}(f) = \sum_{k\in\mathbb{N}^n} e^{-(|k|+n)t} c_k(f) \tilde{H}_k, f \in L^2(\mathbb{R}^n, \gamma_{-1}). \tag{43}$$

We obtain

$$\begin{aligned} \|T_t^{\mathcal{A}}(f)\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})} &= \sum_{k\in\mathbb{N}^n} e^{-2(|k|+n)t} |c_k(f)|^2 \|\tilde{H}_k\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})} \\ &\leq e^{-2nt} \sum_{k\in\mathbb{N}^n} |c_k(f)|^2 \|\tilde{H}_k\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})} \\ &= e^{-2nt} \|f\|^2_{L^2(\mathbb{R}^n,\gamma_{-1})}, f \in L^2(\mathbb{R}^n, \gamma_{-1}), t > 0. \end{aligned} \tag{44}$$

Then,

$$\begin{aligned} \|S_\beta(f)\|_{L^2(\mathbb{R}^n,\gamma_{-1})} &\leq \frac{1}{\Gamma(\beta)} \int_0^\infty \|T_t^{\mathcal{A}}(f)\|_{L^2(\mathbb{R}^n,\gamma_{-1})} t^{\beta-1} dt \\ &\leq \frac{\|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}}{\Gamma(\beta)} \int_0^\infty e^{-nt} t^{\beta-1} dt \\ &\leq \frac{\|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}}{n^\beta}, f \in L^2(\mathbb{R}^n, \gamma_{-1}). \end{aligned} \tag{45}$$

Suppose now that $f \in C_c^\infty(\mathbb{R}^n)$. By using (40), we obtain that there exists $C = C(f) > 0$ such that

$$|T_t^{\mathcal{A}}(f)(x)| \leq C e^{-|x|^2/2} e^{-nt}, x \in \mathbb{R}^n, t > 0. \tag{46}$$

We can write

$$\begin{aligned} S_\beta(f)(x) &= \frac{1}{\Gamma(\beta)} \sum_{k\in\mathbb{N}^n} c_k(f) \tilde{H}_k(x) \int_0^\infty e^{-(|k|+n)t} t^{\beta-1} dt \\ &= \sum_{k\in\mathbb{N}^n} \frac{c_k(f)}{(|k| + n)^\beta} \tilde{H}_k(x) = \mathcal{A}^{-\beta}(f)(x), x \in \mathbb{R}^n. \end{aligned} \tag{47}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \gamma_{-1})$, $\mathcal{A}^{-\beta}(f) = S_\beta(f)$, $f \in L^2(\mathbb{R}^n, \gamma_{-1})$.

According to ([6], Theorem 2.5), we have that, for every $f \in C_c^\infty(\mathbb{R}^n)$

$$\mathcal{A}^{-\beta}(f)(x) = \int_{\mathbb{R}^n} M_\beta(x, y) f(y) dy, \tag{48}$$

for all x outside the support of f, where

$$M_\beta(x, y) = \frac{1}{\Gamma(\beta)} \int_0^\infty T_t^{\mathcal{A}}(x, y) t^{\beta-1} dt, x, y \in \mathbb{R}^n, x \neq y. \tag{49}$$

Proposition 7. *Let $\beta > 0$. The operator $\mathcal{A}^{-\beta}$ can be extended from $L^p(\mathbb{R}^n, \gamma_{-1}) \cap L^2(\mathbb{R}^n, \gamma_{-1})$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.*

Proof. We use the method consisting in decomposing the operator $\mathcal{A}^{-\beta}$ in two parts called local and global parts. This procedure of decomposition was employed by Muckenhoupt ([21, 22]) in the Gaussian setting.

From now on, we consider the function m given by $m(x) = \min\{1, 1/|x|^2\}$, $x \in \mathbb{R}^n \setminus \{0\}$, and $m(0) = 1$ and the region N defined by

$$N = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq n\sqrt{m(x)}\right\}. \tag{50}$$

We decompose $\mathscr{A}^{-\beta}$ as follows

$$\mathscr{A}^{-\beta} = \mathscr{A}_{\text{loc}}^{-\beta} + \mathscr{A}_{\text{glob}}^{-\beta}, \tag{51}$$

where $\mathscr{A}_{\text{loc}}^{-\beta}(f)(x) = \mathscr{A}^{-\beta}(f\chi_N(x,\cdot))(x)$, $x \in \mathbb{R}^n$.

According to ([1], Lemma 3.3.1), we get

$$M_\beta(x,y) \le C\left(\int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{n/2}} t^{\beta-1}dt + \int_{m(x)}^{\infty} \frac{e^{-nt}}{(1-e^{-2t})^{n/2}} t^{\beta-1}dt\right), (x,y) \in N. \tag{52}$$

By choosing $0 < \varepsilon < \min\{2\beta, n\}$, we obtain

$$\begin{aligned} M_\beta(x,y) &\le C\left(\int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{(n-\varepsilon)/2}} t^{\beta-\varepsilon/2-1}dt + \int_{m(x)}^{\infty} \frac{dt}{t^{(n-\varepsilon)/2+1}}\right) \\ &\le C\left(\frac{1}{|x-y|^{n-\varepsilon}} \int_0^1 t^{\beta-\varepsilon/2-1}dt + m(x)^{-(n-\varepsilon)/2}\right) \\ &\le \frac{C}{|x-y|^{n-\varepsilon}}, (x,y) \in N. \end{aligned} \tag{53}$$

In the last inequality, we have taken into account that $\sqrt{m(x)} \sim 1/(1+|x|)$, $x \in \mathbb{R}^n$, and that $(1+|x|)|x-y| \le C$, provided that $(x,y) \in N$.

We have that

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} M_\beta(x,y)\chi_N(x,y)dy &\le C \sup_{x\in\mathbb{R}^n} \int_{|x-y|\le n\sqrt{m(x)}} \frac{dy}{|x-y|^{n-\varepsilon}} \\ &\le C \sup_{x\in\mathbb{R}^n} \int_0^{n\sqrt{m(x)}} r^{\varepsilon-1}dr \\ &\le C \sup_{x\in\mathbb{R}^n} m(x)^{\varepsilon/2} < \infty. \end{aligned} \tag{54}$$

Also, since $\sqrt{m(x)} \sim 1/(1+|x|) \sim 1/(1+|y|) \sim \sqrt{m(y)}$, $(x,y) \in N$,

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} M_\beta(x,y)\chi_N(x,y)dx < \infty. \tag{55}$$

Hence, the operator $M_{\beta,\text{loc}}$ defined by

$$M_{\beta,\text{loc}}(f)(x) = \int_{\mathbb{R}^n} M_\beta(x,y)\chi_N(x,y)f(y)dy, x \in \mathbb{R}^n, \tag{56}$$

is bounded from $L^p(\mathbb{R}^n, dx)$ into itself, for every $1 \le p < \infty$. Since $M_{\beta,\text{loc}}$ is a local operator, by using ([1], Proposition 3.2.5), we deduce that $M_{\beta,\text{loc}}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 \le p < \infty$.

Suppose that $f, g \in C_c^\infty(\mathbb{R}^n)$. As above, we take $0 < \varepsilon < \min\{2\beta, n\}$ and we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} \int_0^\infty t^{\beta-1} T_t^{\mathscr{A}}(x,y)\chi_N(x,y)|g(y)|dtdydx \\ &\quad \le C\int_{\mathbb{R}^n} |f(x)| \int_{\mathbb{R}^n} |g(y)| \frac{\chi_N(x,y)}{|x-y|^{n-\varepsilon}} dydx < \infty. \end{aligned} \tag{57}$$

Then,

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\mathscr{A}_{\text{loc}}^{-\beta}(g)(x)dx &= \int_{\mathbb{R}^n} f(x)S_{\beta,\text{loc}}(g)(x)dx \\ &= \int_{\mathbb{R}^n} f(x)M_{\beta,\text{loc}}(g)(x)dx. \end{aligned} \tag{58}$$

We deduce that $\mathscr{A}_{\text{loc}}^{-\beta}(g)(x) = M_{\beta,\text{loc}}(g)(x)$, for almost all $x \in \mathbb{R}^n$. Since $\mathscr{A}_{\text{loc}}^{-\beta}$ and $M_{\beta,\text{loc}}$ are bounded from $L^2(\mathbb{R}^n, \gamma_{-1})$ into itself, $\mathscr{A}_{\text{loc}}^{-\beta}(f) = M_{\beta,\text{loc}}(f), f \in L^2(\mathbb{R}^n, \gamma_{-1})$. It follows that, for every $1 \le p < \infty$, $\mathscr{A}_{\text{loc}}^{-\beta}$ can be extended from $L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

We now study the operator $M_{\beta,\text{glob}}$ defined by

$$M_{\beta,\text{glob}}(f)(x) = \int_{\mathbb{R}^n} M_\beta(x,y)\chi_{N^c}(x,y)f(y)dy, x \in \mathbb{R}^n. \tag{59}$$

By making the change of variables $s = 1 - e^{-2t}$, $t \in (0,\infty)$, and taking into account that $|x - ry|^2 = |y - rx|^2 + (1 - r^2)(|x|^2 - |y|^2)$, $x, y \in \mathbb{R}^n$, $r \in \mathbb{R}$, we obtain

$$\begin{aligned} M_\beta(x,y) &= \frac{\pi^{-n/2}}{\Gamma(\beta)} \int_0^\infty \frac{e^{-nt}}{(1-e^{-2t})^{n/2}} e^{-\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)} t^{\beta-1}dt \\ &= \frac{\pi^{-n/2}}{2^\beta\Gamma(\beta)} e^{|y|^2-|x|^2} \int_0^1 \frac{e^{-|y-x\sqrt{1-s}|^2/s}}{s^{n/2}} (1-s)^{n/2-1}(-\log(1-s))^{\beta-1}ds, x, y \in \mathbb{R}^n. \end{aligned} \tag{60}$$

We now use some notations that were introduced in [23] and proceed as in the proof of ([23], Proposition 2.2). For every $x, y \in \mathbb{R}^n$, we define

$$a = |x|^2 + |y|^2, b = 2\langle x,y\rangle, s_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \text{ and } u(s) = \frac{\left|y - x\sqrt{1-s}\right|^2}{s}, s \in (0,1). \tag{61}$$

Assume that $(x,y) \notin N$. Suppose first that $b \le 0$. In this case, $u(s) \ge a/s - |x|^2$, $s \in (0,1)$. Furthermore, $s \le C(-\log(1-s)) \le C(1-s)^{-1/(4\beta)}$, $s \in (0,1)$. Then,

$$M_\beta(x,y) \le Ce^{|y|^2} \int_0^1 \frac{e^{-a/s}}{s^{n/2+1}(1-s)^{3/4}} ds. \tag{62}$$

By making the change of variable $r = 1/s - 1$, $s \in (0,1)$, and taking into account that $a \geq 1/2$, we obtain

$$M_\beta(x,y) \leq Ce^{-|x|^2} \int_0^\infty \frac{e^{-r/2}(1+r)^{n/2-1/4}}{r^{3/4}} dr \leq Ce^{-|x|^2}. \quad (63)$$

Suppose now $b > 0$. We write $u_0 = u(s_0) = (|y|^2 - |x|^2)/2 + |x+y||x-y|/2$. By using that (see [24], p. 850)

$$\sup_{s\in(0,1)} \frac{e^{-u(s)}}{s^{n/2}} \sim \frac{e^{-u_0}}{s_0^{n/2}}, \quad (64)$$

we have that

$$\begin{aligned} M_\beta(x,y) &\leq Ce^{|y|^2-|x|^2} \sup_{s\in(0,1)} \frac{e^{-u(s)}}{s^{n/2}} \int_0^1 (1-s)^{n/2-1}(-\log(1-s))^{\beta-1} ds \\ &\leq Ce^{|y|^2-|x|^2} \frac{e^{-u_0}}{s_0^{n/2}}. \end{aligned} \quad (65)$$

Since $s_0 \sim \sqrt{a^2-b^2}/a$ we conclude that, when $(x,y) \notin N$,

$$M_\beta(x,y) \leq C \begin{cases} e^{-|x|^2}, & \text{if } \langle x,y\rangle \leq 0, \\ \left(\frac{|x+y|}{|x-y|}\right)^{n/2} \exp\left(\frac{|y|^2-|x|^2}{2} - \frac{|x-y||x+y|}{2}\right), & \text{if } \langle x,y\rangle > 0. \end{cases} \quad (66)$$

Let $1 < q < \infty$. Since $||y|^2 - |x|^2| \leq |x+y||x-y|$, $x, y \in \mathbb{R}^n$, and $|x-y||x+y| \geq n$, when $(x,y) \in N^c$ and $b > 0$, as in ([25], p. 501), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{|x|^2/q-|y|^2/q} M_\beta(x,y)\chi_{N^c}(x,y)dy \\ &\leq C\left(\int_{\mathbb{R}^n} e^{-|x|^2(1-1/q)}e^{-|y|^2/q}dy + \int_{\mathbb{R}^n} |x+y|^n e^{-|x+y||x-y|(1/2-|1/q-1/2|)}dy\right) \\ &\leq C, x \in \mathbb{R}^n. \end{aligned} \quad (67)$$

Also, we have that

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} e^{|x|^2/q-|y|^2/q} M_\beta(x,y)\chi_{N^c}(x,y)dx < \infty. \quad (68)$$

We deduce that $M_{\beta,glob}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 < p < \infty$.

Next, we are going to see that $M_{\beta,glob}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. We decompose $M_\beta(x,y)$, $x, y \in \mathbb{R}^n$, as follows

$$\begin{aligned} M_\beta(x,y) &= \frac{\pi^{-n/2}}{2^\beta \Gamma(\beta)} e^{|y|^2-|x|^2} \left(\int_0^{1/2} + \int_{1/2}^1\right) \\ &\cdot \frac{e^{-|y-x\sqrt{1-s}|^2/s}}{s^{n/2}} (1-s)^{n/2-1}(-\log(1-s))^{\beta-1} ds \\ &= I_1(x,y) + I_2(x,y). \end{aligned} \quad (69)$$

For every $x, y \in \mathbb{R}^n \setminus \{0\}$, we denote by $\theta(x,y) \in [0,\pi]$ the angle between x and y (we understand $\theta(x,y) = 0$, when $n = 1$). By using ([1], Lemma 3.3.3), we get that, for every $(x,y) \in N^c$, $x, y \neq 0$,

$$\begin{aligned} I_1(x,y) &\leq Ce^{|y|^2-|x|^2} \sup_{s\in(0,1)} \frac{e^{-|y-x\sqrt{1-s}|^2/s}(1-s)^{n/2}}{s^{n/2}} \int_0^{1/2} \frac{(-\log(1-s))^{\beta-1}}{1-s} ds \\ &\leq Ce^{|y|^2-|x|^2} \sup_{r\in(0,1)} \frac{e^{-|(1+r)y-(1-r)x|^2/r}(1-r)^n}{r^{n/2}} \\ &\leq Ce^{|y|^2-|x|^2} \min\{(1+|x|)^n, (|x|\sin\theta(x,y))^{-n}\}. \end{aligned} \quad (70)$$

On the other hand, by proceeding as in ([2], Proposition 5.1), we estimate $I_2(x,y)$, $(x,y) \in N^c$. We first observe that

$$\begin{aligned} I_2(x,y) &\leq Ce^{|y|^2-|x|^2} \int_{1/2}^1 e^{-|y-x\sqrt{1-s}|^2}(1-s)^{n/2-1} \\ &\cdot (-\log(1-s))^{\beta-1} ds, x \text{ and } y \in \mathbb{R}^n. \end{aligned} \quad (71)$$

If $|y| \geq 2|x|$ then $|y - x\sqrt{1-s}| \geq 3|y|/4$, $s \in (1/2, 1)$, and it follows that

$$\begin{aligned} I_2(x,y) &\leq Ce^{-c|y|^2}e^{|y|^2-|x|^2} \int_{1/2}^1 (1-s)^{n/2-1}(-\log(1-s))^{\beta-1} ds \\ &\leq C\frac{e^{|y|^2-|x|^2}}{|y|^{n-1}} \leq C\frac{e^{|y|^2-|x|^2}}{|x|^{n-1}}, x, y \neq 0. \end{aligned} \quad (72)$$

If $x, y \in \mathbb{R}^n \setminus \{0\}$ we define $r_0 = |y||x|^{-1}\cos\theta(x,y)$ and we write $y = y_x + y_\perp$, where y_x is parallel to x and $y_\perp$ is orthogonal to y.

By making the change of variables $r = \sqrt{1-s}$ and since $|y - rx|^2 = |r - r_0|^2|x|^2 + |y_\perp|^2$, it follows that, if $|y| \leq 2|x|$,

$$\begin{aligned} I_2(x,y) &\leq Ce^{-c|y_\perp|^2}e^{|y|^2-|x|^2} \int_0^{1/\sqrt{2}} (|r-r_0|^{n-1} + |r_0|^{n-1})(-\log r)^{\beta-1} e^{-c|r-r_0|^2|x|^2} dr \\ &\leq Ce^{-c|y_\perp|^2}e^{|y|^2-|x|^2} \left(\frac{1}{|x|^{n-1}} + \left(\frac{|y|}{|x|}\right)^{n-1}\right) \int_0^{1/\sqrt{2}} (-\log r)^{\beta-1} dr \\ &\leq Ce^{-c|y_\perp|^2}e^{|y|^2-|x|^2} \left(\frac{1}{|x|^{n-1}} + |x|\left(\frac{|y|}{|x|}\right)^{n-1}\right), (x,y) \in N^c. \end{aligned} \quad (73)$$

In the last inequality, we have used that $|x| \geq C$ when $(x, y) \in N^c$ and $2|x| \geq |y|$.

By combining the above estimates, we obtain

$$M_\beta(x, y) \leq C e^{|y|^2 - |x|^2} \left(\min\{(1 + |x|)^n, (|x| \sin \theta(x, y))^{-n}\} + \frac{1}{|x|^{n-1}} + e^{-c|y_\perp|^2} |x| \left(\frac{|y|}{|x|} \right)^{n-1} \chi_{\{|y| \leq 2|x|\}}(x, y) \right), (x, y) \in N^c. \quad (74)$$

According to ([1], Lemma 3.3.4) and ([2], Lemmas 4.2 and 4.3), we can see that the operator $M_{\beta,glob}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

The estimation (74) allows us to prove that, for every $g \in C_c^\infty(\mathbb{R}^n)$, $\mathscr{A}_{glob}^{-\beta}(f)(x) = M_{\beta,glob}(f)(x)$, for almost all $x \in \mathbb{R}^n$. Then, since $\mathscr{A}_{glob}^{-\beta}$ and $M_{\beta,glob}$ are bounded from $L^2(\mathbb{R}^n, \gamma_{-1})$ into itself, $\mathscr{A}_{glob}^{-\beta}(f) = M_{\beta,glob}(f), f \in L^2(\mathbb{R}^n, \gamma_{-1})$. Hence, $\mathscr{A}_{glob}^{-\beta}$ can be extended from $L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

Thus, the proof is finished. □

3. Higher-Order Riesz Transforms Associated with the Operator $\mathscr{A}$

In this section, we prove Theorems 1 and 2 concerning to the higher-order Riesz transforms in the inverse Gaussian setting.

Proof of Theorem 1. Let $f \in C_c^\infty(\mathbb{R}^n)$. If $\ell \in \mathbb{N}^n$, we have that

$$\partial_x^\ell T_t^{\mathscr{A}}(x, y) = (-1)^{|\ell|} \frac{e^{-nt} e^{-\left|x - e^{-t} y\right|^2 / \left(1 - e^{-2t}\right)}}{\pi^{n/2} \left(1 - e^{-2t}\right)^{(n+|\ell|)/2}} H_\ell \left(\frac{x - e^{-t} y}{\sqrt{1 - e^{-2t}}} \right), x, y \in \mathbb{R}^n, t > 0, \quad (75)$$

and then

$$\left|\partial_x^\ell T_t^{\mathscr{A}}(x, y)\right| \leq C \frac{e^{-nt} e^{-c\left|x - e^{-t} y\right|^2 / \left(1 - e^{-2t}\right)}}{\left(1 - e^{-2t}\right)^{(n+|\ell|)/2}}, x, y \in \mathbb{R}^n, t > 0. \quad (76)$$

Suppose that $k \in \mathbb{N}$ and $\ell \in \mathbb{N}^n$ such that $|\ell| < k$. Then,

$$\int_{\mathbb{R}^n} |f(y)| \int_0^\infty \left|\partial_x^\ell T_t^{\mathscr{A}}(x, y)\right| t^{k/2-1} dt dy < \infty, x \in \mathbb{R}^n. \quad (77)$$

□

Indeed, by considering the function m defined in Proposition 7 and using (76) and ([1], Lemma 3.3.1), we obtain, for $\varepsilon \in (0, 1)$,

$$\begin{aligned} \int_0^\infty \left|\partial_x^\ell T_t^{\mathscr{A}}(x, y)\right| t^{k/2-1} dt &\leq C \left(\int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{(n+|\ell|-k)/2+1}} dt + \int_{m(x)}^\infty \frac{e^{-nt}}{\left(1 - e^{-2t}\right)^{(n+|\ell|)/2}} t^{k/2-1} dt \right) \\ &\leq C \left(\frac{1}{|x - y|^{n-\varepsilon}} \int_0^1 t^{(k-|\ell|-\varepsilon)/2-1} dt + \int_{m(x)}^\infty \frac{dt}{t^{(n-\varepsilon)/2+1}} \right) \\ &\leq C \left(\frac{1}{|x - y|^{n-\varepsilon}} + \frac{1}{m(x)^{(n-\varepsilon)/2}} \right) \\ &\leq \frac{C}{|x - y|^{n-\varepsilon}}, (x, y) \in N. \end{aligned} \quad (78)$$

On the other hand, by reading the proof of ([1], Lemma 3.3.3), we deduce that

$$\begin{aligned} \int_0^\infty \left|\partial_x^\ell T_t^{\mathscr{A}}(x, y)\right| t^{k/2-1} dt &\leq C e^{|y|^2 - |x|^2} \sup_{t \in (0,\infty)} \frac{e^{-c\left|y - e^{-t} x\right|^2 / \left(1 - e^{-2t}\right)}}{\left(1 - e^{-2t}\right)^{(n+|\ell|)/2}} \int_0^\infty e^{-nt} t^{k/2-1} dt \\ &\leq C e^{|y|^2 - |x|^2} (1 + |x|)^{n+|\ell|}, (x, y) \in N^c. \end{aligned} \quad (79)$$

Note that this estimation also holds when $|\ell| = k$.

Since $f \in C_c^\infty(\mathbb{R}^n)$, (77) holds. Hence, according to ([15], Lemma 4.2), we have that, when $k \in \mathbb{N}$, $\ell \in \mathbb{N}^n$ and $|\ell| < k$,

$$\partial_x^\ell \mathscr{A}^{-k/2}(f)(x) = \frac{1}{\Gamma(k/2)} \int_{\mathbb{R}^n} f(y) \int_0^\infty \partial_x^\ell T_t^{\mathscr{A}}(x, y) t^{k/2-1} dt dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \quad (80)$$

We assume $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \setminus \{0\}$, so we can suppose without loss of generality that $\alpha_1 \geq 1$. Let us take $\ell = (\alpha_1 - 1, \alpha_2, \cdots, \alpha_n)$. According to (80), we can write

$$\partial_x^\alpha \mathscr{A}^{-|\alpha|/2}(f)(x) = \partial_{x_1} \left(\frac{1}{\Gamma(|\alpha|/2)} \int_{\mathbb{R}^n} f(y) \cdot \int_0^\infty \partial_x^\ell T_t^{\mathscr{A}}(x, y) t^{|\alpha|/2-1} dt dy \right), \quad \text{for almost all} \quad x \in \mathbb{R}^n. \quad (81)$$

Assume first $n > 1$ and write

$$\int_{\mathbb{R}^n} f(y) \int_0^\infty \partial_x^\ell T_t^{\mathscr{A}}(x, y) t^{|\alpha|/2-1} dt dy = F(x) + G(x), x \in \mathbb{R}^n, \quad (82)$$

where

$$F(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \partial_x^\ell \left[T_t^{\mathscr{A}}(x, y) - W_t(x - y)\right] t^{|\alpha|/2-1} dt dy, x \in \mathbb{R}^n, \quad (83)$$

and

$$G(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \partial_x^\ell W_t(x - y) t^{|\alpha|/2-1} dt dy, x \in \mathbb{R}^n. \quad (84)$$

Here, W_t, $t > 0$, denotes the classical heat kernel $W_t(z) = e^{-|z|^2/(2t)} / (2\pi t)^{n/2}$, $z \in \mathbb{R}^n$.

Next, we show that

$$\partial_{x_1} F(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \partial_x^\alpha \left[T_t^{\mathcal{A}}(x,y) - W_t(x-y) \right] t^{|\alpha|/2-1} dt dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{85}$$

By taking into account (76), we get

$$\begin{aligned} \int_{m(x)}^\infty & |\partial_x^\alpha [T_t^{\mathcal{A}}(x,y) - W_t(x-y)]| t^{|\alpha|/2-1} dt \\ & \leq C \int_{m(x)}^\infty \left(\frac{e^{-nt} t^{|\alpha|/2-1}}{(1-e^{-2t})^{(n+|\alpha|)/2}} + \frac{e^{-c|x-y|^2/t}}{t^{n/2+1}} \right) dt \\ & \leq C \int_{m(x)}^\infty \frac{dt}{t^{n/2+1}} \leq \frac{C}{m(x)^{n/2}} \\ & \leq C(1+|x|)^n, x, y \in \mathbb{R}^n. \end{aligned} \tag{86}$$

Also by using (79), we obtain

$$\begin{aligned} \int_0^{m(x)} & \left| \partial_x^\alpha [T_t^{\mathcal{A}}(x,y) - W_t(x-y)] \right| t^{|\alpha|/2-1} dt \\ & \leq C \left(e^{|y|^2-|x|^2} (1+|x|)^{n+|\alpha|} + \int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{n/2+1}} dt \right) \\ & \leq C \left(e^{|y|^2-|x|^2} (1+|x|)^{n+|\alpha|} + \frac{1}{|x-y|^{n+|\alpha|}} \right) \\ & \leq C e^{|y|^2} (1+|x|)^{n+|\alpha|}, (x,y) \in N^c. \end{aligned} \tag{87}$$

Now we are going to estimate

$$I(x,y) = \int_0^{m(x)} \left| \partial_x^\alpha [T_t^{\mathcal{A}}(x,y) - W_t(x-y)] \right| t^{|\alpha|/2-1} dt, (x,y) \in N. \tag{88}$$

We can write

$$\begin{aligned} \partial_x^\alpha [T_t^{\mathcal{A}}(x,y) - W_t(x-y)] &= \frac{(-1)^{|\alpha|}}{\pi^{n/2}} \left(\frac{e^{-nt}}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\alpha \left(\frac{x-e^{-t}y}{\sqrt{1-e^{-2t}}} \right) - \frac{1}{(2t)^{(n+|\alpha|)/2}} \tilde{H}_\alpha \left(\frac{x-y}{\sqrt{2t}} \right) \right) \\ &= \frac{(-1)^{|\alpha|}}{\pi^{n/2}} \left\{ \frac{e^{-nt}-1}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\alpha \left(\frac{x-e^{-t}y}{\sqrt{1-e^{-2t}}} \right) + \left(\frac{1}{(1-e^{-2t})^{(n+|\alpha|)/2}} - \frac{1}{(2t)^{(n+|\alpha|)/2}} \right) \tilde{H}_\alpha \right. \\ & \quad \left. \cdot \left(\frac{x-e^{-t}y}{\sqrt{1-e^{-2t}}} \right) + \frac{1}{(2t)^{(n+|\alpha|)/2}} \left(\tilde{H}_\alpha \left(\frac{x-e^{-t}y}{\sqrt{1-e^{-2t}}} \right) - \tilde{H}_\alpha \left(\frac{x-y}{\sqrt{2t}} \right) \right) \right\} = \sum_{j=1}^3 I_j(t,x,y), x, y \in \mathbb{R}^n. \end{aligned} \tag{89}$$

Then

$$I(x,y) \leq \sum_{j=1}^3 \int_0^{m(x)} \left| I_j(t,x,y) \right| t^{|\alpha|/2-1} dt = \sum_{j=1}^3 I_j(x,y), x, y \in \mathbb{R}^n. \tag{90}$$

By proceeding as in the proof of ([1], Lemma 3.3.1), we can see that

$$\frac{e^{-c\left|x-e^{(-t)}y\right|^2/\left(1-e^{(-2t)}\right)}}{(1-e^{-2t})^{n/2}} \leq C \frac{e^{-c|x-y|^2/t}}{t^{n/2}}, (x,y) \in N, t \in (0,1). \tag{91}$$

Then, by taking into account that

$$\left| e^{-nt} - 1 \right| \leq Ct \quad \text{and} \quad \left| \frac{1}{(1-e^{-2t})^{(n+|\alpha|)/2}} - \frac{1}{(2t)^{(n+|\alpha|)/2}} \right| \leq \frac{C}{t^{(n+|\alpha|)/2-1}}, t \in (0,1), \tag{92}$$

it follows by using (91) that

$$\begin{aligned} I_1(x,y) + I_2(x,y) & \leq C \int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{n/2}} dt \leq \frac{C}{|x-y|^{n-1/2}} \int_0^1 \frac{dt}{t^{1/4}} \\ & \leq \frac{C}{|x-y|^{n-1/2}}, (x,y) \in N. \end{aligned} \tag{93}$$

Let us analyze the term $I_3(x,y)$, $(x,y) \in N$. For every $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{R}^n$, we write

$$\begin{aligned} \tilde{H}_\alpha(z) - \tilde{H}_\alpha(w) &= \prod_{j=1}^n \tilde{H}_{\alpha_j}(z_j) - \prod_{j=1}^n \tilde{H}_{\alpha_j}(w_j) \\ &= \sum_{k=1}^n \left(\prod_{j=1}^{k-1} \tilde{H}_{\alpha_j}(w_j) \left(\tilde{H}_{\alpha_k}(z_k) - \tilde{H}_{\alpha_k}(w_k) \right) \prod_{j=k+1}^n \tilde{H}_{\alpha_j}(z_j) \right). \end{aligned} \tag{94}$$

Let $(x,y) \in N$, $t \in (0,1)$, and consider $z = (x - e^{-t}y)/\sqrt{1-e^{-2t}}$ and $w = (x-y)/\sqrt{2t}$. By taking into account that $(1+|x|+|y|)|x-y| \leq C$, it follows that $e^{-c|z|^2} \leq Ce^{-c|w|^2}$ and then, for each $k = 1, \cdots, n$,

$$\prod_{j=1}^{k-1}\left|\tilde{H}_{\alpha_j}(w_j)\right|\prod_{j=k+1}^{n}\left|\tilde{H}_{\alpha_j}(z_j)\right|\le C\exp\left(-c\sum_{j=1,j\neq k}^{n}\frac{\left|x_j-y_j\right|^2}{t}\right), \tag{95}$$

and, by considering also (1) and using the mean value theorem, we get

$$\begin{aligned}\left|\tilde{H}_{\alpha_k}(z_k)-\tilde{H}_{\alpha_k}(w_k)\right| &\le Ce^{-c|x_k-y_k|^2/t}|z_k-w_k|\\ &\le Ce^{-c|x_k-y_k|^2/t}\left(\frac{(1-e^{-t})|y_k|}{\sqrt{1-e^{-2t}}}+|x_k-y_k|\left|\frac{1}{\sqrt{1-e^{-2t}}}-\frac{1}{\sqrt{2t}}\right|\right)\\ &\le Ce^{-c|x_k-y_k|^2/t}\sqrt{t}(|y_k|+|x_k-y_k|)\\ &\le Ce^{-c|x_k-y_k|^2/t}\sqrt{t}(|x_k|+|x_k-y_k|)\le Ce^{-c|x_k-y_k|^2/t}\sqrt{t}(1+|x|).\end{aligned} \tag{96}$$

Then,

$$\begin{aligned}&\left|\tilde{H}_\alpha\left(\frac{x-e^{-t}y}{\sqrt{1-e^{-2t}}}\right)-\tilde{H}_\alpha\left(\frac{x-y}{2t}\right)\right|\\ &\quad\le Ce^{-c|x-y|^2/t}\sqrt{t}(1+|x|),(x,y)\in N, t\in(0,1),\end{aligned} \tag{97}$$

and thus,

$$\begin{aligned}I_3(x,y)&\le C(1+|x|)\int_0^{m(x)}\frac{e^{-c|x-y|^2/t}}{t^{(n+1)/2}}dt\le C\frac{1+|x|}{|x-y|^{n-1/2}}\int_0^{m(x)}t^{-3/4}dt\\ &=C\frac{(1+|x|)m(x)^{1/4}}{|x-y|^{n-1/2}}\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},(x,y)\in N.\end{aligned} \tag{98}$$

We deduce that

$$I(x,y)\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},(x,y)\in N. \tag{99}$$

This estimation, jointly with (86) and (87), leads to

$$\begin{aligned}&\int_{\mathbb{R}^n}|f(y)|\int_0^\infty\left|\partial_x^\alpha\left[T_t^{\mathscr{A}}(x,y)-W_t(x-y)\right]\right|t^{|\alpha|/2-1}dtdy\\ &\quad\le C\int_{\mathbb{R}^n}|f(y)|\left(e^{|y|^2}(1+|x|)^{n+|\alpha|}+\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}}\right)dy\\ &\quad\le C(1+|x|)^{n+|\alpha|}<\infty,x\in\mathbb{R}^n,\end{aligned} \tag{100}$$

where we have used that f has compact support. According to ([15], Lemma 4.2), (85) is then established.

We are going to evaluate $\partial_{x_1}G(x)$. We write $G(x)=\int_{\mathbb{R}^n}f(x-y)\Phi(y)dy$, $x\in\mathbb{R}^n$, where

$$\Phi(z)=\int_0^\infty\partial_z^\ell(W_t(z))t^{|\alpha|/2-1}dt,z\in\mathbb{R}^n. \tag{101}$$

Since $n>1$, we have that

$$|\Phi(z)|\le C\int_0^\infty\frac{e^{-c|z|^2/t}}{t^{(n+1)/2}}dt\le\frac{C}{|z|^{n-1}},z\in\mathbb{R}^n\setminus\{0\}. \tag{102}$$

According to ([15], Lemma 4.2), we can derivate under the integral sign obtaining

$$\begin{aligned}\partial_{x_1}G(x)&=\int_{\mathbb{R}^n}\partial_{x_1}(f(x-y))\Phi(y)dy\\ &=-\int_{\mathbb{R}^n}\partial_{y_1}(f(x-y))\Phi(y)dy,\quad\text{for almost all}\quad x\in\mathbb{R}^n,\end{aligned} \tag{103}$$

where the last integral is absolutely convergent.

For every $z=(z_1,\cdots,z_n)\in\mathbb{R}^n$, we define $\bar{z}=(z_2,\cdots,z_n)$. Partial integration leads to

$$\begin{aligned}\int_{\mathbb{R}^n}\partial_{y_1}(f(x-y))\Phi(y)dy&=\lim_{\varepsilon\longrightarrow0^+}\int_{|y|>\varepsilon}\partial_{y_1}(f(x-y))\Phi(y)dy\\ &=\lim_{\varepsilon\longrightarrow0^+}\left(\int_{|\bar{y}|<\varepsilon}\int_{-\infty}^{-\sqrt{\varepsilon^2-|\bar{y}|^2}}+\int_{|\bar{y}|<\varepsilon}\int_{\sqrt{\varepsilon^2-|\bar{y}|^2}}^{+\infty}+\int_{|\bar{y}|>\varepsilon}\int_{\mathbb{R}}\right)\partial_{y_1}(f(x-y))\Phi(y)dy_1d\bar{y}\\ &=\lim_{\varepsilon\longrightarrow0^+}\left(-\int_{|y|>\varepsilon}f(x-y)\partial_{y_1}\Phi(y)dy+\int_{|\bar{y}|<\varepsilon}f(x-y)\Phi(y)\Big]_{y_1=-\infty}^{y_1=-\sqrt{\varepsilon^2-|\bar{y}|^2}}d\bar{y}+\int_{|\bar{y}|<\varepsilon}f(x-y)\Phi(y)\Big]_{y_1=\sqrt{\varepsilon^2-|\bar{y}|^2}}^{y_1=+\infty}d\bar{y}\right)\\ &=\lim_{\varepsilon\longrightarrow0^+}\left(-\int_{|\bar{y}|>\varepsilon}f(x-y)\partial_{y_1}\Phi(y)dy+J_\varepsilon(x)\right),x\in\mathbb{R}^n,\end{aligned} \tag{104}$$

where

$$J_\varepsilon(x) = \int_{|\bar{y}|<\varepsilon} f\left(x_1 + \sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{x} - \bar{y}\right) \Phi\left(-\sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{y}\right) d\bar{y} - \int_{|\bar{y}|<\varepsilon} f\left(x_1 - \sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{x} - \bar{y}\right) \Phi\left(\sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{y}\right) d\bar{y}, x \in \mathbb{R}^n. \tag{105}$$

Let us estimate $\lim_{\varepsilon \longrightarrow 0^+} J_\varepsilon(x)$, $x \in \mathbb{R}^n$. We recall that Φ can be written as follows

$$\Phi(z) = \frac{(-1)^{|\alpha|-1}}{2^{|\ell|/2}(2\pi)^{n/2}} \int_0^\infty \tilde{H}_\ell\left(\frac{z}{\sqrt{2t}}\right) \frac{dt}{t^{(n+1)/2}}, z \in \mathbb{R}^n. \tag{106}$$

Suppose now that α_1 is odd. Then,

$$\Phi\left(-\sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{y}\right) = \Phi\left(\sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{y}\right), \bar{y} \in \mathbb{R}^{n-1}, |\bar{y}| < \varepsilon. \tag{107}$$

We have that, for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$\begin{aligned} J_\varepsilon(x) &= \int_{|\bar{y}|<\varepsilon} \left(f\left(x_1 + \sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{x} - \bar{y}\right) - f\left(x_1 - \sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{x} - \bar{y}\right)\right) \Phi\left(\sqrt{\varepsilon^2 - |\bar{y}|^2}, \bar{y}\right) d\bar{y} \\ &= \varepsilon^{n-1} \int_{|\bar{z}|<1} \left(f\left(x_1 + \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right) - f\left(x_1 - \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right)\right) \Phi\left(\varepsilon\sqrt{1 - |\bar{z}|^2}, \varepsilon\bar{z}\right) d\bar{z}. \end{aligned} \tag{108}$$

On the other hand, by performing the change of variable $s = \varepsilon^2/(2t)$, we get

$$\begin{aligned} \Phi\left(\varepsilon\sqrt{1 - |\bar{z}|^2}, \varepsilon\bar{z}\right) &= \frac{(-1)^{|\alpha|-1}}{2^{|\ell|/2}(2\pi)^{n/2}} \int_0^\infty \tilde{H}_{\alpha_1-1}\left(\frac{\varepsilon\sqrt{1 - |\bar{z}|^2}}{\sqrt{2t}}\right) \prod_{i=2}^n \tilde{H}_{\alpha_i} \cdot \left(\frac{\varepsilon z_i}{\sqrt{2t}}\right) \frac{dt}{t^{(n+1)/2}} \\ &= \frac{(-1)^{|\alpha|-1}\varepsilon^{1-n}}{2^{|\alpha|/2}\pi^{n/2}} \int_0^\infty \tilde{H}_{\alpha_1-1}\left(\sqrt{s\left(1 - |\bar{z}|^2\right)}\right) \prod_{i=2}^n \tilde{H}_{\ell_i} \cdot \left(z_i\sqrt{s}\right) s^{(n-3)/2} ds, \varepsilon > 0, \bar{z} \in \mathbb{R}^{n-1}, |z| < 1. \end{aligned} \tag{109}$$

It follows that

$$J_\varepsilon(x) = \frac{(-1)^{|\alpha|-1}}{2^{|\alpha|/2}\pi^{n/2}} \int_{|\bar{z}|<1} \left(f\left(x_1 + \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right) - f\left(x_1 - \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right)\right) \times \int_0^\infty H_{\alpha_1-1} \cdot \left(\sqrt{s\left(1 - |\bar{z}|^2\right)}\right) \prod_{i=2}^n H_{\ell_i}\left(z_i\sqrt{s}\right) e^{-s} s^{(n-3)/2} ds d\bar{z}, x \in \mathbb{R}^n, \varepsilon > 0. \tag{110}$$

Then, by using the dominated convergence theorem, we obtain

$$\lim_{\varepsilon \longrightarrow 0^+} J_\varepsilon(x) = 0, x \in \mathbb{R}^n. \tag{111}$$

Suppose now that α_1 is even. Then,

$$\Phi\left(-\varepsilon\sqrt{1 - |\bar{y}|^2}, \bar{y}\right) = -\Phi\left(\varepsilon\sqrt{1 - |\bar{y}|^2}, \bar{y}\right), \varepsilon > 0, \bar{y} \in \mathbb{R}^{n-1}, |\bar{y}| < \varepsilon, \tag{112}$$

and proceeding as above, we get

$$J_\varepsilon(x) = \frac{(-1)^{|\alpha|}}{2^{|\alpha|/2}\pi^{n/2}} \int_{|\bar{z}|<1} \left(f\left(x_1 + \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right) + f\left(x_1 - \varepsilon\sqrt{1 - |\bar{z}|^2}, \bar{x} - \varepsilon\bar{z}\right)\right) \times \int_0^\infty H_{\alpha_1-1}\left(\sqrt{s\left(1 - |\bar{z}|^2\right)}\right) \prod_{i=2}^n H_{\alpha_i}\left(z_i\sqrt{s}\right) e^{-s} s^{(n-3)/2} ds d\bar{z}, x \in \mathbb{R}^n, \varepsilon > 0. \tag{113}$$

It follows that

$$\begin{aligned} \lim_{\varepsilon \longrightarrow 0^+} J_\varepsilon(x) &= \frac{(-1)^{|\alpha|}}{2^{|\alpha|/2-1}\pi^{n/2}} f(x) \int_{|\bar{z}|<1} \int_0^\infty H_{\alpha_1-1} \cdot \left(\sqrt{s\left(1 - |z|^2\right)}\right) \prod_{i=2}^{n-1} H_{\alpha_i}\left(z_i\sqrt{s}\right) e^{-s} s^{(n-3)/2} ds d\bar{z} \\ &= -c_\alpha f(x), x \in \mathbb{R}^n. \end{aligned} \tag{114}$$

We note that if α_i is odd for some $i = 2, \cdots, n$, then $c_\alpha = 0$. Thus, we conclude that

$$\begin{aligned} \partial_{x_1} G(x) &= \lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} f(y) \partial_{x_1} \Phi(x - y) dy + c_\alpha f(x) \\ &= \lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} f(y) \int_0^\infty \partial_x^\alpha W_t(x - y) t^{|\alpha|/2-1} dt dy + c_\alpha f(x), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \end{aligned} \tag{115}$$

where $c_\alpha = 0$ when α_i is odd for some $i = 1, \cdots, n$.

By putting together (85) and (115), we obtain

$$
\begin{aligned}
\partial_x^\alpha \mathscr{A}^{-|\alpha|/2}(f)(x) &= \frac{1}{\Gamma(|\alpha|/2)}\partial_{x_1}(F(x)+G(x)) \\
&= \frac{1}{\Gamma(|\alpha|/2)}\left(\int_{\mathbb{R}^n} f(y)\int_0^\infty \partial_x^\alpha\left[T_t^{\mathscr{A}}(x,y)-W_t(x-y)\right]t^{|\alpha|/2-1}dtdy\right. \\
&\quad \left.+\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} f(y)\int_0^\infty \partial_x^\alpha W_t(x-y)t^{|\alpha|/2-1}dtdy + c_\alpha f(x)\right) \\
&= \frac{1}{\Gamma(|\alpha|/2)}\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} f(y)\int_0^\infty \partial_x^\alpha T_t^{\mathscr{A}}(x,y))t^{|\alpha|/2-1}dtdy \\
&\quad + c_\alpha f(x), \quad \text{for almost all} \quad x\in\mathbb{R}^n.
\end{aligned} \tag{116}
$$

We now deal with the case $n=1$. We have $\alpha\in\mathbb{N}$, $\alpha\geq 1$. According to (80), we can write

$$
\begin{aligned}
\frac{d^\alpha}{dx^\alpha}\mathscr{A}^{-\alpha/2}(f)(x) &= \frac{1}{\Gamma(\alpha/2)}\frac{d}{dx}\int_{\mathbb{R}^n} f(y)\int_0^\infty \partial_x^{\alpha-1}T_t^{\mathscr{A}}(x,y)t^{\alpha/2-1}dtdy \\
&= \frac{1}{\Gamma(\alpha/2)}\frac{d}{dx}\left(\bar{F}(x)+\bar{G}(x)\right), x\in\mathbb{R},
\end{aligned} \tag{117}
$$

where

$$
\begin{aligned}
\bar{F}(x) = \int_{\mathbb{R}} f(y)\int_0^\infty &\left[\partial_x^{\alpha-1}T_t^{\mathscr{A}}(x,y) - \left(\left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(x-y)\right.\right. \\
&\left.\left. - \left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(0)\chi_{(1,\infty)}(t)\right)\right]t^{\alpha/2-1}dtdy, x\in\mathbb{R},
\end{aligned} \tag{118}
$$

and

$$
\begin{aligned}
\bar{G}(x) = \int_{\mathbb{R}} f(y)\int_0^\infty &\left(\left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(x-y)\right. \\
&\left. - \left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(0)\chi_{(1,\infty)}(t)\right)t^{\alpha/2-1}dtdy, x\in\mathbb{R}.
\end{aligned} \tag{119}
$$

Note that if α is even, then $(d^{\alpha-1}/dx^{\alpha-1})(W_t)(0)=0$.

By proceeding as in the case of $n>1$ and taking into account that

$$
\begin{aligned}
\left|\left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(x-y) - \left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(0)\right| &\leq \frac{C}{t^{\alpha/2}}\left|\tilde{H}_{\alpha-1}\left(\frac{x-y}{\sqrt{2t}}\right)-\tilde{H}_{\alpha-1}(0)\right| \\
&\leq C\frac{|x-y|}{t^{(\alpha+1)/2}}, x,y\in\mathbb{R}, t>1,
\end{aligned} \tag{120}
$$

we can see that the integral defining $\bar{F}(x)$ is absolutely convergent for every $x\in\mathbb{R}$ and

$$
\frac{d}{dx}\bar{F}(x) = \int_{\mathbb{R}} f(y)\int_0^\infty \partial_x^\alpha\left[T_t^{\mathscr{A}}(x,y)-W_t(x-y)\right]t^{\alpha/2-1}dtdy, \tag{121}
$$

being also this integral absolutely convergent for almost $x\in\mathbb{R}$.

On the other hand, by considering

$$
\begin{aligned}
\bar{\Phi}(y) &= \int_0^\infty \left(\left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(y) - \left(\frac{d^{\alpha-1}}{dx^{\alpha-1}}W_t\right)(0)\chi_{(1,\infty)}(t)\right)t^{\alpha/2-1}dt \\
&= \frac{(-1)^{\alpha-1}}{2^{\alpha/2}\sqrt{\pi}}\int_0^\infty \left(\tilde{H}_{\alpha-1}\left(\frac{y}{\sqrt{2t}}\right)-\tilde{H}_{\alpha-1}(0)\chi_{(1,\infty)}(t)\right)\frac{dt}{t}, y\in\mathbb{R}\setminus\{0\},
\end{aligned} \tag{122}
$$

we can write

$$
\bar{G}(x) = \int_{\mathbb{R}} f(x-y)\bar{\Phi}(y)dy, x\in\mathbb{R}^n, \tag{123}
$$

and according to ([15], Lemma 4.2), we get

$$
\begin{aligned}
\frac{d}{dx}\bar{G}(x) &= -\int_{\mathbb{R}} \partial_y(f(x-y))\bar{\Phi}(y)dy \\
&= -\lim_{\varepsilon\longrightarrow 0^+}\int_{|y|>\varepsilon} \partial_y(f(x-y))\bar{\Phi}(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}.
\end{aligned} \tag{124}
$$

By partial integration, we obtain

$$
\begin{aligned}
\int_{|y|>\varepsilon} \partial_y(f(x-y))\bar{\Phi}(y)dy &= -\int_{|y|>\varepsilon} f(x-y)\bar{\Phi}'(y)dy \\
&\quad + f(x+\varepsilon)\bar{\Phi}(-\varepsilon) - f(x-\varepsilon)\bar{\Phi}(\varepsilon), \varepsilon>0, x\in\mathbb{R}^n.
\end{aligned} \tag{125}
$$

Then

$$
\frac{d}{dx}\bar{G}(x) = \lim_{\varepsilon\longrightarrow 0^+}\left(\int_{|y|>\varepsilon} f(x-y)\bar{\Phi}'(y)dy + \bar{J}_\varepsilon(x)\right), \quad \text{for almost all} \quad x\in\mathbb{R}, \tag{126}
$$

where

$$
\bar{J}_\varepsilon(x) = f(x-\varepsilon)\bar{\Phi}(\varepsilon) - f(x+\varepsilon)\bar{\Phi}(-\varepsilon), \varepsilon>0, x\in\mathbb{R}. \tag{127}
$$

By taking into account (10) and that $\tilde{H}_\alpha(z)\leq Ce^{-cz^2}$, $z\in\mathbb{R}$, it follows that

$$
\begin{aligned}
\bar{\Phi}'(y) &= \frac{(-1)^\alpha}{2^{(\alpha+1)/2}\sqrt{\pi}}\int_0^\infty \tilde{H}_\alpha\left(\frac{y}{\sqrt{2t}}\right)t^{-3/2}dt \\
&= \int_0^\infty \left(\frac{d^\alpha}{dx^\alpha}W_t\right)(y)t^{\alpha/2-1}dt, y\in\mathbb{R}\setminus\{0\}.
\end{aligned} \tag{128}
$$

On the other hand, we have that

$$\bar{\Phi}(y)=\frac{(-1)^{\alpha-1}}{2^{\alpha/2}\sqrt{\pi}}\left(\int_0^1\tilde{H}_{\alpha-1}\left(\frac{y}{\sqrt{2t}}\right)\frac{dt}{t}+\int_1^\infty\left(\tilde{H}_{\alpha-1}\left(\frac{y}{\sqrt{2t}}\right)-\tilde{H}_{\alpha-1}(0)\right)\frac{dt}{t}\right)$$
$$=\frac{(-1)^{\alpha-1}}{2^{\alpha/2}\sqrt{\pi}}\left(\int_{\frac{y^2}{2}}^\infty\tilde{H}_{\alpha-1}(\sqrt{s})\frac{ds}{s}+\int_0^{y^2/2}\left(\tilde{H}_{\alpha-1}(\sqrt{s})-\tilde{H}_{\alpha-1}(0)\right)\frac{ds}{s}\right), y>0, \tag{129}$$

and,

$$\bar{\Phi}(y)=\frac{1}{2^{\alpha/2}\sqrt{\pi}}\left(\int_{y^2/2}^\infty\tilde{H}_{\alpha-1}(\sqrt{s})\frac{ds}{s}+\int_0^{y^2/2}\left(\tilde{H}_{\alpha-1}(\sqrt{s})-\tilde{H}_{\alpha-1}(0)\right)\frac{ds}{s}\right), y<0. \tag{130}$$

Let $x\in\mathbb{R}$. Since $f\in C_c^\infty(\mathbb{R}^n)$, we get

$$|f(x+\varepsilon)-f(x-\varepsilon)|\le C\varepsilon, \varepsilon>0. \tag{131}$$

If α is odd, then $\bar{\Phi}$ is even and, for every $\varepsilon\in(0,1)$,

$$|\bar{J}_\varepsilon(x)|=|(f(x+\varepsilon)-f(x-\varepsilon))\bar{\Phi}(\varepsilon)|$$
$$\le C\varepsilon\left(\int_{\varepsilon^2/2}^\infty|\tilde{H}_{\alpha-1}(\sqrt{s})|\frac{ds}{s}+\int_0^{\varepsilon^2/2}|\tilde{H}_{\alpha-1}(\sqrt{s})-\tilde{H}_{\alpha-1}(0)|\frac{ds}{s}\right)$$
$$\le C\varepsilon\left(\int_1^\infty e^{-s}ds+\int_{\varepsilon^2/2}^1\frac{ds}{s}+\int_0^{\varepsilon^2/2}\sqrt{s}\frac{ds}{s}\right)\le C\varepsilon(1+|\log\varepsilon|+\varepsilon). \tag{132}$$

Hence, if α is odd, $\lim_{\varepsilon\longrightarrow0^+}\bar{J}_\varepsilon(x)=0$.
If α is even, then $\bar{\Phi}$ is odd, $H_{\alpha-1}(0)=0$, and

$$\bar{\Phi}(\varepsilon)=-\frac{1}{2^{\alpha/2}\sqrt{\pi}}\int_0^\infty\tilde{H}_{\alpha-1}(\sqrt{s})\frac{ds}{s}, \varepsilon>0. \tag{133}$$

We obtain

$$\lim_{\varepsilon\longrightarrow0^+}\bar{J}_\varepsilon(x)=-\frac{f(x)}{2^{\alpha/2-1}\sqrt{\pi}}\int_0^\infty\tilde{H}_{\alpha-1}(\sqrt{s})\frac{ds}{s}, \tag{134}$$

provided that α is even.
We conclude that, for certain $c_\alpha\in\mathbb{R}$

$$\frac{d}{dx}G(x)=\lim_{\varepsilon\longrightarrow0^+}\int_{|x-y|>\varepsilon}f(y)\int_0^\infty\left(\frac{d^\alpha}{dx^\alpha}W_t\right)(x-y)t^{\alpha/2-1}dtdy$$
$$+c_\alpha f(x),\quad \text{for almost all}\quad x\in\mathbb{R}. \tag{135}$$

Then, we get

$$\frac{d^\alpha}{dx^\alpha}\mathscr{A}^{-\alpha/2}(f)(x)=\frac{1}{\Gamma(\alpha/2)}\lim_{\varepsilon\longrightarrow0^+}\int_{|x-y|>\varepsilon}f(y)\int_0^\infty\partial_x^\alpha T_t^{\mathscr{A}}(x,y)t^{\alpha/2-1}dtdy$$
$$+c_\alpha f(x),\quad \text{for almost all}\quad x\in\mathbb{R}, \tag{136}$$

where $c_\alpha=0$ when α is odd.

Finally, we observe that when α is even

$$\int_0^\infty\left(\frac{d^\alpha}{dz^\alpha}W_t\right)(z)t^{\alpha/2-1}dt=\frac{1}{2^{(\alpha+1)/2}\sqrt{\pi}}\int_0^\infty e^{-z^2/2t}H_\alpha\left(\frac{|z|}{\sqrt{2t}}\right)t^{-3/2}dt$$
$$=\frac{1}{2^{\alpha/2-1}\sqrt{\pi}}\frac{1}{|z|}\int_0^\infty e^{-s^2}H_\alpha(s)ds=0. \tag{137}$$

By taking into account the arguments developed in this proof, we can see that

$$\int_{\mathbb{R}}\left|f(y)\int_0^\infty\partial_x^\alpha T_t^{\mathscr{A}}(x,y)t^{\alpha/2-1}dt\right|dy$$
$$=\int_{\mathbb{R}}\left|f(y)\int_0^\infty\partial_x^\alpha\left(T_t^{\mathscr{A}}(x,y)-W_t(x-y)\right)t^{\alpha/2-1}dt\right|dy<\infty, \tag{138}$$

for every $x\in\mathbb{R}$. Then,

$$\frac{d^\alpha}{dx^\alpha}\mathscr{A}^{-\alpha/2}(f)(x)=\frac{1}{\Gamma(\alpha/2)}\int_{\mathbb{R}}f(y)\int_0^\infty\partial_x^\alpha T_t^{\mathscr{A}}(x,y)t^{\alpha/2-1}dtdy$$
$$+c_\alpha f(x),\quad \text{for almost all}\quad x\in\mathbb{R}. \tag{139}$$

The proof of Theorem 1 is completed.

Proof of Theorem 2. For every $f\in C_c^\infty(\mathbb{R}^n)$, we have that

$$\mathscr{A}^{-|\alpha|/2}(f)(x)=\sum_{k\in\mathbb{N}^n}\frac{c_k(f)}{(|k|+n)^{|\alpha|/2}}\tilde{H}_k(x), x\in\mathbb{R}^n, \tag{140}$$

and according to (36), (37), and (40), the last series is pointwise absolutely convergent, it defines a smooth function on $\mathbb{R}^n$ and

$$\partial_x^\alpha\mathscr{A}^{-|\alpha|/2}(f)(x)=\sum_{k\in\mathbb{N}^n}\frac{c_k(f)}{(|k|+n)^{|\alpha|/2}}\partial_x^\alpha\tilde{H}_k(x)$$
$$=(-1)^{|\alpha|}\sum_{k\in\mathbb{N}^n}\frac{c_k(f)}{(|k|+n)^{|\alpha|/2}}\tilde{H}_{k+\alpha}(x), x\in\mathbb{R}^n. \tag{141}$$

Then, according to Theorem 1, for each $f\in C_c^\infty(\mathbb{R}^n)$,

$$R_\alpha(f)(x)=\lim_{\varepsilon\longrightarrow0^+}\int_{|x-y|>\varepsilon}R_\alpha(x,y)f(y)dy+c_\alpha f(x),\quad \text{for almost all}\quad x\in\mathbb{R}^n. \tag{142}$$

Here, $c_\alpha=0$ when α_i is odd for some $i=1,\cdots,n$.
To establish our result, it is sufficient to show that for every $f\in L^p(\mathbb{R}^n,\gamma_{-1})$ the limit

$$\lim_{\varepsilon\longrightarrow0^+}\int_{|x-y|>\varepsilon}R_\alpha(x,y)f(y)dy, \tag{143}$$

exists for almost all $x\in\mathbb{R}^n$ and the operator L_α defined by

$$L_\alpha(f)(x) = \lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} R_\alpha(x,y) f(y)dy + c_\alpha f(x), \quad (144)$$

is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself. Thus, L_α is the unique extension of R_α from $L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

For every $\beta > 0$ we define the set

$$N_\beta = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \le \beta n \min\left\{1, \frac{1}{|x|}\right\}\right\}. \quad (145)$$

Observe that $N_1 = N$ and that if $\beta > 0$, $a \in (0,1)$ and $(x,y) \in N_\beta^c$, then

$$|ax - ay| \ge a\beta n \min\left\{1, \frac{1}{|x|}\right\} \ge a^2 \beta n \min\left\{1, \frac{1}{|ax|}\right\}, \quad (146)$$

that is, $(ax, ay) \in N_{a^2\beta}^c$. In particular, we have that if $a \in (0,1)$ then $(ax, ay) \in N^c$ provided that $(x,y) \in N_{1/a^2}^c$.

Let $\beta > 0$. We consider the operators $R_{\alpha,\text{loc}}$ and $R_{\alpha,glob}$ defined on $C_c^\infty(\mathbb{R}^n)$ by

$$\begin{aligned} R_{\alpha,\text{loc}}(f)(x) &= R_\alpha\left(f\chi_{N_\beta}(x,\cdot)\right)(x), R_{\alpha,\text{glob}}(f)(x) \\ &= R_\alpha\left(f\chi_{N_\beta^c}(x,\cdot)\right)(x), x \in \mathbb{R}^n. \end{aligned} \quad (147)$$

□

We recall that

$$\begin{aligned} R_\alpha(x,y) &= \frac{(-1)^{|\alpha|}}{\pi^{n/2}\Gamma(|\alpha|/2)} \int_0^\infty \frac{e^{-nt}}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\ell \\ &\cdot \left(\frac{x - e^{-t}y}{\sqrt{1-e^{-2t}}}\right) t^{|\alpha|/2-1} dt, x, y \in \mathbb{R}^n, x \ne y. \end{aligned} \quad (148)$$

Then,

$$|R_\alpha(x,y)| \le C \int_0^\infty \frac{e^{-nt} e^{-\eta\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)}}{\left(1-e^{-2t}\right)^{(n+|\alpha|)/2}} t^{|\alpha|/2-1} dt, x, y \in \mathbb{R}^n, x \ne y, \quad (149)$$

for every $\eta \in (0,1)$. From now on we consider $1/p < \eta < 1$ and $\beta = 1/\eta$. Since $|x - ry|^2 = |y - rx|^2 + (1 - r^2)(|x|^2 - |y|^2)$, $x, y \in \mathbb{R}^n$, $r \in \mathbb{R}$, by making the change of variables $s = 1 - e^{-2t}$ in the last integral, we get

$$\begin{aligned} |R_\alpha(x,y)| &\le C e^{\eta\left(|y|^2 - |x|^2\right)} \int_0^1 \frac{e^{-\eta|y - x\sqrt{1-s}|^2/s}}{s^{(n+|\alpha|)/2}} (1-s)^{n/2-1} \\ &\cdot (-\log(1-s))^{|\alpha|/2-1} ds, x, y \in \mathbb{R}^n, x \ne y. \end{aligned} \quad (150)$$

Assume first that $(x,y) \in N_\beta^c$. Then, $(\sqrt{\eta}x, \sqrt{\eta}y) \in N^c$. By proceeding as in the proof of (66), it follows that, when $\langle x, y\rangle \le 0$,

$$|R_\alpha(x,y)| \le C e^{-\eta|x|^2} \int_0^\infty \frac{e^{-r/2}(1+r)^{(n+|\alpha|)/2-1/4}}{r^{3/4}} dr \le C e^{-\eta|x|^2}. \quad (151)$$

Suppose now that $\langle x, y\rangle > 0$. Again, as in the estimation in (66), since $(\sqrt{\eta}x, \sqrt{\eta}y) \in N^c$, we obtain

$$\begin{aligned} &\int_{1/2}^1 \frac{e^{-\eta|y - x\sqrt{1-s}|^2/s}}{s^{(n+|\alpha|)/2}} (1-s)^{n/2-1} (-\log(1-s))^{|\alpha|/2-1} ds \\ &\quad \le C\left(\frac{|x+y|}{|x-y|}\right)^{n/2} \exp\left(\frac{\eta}{2}\left(|y|^2 - |x|^2 - |x+y||x-y|\right)\right). \end{aligned} \quad (152)$$

On the other hand, proceeding as in ([23], p. 862) and considering the notation in (61) and ([23], Lemma 2.3), we have

$$\begin{aligned} &\int_0^{1/2} \frac{e^{-\eta|y - x\sqrt{1-s}|^2/s}}{s^{(n+|\alpha|)/2}} (1-s)^{n/2-1} (-\log(1-s))^{|\alpha|/2-1} ds \\ &\quad \le C \sup_{s \in (0,1)} \left(\frac{e^{-\eta u(s)}}{s^{n/2}}\right)^{1-1/n} \int_0^{1/2} \frac{e^{-\eta u(s)/n}}{\sqrt{s}} \frac{(-\log(1-s))^{|\alpha|/2-1}}{s^{|\alpha|/2}} ds \\ &\quad \le C\left(\frac{e^{-\eta u_0}}{s_0^{n/2}}\right)^{1-1/n} \int_0^{1/2} \frac{e^{-u(s)/n}}{s^{3/2}} ds \\ &\quad \le C\left(\frac{e^{-\eta u_0}}{s_0^{n/2}}\right)^{1-1/n} \int_0^1 \frac{e^{-u(s)/n}}{s^{3/2}\sqrt{1-s}} ds \\ &\quad \le C\left(\frac{e^{-\eta u_0}}{s_0^{n/2}}\right)^{1-1/n} \frac{e^{-\eta u_0/n}}{\sqrt{s_0}} \le C \frac{e^{-\eta u_0}}{s_0^{n/2}} \\ &\quad \le C\left(\frac{|x+y|}{|x-y|}\right)^{n/2} \exp\left(\frac{\eta}{2}\left(|y|^2 - |x|^2 - |x+y||x-y|\right)\right). \end{aligned} \quad (153)$$

From the above estimates, we conclude that, when $(x,y) \in N_\beta^c$,

$$|R_\alpha(x,y)| \le C \begin{cases} e^{-\eta|x|^2}, & \langle x,y\rangle \le 0, \\ \left(\frac{|x+y|}{|x-y|}\right)^{n/2} \exp\left(\eta\left(\frac{|y|^2 - |x|^2}{2} - \frac{|x-y||x+y|}{2}\right)\right), & \langle x,y\rangle > 0. \end{cases} \quad (154)$$

On the other hand, we consider the kernel

$$\mathbb{R}_\alpha(x,y) = \frac{1}{\Gamma(|\alpha|/2)} \int_0^\infty \partial_x^\alpha W_t(x-y) t^{|\alpha|/2-1} dt, x, y \in \mathbb{R}^n, x \ne y. \quad (155)$$

Let us show that

$$|R_\alpha(x,y)-\mathbb{R}_\alpha(x,y)|\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta, x\neq y. \tag{156}$$

For every $x\in\mathbb{R}^n$, we have that

$$\begin{aligned}R_\alpha(x,y)-\mathbb{R}_\alpha(x,y)&=\frac{1}{\Gamma(|\alpha|/2)}\left(\int_0^{m(x)}+\int_{m(x)}^\infty\right)\partial_x^\alpha\left(T_t^{\mathcal{A}}(x,y)\right.\\&\quad\left.-W_t(x-y)\right)t^{|\alpha|/2-1}dt\\&=I(x,y)+J(x,y), x,y\in\mathbb{R}^n, x\neq y.\end{aligned} \tag{157}$$

The same proof of (99) allows us to obtain that

$$|I(x,y)|\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta, x\neq y. \tag{158}$$

Also, we get

$$\begin{aligned}|J(x,y)|&\le C\left(\int_{m(x)}^\infty\frac{e^{-nt}}{\left(1-e^{-2t}\right)^{(n+|\alpha|)/2}}\left|\tilde{H}_\alpha\left(\frac{x-ye^{-t}}{\sqrt{1-e^{-2t}}}\right)\right|t^{|\alpha|/2-1}dt\right.\\&\quad\left.+\int_{m(x)}^\infty\left|\tilde{H}_\alpha\left(\frac{x-y}{\sqrt{2t}}\right)\right|\frac{dt}{t^{n/2+1}}\right)\\&\le C\left(\int_{m(x)}^\infty e^{-nt}\frac{e^{-c\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)}}{\left(1-e^{-2t}\right)^{(n+|\alpha|)/2}}t^{|\alpha|/2-1}dt+\int_{m(x)}^\infty\frac{e^{-c|x-y|^2/(2t)}}{t^{n/2+1}}dt\right)\\&\le C\int_{m(x)}^\infty\frac{dt}{t^{n/2+1}}=\frac{C}{m(x)^{n/2}}\le C(1+|x|)^n\\&\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta, x\neq y,\end{aligned} \tag{159}$$

and thus, (156) is established.

From (154), we obtain that $R_{\alpha,glob}$ can be extended to $L^p(\mathbb{R}^n,\gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself, and the extension is given by (147). Indeed, when $(x,y)\in N_\beta^c$, we have that $|x-y||x+y|\ge C$. By taking into account also that $\left||y|^2-|x|^2\right|\le|x+y||x-y|$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}|R_\alpha(x,y)|e^{\left(|x|^2-|y|^2\right)/p}\chi_{N_\beta^c}(x,y)dy\le C\left(\int_{\mathbb{R}^n}e^{-(\eta-1/p)|x|^2-|y|^2/p}dy\right.\\&\quad\left.+\int_{\mathbb{R}^n}|x+y|^n\exp\left(\left(\frac{\eta}{2}-\frac{1}{p}\right)\left(|y|^2-|x|^2\right)-\frac{\eta}{2}|x-y||x+y|\right)dy\right)\\&\le C\left(e^{-(\eta-1/p)|x|^2}\int_{\mathbb{R}^n}e^{-|y|^2/p}dy\right.\\&\quad\left.+\int_{\mathbb{R}^n}|x+y|^n\exp\left(-|x-y||x+y|\left(\frac{\eta}{2}-\left|\frac{1}{p}-\frac{\eta}{2}\right|\right)\right)dy\right), x\in\mathbb{R}^n.\end{aligned} \tag{160}$$

Since $\eta>1/p$ it follows that

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|R_\alpha(x,y)|e^{\left(|x|^2-|y|^2\right)/p}\chi_{N_\beta^c}(x,y)dy<\infty, \tag{161}$$

and in a similar way

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|R_\alpha(x,y)|e^{\left(|x|^2-|y|^2\right)/p}\chi_{N_\beta^c}(x,y)dx<\infty, \tag{162}$$

from which we deduce that $R_{\alpha,glob}$ is bounded on $L^p(\mathbb{R}^n,\gamma_{-1})$.

On the other hand, by using (156) and that $\sqrt{m(x)}\sim 1/(1+|x|)$, $x\in\mathbb{R}^n$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^n}|R_\alpha(x,y)-\mathbb{R}_\alpha(x,y)|\chi_{N_\beta}(x,y)dy\\&\quad\le C\sqrt{1+|x|}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1/2}}\chi_{N_\beta}(x,y)dy\\&\quad=C\sqrt{1+|x|}\int_0^{\beta n\sqrt{m(x)}}r^{-1/2}dr\\&\quad=C\sqrt{(1+|x|)\sqrt{m(x)}}\le C, x\in\mathbb{R}^n.\end{aligned} \tag{163}$$

Also, since $m(x)\sim m(y)$, $(x,y)\in N_\beta$, we get

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|R_\alpha(x,y)-\mathbb{R}_\alpha(x,y)|\chi_{N_\beta}(x,y)dx<\infty. \tag{164}$$

Hence, the operator S_α defined by

$$S_\alpha(f)(x)=\int_{\mathbb{R}^n}\left(R_\alpha(x,y)-\mathbb{R}_\alpha(x,y)\right)\chi_{N_\beta}(x,y)f(y)dy, x\in\mathbb{R}^n, \tag{165}$$

is a bounded operator from $L^p(\mathbb{R}^n,dx)$ into itself. Since S_α is a local operator, by ([1], Proposition 3.2.5) S_α is bounded from $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself.

We now observe that the kernel $\mathbb{R}_\alpha$ is a standard Calderón-Zygmund kernel. Indeed, we get

$$\begin{aligned}|\mathbb{R}_\alpha(x,y)|&\le C\int_0^\infty\left|H_\alpha\left(\frac{x-y}{\sqrt{2t}}\right)\right|\frac{e^{-|x-y|^2/(2t)}}{t^{n/2+1}}dt\le C\int_0^\infty\frac{e^{-c|x-y|^2/t}}{t^{n/2+1}}dt\\&\le\frac{C}{|x-y|^n}, x,y\in\mathbb{R}^n, x\neq y.\end{aligned} \tag{166}$$

Let $i=1,\cdots,n$ and denote $\alpha^i=(\alpha_1,\cdots,\alpha_i+1,\cdots,\alpha_n)$. We have that

$$\partial_{x_i}\mathbb{R}_\alpha(x,y)=\frac{(-1)^{|\alpha|+1}}{(2\pi)^{n/2}2^{(|\alpha|+1)/2}\Gamma(|\alpha|/2)}\int_0^\infty H_{\alpha^i}\left(\frac{x-y}{\sqrt{2t}}\right)\frac{e^{-|x-y|^2/(2t)}}{t^{(n+3)/2}}dt, x,y\in\mathbb{R}^n, x\neq y. \tag{167}$$

Then,

$$\left|\partial_{x_i}\mathbb{R}_\alpha(x,y)\right| \le \frac{C}{|x-y|^{n+1}}, x, y \in \mathbb{R}^n, x \ne y. \tag{168}$$

The Euclidean α-order Riesz transform $\mathbb{R}_\alpha$ is bounded from $L^q(\mathbb{R}^n, dx)$ into itself, for every $1 < q < \infty$. According to ([1], Proposition 3.2.5), the operator $\mathbb{R}_{\alpha,loc}$ defined by

$$\mathbb{R}_{\alpha,\mathrm{loc}}(f)(x) = \mathbb{R}_\alpha\left(f\chi_{N_\beta}(x,\cdot)\right)(x), \tag{169}$$

is bounded from $L^q(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 < q < \infty$.

We can write $R_{\alpha,loc} = S_\alpha(f) + \mathbb{R}_{\alpha,loc}(f)$ on $C_c^\infty(\mathbb{R}^n)$. Then, $R_{\alpha,loc}$ can be extended from $C_c^\infty(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

Since $R_\alpha = R_{\alpha,\mathrm{loc}} + R_{\alpha,glob}$, we conclude that R_α can be extended from $C_c^\infty(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

Let us consider the maximal operator

$$R_{\alpha,*}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} R_\alpha(x,y)f(y)dy\right|, f \in L^p(\mathbb{R}^n, \gamma_{-1}), 1 < p < \infty. \tag{170}$$

Let $f \in L^p(\mathbb{R}^n, \gamma_{-1})$, $1 < p < \infty$. For every $\varepsilon > 0$, by using the above estimates, we can write

$$\begin{aligned}\left|\int_{|x-y|>\varepsilon} R_\alpha(x,y)f(y)dy\right| &\le \int_{|x-y|>\varepsilon} |R_\alpha(x,y) - \mathbb{R}_\alpha(x,y)|\chi_{N_\beta}(x,y)|f(y)|dy \\ &\quad + \int_{|x-y|>\varepsilon} |R_\alpha(x,y)|\chi_{N_\beta^c}(x,y)|f(y)|dy \\ &\quad + \int_{|x-y|>\varepsilon} |\mathbb{R}_\alpha(x,y)|\chi_{N_\beta}(x,y)|f(y)|dy < \infty, x \in \mathbb{R}^n.\end{aligned} \tag{171}$$

We also have that

$$\begin{aligned}R_{\alpha,*}(f)(x) \le C\Bigg(&\int_{\mathbb{R}^n} \frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}}\chi_{N_\beta}(x,y)|f(y)|dy \\ &+ \int_{\mathbb{R}^n} |R_\alpha(x,y)|\chi_{N_\beta^c}(x,y)|f(y)|dy \\ &+ \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \mathbb{R}_\alpha(x,y)\chi_{N_\beta}(x,y)f(y)dy\right|\Bigg), x \in \mathbb{R}^n.\end{aligned} \tag{172}$$

Since the maximal operator

$$\mathbb{R}_{\alpha,*}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \mathbb{R}_\alpha(x,y)f(y)dy\right|, \tag{173}$$

is bounded from $L^p(\mathbb{R}^n, dx)$ into itself, by using a vector-valued version of ([1], Proposition 3.2.5) (see ([16], Proposition 2.3)), we deduce that the local maximal operator

$$\mathbb{R}_{\alpha,loc,*}(f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \mathbb{R}_\alpha(x,y)\chi_{N_\beta}(x,y)f(y)dy\right|, \tag{174}$$

is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

By using the same arguments as above, we conclude that $R_{\alpha,*}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

From (142) and since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \gamma_{-1})$ and $\mathbb{R}_{\alpha,*}$ is bounded in $L^p(\mathbb{R}^n, \gamma_{-1})$, by using a standard procedure, we can conclude that the limit

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_\alpha(x,y)f(y)dy, \tag{175}$$

exists for almost all $x \in \mathbb{R}^n$ and L_α (defined in (144)) is a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

Remark 10. The $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness of the local part $R_{\alpha,loc}$ of R_α can be proved also by using Calderón-Zygmund theory. We have preferred to do it by comparing $R_{\alpha,loc}$ with the classical local Riesz transform $\mathbb{R}_{\alpha,loc}$ because in this way we can know how the singularity of $R_{\alpha,loc}$ is. Furthermore, these comparative results will be useful in the proof of Theorems 1.4 and 1.5.

4. Riesz Transform Associated with the Operator $\bar{\mathscr{A}}$

Our objective in this section is to prove Theorem 3.

We define $L_0^2(\mathbb{R}^n, \gamma_{-1})$ as the space that consists of all those $f \in L^2(\mathbb{R}^n, \gamma_{-1})$ such that $c_0(f) = \int_{\mathbb{R}^n} f(x)dx = 0$. Let $\beta > 0$. For every $f \in L_0^2(\mathbb{R}^n, \gamma_{-1})$, $\bar{\mathscr{A}}^{-\beta}f$ is defined by

$$\bar{\mathscr{A}}^{-\beta}(f) = \sum_{k\in\mathbb{N}^n\setminus\{0\}} \frac{c_k(f)}{|k|^\beta}\tilde{H}_k. \tag{176}$$

We have that

$$\begin{aligned}\left\|\bar{\mathscr{A}}^{-\beta}(f)\right\|_{L^2(\mathbb{R}^n,\gamma_{-1})}^2 &= \sum_{k\in\mathbb{N}^n\setminus\{0\}} \frac{|c_k(f)|^2\left\|\tilde{H}_k\right\|_{L^2(\mathbb{R}^n,\gamma_{-1})}^2}{|k|^{2\beta}} \\ &\le \|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}^2, f \in L_0^2(\mathbb{R}^n, \gamma_{-1}).\end{aligned} \tag{177}$$

We introduce the operator $\bar{S}_\beta$ defined by

$$\bar{S}_\beta(f) = \frac{1}{\Gamma(\beta)}\int_0^\infty \left(T_t^{\bar{\mathscr{A}}}(f) - c_0(f)e^{|\cdot|^2}\right)t^{\beta-1}dt, f \in L^2(\mathbb{R}^n, \gamma_{-1}). \tag{178}$$

Let $f \in L^2(\mathbb{R}^n, \gamma_{-1})$. We have that

$$T_t^{\bar{\mathscr{A}}}(f) - c_0(f)e^{-|\cdot|^2} = \sum_{k\in\mathbb{N}^n} e^{-|k|t} c_k(f)\tilde{H}_k - e^{|\cdot|^2} c_0(f) = \sum_{k\in\mathbb{N}^n\setminus\{0\}} e^{-|k|t} c_k(f)\tilde{H}_k, t > 0. \tag{179}$$

Then,

$$\left\| T_t^{\bar{\mathscr{A}}}(f) - c_0(f)e^{-|\cdot|^2} \right\|_{L^2(\mathbb{R}^n,\gamma_{-1})} \le e^{-t} \|f\|_{L^2(\mathbb{R}^n,\gamma_{-1})}, t > 0. \tag{180}$$

Hence, the integral defining $\bar{S}_\beta(f)$ converges in the $L^2(\mathbb{R}^n, \gamma_{-1})$-Bochner sense.

Let $f \in C_c^\infty(\mathbb{R}^n)$. According to (36), (37), and (40), we get

$$\sum_{k\in\mathbb{N}^n\setminus\{0\}} e^{-|k|t} |c_k(f)| \left|\tilde{H}_k(x)\right| \le Ce^{-t} e^{-|x|^2/2} \sum_{k\in\mathbb{N}^n\setminus\{0\}} \frac{1}{|k|^2} \le Ce^{-t} e^{-|x|^2/2}, t > 0, x \in \mathbb{R}^n. \tag{181}$$

It follows that the series that defines $T_t^{\bar{\mathscr{A}}}(f)(x) - c_0(f) e^{-|x|^2}$ converges pointwisely and absolutely and

$$\bar{S}_\beta(f)(x) = \frac{1}{\Gamma(\beta)} \sum_{k\in\mathbb{N}^n\setminus\{0\}} c_k(f)\tilde{H}_k(x) \int_0^\infty e^{-|k|t} t^{\beta-1} dt = \sum_{k\in\mathbb{N}^n\setminus\{0\}} \frac{c_k(f)}{|k|^\beta} \tilde{H}_k(x) = \bar{\mathscr{A}}^{-\beta}(f_0)(x), x \in \mathbb{R}^n, \tag{182}$$

where $f_0(x) = f(x) - c_0(f)e^{-|x|^2}$, $x \in \mathbb{R}^n$.

On the other hand, since supp f is compact, we have that

$$\begin{aligned} &\int_{m(x)}^\infty \int_{\mathbb{R}^n} \left| T_t^{\bar{\mathscr{A}}}(x,y) - e^{-|x|^2} \right| |f(y)| dy t^{\beta-1} dt \\ &\le C \int_{m(x)}^\infty \left(\int_{\mathbb{R}^n} e^{-c\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)} \left| \left(1-e^{-2t}\right)^{-n/2} - 1 \right| |f(y)| dy \right. \\ &\quad + \int_{\mathbb{R}^n} \left| e^{-c\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)} - e^{-\left|x-e^{-t}y\right|^2} \right| |f(y)| dy \\ &\quad \left. + \int_{\mathbb{R}^n} \left| e^{-\left|x-e^{-t}y\right|^2} - e^{-|x|^2} \right| |f(y)| dy \right) t^{\beta-1} dt \\ &\le C \int_0^\infty e^{-t} t^{\beta-1} dt, x \in \mathbb{R}^n, \end{aligned} \tag{183}$$

and, taking $0 < \varepsilon < \min\{2\beta, n\}$,

$$\begin{aligned} &\int_0^{m(x)} t^{\beta-1} \int_{\mathbb{R}^n} \left| T_t^{\bar{\mathscr{A}}}(x,y) - e^{-|x|^2} \right| |f(y)| dy dt \\ &\le C \left(\int_0^{m(x)} t^{\beta-1} \int_{\mathbb{R}^n} \left(e^{c|x|} \frac{e^{-c|x-y|^2/t}}{t^{n/2}} + e^{-|x|^2} \right) |f(y)| dy dt \right) \\ &\le C \left(e^{c|x|} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\varepsilon}} dy + m(x)^\beta e^{-|x|^2} \int_{\mathbb{R}^n} |f(y)| dy \right) \\ &\le C \left(e^{c|x|} + m(x)^\beta e^{-|x|^2} \right), x \in \mathbb{R}^n. \end{aligned} \tag{184}$$

Then, we obtain

$$\bar{S}_\beta(f)(x) = \int_{\mathbb{R}^n} \bar{K}_\beta(x,y) f(y) dy, x \in \mathbb{R}^n, \tag{185}$$

where

$$\bar{K}_\beta(x,y) = \frac{1}{\Gamma(\beta)} \int_0^\infty \left(T_t^{\bar{A}}(x,y) - e^{-|x|^2} \right) t^{\beta-1} dt, x, y \in \mathbb{R}^n, \tag{186}$$

and

$$T_t^{\bar{\mathscr{A}}}(x,y) = e^{nt} T_t^{\mathscr{A}}(x,y) = \frac{e^{|y|^2-|x|^2}}{\pi^{n/2}\left(1-e^{-2t}\right)^{n/2}} \exp\left(-\frac{\left|y-e^{-t}x\right|^2}{1-e^{-2t}} \right), x, y \in \mathbb{R}^n, t > 0. \tag{187}$$

By denoting Π_0 the projection from $L^2(\mathbb{R}^n, \gamma_{-1})$ to $L_0^2(\mathbb{R}^n, \gamma_{-1})$, we have proved that, for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$\bar{\mathscr{A}}^{-\beta} \Pi_0(f)(x) = \int_{\mathbb{R}^n} \bar{K}_\beta(x,y) f(y) dy, x \in \mathbb{R}^n. \tag{188}$$

Let $f \in C_c^\infty(\mathbb{R}^n)$. Next, we show that

$$\delta_x^\alpha \bar{\mathscr{A}}^{-|\alpha|/2} \Pi_0(f)(x) = \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\Gamma(|\alpha|/2)} \int_{|x-y|>\varepsilon} f(y) \int_0^\infty \delta_x^\alpha \mathrm{T}_t^{\bar{A}}(x,y) t^{|\alpha|/2-1} dt dy + c_\alpha f(x), \tag{189}$$

for almost all $x \in \mathbb{R}^n$. Here, $c_\alpha \in \mathbb{R}$ and, when α_i is odd for some $i = 1, \cdots, n$, $c_\alpha = 0$.

For every $\ell \in \mathbb{N}^n$,

$$\delta_x^\ell T_t^{\bar{\mathscr{A}}}(x,y) = \frac{(-1)^{|\ell|} e^{-|x|^2}}{2^{|\ell|}} \partial_x^\ell \left(e^{|x|^2} T_t^{\bar{\mathscr{A}}}(x,y) \right) = \frac{(-1)^{|\ell|} e^{|y|^2-|x|^2}}{2^{|\ell|} \pi^{n/2} \left(1-e^{-2t}\right)^{(n+|\ell|)/2}} e^{-|\ell|t} \tilde{H}_\ell \left(\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} \right), x, y \in \mathbb{R}^n, t > 0, \tag{190}$$

Then, for each $\ell \in \mathbb{N}^n$,

$$\left| \delta_x^\ell T_t^{\bar{A}}(x,y) \right| \le Ce^{|y|^2-|x|^2} \frac{e^{-|\ell|t} e^{-c\left|y-e^{-t}x\right|^2/\left(1-e^{-2t}\right)}}{\left(1-e^{-2t}\right)^{(n+|\ell|)/2}}, x, y \in \mathbb{R}^n, t > 0. \tag{191}$$

Since $\delta_x^\ell(e^{-|x|^2}) = 0$, $x \in \mathbb{R}^n$, when $\ell \in \mathbb{N}^n \setminus \{0\}$, by proceeding as in the proof of (80), we get that for every $\ell \in \mathbb{N}^n \setminus \{0\}$ and $k \in \mathbb{N}$ being $|\ell| < k$,

$$\delta_x^\ell \bar{\mathscr{A}}^{-k/2} \Pi_0(f)(x) = \frac{1}{\Gamma(k/2)} \int_{\mathbb{R}^n} f(y) \cdot \int_0^\infty \delta_x^\ell T_t^{\bar{A}}(x,y) t^{k/2-1} dt dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{192}$$

Without loss of generality we can assume that $\alpha_1 \ge 1$ and consider $\ell = (\alpha_1 - 1, \alpha_2, \cdots, \alpha_n)$. When $\ell \in \mathbb{N}^n \setminus \{0\}$, we can proceed as in the proof of Theorem 1. For $n > 1$, we write

$$\begin{aligned}\delta_x^\alpha \bar{\mathscr{A}}^{-|\alpha|/2} \Pi_0(f)(x) &= \frac{1}{\Gamma(|\alpha|/2)} \delta_{x_1} \Bigg(\int_{\mathbb{R}^n} f(y) \int_0^\infty \delta_x^\ell \left(T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right) t^{|\alpha|/2-1} dt dy \\ &\quad + \int_{\mathbb{R}^n} f(y) \int_0^\infty \delta_x^\ell \left(e^{|y|^2-|x|^2} W_t(y-x) \right) t^{|\alpha|/2-1} dt dy \Bigg) \\ &= \frac{1}{\Gamma(|\alpha|/2)} \delta_{x_1} (F(x) + G(x)), x \in \mathbb{R}^n.\end{aligned} \tag{193}$$

We observe that, for every $r \in \mathbb{N}^n$,

$$\begin{aligned}\delta_x^r \left(e^{|y|^2-|x|^2} W_t(y-x) \right) &= \frac{(-1)^{|r|}}{2^{|r|}} e^{|y|^2-|x|^2} \partial_x^r (W_t(y-x)) \\ &= \frac{(-1)^{|r|}}{2^{|r|}\pi^{n/2}} \frac{e^{|y|^2-|x|^2}}{(2t)^{(n+|r|)/2}} \tilde{H}_r\left(\frac{y-x}{\sqrt{2t}} \right), x, y \in \mathbb{R}^n, t > 0.\end{aligned} \tag{194}$$

By considering the decomposition

$$\begin{aligned}\delta_x^\alpha \left(T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right) &= \frac{(-1)^{|\alpha|}}{2^{|\alpha|}\pi^{n/2}} e^{|y|^2-|x|^2} \cdot \left(\frac{e^{-|\alpha|t}}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\ell\left(\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} \right) - \frac{1}{(2t)^{(n+|\alpha|)/2}} \tilde{H}_\alpha\left(\frac{y-x}{\sqrt{2t}} \right) \right) \\ &= \frac{(-1)^{|\alpha|}}{2^{|\alpha|}\pi^{n/2}} e^{|y|^2-|x|^2} \Bigg\{ \frac{e^{-|\alpha|t}-1}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\alpha\left(\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} \right) \\ &\quad + \left(\frac{1}{(1-e^{-2t})^{(n+|\alpha|)/2}} - \frac{1}{(2t)^{(n+|\alpha|)/2}} \right) \tilde{H}_\alpha\left(\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} \right) \\ &\quad + \frac{1}{(2t)^{(n+|\alpha|)/2}} \left(\tilde{H}_\alpha\left(\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}} \right) - \tilde{H}_\alpha\left(\frac{y-x}{\sqrt{2t}} \right) \right) \Bigg\}, x, y \in \mathbb{R}^n.\end{aligned} \tag{195}$$

We can argue as in the proof of (85) to obtain that

$$\delta_{x_1} F(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \delta_x^\alpha \cdot \left(T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right) t^{|\alpha|/2-1} dt dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{196}$$

On the other hand, to deal with $\delta_{x_1} G(x)$, we consider $g(x) = f(x) e^{|x|^2}$ and $\Psi(x) = \int_0^\infty \partial_x^\ell (W_t)(-x) t^{|\alpha|/2-1} dt$, $x \in \mathbb{R}^n$, and write

$$G(x) = \frac{e^{-|x|^2}}{2^{|\ell|}} \int_{\mathbb{R}^n} g(x-y) \Psi(y) dy, x \in \mathbb{R}^n. \tag{197}$$

We proceed as in the proof of (115) to get

$$\delta_{x_1} G(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} f(y) \int_0^\infty \delta_x^\alpha \left(e^{|y|^2-|x|^2} W_t(x-y) \right) t^{|\alpha|/2-1} dt dy + c_\alpha f(x), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{198}$$

where $c_\alpha \in \mathbb{R}$ and $c_\alpha = 0$ if α_i is odd for some $i = 1, \cdots, n$. Thus, (189) is established when $n > 1$.

If $n = 1$, we can also follow the proof of Theorem 1 by using the decomposition

$$\delta_x^\alpha \bar{\mathscr{A}}^{-a/2} \Pi_0(f)(x) = \frac{1}{\Gamma(|\alpha|/2)} \delta_x \left(\bar{F}(x) + \bar{G}(x) \right), x \in \mathbb{R}^n, \tag{199}$$

where

$$\bar{F}(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \delta_x^\ell T_t^{\bar{\mathscr{A}}}(x,y) - \delta_x^\ell \left(e^{|y|^2-|x|^2} \left[W_t(y-x) - \frac{1}{2^\ell} \frac{d^\ell}{dx^\ell} W_t(0) \chi_{(1,\infty)}(t) \right] \right) t^{|\alpha|/2-1} dt dy, x \in \mathbb{R}^n, \tag{200}$$

and

$$\bar{G}(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \delta_x^\ell \cdot \left(e^{|y|^2-|x|^2} \left[W_t(y-x) - \frac{1}{2^\ell} \frac{d^\ell}{dx^\ell} W_t(0) \chi_{(1,\infty)}(t) \right] \right) t^{|\alpha|/2-1} dt dy, x \in \mathbb{R}^n. \tag{201}$$

When $\ell = 0$, that is, $\alpha = (1, 0, \cdots, 0)$, we can replace F in (193) and $\bar{F}$ in (199) by

$$F(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \left(T_t^{\bar{\mathscr{A}}}(x,y) - e^{-|x|^2} - e^{|y|^2-|x|^2} W_t(y-x) \right) \frac{dt dy}{\sqrt{t}}, x \in \mathbb{R}^n, \tag{202}$$

and

$$\bar{F}(x) = \int_{\mathbb{R}^n} f(y) \int_0^\infty \Bigg(T_t^{\bar{\mathscr{A}}}(x,y) - e^{-|x|^2} - \left(e^{|y|^2-|x|^2} \left[W_t(y-x) - \frac{1}{2^\ell} \frac{d^\ell}{dx^\ell} W_t(0) \chi_{(1,\infty)}(t) \right] \right) \Bigg) \frac{dt dy}{\sqrt{t}}, x \in \mathbb{R}^n, \tag{203}$$

respectively, and proceed as above to obtain (189).

According to (36), (37), and (40), we get

$$\bar{R}_\alpha f(x) = \delta^\alpha \mathscr{A}^{-|\alpha|/2} \Pi_0 f(x), x \in \mathbb{R}^n, \tag{204}$$

and then,

$$\bar{R}_\alpha f(x) = \lim_{\varepsilon \longrightarrow 0^+} \frac{1}{\Gamma(|\alpha|/2)} \int_{|x-y|>\varepsilon} f(y) \int_0^\infty \delta_x^\alpha T_t^{\bar{A}}(x,y) t^{|\alpha|/2-1} dt dy + c_\alpha f(x), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{205}$$

with $c_\alpha = 0$, when α_i is odd for some $i = 1, \cdots, n$.

We are going to show the $L^p(\mathbb{R}^n, \gamma_{-1})$-boundedness properties of $\bar{R}_\alpha$. We recall that

$$\bar{R}_\alpha(x,y) = \frac{(-1)^{|\alpha|}}{2^{|\alpha|} \pi^{n/2} \Gamma(|\alpha|/2)} e^{|y|^2-|x|^2} \int_0^\infty \frac{e^{-|\alpha|t}}{(1-e^{-2t})^{(n+|\alpha|)/2}} \tilde{H}_\alpha \cdot \left(\frac{y - e^{-t}x}{\sqrt{1-e^{-2t}}} \right) t^{|\alpha|/2-1} dt, x, y \in \mathbb{R}^n, x \neq y. \tag{206}$$

Consider first $1 < p < \infty$ and choose $1 - 1/p < \eta < 1$. By making the change of variables $s = 1 - e^{-2t}$, $t \in (0,\infty)$, we obtain

$$\left|\bar{R}_\alpha(x,y)\right| \leq C e^{|y|^2-|x|^2} \int_0^1 \frac{e^{-\eta|y-x\sqrt{1-s}|^2/s}}{s^{(n+|\alpha|)/2}} \cdot (1-s)^{|\alpha|/2-1} (-\log(1-s))^{|\alpha|/2-1} ds, x, y \in \mathbb{R}^n, x \neq y. \tag{207}$$

Let $\beta = \eta^{-1}$ and consider the local and global operators defined on $C_c^\infty(\mathbb{R}^n)$ by

$$\bar{R}_{\alpha,loc}(f)(x) = \bar{R}_\alpha\left(f \chi_{N_\beta}(x,\cdot)\right)(x), \quad \text{and} \quad \bar{R}_{\alpha,glob}(f)(x) = \bar{R}_\alpha\left(f \chi_{N_\beta^c}(x,\cdot)\right)(x), x \in \mathbb{R}^n. \tag{208}$$

By proceeding as in the proof of (154) it follows that, for each $(x,y) \in N_\beta^c$,

$$\bar{R}_\alpha(x,y) \leq C \begin{cases} e^{(1-\eta)|y|^2-|x|^2}, & \langle x,y\rangle \leq 0, \\ \left(\frac{|x+y|}{|x-y|}\right)^n \exp\left(\left(1-\frac{\eta}{2}\right)\left(|y|^2-|x|^2\right) - \frac{\eta}{2}|x+y||x-y|\right), & \langle x,y\rangle > 0. \end{cases} \tag{209}$$

We have that

$$\int_{\mathbb{R}^n} e^{-|y|^2/p+|x|^2/p} \left|\bar{R}_\alpha(x,y)\right| \chi_{N_\beta^c}(x,y) dy \leq C \left(\int_{\mathbb{R}^n} e^{|y|^2(1-\eta-1/p)} e^{(1/p-1)|x|^2} dy + \int_{\mathbb{R}^n} |x+y|^n \exp \cdot \left(-|x+y||x-y| \left(\frac{\eta}{2} - \left| 1 - \frac{1}{p} - \frac{\eta}{2} \right| \right) \right) dy \right), x \in \mathbb{R}^n. \tag{210}$$

Then, since $\eta > 1 - 1/p$,

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-|y|^2/p+|x|^2/p} \left|\bar{R}_\alpha(x,y)\right| \chi_{N_\beta^c}(x,y) dy < \infty. \tag{211}$$

Also, we get

$$\sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-|y|^2/p+|x|^2/p} \left|\bar{R}_\alpha(x,y)\right| \chi_{N_\beta^c}(x,y) dx < \infty. \tag{212}$$

We conclude that $\bar{R}_{\alpha,glob}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself.

We are going to study the operator $\bar{R}_{\alpha,\mathrm{loc}}$. We write

$$\bar{R}_\alpha(x,y) = \frac{1}{\Gamma(|\alpha|/2)} \left(\int_0^\infty \delta_x^\alpha \left[T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right] t^{|\alpha|/2-1} dt + \int_0^\infty \delta_x^\alpha \left[e^{|y|^2-|x|^2} W_t(y-x) \right] t^{|\alpha|/2-1} dt \right) = I(x,y) + J(x,y), x, y \in \mathbb{R}^n, x \neq y. \tag{213}$$

By taking into account that

$$\left| \delta_x^\alpha \left[T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right] \right| \leq C e^{|y|^2-|x|^2} \left(\frac{e^{-|\alpha|t}}{(1-e^{-2t})^{(n+|\alpha|)/2}} + \frac{e^{-c|x-y|^2/t}}{t^{(n+|\alpha|)/2}} \right), x, y \in \mathbb{R}^n, t > 0, \tag{214}$$

we get (see (86))

$$\int_{m(x)}^\infty \left| \delta_x^\alpha \left[T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right] \right| t^{|\alpha|/2-1} dt \leq C e^{|y|^2-|x|^2} (1+|x|)^n, x, y \in \mathbb{R}^n. \tag{215}$$

Also, from (195) and proceeding as in the proof of (99), we can see that

$$\int_0^{m(x)} \left| \delta_x^\alpha \left[T_t^{\bar{\mathscr{A}}}(x,y) - e^{|y|^2-|x|^2} W_t(y-x) \right] \right| t^{|\alpha|/2-1} dt \leq C e^{|y|^2-|x|^2} \frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}}, (x,y) \in N_\beta. \tag{216}$$

Since $||y|^2 - |x|^2| \le C$ when $(x, y) \in N_\beta$, we obtain that

$$|I(x, y)| \le C \frac{\sqrt{1 + |x|}}{|x - y|^{n-1/2}}, (x, y) \in N_\beta, x \ne y. \tag{217}$$

On the other hand, we observe that

$$\begin{aligned} J(x, y) &= \frac{e^{|y|^2-|x|^2}}{2^{|\alpha|}\Gamma(|\alpha|/2)} \int_0^\infty \partial_y^\alpha (W_t(y-x)) t^{|\alpha|/2-1} dt \\ &= 2^{-|\alpha|} e^{|y|^2-|x|^2} \mathbb{R}_\alpha(y, x), x, y \in \mathbb{R}^n, x \ne y, \end{aligned} \tag{218}$$

where $\mathbb{R}_\alpha(\cdot, \cdot)$ is the classical kernel considered in (155). The result can be established by proceeding as in the proof of Theorem 2 by taking into account that the Euclidean Riesz transform $\mathbb{R}_\alpha$ is bounded from $L^p(\mathbb{R}^n, dx)$ into itself.

To deal with the case $p = 1$, we consider the local and global operators $\bar{R}_{\alpha,loc}$ and $\bar{R}_{\alpha,glob}$ defined above with N instead of N_β. Since the classical Riesz transform $\mathbb{R}_\alpha$ is bounded from $L^1(\mathbb{R}^n, dx)$ into $L^{1,\infty}(\mathbb{R}^n, dx)$, we can use (217) and argue as in the proof of Theorem 2 to obtain that $\bar{R}_{\alpha,loc}$ defines a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

In order to prove that $\bar{R}_{\alpha,glob}$ defines a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1})$ into itself, we make the change of variables $r = e^{-t}$, $t \in (0,\infty)$, and write $\bar{R}_\alpha(x, y) = \bar{R}_{\alpha,1}(x, y) + \bar{R}_{\alpha,2}(x, y)$, where

$$\begin{aligned} \bar{R}_{\alpha,1}(x, y) = {} & \frac{(-1)^{|\alpha|}}{2^{|\alpha|}\pi^{n/2}\Gamma(|\alpha|/2)} e^{|y|^2-|x|^2} \int_0^{\frac{1}{2}} \frac{r^{|\alpha|-1}}{(1-r^2)^{(n+|\alpha|)/2}} \tilde{H}_\alpha \\ & \cdot \left(\frac{y - rx}{\sqrt{1-r^2}} \right) (-\log r)^{|\alpha|/2-1} dr, x, y \in \mathbb{R}^n, x \ne y. \end{aligned} \tag{219}$$

Suppose that $n = 1$ or $|\alpha| \ge n + 1$ when $n > 1$. By using ([1], Lemma 3.3.3), it follows that

$$\begin{aligned} \left|\bar{R}_{\alpha,1}(x, y)\right| &\le C e^{|y|^2-|x|^2} \int_0^{1/2} r^{|\alpha|-1} \frac{e^{-c|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2}} (-\log r)^{|\alpha|/2-1} dr \\ &\le C e^{|y|^2-|x|^2} \sup_{r \in (0,1)} \frac{r^n e^{-c|y-rx|^2/(1-r^2)}}{(1-r^2)^{n/2}} \int_0^1 (-\log r)^{|\alpha|/2-1} dr \\ &\le C e^{|y|^2-|x|^2} \min \{(1 + |x|)^n, (|x| \sin \theta(x, y))^{-n}\}, (x, y) \in N^c. \end{aligned} \tag{220}$$

On the other hand, by proceeding as in the estimation of $K_2^0(x, y)$ in ([2], proof of Proposition 5.1), we obtain, for every $(x, y) \in N^c$,

$$\begin{aligned} \left|\bar{R}_{\alpha,2}(x, y)\right| &\le C e^{|y|^2-|x|^2} \int_{1/2}^1 \frac{e^{-c|y-rx|^2/(1-r^2)}}{(1-r^2)^{(n+2)/2}} dr \\ &\le C e^{|y|^2-|x|^2} (|x|^{-n} + \min \{(1 + |x|)^n, (|x| \sin \theta(x, y))^{-n}\}). \end{aligned} \tag{221}$$

From ([1], Lemma 3.3.4) and ([2], Lemma 4.2), we deduce that $\bar{R}_{\alpha,glob}$ defines a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

Thus, we conclude that R_α can be extended to $L^1(\mathbb{R}^n, \gamma_{-1})$ as a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$.

5. UMD Spaces and Riesz Transforms in the Inverse Gaussian Setting

Proof of Theorem 4. For every $i = 1, \cdots, n$, by $\mathbb{R}_{e^i}$, we denote the i-th Euclidean Riesz transform defined, for every $f \in L^p(\mathbb{R}^n, dx)$, $1 < p < \infty$, by

$$\mathbb{R}_{e^i}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \mathbb{R}_{e^i}(x - y) f(y) dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n, \tag{222}$$

where

$$\mathbb{R}_{e^i}(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \partial_{x_i} W_t(z) \frac{dt}{\sqrt{t}}, z \in \mathbb{R}^n, z \ne 0. \tag{223}$$

Observe that

$$\mathbb{R}_{e^i}(z) = -\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{z_i}{|z|^{n+1}}, z \in \mathbb{R}^n, z \ne 0, i = 1, \cdots, n. \tag{224}$$

□

Let X be a Banach space. For every $i = 1, \cdots, n$, we define $\mathbb{R}_{e^i}$ on $L^p(\mathbb{R}^n, dx) \otimes X$, $1 \le p < \infty$, in the obvious way.

The UMD-property for X can be characterized by using $\mathbb{R}_{e^i}$, $i = 1, \cdots, n$. The properties stated in Theorems 4 and 5 hold when R_{e^i} is replaced by $\mathbb{R}_{e^i}$, $i = 1, \cdots, n$. The estimations established in the proofs of Theorems 1 and 2 allow us to pass from $\mathbb{R}_{e^i}$ to R_{e^i}, $i = 1, \cdots, n$.

Let $i = 1, \cdots, n$ and $1 < p < \infty$. We are going to see that the following two assertions are equivalent:

(i) R_{e^i} can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \bigcap L^p(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself

(ii) $\mathbb{R}_{e^i}$ can be extended from $(L^2(\mathbb{R}^n, dx) \bigcap L^p(\mathbb{R}^n, dx)) \otimes X$ to $L^p(\mathbb{R}^n, dx, X)$ as a bounded operator from $L^p(\mathbb{R}^n, dx, X)$ into itself

We choose $1/p < \eta < 1$ and consider the global and local operators as in the previous sections according to the region N_β, with $\beta = \eta^{-1}$.

Suppose that (ii) holds. We can write $R_{e^i} = (R_{e^i,\text{loc}} - \mathbb{R}_{e^i,\text{loc}}) + \mathbb{R}_{e^i,\text{loc}} + R_{e^i,glob}$. Since $\mathbb{R}_{e^i}$ is a Calderón-Zygmund operator, by using a vectorial version of ([1], Proposition 3.2.5) (see ([16], Proposition 2.3)), we deduce that $\mathbb{R}_{e^i,\text{loc}}$ can be extended from $(L^2(\mathbb{R}^n, \gamma_{-1}) \cap L^p(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to L^p

$(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself.

According to (154) and (156), we have that

$$|R_{e^i}(x,y) - \mathbb{R}_{e^i}(x-y)| \leq L_i(x,y), (x,y) \in N_\beta, \quad \text{and} \quad |R_{e^i}(x,y)| \leq M_i(x,y), (x,y) \in N_\beta^c, \tag{225}$$

and the integral operators

$$L_i(f)(x) = \int_{\mathbb{R}^n} L_i(x,y)\chi_{N_\beta}(x,y)f(y)dy, \quad \text{and} \quad M_i(f)(x) = \int_{\mathbb{R}^n} M_i(x,y)\chi_{N_\beta^c}(x,y)f(y)dy, \tag{226}$$

are bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself. Then, L_i and M_i define bounded operators from $L^p(\mathbb{R}^n, \gamma_{-1}, X)$ into itself, and the same property holds for the operators $R_{e^i,loc} - \mathbb{R}_{e^i,loc}$ and $R_{e^i,glob}$. We conclude that (i) holds.

Suppose now that (i) holds. By (149), we get

$$\begin{aligned} |R_{e^i}(x,y)| &\leq C\int_0^\infty e^{-nt} \frac{e^{-c\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)}}{\left(1-e^{-2t}\right)^{(n+1)/2}} \frac{dt}{\sqrt{t}} \\ &\leq C\left(\int_0^{m(x)} \frac{e^{-c|x-y|^2/t}}{t^{n/2+1}}dt + \int_{m(x)}^\infty \frac{dt}{t^{n/2+1}}\right) \\ &\leq C\left(\frac{1}{|x-y|^n} + \frac{1}{m(x)^{n/2}}\right) \\ &\leq \frac{C}{|x-y|^n}, (x,y) \in N_\beta, x \neq y. \end{aligned} \tag{227}$$

In a similar way, we get, for each $k = 1, \cdots, n$,

$$\begin{aligned} \left|\partial_{x_k} R_{e^i}(x,y)\right| &\leq C\int_0^\infty e^{-nt} \frac{e^{-c\left|x-e^{-t}y\right|^2/\left(1-e^{-2t}\right)}}{\left(1-e^{-2t}\right)^{n/2+1}} \frac{dt}{\sqrt{t}} \\ &\leq \frac{C}{|x-y|^{n+1}}, (x,y) \in N_\beta, x \neq y. \end{aligned} \tag{228}$$

Then, according to a vector-valued version of ([1], Propositions 3.2.5 and 3.2.7) (see ([16], Propositions 2.3 and 2.4), we deduce that $R_{e^i,\mathrm{loc}}$ defines a bounded operator from $L^p(\mathbb{R}^n, dx, X)$ into itself. Also, $R_{e^i,\mathrm{loc}} - \mathbb{R}_{e^i,\mathrm{loc}}$ defines a bounded operator from $L^p(\mathbb{R}^n, dx, X)$ into itself. We conclude that $\mathbb{R}_{e^i,\mathrm{loc}}$ defines a bounded operator from $L^p(\mathbb{R}^n, dx, X)$ into itself. Since $\mathbb{R}_{e^i}$ is dilatation invariant, by proceeding as in the proof of ([16], Theorem 1.10, (ii)⟹(i)), it follows that $\mathbb{R}_{e^i}$ can be extended from $(L^2(\mathbb{R}^n, dx) \bigcap L^p(\mathbb{R}^n, dx) \otimes X$ to $L^p(\mathbb{R}^n, dx, X)$ as a bounded operator from $L^p(\mathbb{R}^n, dx, X)$ into itself.

The same arguments allow us to prove that the following assertions are equivalent.

(i) R_{e^i} can be extended from $(L^1(\mathbb{R}^n, \gamma_{-1}) \bigcap L^2(\mathbb{R}^n, \gamma_{-1})) \otimes X$ to $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ as a bounded operator from $L^1(\mathbb{R}^n, \gamma_{-1}, X)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1}, X)$

(ii) $\mathbb{R}_{e^i}$ can be extended from $(L^1(\mathbb{R}^n, dx) \bigcap L^2(\mathbb{R}^n, dx)) \otimes X$ to $L^1(\mathbb{R}^n, dx, X)$ as a bounded operator from $L^1(\mathbb{R}^n, dx, X)$ into $L^{1,\infty}(\mathbb{R}^n, dx, X)$

Furthermore, in a similar way, we can see that (i)⟺(ii) and (iii)⟺(iv) when R_{e^i} and $\mathbb{R}_{e^i}$ are replaced by $R_{e^i,*}$ and $\mathbb{R}_{e^i,*}$, respectively.

The proof of Theorem 4 is thus finished.

Proof of Theorem 5. Let $1 < p < \infty$ and $i = 1, \cdots, n$. We are going to see that the following two assertions are equivalent.

(a) For every $f \in L^p(\mathbb{R}^n, \gamma_{-1}, X)$, there exists

$$\lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} R_{e^i}(x,y)f(y)dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{229}$$

(b) For every $f \in L^p(\mathbb{R}^n, dx, X)$, there exists

$$\lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} \mathbb{R}_{e^i}(x-y)f(y)dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{230}$$

□

We consider again $1/p < \eta < 1$ and $\beta = \eta^{-1}$. Suppose that (a) is true. Let $f \in L^p(\mathbb{R}^n, dx, X)$. We can write

$$\begin{aligned} \int_{|x-y|>\varepsilon} \mathbb{R}_{e^i}(x-y)f(y)dy = &\int_{|x-y|>\varepsilon} (\mathbb{R}_{e^i}(x-y) - R_{e^i}(x-y))\chi_{N_\beta}(x,y)f(y)dy \\ &+ \int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)f(y)dy \\ &+ \int_{|x-y|>\varepsilon} \mathbb{R}_{e^i}(x-y)\chi_{N_\beta^c}(x,y)f(y)dy, x \in \mathbb{R}^n, \varepsilon > 0. \end{aligned} \tag{231}$$

Since (225) holds and the operator L_i is bounded from $L^p(\mathbb{R}^n, dx, X)$ into itself, there exists the limit

$$\lim_{\varepsilon \longrightarrow 0^+} \int_{|x-y|>\varepsilon} (R_{e^i}(x,y) - \mathbb{R}_{e^i}(x-y))\chi_{N_\beta}(x,y)f(y)dy, \quad \text{for almost all} \quad x \in \mathbb{R}^n. \tag{232}$$

On the other hand, we get

$$\int_{\mathbb{R}^n} |\mathbb{R}_{e^i}(x-y)|\chi_{N_\beta^c}(x,y)\|f(y)\|dy$$
$$\leq C\int_{\mathbb{R}^n} \frac{1}{|x-y|^n}\chi_{N_\beta^c}(x,y)\|f(y)\|dy$$
$$\leq C\left(\int_{|x-y|>\beta n\sqrt{m(x)}} \frac{dy}{|x-y|^{np'}}\right)^{1/p'}\|f\|_{L^p(\mathbb{R}^n,dx,X)}$$
$$\leq C\left(\int_{\beta n\sqrt{m(x)}}^{\infty} \frac{dr}{r^{n(p'-1)+1}}\right)^{1/p'}\|f\|_{L^p(\mathbb{R}^n,dx,X)}$$
$$\leq \frac{C}{m(x)^{n/(2p)}}\|f\|_{L^p(\mathbb{R}^n,dx,X)}, x\in\mathbb{R}^n. \tag{233}$$

Then, there exists the limit

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{\mathbb{R}^n} \mathbb{R}_{e^i}(x-y)\chi_{N_\beta^c}(x,y)f(y)dy, x\in\mathbb{R}^n. \tag{234}$$

Suppose that $g\in L^p(\mathbb{R}^n,\gamma_{-1},X)$. It was seen in the proof of Theorem 2 that

$$\int_{\mathbb{R}^n} |R_{e^i}(x,y)|\chi_{N_\beta^c}(x,y)\|g(y)\|dy \in L^p(\mathbb{R}^n,\gamma_{-1}). \tag{235}$$

Then,

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{\mathbb{R}^n} R_{e^i}(x,y)\chi_{N_\beta^c}(x,y)g(y)dy = \int_{\mathbb{R}^n} R_{e^i}(x,y)\chi_{N_\beta^c}(x,y)g(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}^n. \tag{236}$$

Since (a) holds, there also exists the limit

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)g(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}^n. \tag{237}$$

Let $k\in\mathbb{N}$. We have that $|y|\leq\beta n+k$ provided that $|x|\leq k$ and $|x-y|\leq\beta n \min\{1,|x|^{-1}\}$. Then, for every $\varepsilon>0$,

$$\int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)f(y)dy = \int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)\chi_{B(0,\beta n+k)}(y)f(y)dy, |x|\leq k. \tag{238}$$

Since $f\in L^p(\mathbb{R}^n,dx,X)$, $\chi_{B(0,\beta n+k)}f\in L^p(\mathbb{R}^n,\gamma_{-1},X)$ and then there exists

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)f(y)dy, \quad \text{for almost all} \quad x\in B(0,k). \tag{239}$$

Hence, we get that there exists

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta}(x,y)f(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}^n. \tag{240}$$

We conclude that there exists

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} \mathbb{R}_{e^i}(x-y)f(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}^n. \tag{241}$$

In a similar way, we can see that (a) holds provided that (b) is true. Note that $L^p(\mathbb{R}^n,\gamma_{-1},X)\subset L^p(\mathbb{R}^n,dx,X)$.

As it was proved in [2] the operator S_i defined by

$$S_i(f)(x) = \int_{\mathbb{R}^n} |R_{e^i}(x,y)|\chi_{N_\beta^c}(x,y)f(y)dy, x\in\mathbb{R}^n, \tag{242}$$

is bounded from $L^1(\mathbb{R}^n,\gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n,\gamma_{-1})$. Then, for every $f\in L^1(\mathbb{R}^n,\gamma_{-1})$, there exists

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{|x-y|>\varepsilon} R_{e^i}(x,y)\chi_{N_\beta^c}(x,y)f(y)dy = \int_{\mathbb{R}^n} R_{e^i}(x,y)\chi_{N_\beta^c}(x,y)f(y)dy, \quad \text{for almost all} \quad x\in\mathbb{R}^n. \tag{243}$$

The same arguments allow us to prove that (a) $\Longleftrightarrow$(b) when $p=1$.

By using a n-dimensional version of ([26], Theorem D), we deduce that the properties (i), (ii), and (iii) in Theorem 5 are equivalent.

By proceeding as above, we can see that the properties in (a) and (b) continue being equivalent when we replace the principal values by the maximal operators $R_{e^i,*}$ and $\mathbb{R}_{e^i,*}$ in (a) and (b), respectively. Then, the property (b) is equivalent to the property UMD for X (see the comments before the proof of ([16], Theorem 1.10, p. 19).

Thus, the proof of Theorem 5 is finished.

6. UMD Spaces and the Imaginary Powers of $\mathcal{A}$

In this section, we prove Theorem 6.

According to ([18], (11)), we have that, for every $f\in C_c^\infty(\mathbb{R}^n)$,

$$\left(-\frac{\Delta}{2}\right)^{i\sigma} f(x) = \lim_{\varepsilon\longrightarrow 0^+} \cdot\left(\alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} K_\sigma(x-y)f(y)dy\right), \quad \text{for almost all} \quad x\in\mathbb{R}^n, \tag{244}$$

where

$$K_\sigma(z) = -\int_0^\infty \phi_\sigma(t)\partial_t W_t(z)dt, z \in \mathbb{R}^n \setminus \{0\}, \tag{245}$$

$$\alpha(\varepsilon) = \frac{1}{\Gamma(n/2)}\int_0^\infty \phi_\sigma\left(\frac{\varepsilon^2}{4u}\right)e^{-u}u^{n/2-1}du, \varepsilon \in (0,\infty), \tag{246}$$

and

$$\phi_\sigma(t) = \frac{t^{-i\sigma}}{\Gamma(1-i\sigma)}, t \in (0,\infty). \tag{247}$$

Note that the limit $\lim_{t\longrightarrow 0^+}\phi_\sigma(t)$ does not exist.

Let $f, g \in C_c^\infty(\mathbb{R}^n)$. By proceeding as in ([18], p. 213), we can see that

$$\begin{aligned}\langle \mathscr{A}^{i\sigma} f, g\rangle_{L^2(\mathbb{R}^n,\gamma_{-1})} &= \int_0^\infty \phi_\sigma(t)\left(-\frac{d}{dt}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} T_t^{\mathscr{A}}(x,y)f(y)dy\bar{g}(x)\gamma_{-1}(x)dxdt \\ &= \int_0^\infty \phi_\sigma(t)\left(-\frac{d}{dt}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_t(x-y)f(y)dy\bar{g}(x)\gamma_{-1}(x)dxdt \\ &\quad + \int_0^\infty \phi_\sigma(t)\left(-\frac{d}{dt}\right)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(T_t^{\mathscr{A}}(x,y)\right. \\ &\quad \left. - W_t(x-y)\right)f(y)dy\bar{g}(x)\gamma_{-1}(x)dxdt.\end{aligned} \tag{248}$$

We have that

$$\begin{aligned}\partial_t T_t^{\mathscr{A}}(x,y) &= \left(-n - 2e^{-t}\sum_{i=1}^n y_i\left(x_i - e^{-t}y_i\right) + \frac{2e^{-2t}|x-e^{-t}y|^2}{1-e^{-2t}}\right)\frac{T_t^{\mathscr{A}}(x,y)}{1-e^{-2t}} \\ &= \left(-n - 2e^{-t}\left(\langle x,y\rangle - e^{-t}|y|^2\right) + \frac{2e^{-2t}|x-e^{-t}y|^2}{1-e^{-2t}}\right) \\ &\quad \frac{T_t^{\mathscr{A}}(x,y)}{1-e^{-2t}}, x,y \in \mathbb{R}^n, t > 0,\end{aligned} \tag{249}$$

and writing $2e^{-t}\langle x,y\rangle = |x|^2 + e^{-2t}|y|^2 - |x-e^{-t}y|^2$, we get

$$\begin{aligned}&\partial_t T_t^{\mathscr{A}}(x,y) \\ &\quad = \left(-n + e^{-2t}\left(|y|^2 - |x|^2\right) - \left(1-e^{-2t}\right)|x|^2 + \frac{1+e^{-2t}}{1-e^{-2t}}\left|x-e^{-t}y\right|^2\right)\right) \\ &\quad \frac{T_t^{\mathscr{A}}(x,y)}{1-e^{-2t}}, x,y \in \mathbb{R}^n, t > 0.\end{aligned} \tag{250}$$

Also, we have that

$$\partial_t W_t(x-y) = \left(-n + \frac{|x-y|^2}{t}\right)\frac{W_t(x-y)}{2t}, x,y \in \mathbb{R}^n, t > 0. \tag{251}$$

We can write, for every $x, y \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}&\partial_t T_t^{\mathscr{A}}(x,y) - \partial_t W_t(x-y) \\ &\quad = \left(-n + \frac{|x-y|^2}{t}\right)\frac{T_t^{\mathscr{A}}(x,y) - W_t(x-y)}{2t} \\ &\quad + \left[-n\left(\frac{1}{1-e^{-2t}} - \frac{1}{2t}\right) + \frac{e^{-2t}}{1-e^{-2t}}\left(|y|^2 - |x|^2\right)\right. \\ &\quad \left. - |x|^2 - \frac{|x-e^{-t}y|^2}{1-e^{-2t}} + 2\left(\frac{|x-e^{-t}y|^2}{\left(1-e^{-2t}\right)^2} - \frac{|x-y|^2}{(2t)^2}\right)\right]T_t^{\mathscr{A}}(x,y).\end{aligned} \tag{252}$$

The derivative under the integral sign is justified and we get

$$\begin{aligned}\langle \mathscr{A}^{i\sigma} f, g\rangle_{L^2(\mathbb{R}^n,\gamma_{-1})} &= -\int_0^\infty \phi_\sigma(t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\partial_t W_t(x-y)f(y)dy\bar{g}(x)\gamma_{-1}(x)dxdt \\ &\quad - \int_0^\infty \phi_\sigma(t)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\partial_t\left(T_t^{\mathscr{A}}(x,y)\right. \\ &\quad \left. - W_t(x-y)\right)f(y)dy\bar{g}(x)\gamma_{-1}(x)dxdt.\end{aligned} \tag{253}$$

To ensure the change on the order of integration, we are going to see that

$$J(x) = \int_0^\infty \int_{\mathbb{R}^n}\left|\partial_t\left(T_t^{\mathscr{A}}(x,y) - W_t(x-y)\right)\right||f(y)|dydt < \infty, x \in \mathbb{R}^n, \tag{254}$$

and

$$\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\partial_t\left(T_t^{\mathscr{A}}(x,y) - W_t(x-y)\right)\right||f(y)||g(x)|\gamma_{-1}(x)dydxdt < \infty. \tag{255}$$

We observe that, since $g \in C_c^\infty(\mathbb{R}^n)$, it is sufficient to see that $J(x) \le h(x)$, $x \in \mathbb{R}^n$, for certain continuous function h.

We consider the decomposition

$$\begin{aligned}J(x) &= \left(\int_0^{m(x)} + \int_{m(x)}^\infty\right)\int_{\mathbb{R}^n}\left|\partial_t\left(T_t^{\mathscr{A}}(x,y) - W_t(x-y)\right)\right||f(y)|dydt \\ &= J_1(x) + J_2(x), x \in \mathbb{R}^n.\end{aligned} \tag{256}$$

Since $f \in C_c^\infty(\mathbb{R}^n)$ by (252), we have that

$$J_2(x) \le C(1+|x|)^2\int_{m(x)}^\infty \int_{suppf}\frac{1}{t^{n/2+1}}dtdy \le C\frac{(1+|x|)^2}{m(x)^{n/2}}, x \in \mathbb{R}^n. \tag{257}$$

We now estimate $J_1(x)$, $x \in \mathbb{R}^n$. We take into account that, if supp $f \subset B(0, r_f)$, with $r_f > 0$, then

$$\left|x - e^{-t}y\right|^2 \geq |x - y|^2 + \left(1 - e^{-t}\right)^2|y|^2 - 2|x - y||y|\left(1 - e^{-t}\right) \geq |x - y|^2 - d(x)\left(1 - e^{-t}\right), x \in \mathbb{R}^n, y \in suppf, t > 0, \quad (258)$$

where $d(x) = 2r_f(r_f + |x|)$, $x \in \mathbb{R}^n$. By considering the decomposition (89) for $\alpha = 0$, the estimations (92) and the proof of (97), we can see that

$$\left|T_t^{\mathscr{A}}(x, y) - W_t(x - y)\right| \leq Ce^{d(x)}e^{-c|x-y|^2/t} \cdot \left(\frac{1}{t^{n/2-1}} + (|y| + |x - y|)^n\right), x \in \mathbb{R}^n, y \in suppf, t \in (0, 1). \quad (259)$$

Then, according to (252), we have that

$$\left|\partial_t\left(T_t^{\mathscr{A}}(x, y) - W_t(x - y)\right)\right| \leq Ce^{d(x)}e^{-c|x-y|^2/t} \cdot\left(\frac{1 + |x|^2}{t^{n/2}} + \frac{(|y| + |x - y|)^n}{t} + \frac{|x - y||y + x|}{t^{n/2+1}} + \frac{|y| + |x - y|}{t^{n/2}}\right) \leq Ce^{d(x)}e^{-c|x-y|^2/t}\left(\frac{1 + |x|^2}{t^{n/2}} + \frac{(1 + |x|)^n}{t} + \frac{1 + |x|}{t^{(n+1)/2}}\right), x \in \mathbb{R}^n, y \in suppf, t \in (0, 1). \quad (260)$$

Since $m(x) \sim (1 + |x|)^{-2}$, $x \in \mathbb{R}^n$, we get when $t \in (0, m(x))$,

$$\left|\partial_t\left(T_t^{\mathscr{A}}(x, y) - W_t(x - y)\right)\right| \leq Ce^{d(x)}e^{-c|x-y|^2/t}\left(\frac{1 + |x|}{t^{(n+1)/2}} + \frac{(1 + |x|)^n}{t}\right), x \in \mathbb{R}^n, y \in suppf. \quad (261)$$

We deduce that

$$J_1(x) \leq Ce^{d(x)}\int_0^{m(x)}\int_{\text{supp } f}\left(\frac{1 + |x|}{t^{3/4}} + (1 + |x|)^n t^{n/2-5/4}\right)\frac{dy}{|x - y|^{n-1/2}} \leq Ce^{d(x)}\left((1 + |x|)m(x)^{1/4} + (1 + |x|)^n m(x)^{n/2-1/4}\right) \leq Ce^{d(x)}\sqrt{1 + |x|}, x \in \mathbb{R}^n. \quad (262)$$

Then, $|J(x)| \leq C((1 + |x|)^{n+2} + e^{d(x)}\sqrt{1 + |x|})$, $x \in \mathbb{R}^n$ and (254) and (255) are established.

Note that the estimations that we have just proved are depending on f.

By interchanging the order of integration, we get

$$\left\langle \mathscr{A}^{i\sigma}f, g\right\rangle_{L^2(\mathbb{R}^n,\gamma_{-1})} = \left\langle\left(-\frac{\Delta}{2}\right)^{i\sigma}f, g\right\rangle_{L^2(\mathbb{R}^n,\gamma_{-1})} - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(y)\int_0^\infty\phi_\sigma(t)\left(\partial_t T_t^{\mathscr{A}}(x, y) - \partial_t W_t(x - y)\right)dtdy\bar{g}(x)\gamma_{-1}(x)dx. \quad (263)$$

It follows that

$$\mathscr{A}^{i\sigma}(f)(x) = \left(-\frac{\Delta}{2}\right)^{i\sigma}f(x) - \int_{\mathbb{R}^n}f(y)\int_0^\infty\phi_\sigma(t)\left(\partial_t T_t^{\mathscr{A}}(x, y) - \partial_t W_t(x - y)\right)dtdy = \left(-\frac{\Delta}{2}\right)^{i\sigma}f(x) - \lim_{\varepsilon\longrightarrow0^+}\int_{|x-y|>\varepsilon}f(y) \cdot\int_0^\infty\phi_\sigma(t)\left(\partial_t T_t^{\mathscr{A}}(x, y) - \partial_t W_t(x - y)\right)dtdy, \quad (264)$$

for almost all $x \in \mathbb{R}^n$.

We conclude that

$$\mathscr{A}^{i\sigma}(f)(x) = \lim_{\varepsilon\longrightarrow0^+}\left(\int_{|x-y|>\varepsilon}K_\sigma^{\mathscr{A}}(x, y)f(y)dy + \alpha(\varepsilon)f(x)\right), \quad \text{for almost all} \quad x \in \mathbb{R}^n, \quad (265)$$

where

$$K_\sigma^{\mathscr{A}}(x, y) = -\int_0^\infty\phi_\sigma(t)\partial_t T_t^{\mathscr{A}}(x, y)dt, x, y \in \mathbb{R}^n, t > 0. \quad (266)$$

Salogni ([1], Theorem 3.4.3) proved that $\mathscr{A}^{i\sigma}$ is bounded from $L^p(\mathbb{R}^n, \gamma_{-1})$ into itself, for every $1 < p < \infty$, and Bruno ([6], Theorem 4.1, (i)) established that $\mathscr{A}^{i\sigma}$ is bounded from $L^1(\mathbb{R}^n, \gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma_{-1})$. In order to extend $\mathscr{A}^{i\sigma}$ to functions taking values in a Banach space, we need to prove these results in a different way by making a comparation between $\mathscr{A}^{i\sigma}$ and $(-\Delta/2)^{i\sigma}$.

Let $\beta > 0$. We define the local and global part as follows

$$\mathscr{A}_{loc}^{i\sigma}(f)(x) = \lim_{\varepsilon\longrightarrow0^+}\left(\alpha(\varepsilon)f(x) + \int_{|x-y|>\varepsilon}K_\sigma^{\mathscr{A}}(x, y)\chi_{N_\beta}(x, y)f(y)dy\right), x \in \mathbb{R}^n, \quad (267)$$

and

$$\mathscr{A}_{glob}^{i\sigma}(f)(x) = \int_{|x-y|>\varepsilon}K_\sigma^{\mathscr{A}}(x, y)\chi_{N_\beta^c}(x, y)f(y)dy, x \in \mathbb{R}^n. \quad (268)$$

The operators $(-\Delta/2)_{loc}^{i\sigma}$ and $(-\Delta/2)_{glob}^{i\sigma}$ are defined in analogous way.

We are going to see that

$$\left|K_\sigma^{\mathscr{A}}(x, y) - K_\sigma(x - y)\right| \leq C\frac{\sqrt{1 + |x|}}{|x - y|^{n-1/2}}, (x, y) \in N_\beta. \quad (269)$$

We can write

$$\begin{aligned}&\left|K_\sigma^{\mathscr{A}}(x,y)-K_\sigma(x-y)\right|\\&\le C\left(\int_0^{m(x)}\left|\partial_t\left(T_t^{\mathscr{A}}(x,y)-W_t(x-y)\right)\right|dt\right.\\&\left.+\int_{m(x)}^{\infty}\left|\partial_t T_t^{\mathscr{A}}(x,y)\right|dt+\int_{m(x)}^{\infty}\left|\partial_t W_t(x-y)\right|dt\right)\\&=\sum_{i=1}^{3}I_i(x,y),\ x,y\in\mathbb{R}^n.\end{aligned}\tag{270}$$

First, we observe that

$$\begin{aligned}I_3(x,y)&\le C\int_{m(x)}^{\infty}\frac{e^{-c|x-y|^2/t}}{t^{n/2+1}}dt\le\frac{C}{m(x)^{n/2}}\\&\le C(1+|x|)^n\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta.\end{aligned}\tag{271}$$

By using (250) and since $||y|^2-|x|^2|\le|x-y||x+y|\le C$ when $(x,y)\in N_\beta$, we have that

$$\begin{aligned}I_2(x,y)&\le C\left(1+|x|^2\right)\int_{m(x)}^{\infty}\frac{e^{-n/2t}}{t^{n/2+1}}dt\le C\frac{\left(1+|x|^2\right)}{m(x)^{n/2+1}}\\&\le C(1+|x|)^n\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta.\end{aligned}\tag{272}$$

Finally, from (252), proceeding as above for the estimation of J_1, but now by taking into account that $(x,y)\in N_\beta$, we obtain that

$$\begin{aligned}I_1(x,y)&\le C\int_0^{m(x)}e^{-c|x-y|^2/t}\left(\frac{1+|x|}{t^{(n+1)/2}}+\frac{(1+|x|)^n}{\mathrm{t}}\right)dt\\&\le C\frac{(1+|x|)m(x)^{1/4}+(1+|x|)^n m(x)^{n/2-1/4}}{|x-y|^{n-1/2}}\\&\le C\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}},\ (x,y)\in N_\beta.\end{aligned}\tag{273}$$

Thus, (269) is proved.

As it was established in Section 3.2,

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}}\chi_{N_\beta}(x,y)dy<\infty\text{and}\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\sqrt{1+|x|}}{|x-y|^{n-1/2}}\chi_{N_\beta}(x,y)dx<\infty.\tag{274}$$

Then, the operator L_β defined by

$$L_\beta(f)(x)=\int_{\mathbb{R}^n}\left|K_\sigma^{\mathscr{A}}(x,y)-K_\sigma(x,y)\right|\chi_{N_\beta}(x,y)f(y)dy,\ x\in\mathbb{R}^n,\tag{275}$$

is bounded from $L^p(\mathbb{R}^n,dx)$ into itself for every $1\le p<\infty$. From ([1], Proposition 3.2.5), it follows that L_β is bounded from $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself, for every $1\le p<\infty$.

Note that

$$\left|\mathscr{A}_{\mathrm{loc}}^{i\sigma}(f)-\left(-\frac{\Delta}{2}\right)_{\mathrm{loc}}^{i\sigma}(f)\right|\le CL_\beta(|f|).\tag{276}$$

We now study the global operator $\mathscr{A}_{glob}^{i\sigma}$. We recall that $\mathscr{A}_{glob}^{i\sigma}$ is the integral operator defined by

$$\mathscr{A}_{glob}^{i\sigma}(f)(x)=-\int_{\mathbb{R}^n}\int_0^{\infty}\phi_\sigma(t)\partial_t T_t^{\mathscr{A}}(x,y)dt\chi_{N_\beta^c}(x,y)f(y)dy,\ x\in\mathbb{R}^n.\tag{277}$$

We have, by making $r=e^{-t}$, $t\in(0,\infty)$, that

$$\begin{aligned}&\int_0^{\infty}\left|\phi_\sigma(t)\partial_t T_t^{\mathscr{A}}(x,y)\right|dt\le C\int_0^1\left|\partial_r\left[r^n\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}\right]\right|dr\\&\le C\int_0^1\left|\frac{n\left(1-r^2\right)+2r\left(1-r^2\right)\sum_{i=1}^n y_i\left(x_i-ry_i\right)-2r^2|x-ry|^2}{\left(1-r^2\right)^2}\right|r^{n-1}\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}dr\\&=C\int_0^1\left|\frac{\left(1-r^2\right)\left(n+|x|^2-r^2|y|^2\right)-\left(1+r^2\right)|x-ry|^2}{\left(1-r^2\right)^2}\right|r^{n-1}\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}dr.\end{aligned}\tag{278}$$

For every $x,y\in\mathbb{R}^n$, there exists a polynomial $P_{x,y}$ with degree 4 and a positive function $Q_{x,y}$ such that

$$\partial_r\left[r^n\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}\right]=P_{x,y}(r)Q_{x,y}(r),\ r\in(0,1).\tag{279}$$

Then, for every $x,y\in\mathbb{R}^n$, the function

$$\partial_r\left[r^n\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}\right]\tag{280}$$

changes the sign at most four times in $(0,1)$. We deduce that

$$\begin{aligned}\int_0^{\infty}|\phi_\sigma(t)|\left|\partial_t T_t^{\mathscr{A}}(x,y)\right|dt&\le C\int_0^1\left|\partial_r\left[r^n\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}}\right]\right|dr\\&\le C\sup_{r\in(0,1)}r^n\frac{e^{-|x-ry|^2/\left(1-r^2\right)}}{\left(1-r^2\right)^{n/2}},\ x,y\in\mathbb{R}^n.\end{aligned}\tag{281}$$

If $(u,v)\in N_\lambda$, with $\lambda>0$, we have that

$$\frac{1}{(1+\lambda n)(1+|v|)}\le\frac{1}{1+|u|}\le\frac{1+\lambda n}{1+|v|}.\tag{282}$$

Also, $\min\{1,|x|^{-1}\}/2\le(1+|x|)^{-1}\le\min\{1,|x|^{-1}\}$, $x\in\mathbb{R}^n$. Then, if $(y,x)\in N$, then $(x,y)\in N_{2(1+n)}$.

We take $\beta = 2(1+n)$. Since $(y,x) \notin N$ when $(x,y) \notin N_\beta$, according to ([24], Proposition 2.1), we get, for $(x,y) \notin N_\beta$,

$$\sup_{r\in(0,1)} \frac{e^{-|x-ry|^2/(1-r^2)}}{(1-r^2)^{n/2}} \le C \begin{cases} e^{-|x|^2}, & \langle x,y\rangle \le 0, \\ \left(\frac{|x+y|}{|x-y|}\right)^{n/2} \exp\left(\frac{|y|^2-|x|^2}{2} - \frac{|x-y||x+y|}{2}\right), & \langle x,y\rangle > 0. \end{cases} \tag{283}$$

By proceeding as in the proof of Theorem 2, we can see that, for every $1<p<\infty$,

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\left(|x|^2-|y|^2\right)/p}\chi_{N_\beta^c}(x,y)\int_0^\infty |\phi_\sigma(t)|\left|\partial_t T_t^{\mathscr{A}}(x,y)\right| dt dy < \infty, \tag{284}$$

and

$$\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\left(|x|^2-|y|^2\right)/p}\chi_{N_\beta^c}(x,y)\int_0^\infty |\phi_\sigma(t)|\left|\partial_t T_t^{\mathscr{A}}(x,y)\right| dt dy < \infty. \tag{285}$$

Hence, the operator $\mathscr{L}_\beta$ defined by

$$\mathscr{L}_\beta(f)(x) = \int_{\mathbb{R}^n} f(y)\chi_{N_\beta^c}(x,y)\int_0^\infty |\phi_\sigma(t)|\left|\partial_t T_t^{\mathscr{A}}(x,y)\right| dt dy, x\in\mathbb{R}^n, \tag{286}$$

is bounded from $L^p(\mathbb{R}^n,\gamma_{-1})$ into itself, for every $1<p<\infty$.

On the other hand, according to ([1], Lemma 3.3.3)

$$\sup_{t>0} T_t^{\mathscr{A}}(x,y) \le Ce^{-|x|^2}\min\left\{(1+|x|)^n, (|x|\sin\theta(x,y))^{-n}\right\}, (x,y)\in N_\beta^c, x,y\neq 0. \tag{287}$$

We recall that $\theta(x,y)\in[0,\pi)$ represents the angle between the nonzero vectors x and y when $n>1$ and $\theta(x,y)=0$, $x,y\in\mathbb{R}^n$, when $n=1$. By ([1], Lemma 3.3.4) the operator $\mathscr{L}_\beta$ is bounded from $L^1(\mathbb{R}^n,\gamma_{-1})$ into $L^{1,\infty}(\mathbb{R}^n,\gamma_{-1})$.

Note that $|\mathscr{A}^{i\sigma}_{glob}(f)| \le \mathscr{L}_\beta(|f|)$.

By taking into account that $(-\Delta/2)^{i\sigma}$ is a Calderón-Zygmund operator, the arguments developed in the proofs of Theorems 2 and 3 allow us to finish the proof of this theorem.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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