Research Article

On a More Accurate Half-Discrete Mulholland-Type Inequality Involving One Multiple Upper Limit Function

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By the use of the weight functions, the symmetry property, and Hermite-Hadamard's inequality, a more accurate half-discrete Mulholland-type inequality involving one multiple upper limit function is given. The equivalent conditions of the best possible constant factor related to multiparameters are studied. Furthermore, the equivalent forms, several inequalities for the particular parameters, and the operator expressions are provided.

1. Introduction

Suppose that $p > 1$, $(1/p) + (1/q) = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. The following well-known Hardy-Hilbert's inequality was provided (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \quad (1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Replacing $1/(m+n)$ and $1/(m+n-1)$ in (1), we have a more accurate form of (1) (cf. [1], Theorem 323). With regard to the similar assumptions, the following Mulholland's inequality was obtained (cf. [1], Theorem 343):

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=2}^{\infty} m^{p-1} a_m^p \right)^{1/p} \left(\sum_{n=2}^{\infty} n^{q-1} b_n^q \right)^{1/q}. \quad (2)$$

In 2006, by using Euler-Maclaurin's summation formula and the technique of real analysis, Krnić and Pečarić [2] obtained an extension of (1) as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}, \quad (3)$$

where $\lambda_i \in (0, 2]$ ($i = 1, 2$), $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$, the constant factor $B(\lambda_1, \lambda_2)$ in (3) is the best possible, and

$$B(u, v) = \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \quad (u, v > 0) \quad (4)$$

is the beta function. For $\lambda = 1$, $\lambda_1 = 1/q$, $\lambda_2 = 1/p$ in (3), it reduces to (1); for $p = q = 2$ and $\lambda_1 = \lambda_2 = \lambda/2$ in (3), it reduces to Yang's inequality in [3].

Using Abel's partial sum formula and (2), in 2019, Adiyasuren et al. [4] provided a discrete Hardy-Hilbert's inequality with the kernel as (3) involving two partial sums. We find that inequalities (1)–(3) played an important role in analysis and its applications (cf. [5–14]).

In 2016, Hong and Wen [15] gave an equivalent condition of the best possible constant factor related to a few parameters in the general form of (1). In the same way, the other similar works of (1) and the integral analogues were given in [16–19].

In 1934, a half-discrete Hilbert-type inequality was published in Hardy et al. [1] (Theorem 351) as follows: if $K(t)(t > 0)$ is decreasing, $p > 1$, $(1/p) + (1/q) = 1$, $0 < \phi(s) =$

$\int_0^\infty K(t)t^{s-1}dt < \infty$, and $a_n \geq 0$, satisfying $0 < \sum_{n=1}^\infty a_n^p < \infty$, then, we have

$$\int_0^\infty x^{p-2} \left(\sum_{n=1}^\infty K(nx)a_n \right)^p dx < \phi^p \left(\frac{1}{q} \right) \sum_{n=1}^\infty a_n^p. \quad (5)$$

Some new extensions of (5) with their applications were provided by [20–24].

In this article, following the way of [4, 15], by the use of the weight functions, the symmetry property, and Hermite-Hadamard's inequality, a more accurate half-discrete Mulholland-type inequality with the kernel as $1/[x + \ln^\alpha(n - \xi)]^\lambda$ involving one multiple upper limit function is obtained. The equivalent conditions of the best possible constant factor related to multiparameters are given. Furthermore, the equivalent forms, several new inequalities for the particular parameters, and the operator expressions are considered. The lemmas and theorems of this paper provide an extensive account of this type of inequalities.

2. Some Lemmas

In what follows, we suppose that $p > 1$, $(1/p) + (1/q) = 1$, $N = \{1, 2, \dots\}$, $\xi \in [0, 1/2]$, $\lambda > 0$, $\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \lambda + m) \cap (0, 1/\alpha]$, $\hat{\lambda}_1 := ((\lambda - \lambda_2)/p) + (\lambda_1/q)$, $\hat{\lambda}_2 := ((\lambda - \lambda_1)/q) + (\lambda_2/p)$. We also assume that $f(x) := F_0(x)$ is a nonnegative Lebesgue integrable function in any interval $(0, b)$ ($b > 0$). Define a multiple upper limit function as follows: $F_i(x) := \int_0^x F_{i-1}(t)dt$ ($x \geq 0$), satisfying

$$F_i(0) = 0, F_i(x) = o(e^{tx}) \quad (t > 0, x \rightarrow \infty; i = 1, \dots, m), \quad (6)$$

and for any $F_m(x)$, $a_n \geq 0$, it follows that

$$\begin{aligned} 0 &< \int_0^\infty x^{-p(m-1+\lambda\wedge_1)-1} F_m^p(x) dx < \infty, \\ 0 &< \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda\wedge_2)-1} (n - \xi)^{q-1} a_n^q < \infty. \end{aligned} \quad (7)$$

Lemma 1. Assuming that $s \in (0, \infty)$, $s_2 \in (0, s) \cap (0, 1/\alpha]$, $k_s(s_2) := B(s_2, s - s_2)$, define the following weight function:

$$\omega(s_2, x) := \alpha x^{s-s_2} \sum_{n=2}^\infty \frac{[\ln(n - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(n - \xi)]^s (n - \xi)} \quad (x \in R_+ = (0, \infty)). \quad (8)$$

The following inequality is valid:

$$\omega(s_2, x) < k_s(s_2) (x \in R_+). \quad (9)$$

Proof. For fixed $x \in R_+$, the function $[\ln(t - \xi)]^{\alpha s_2 - 1} / [x + \ln^\alpha(t - \xi)]^s (t - \xi)$ is strictly decreasing and strictly convex in the interval $(3/2, \infty)$. In fact, for $\alpha \in (0, 1]$, $s_2 \in (0, 1/\alpha]$, $\xi \in [0, 1/2]$, $t \in (3/2, \infty)$, it follows that

$$\begin{aligned} \frac{\partial}{\partial t} \frac{[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)} &= - \frac{(1 - \alpha s_2)[\ln(t - \xi)]^{\alpha s_2 - 2}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)^2} \\ &\quad - \frac{s\alpha[\ln(t - \xi)]^{\alpha s_2 + \alpha - 2}}{[x + \ln^\alpha(t - \xi)]^{s+1} (t - \xi)^2} \\ &\quad - \frac{[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)^2} < 0, \\ \frac{\partial^2}{\partial t^2} \frac{[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)} &= \frac{(1 - \alpha s_2)(2 - \alpha s_2)[\ln(t - \xi)]^{\alpha s_2 - 3}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)^3} \\ &\quad + \frac{s\alpha(3 - 2\alpha s_2 - \varepsilon)[\ln(t - \xi)]^{\alpha s_2 + \alpha - 3}}{[x + \ln^\alpha(t - \xi)]^{s+1} (t - \xi)^3} \\ &\quad + \frac{3(1 - \alpha s_2)[\ln(t - \xi)]^{\alpha s_2 - 2}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)^3} \\ &\quad + \frac{s\alpha^2(s + 1)[\ln(t - \xi)]^{\alpha s_2 + 2\alpha - 3}}{[x + \ln^\alpha(t - \xi)]^{s+2} (t - \xi)^3} \\ &\quad + \frac{3s\alpha[\ln(t - \xi)]^{\alpha s_2 + \alpha - 2}}{[x + \ln^\alpha(t - \xi)]^{s+1} (t - \xi)^3} \\ &\quad + \frac{2[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)^3} > 0. \end{aligned} \quad (10)$$

Applying Hermite-Hadamard's inequality (cf. [25]), we have

$$\sum_{n=2}^\infty \frac{[\ln(n - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(n - \xi)]^s (n - \xi)} < \int_{3/2}^\infty \frac{[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)} dt. \quad (11)$$

Setting $v = \ln^\alpha(t - \xi)/x((1/(t - \xi))dt = (1/\alpha)x^{1/\alpha}v^{1/\alpha-1}dv)$, it follows that

$$\begin{aligned} \int_{3/2}^\infty \frac{[\ln(t - \xi)]^{\alpha s_2 - 1}}{[x + \ln^\alpha(t - \xi)]^s (t - \xi)} dt &= \frac{1}{\alpha x^s} \int_{\ln^\alpha((3/2) - \xi)/x}^\infty \frac{(xv)^{1/\alpha(\alpha s_2 - 1)}}{(1 + v)^s} x^{1/\alpha} v^{1/\alpha-1} dv \\ &\leq \frac{1}{\alpha x^{s-s_2}} \int_0^\infty \frac{v^{s_2-1} dv}{(1 + v)^s} = \frac{1}{\alpha x^{s-s_2}} k_s(s_2). \end{aligned} \quad (12)$$

By (8) and (11), we have the following inequality:

$$\omega(s_2, x) < \alpha x^{s-s_2} \frac{1}{\alpha x^{s-s_2}} k_s(s_2) = k_s(s_2), \quad (13)$$

namely, (9) follows.

The lemma is proved. $\square$

Lemma 2. For $s_1 \in (0, s)$ and $k_s(s_1) = B(s_1, s - s_1)$, if

$$0 < \int_0^\infty x^{p[1-(((s-s_2)/p)+(s_1/q))]-1} f^p(x) dx < \infty, \quad (14)$$

$$0 < \sum_{n=2}^{\infty} [\ln(n-\xi)]^{q[1-\alpha((s_2/p)+((s-s_1)/q))]-1} (n-\xi)^{q-1} a_n^q < \infty, \quad (15)$$

then we have the following more accurate half-discrete Mulholland-type inequality:

$$\begin{aligned} I_0 &:= \int_0^{\infty} \sum_{n=2}^{\infty} \frac{a_n}{[x + \ln^{\alpha}(n-\xi)]^s} f(x) dx \\ &< \left(\frac{1}{\alpha} k_s(s_2) \right)^{1/p} (k_s(s_1))^{1/q} \\ &\quad \times \left\{ \int_0^{\infty} x^{p[1-((s-s_2)/p)+(s_1/q)]-1} f^p(x) dx \right\}^{1/p} \\ &\quad \cdot \left\{ \sum_{n=2}^{\infty} [\ln(n-\xi)]^{q[1-\alpha((s_2/p)+((s-s_1)/q))]-1} (n-\xi)^{q-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (16)$$

Proof. Setting $v = x/\ln^{\alpha}(n-\xi)$, we obtain the expression of the following another weight function:

$$\begin{aligned} \omega(s_1, n) &:= [\ln(n-\xi)]^{\alpha(s-s_1)} \int_0^{\infty} \frac{x^{s_1-1} dx}{[x + \ln^{\alpha}(n-\xi)]^s} \\ &= \int_0^{\infty} \frac{v^{s_1-1} dv}{(v+1)^s} = k_s(s_1) (n \in N \setminus \{1\}). \end{aligned} \quad (17)$$

Using Hölder's inequality (cf. [25]), it follows that

$$\begin{aligned} I_0 &= \int_0^{\infty} \sum_{n=2}^{\infty} \frac{1}{[x + \ln^{\alpha}(n-\xi)]^s} \left[\frac{x^{(1-s_1)/q} (n-\xi)^{-1/p}}{[\ln(n-\xi)]^{(1-\alpha s_2)/p}} f(x) \right] \\ &\quad \cdot \left[\frac{[\ln(n-\xi)]^{(1-\alpha s_2)/p}}{x^{(1-s_1)/q} (n-\xi)^{-1/p}} a_n \right] dx \\ &\leq \left\{ \int_0^{\infty} \sum_{n=2}^{\infty} \frac{(n-\xi)^{-1}}{[x + \ln^{\alpha}(n-\xi)]^s} \frac{x^{(1-s_1)(p-1)}}{[\ln(n-\xi)]^{1-\alpha s_2}} f^p(x) dx \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=2}^{\infty} \int_0^{\infty} \frac{1}{[x + \ln^{\alpha}(n-\xi)]^s} \frac{[\ln(n-\xi)]^{(1-\alpha s_2)(q-1)} a_n^q}{x^{1-s_1} (n-\xi)^{1-q}} dx \right\}^{1/q} \\ &= \left\{ \frac{1}{\alpha} \int_0^{\infty} \omega(s_2, x) x^{p[1-((s-s_2)/p)+(s_1/q)]-1} f^p(x) dx \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=2}^{\infty} \omega(s_1, n) [\ln(n-\xi)]^{q[1-\alpha((s-s_1)/q)+(s_2/p)]-1} (n-\xi)^{q-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (18)$$

We show that the above inequality does not keep the form of equality. Otherwise (cf. [25]), there exist constants

A and B , such that they are not both zero and

$$\begin{aligned} A \frac{x^{(1-s_1)(p-1)} (n-\xi)^{-1}}{[\ln(n-\xi)]^{1-\alpha s_2}} f^p(x) \\ = B \frac{[\ln(n-\xi)]^{(1-\alpha s_2)(q-1)} a_n^q}{x^{1-s_1} (n-\xi)^{1-q}} \text{a.e. in } R_+ \times N \setminus \{1\}. \end{aligned} \quad (19)$$

Suppose that $A \neq 0$. There exists a $n \in N \setminus \{1\}$, such that

$$\begin{aligned} x^{p[1-\alpha((s-s_2)/p)+(s_1/q)]-1} f^p(x) &= \frac{B}{A} [\ln(n-\xi)]^{q(1-\alpha s_2)} \\ &\quad \cdot (n-\xi)^q a_n^q \frac{1}{x^{1+(s-s_1-s_2)}} \text{a.e. in } R_+, \end{aligned} \quad (20)$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p[1-\alpha((s-s_2)/p)+(s_1/q)]-1} f^p(x) dx < \infty, \quad (21)$$

based on $\int_0^{\infty} (1/x^{1+(s-s_1-s_2)}) dx = \infty$. Then, by (9) and (17), we have (16).

The lemma is proven. $\square$

Remark 1. Replacing $f(x)$ by $F_m(x)$ and setting $s = \lambda + m \in (m, \infty)$, $s_1 = \lambda_1 + m \in (m, s)$, and $s_2 = \lambda_2 \in (0, s) \cap (0, 1/\alpha]$ in (16), we find $\lambda \in (0, \infty)$, $\lambda_1 \in (0, \lambda)$, and $\lambda_2 \in (0, \lambda + m) \cap (0, 1/\alpha]$ and obtain the following more accurate half-discrete Mulholland-type inequality with new parameters:

$$\begin{aligned} \int_0^{\infty} \sum_{n=2}^{\infty} \frac{a_n}{[x + \ln^{\alpha}(n-\xi)]^{\lambda+m}} F_m(x) dx \\ < \left(\frac{1}{\alpha} k_{\lambda+m}(\lambda_2) \right)^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \\ &\quad \times \left[\int_0^{\infty} x^{-p(m-1+\lambda_{\lambda_1})-1} F_m^p(x) dx \right]^{1/p} \\ &\quad \times \left\{ \sum_{n=2}^{\infty} [\ln(n-\xi)]^{q(1-\alpha \lambda_2)-1} (n-\xi)^{q-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (22)$$

Lemma 3. For $t > 0$, the following expression is valid:

$$\int_0^{\infty} e^{-tx} f(x) dx = t^m \int_0^{\infty} e^{-tx} F_m(x) dx. \quad (23)$$

Proof. For $m = 0$ since $F_0(x) = f(x)$, (23) is naturally valid; for $m \in N$, integration by parts, since $F_i(0) = 0$, $F_i(x) = o(e^{tx})$ ($t > 0$, $i = 1, \dots, m$; $x \rightarrow \infty$), we find

$$\int_0^{\infty} e^{-tx} F_{i-1}(x) dx = \int_0^{\infty} e^{-tx} dF_i(x) = e^{-tx} F_i(x) \Big|_0^{\infty} - \int_0^{\infty} F_i(x) d e^{-tx} = \lim_{x \rightarrow \infty} e^{-tx} F_i(x) + t \int_0^{\infty} e^{-tx} F_i(x) dx = t \int_0^{\infty} e^{-tx} F_i(x) dx. \quad (24)$$

By substitution of $i = 1, \dots, m$ in the above expression, we have (23).

The lemma is proven. $\square$

3. Main Results

Theorem 1. *The following more accurate half-discrete Mulholland-type inequality involving one multiple upper limit function is valid:*

$$\begin{aligned} I &:= \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n - \xi)]^\lambda} f(x) dx \\ &< \frac{\Gamma(\lambda + m)}{\Gamma(\lambda)} \left(\frac{1}{\alpha} k_{\lambda+m}(\widehat{\lambda}_2) \right)^{1/p} \left(k_{\lambda+m}(\widehat{\lambda}_1 + m) \right)^{1/q} \\ &\quad \times \left[\int_0^\infty x^{-p(m-1+\widehat{\lambda}_1)-1} F_m^p(x) dx \right]^{1/p} \\ &\quad \cdot \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\widehat{\lambda}_2)-1} (n - \xi)^{q-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (25)$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ ($\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \lambda) \cap (0, 1/\alpha]$), we have

$$0 < \int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx < \infty, \quad 0 < \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{q-1} a_n^q < \infty, \quad (26)$$

and the following inequality:

$$\begin{aligned} I &= \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n - \xi)]^\lambda} f(x) dx \\ &< \frac{\Gamma(\lambda + m)}{\alpha^{1/p} \Gamma(\lambda)} B(\lambda_1 + m, \lambda_2) \\ &\quad \times \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \\ &\quad \cdot \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{q-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (27)$$

Proof. In view of the following expression:

$$\frac{1}{[x + \ln^\alpha(n - \xi)]^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-[x + \ln^\alpha(n - \xi)]t} dt, \quad (28)$$

by the Lebesgue term by the term integration theorem (cf. [26]) and (23), we obtain

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty a_n f(x) \int_0^\infty t^{\lambda-1} e^{-[x + \ln^\alpha(n - \xi)]t} dt dx \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(\int_0^\infty e^{-xt} f(x) dx \right) \sum_{n=2}^\infty e^{-t \ln^\alpha(n - \xi)} a_n dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} \left(t^m \int_0^\infty e^{-xt} F_m(x) dx \right) \sum_{n=2}^\infty e^{-t \ln^\alpha(n - \xi)} a_n dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty \sum_{n=2}^\infty a_n F_m(x) \left\{ \int_0^\infty t^{(\lambda+m)-1} e^{-[x + \ln^\alpha(n - \xi)]t} dt \right\} dx \\ &= \frac{\Gamma(\lambda + m)}{\Gamma(\lambda)} \int_0^\infty \sum_{n=1}^\infty \frac{a_n}{[x + \ln^\alpha(n - \xi)]^{\lambda+m}} F_m(x) dx. \end{aligned} \quad (29)$$

Then, by (22), we have (25).

The theorem is proven. $\square$

Theorem 2. *If $\lambda_1 + \lambda_2 = \lambda$ ($\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \lambda) \cap (0, 1/\alpha]$), then the constant factor*

$$\frac{\Gamma(\lambda + m)}{\Gamma(\lambda)} \left(\frac{1}{\alpha} k_{\lambda+m}(\lambda_2) \right)^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \quad (30)$$

is the best possible. On the other hand, if we add $\lambda_2 < \lambda$, $\lambda - \lambda_1 \leq 1/\alpha$ and the same constant factor in (25) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. If $\lambda_1 + \lambda_2 = \lambda$, then (25) reduces to (27). For any $0 < \varepsilon < p\lambda_1$, we set

$$\begin{aligned} \tilde{f}(x) &:= \begin{cases} 0, & 0 < x < 1, \\ x^{\lambda_1 - \varepsilon/p - 1}, & x \geq 1, \end{cases} \quad \tilde{a}_n \\ &:= [\ln(n - \xi)]^{\alpha(\lambda_2 - \varepsilon/q) - 1} (n - \xi)^{-q} (n \in N \setminus \{1\}). \end{aligned} \quad (31)$$

We have $\tilde{F}_0(x) = \tilde{f}(x)$ and

$$\begin{aligned} \tilde{F}_i(x) &:= \int_0^x \tilde{F}_{i-1}(t) dt \\ &\leq \begin{cases} 0, & 0 < x < 1, \\ \frac{1}{\prod_{j=0}^{i-1} (\lambda_1 + j - \varepsilon/p)} x^{\lambda_1 + i - 1 - \varepsilon/p}, & x \geq 1, \end{cases} \quad (i \in \{1, \dots, m\}), \end{aligned} \quad (32)$$

satisfying $\tilde{F}_i(0) = 0$, $\tilde{F}_i(x) = o(e^{tx})(t > 0, x \rightarrow \infty; i = 1, \dots, m)$.

If there exists a positive constant $M(\leq (\Gamma(\lambda + m)/\alpha^{1/p} \Gamma(\lambda)) k_{\lambda+m}(\lambda_1 + m))$, such that (27) is valid when we replace $(\Gamma(\lambda + m)/\alpha^{1/p} \Gamma(\lambda)) k_{\lambda+m}(\lambda_1 + m)$ by M , then in particular, it follows that

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \sum_{n=2}^\infty \frac{\tilde{a}_n \tilde{f}(x) dx}{[x + \ln^\alpha(n - \xi)]^\lambda} \\ &< M \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} \tilde{F}_m^p(x) dx \right]^{1/p} \\ &\quad \cdot \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{q-1} \tilde{a}_n^q \right\}^{1/q}. \end{aligned} \quad (33)$$

For $m = 0, \varepsilon \geq 0$, we define $\prod_{i=0}^{m-1} (\lambda_1 + i - (\varepsilon/p)) = 1$. By the decreasing property of the series, we find

$$\begin{aligned} \tilde{I} &< \frac{M}{\prod_{i=0}^{m-1} (\lambda_1 + i - \varepsilon/p)} \left[\int_1^\infty x^{-p(m-1+\lambda_1)-1} x^{p(\lambda_1+m-1)-\varepsilon} dx \right]^{1/p} \\ &\quad \times \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{q-1} [\ln(n - \xi)]^{q\alpha\lambda_2 - \alpha\varepsilon - q} (n - \xi)^{-q} \right\}^{1/q} \\ &= \frac{M}{\prod_{i=0}^{m-1} (\lambda_1 + i - \varepsilon/p)} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{1/p} \\ &\quad \times \left\{ [\ln(2 - \xi)]^{-\alpha\varepsilon-1} (2 - \xi)^{-1} + \sum_{n=3}^\infty [\ln(n - \xi)]^{-\alpha\varepsilon-1} (n - \xi)^{-1} \right\}^{1/q} \\ &\leq \frac{M}{\prod_{i=0}^{m-1} (\lambda_1 + i - \varepsilon/p)} \left(\int_1^\infty x^{-\varepsilon-1} dx \right)^{1/p} \\ &\quad \times \left\{ [\ln(2 - \xi)]^{-\alpha\varepsilon-1} (2 - \xi)^{-1} + \int_2^\infty [\ln(t - \xi)]^{-\alpha\varepsilon-1} (t - \xi)^{-1} dt \right\}^{1/q} \\ &= \frac{M}{\varepsilon \prod_{i=0}^{m-1} (\lambda_1 + i - \varepsilon/p)} \left\{ \varepsilon [\ln(2 - \xi)]^{-\alpha\varepsilon-1} (2 - \xi)^{-1} + \frac{1}{\alpha} [\ln(2 - \xi)]^{-\alpha\varepsilon} \right\}^{1/q}. \end{aligned} \quad (34)$$

By (17), setting $s = \lambda$ and $s_1 = \tilde{\lambda}_1 := \lambda_1 - (\varepsilon/p) \in (0, \lambda)$ ($0 < \tilde{\lambda}_2 := \lambda_2 + (\varepsilon/p) < \lambda$), it follows that

$$\begin{aligned} \tilde{I} &= \sum_{n=2}^\infty \left\{ [\ln(n - \xi)]^{\alpha(\lambda_2 + \varepsilon/p)} \int_1^\infty \frac{x^{(\lambda_1 - \varepsilon/p)-1}}{[x + \ln^\alpha(n - \xi)]^\lambda} dx \right\} [\ln(n - \xi)]^{-\alpha\varepsilon-1} (n - \xi)^{-1} \\ &= \sum_{n=2}^\infty \omega(\tilde{\lambda}_1, n) [\ln(n - \xi)]^{-\alpha\varepsilon-1} (n - \xi)^{-1} \\ &\quad - \sum_{n=2}^\infty [\ln(n - \xi)]^{\alpha(\tilde{\lambda}_2 - \varepsilon)-1} (n - \xi)^{-1} \int_0^1 \frac{x^{\tilde{\lambda}_1-1}}{[x + \ln^\alpha(n - \xi)]^\lambda} dx \\ &> B(\tilde{\lambda}_1, \tilde{\lambda}_2) \sum_{n=2}^\infty [\ln(n - \xi)]^{-\alpha\varepsilon-1} (n - \xi)^{-1} \\ &\quad - \sum_{n=2}^\infty [\ln(n - \xi)]^{\alpha(\tilde{\lambda}_2 - \varepsilon)-1} (n - \xi)^{-1} \int_0^1 \frac{x^{\tilde{\lambda}_1-1} dx}{[\ln(n - \xi)]^{\alpha\lambda}} \\ &\geq B(\tilde{\lambda}_1, \tilde{\lambda}_2) \int_2^\infty [\ln(y - \xi)]^{-\alpha\varepsilon-1} (y - \xi)^{-1} dy \\ &\quad - \frac{1}{\tilde{\lambda}_1} \sum_{n=2}^\infty [\ln(n - \xi)]^{-\alpha(\tilde{\lambda}_1 + \varepsilon)-1} (n - \xi)^{-1} \\ &= \frac{1}{\varepsilon} \left\{ \frac{1}{\alpha} B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) [\ln(2 - \xi)]^{-\alpha\varepsilon} - \varepsilon O(1) \right\}. \end{aligned} \quad (35)$$

In virtue of the above results, we have

$$\begin{aligned} &\frac{1}{\alpha} B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) [\ln(2 - \xi)]^{-\alpha\varepsilon} - \varepsilon O(1) \\ &< \tilde{\varepsilon} I < \frac{M}{\prod_{i=0}^{m-1} (\lambda_1 + i - (\varepsilon/p))} \\ &\quad \times \left\{ \varepsilon [\ln(2 - \xi)]^{-\alpha\varepsilon-1} (2 - \xi)^{-1} + \frac{1}{\alpha} [\ln(2 - \xi)]^{-\alpha\varepsilon} \right\}^{1/q}. \end{aligned} \quad (36)$$

For $\varepsilon \rightarrow 0^+$, by the continuity of the beta function, we have

$$\begin{aligned} \frac{\Gamma(\lambda + m)}{\alpha^{1/p} \Gamma(\lambda)} B(\lambda_1 + m, \lambda_2) &= \frac{\Gamma(\lambda_1 + m) \Gamma(\lambda_2)}{\alpha^{1/p} \Gamma(\lambda)} \\ &= \frac{1}{\alpha^{1/p}} B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \\ &\leq M. \end{aligned} \quad (37)$$

Hence, $M = (\Gamma(\lambda + m)/\alpha^{1/p} \Gamma(\lambda)) B(\lambda_1 + m, \lambda_2)$ is the best possible constant factor in (27).

On the other hand, for $\hat{\lambda}_1 = ((\lambda - \lambda_2)/p) + (\lambda_1/q)$, $\hat{\lambda}_2 = ((\lambda - \lambda_1)/q) + (\lambda_2/p)$, in view of $0 < \lambda_2 < \lambda$, $\lambda - \lambda_1 \leq 1/\alpha$, we find $\hat{\lambda}_1 + \hat{\lambda}_2 = ((\lambda - \lambda_2)/p) + (\lambda_1/q) + ((\lambda - \lambda_1)/q) + (\lambda_2/p) = \lambda$, $0 < \hat{\lambda}_1, \hat{\lambda}_2 < (\lambda/p) + (\lambda/q) = \lambda$, $\hat{\lambda}_2 \leq ((1/\alpha)/q) + ((1/\alpha)/p) = 1/\alpha$, and

$$\frac{\Gamma(\lambda + m)}{\alpha^{1/p} \Gamma(\lambda)} B(\hat{\lambda}_1 + m, \hat{\lambda}_2) \in R_+.. \quad (38)$$

By substitution of $\hat{\lambda}_i = \lambda_i (i = 1, 2)$ in (27), we find

$$\begin{aligned} I &= \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n - \xi)]^\lambda} f(x) dx \\ &< \frac{\Gamma(\lambda + m)}{\alpha^{1/p} \Gamma(\lambda)} k_{\lambda+m}(\hat{\lambda}_1 + m) \\ &\quad \times \left[\int_0^\infty x^{-p(m-1+\lambda\wedge_1)-1} F_m^p(x) dx \right]^{1/p} \\ &\quad \cdot \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda\wedge_2)-1} (n - \xi)^{-1} a_n^q \right\}^{1/q}. \end{aligned} \quad (39)$$

By means of Hölder's inequality (cf. [25]), it follows that

$$\begin{aligned}
 k_{\lambda+m}(\widehat{\lambda}_1 + m) &= k_{\lambda+m} \left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + m \right) \\
 &= \int_0^\infty \frac{1}{(1+u)^{\lambda+m}} u^{((\lambda-\lambda_2+m)/p) + ((\lambda_1+m)/q) - 1} du \\
 &= \int_0^\infty \frac{1}{(1+u)^{\lambda+m}} \left(u^{(\lambda-\lambda_2+m-1)/p} \right) \left(u^{(\lambda_1+m-1)/q} \right) du \\
 &\leq \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+m}} u^{\lambda-\lambda_2+m-1} du \right]^{1/p} \\
 &\quad \cdot \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+m}} u^{\lambda_1+m-1} du \right]^{1/q} \\
 &= \left[\int_0^\infty \frac{1}{(1+v)^{\lambda+m}} v^{\lambda_2-1} dv \right]^{1/p} \\
 &\quad \cdot \left[\int_0^\infty \frac{1}{(1+u)^{\lambda+m}} u^{(\lambda_1+m)-1} du \right]^{1/q} \\
 &= (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q}.
 \end{aligned} \tag{40}$$

Since $(\Gamma(\lambda + m)/\alpha^{1/p}\Gamma(\lambda))(k_{\lambda+m}(\lambda_2))^{1/p}(k_{\lambda+m}(\lambda_1 + m))^{1/q}$ is the best possible constant factor in (25), by (39), we have the following inequality:

$$\begin{aligned}
 &\frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \\
 &\leq \frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} k_{\lambda+m}(\widehat{\lambda}_1 + m) \in R_+,
 \end{aligned} \tag{41}$$

namely, $k_{\lambda+m}(\widehat{\lambda}_1 + m) \geq (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q}$, and then, (40) keeps the form of equality.

We observe that (40) keeps the form of equality if and only if there exist constants A and B (cf. [25]), such that they are not both zero and $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ a.e. in R_+ . Suppose that $A \neq 0$. We find $u^{\lambda-\lambda_2-\lambda_1} = (B/A)$ a.e. in R_+ , namely, $\lambda - \lambda_2 - \lambda_1 = 0$. Hence, it follows that $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proven. $\square$

4. Equivalent Forms and Some Particular Inequalities

Theorem 3. *The following half-discrete more accurate Mulholland-type inequality involving a multiple upper limit function is equivalent to (25):*

$$\begin{aligned}
 J &:= \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{p\alpha\lambda_2-1} (n - \xi)^{-1} \left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n - \xi)]^\lambda} dx \right]^p \right\}^{1/p} \\
 &< \frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \\
 &\quad \cdot \left[\int_0^\infty x^{-p(m-1+\lambda\wedge_1)-1} F_m^p(x) dx \right]^{1/p}.
 \end{aligned} \tag{42}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$ ($\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \lambda) \cap (0, 1/\alpha]$), we have the following Mulholland-type inequality equivalent to (27):

$$\begin{aligned}
 &\left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{p\alpha\lambda_2-1} (n - \xi)^{-1} \left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n - \xi)]^\lambda} dx \right]^p \right\}^{1/p} \\
 &< \frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} B(\lambda_1 + m, \lambda_2) \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p}.
 \end{aligned} \tag{43}$$

Proof. By assuming that (42) is valid, by Hölder's inequality (cf. [25]), it follows that

$$\begin{aligned}
 I &= \sum_{n=2}^\infty \left\{ [\ln(n - \xi)]^{\alpha\lambda_2-1/p} (n - \xi)^{-1/p} \int_0^\infty \frac{f(x) dx}{[x + \ln^\alpha(n - \xi)]^\lambda} \right\} \\
 &\quad \times \left\{ [\ln(n - \xi)]^{-\alpha\lambda_2+1/p} (n - \xi)^{1/p} a_n^q \right\} \\
 &\leq J \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{-1} a_n^q \right\}^{1/q}.
 \end{aligned} \tag{44}$$

Then, by (42), we have (25).

On the other hand, suppose that (25) is valid. We set

$$a_n := [\ln(n - \xi)]^{p\alpha\lambda_2-1} (n - \xi)^{-1} \left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n - \xi)]^\lambda} dx \right]^{p-1}, \quad n \in N \setminus \{1\}. \tag{45}$$

If $J = 0$, then (42) is naturally valid; if $J = \infty$, then it is impossible that makes (42) valid, i.e., $J < \infty$. By assuming that $0 < J < \infty$, by (25), it follows that

$$\begin{aligned}
 0 &< \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{-1} a_n^q = J^p \\
 &= I < \frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \\
 &\quad \cdot \left[\int_0^\infty x^{-p(m-1+\lambda\wedge_1)-1} F_m^p(x) dx \right]^{1/p} J^{p-1} < \infty, \\
 J &= \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{-1} a_n^q \right\}^{1/p} \\
 &< \frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \\
 &\quad \cdot \left[\int_0^\infty x^{-p(m-1+\lambda\wedge_1)-1} F_m^p(x) dx \right]^{1/p},
 \end{aligned} \tag{46}$$

namely, (42) follows, which is equivalent to (25).

The theorem is proven. $\square$

Theorem 4. *If $\lambda_1 + \lambda_2 = \lambda$ ($\lambda_1 \in (0, \lambda)$, $\lambda_2 \in (0, \lambda) \cap (0, 1/\alpha]$), then the constant factor*

$$\frac{\Gamma(\lambda + m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \tag{47}$$

in (42) is the best possible. On the other hand, if we add $\lambda_2 < \lambda$, $\lambda - \lambda_1 \leq 1/\alpha$ and the same constant factor in (42) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. If $\lambda_1 + \lambda_2 = \lambda$, then by Theorem 2, the constant factor

$$\frac{\Gamma(\lambda + m)}{\alpha^{1/p} \Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1 + m))^{1/q} \quad (48)$$

in (25) is the best possible. By (44), the constant factor in (42) is still the best possible. Otherwise, we would reach a contradiction that the constant factor in (25) is not the best possible.

On the other hand, if the constant factor in (42) is the best possible, then by the equivalency of (42) and (25), in view of $J^p = I$ (see the proof of Theorem 3), we still can show that the constant factor in (25) is the best possible. By the assumptions and Theorem 2, it follows that $\lambda_1 + \lambda_2 = \lambda$.

The theorem is proven. $\square$

Example 1.

(i) For $\alpha = 1$, (27) and (43) reduce to the following equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{\ln^\lambda e^x (n - \xi)} f(x) dx \\ & < \times \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \\ & \times \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{q(1-\lambda_2)-1} (n - \xi)^{q-1} a_n^q \right\}^{1/q}, \end{aligned} \quad (49)$$

$$\begin{aligned} & \left\{ \sum_{n=2}^\infty [\ln(n - \xi)]^{p\lambda_2-1} (n - \xi)^{-1} \left[\int_0^\infty \frac{f(x)}{\ln^\lambda e^x (n - \xi)} dx \right]^p \right\}^{1/p} \\ & < B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p}. \end{aligned} \quad (50)$$

(ii) For $\xi = 0$, (27) and (43) reduce to the following equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{(x + \ln^\alpha n)^\lambda} f(x) dx < \frac{1}{\alpha^{1/p}} B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \\ & \times \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \\ & \cdot \left[\sum_{n=2}^\infty (\ln n)^{q(1-\alpha\lambda_2)-1} n^{q-1} a_n^q \right]^{1/q}, \end{aligned} \quad (51)$$

$$\begin{aligned} & \left\{ \sum_{n=2}^\infty \frac{(\ln n)^{p\alpha\lambda_2-1}}{n} \left[\int_0^\infty \frac{f(x)}{(x + \ln^\alpha n)^\lambda} dx \right]^p \right\}^{1/p} \\ & < \frac{1}{\alpha^{1/p}} B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p}. \end{aligned} \quad (52)$$

Hence, (27) (resp. (43)) is a more accurate form of (51) (resp. (52)).

(iii) For $\xi = 1/2$, (27) and (43) reduce to the following equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \sum_{n=2}^\infty \frac{a_n}{[x + \ln^\alpha(n - 1/2)]^\lambda} f(x) dx < \frac{1}{\alpha^{1/p}} B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \\ & \times \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p} \\ & \cdot \left\{ \sum_{n=2}^\infty \left[\ln \left(n - \frac{1}{2} \right) \right]^{q(1-\alpha\lambda_2)-1} \left(n - \frac{1}{2} \right)^{q-1} a_n^q \right\}^{1/q}, \\ & \left\{ \sum_{n=2}^\infty \left[\ln \left(n - \frac{1}{2} \right) \right]^{p\alpha\lambda_2-1} \left(n - \frac{1}{2} \right)^{-1} \left[\int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n - 1/2)]^\lambda} dx \right]^p \right\}^{1/p} \\ & < \frac{1}{\alpha^{1/p}} B(\lambda_1, \lambda_2) \prod_{i=0}^{m-1} (\lambda_1 + i) \left[\int_0^\infty x^{-p(m-1+\lambda_1)-1} F_m^p(x) dx \right]^{1/p}. \end{aligned} \quad (53)$$

The constant factors in the above inequalities are still the best possible.

5. Operator Expressions

Setting $\phi_m(x) := x^{-p(m-1+\lambda_1)-1}$, $\psi(n) := [\ln(n - \xi)]^{q(1-\alpha\lambda_2)-1} (n - \xi)^{q-1}$, from

$$\psi^{1-p}(n) = [\ln(n - \xi)]^{p\alpha\lambda_2-1} (n - \xi)^{-1} \quad (x \in R_+, n \in N \setminus \{1\}), \quad (54)$$

we define the following normed linear spaces:

$$\begin{aligned} L_{p,\phi_m}(R_+) &:= \left\{ f = f(x) ; \|f\|_{p,\phi_m} := \left(\int_0^\infty \phi_m(x) |f(x)|^p dx \right)^{1/p} < \infty \right\}, \\ l_{q,\psi} &:= \left\{ a = \{a_n\}_{n=2}^\infty ; \|a\|_{q,\psi} := \left(\sum_{n=2}^\infty \psi(n) |a_n|^q \right)^{1/q} < \infty \right\}, \\ l_{p,\psi^{1-p}} &:= \left\{ c = \{c_n\}_{n=2}^\infty ; \|c\|_{p,\psi^{1-p}} := \left(\sum_{n=2}^\infty \psi^{1-p}(n) |c_n|^p \right)^{1/p} < \infty \right\}. \end{aligned} \quad (55)$$

For any $f = F_0(x) \in L_{p,\phi_0}(R_+)$, setting

$$c = \{c_n\}_{n=2}^\infty : c_n := \int_0^\infty \frac{f(x)}{[x + \ln^\alpha(n - \xi)]^\lambda} dx, \quad (56)$$

we can rewrite (42) as follows:

$$\|c\|_{p,\psi^{1-p}} < \frac{\Gamma(\lambda+m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1+m))^{1/q} \|F_m\|_{p,\phi_m} < \infty, \quad (57)$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 1. Define a half-discrete Mulholland-type operator $T : L_{p,\phi_0}(R_+) \rightarrow l_{p,\psi^{1-p}}$ as follows: for any $f \in L_{p,\phi_0}(R_+)$, there exists a unique representation $c = Tf \in l_{p,\psi^{1-p}}$, satisfying for any $n \in N \setminus \{1\}$ $Tf(n) = c_n$. Define the formal inner product of Tf and $a \in l_{q,\psi}$ and the norm of T as follows:

$$(Tf, a) := \sum_{n=2}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{[x + \ln^{\alpha}(n - \xi)]^{\lambda}} dx = I, \quad (58)$$

$$\|T\| := \sup_{f(\neq 0) \in L_{p,\phi_0}(R_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|F_m\|_{p,\phi_m}}.$$

By Theorem 3 and Theorem 4, it follows that.

Theorem 5. If $f(\geq 0) \in L_{p,\phi_0}(R_+)$, $a(\geq 0) \in l_{q,\psi}$, $\|f\|_{p,\phi_0} > 0$, $\|a\|_{q,\psi} > 0$, then we have the following equivalent inequalities:

$$(Tf, a) < \frac{\Gamma(\lambda+m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1+m))^{1/q} \|F_m\|_{p,\phi_m} \|a\|_{q,\psi}, \quad (59)$$

$$\|Tf\|_{p,\psi^{1-p}} < \frac{\Gamma(\lambda+m)}{\alpha^{1/p}\Gamma(\lambda)} (k_{\lambda+m}(\lambda_2))^{1/p} (k_{\lambda+m}(\lambda_1+m))^{1/q} \|F_m\|_{p,\phi_m}. \quad (60)$$

Moreover, if $\lambda_1 + \lambda_2 = \lambda$, then the constant factor $(\Gamma(\lambda+m)/\alpha^{1/p}\Gamma(\lambda))(k_{\lambda+m}(\lambda_2))^{1/p}(k_{\lambda+m}(\lambda_1+m))^{1/q}$ in (59) and (60) is the best possible, namely,

$$\|T\| = \frac{\Gamma(\lambda+m)}{\alpha^{1/p}\Gamma(\lambda)} B(\lambda_1+m, \lambda_2). \quad (61)$$

On the other hand, if we add $\lambda_2 < \lambda$, $\lambda - \lambda_1 \leq 1/\alpha$ and the constant factor in (59) and (60) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interest.

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