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Research Article

Iterative Approximation of Fixed Points by Using F Iteration Process in Banach Spaces

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We connect the F iteration process with the class of generalized α-nonexpansive mappings. Under some appropriate assumption, we establish some weak and strong convergence theorems in Banach spaces. To show the numerical efficiency of our established results, we provide a new example of generalized α-nonexpansive mappings and show that its F iteration process is more efficient than many other iterative schemes. Our results are new and extend the corresponding known results of the current literature.

1. Introduction and Preliminaries

Once an existence of a solution for an operator equation is established then in many cases, such solution cannot be obtained by using ordinary analytical methods. To overcome such cases, one needs the approximate value of this solution. To do this, we first rearrange the operator equation in the form of fixed-point equation. We apply the most suitable iterative algorithm on the fixed point equation, and the limit of the sequence generated by this most suitable algorithm is in fact the value of the desired fixed point for the fixed point equation and the solution for the operator equation. The Banach Fixed Point Theorem [1] (BFPT, for short) suggests the elementry Picard iteration $w_{t+1} = \mathscr{G}w_t$ in the case of contraction mappings. Since for the class of nonexpansive mappings, Picard iterates do not always converge to a fixed point of a certain nonexpansive mapping, we, therefore use some other iterative processes involving different steps and set of parameters. Among the other things, Mann [2], Ishikawa [3], Noor [4], S iteration of Agarwal et al. [5], SP iteration of Phuengrattana and Suantai [6], S^* iteration of Karahan and Ozdemir [7], Normal-S [8], Picard-Mann hybrid [9], Krasnoselskii-Mann [10], Abbas [11], Thakur [12], and Picard-S [13] are the most studied iterative processes. In 2018, Ullah and Arshad introduced M [14] iteration process for Suzuki mappings and proved that it converges faster than all of these iteration processes.

Very recently, Ali and Ali [15] introduced the novel iteration process, namely, F iterative scheme for generalized contractions as follows:

$$\begin{cases} w_1 \in \mathscr{P}, \\ u_t = G((1-\alpha_t)w_t + \alpha_t G w_t), \\ v_t = G u_t, \\ w_{t+1} = G v_t, t \geq 1, \end{cases} \tag{1}$$

where $\alpha_t \in (0, 1)$.

They showed that the F iteration (1) is stable and has a better rate of convergence when compared with the other iterations in the setting of generalized contractions.

Definition 1. Let $\mathscr{G} : \mathscr{P} \to \mathscr{P}$. Then $\mathscr{G}$ is said to be

(i) nonexpansive provided that $\|\mathcal{G}p' - \mathcal{G}p''\| \le \|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$

(ii) endowed with condition (C) provided that $1/2\|p' - \mathcal{G}p'\| \le \|p' - p''\|$ implies $\|\mathcal{G}p' - \mathcal{G}p''\| \le \|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$

(iii) generalized α-nonexpansive provided that $1/2\|p' - \mathcal{G}p'\| \le \|p' - p''\|$ implies $\|\mathcal{G}p' - \mathcal{G}p''\| \le \alpha\|p' - \mathcal{G}p''\| + \alpha\|p'' - \mathcal{G}p'\| + (1 - 2\alpha)\|p' - p''\|$, for every two $p', p'' \in \mathcal{P}$ and $\alpha \in [0, 1)$

(iv) endowed with condition I [16] if one has a nondecreasing function f such that $f(0) = 0$ and $f(a) > 0$ at $a > 0$ and $\|p' - \mathcal{G}p'\| \ge f(d(p', F_{\mathcal{G}})$ for all $p' \in \mathcal{P}$

In 1965, Browder [17] and Gohde [18] are in a uniformly convex Banach space (UCBS), while Kirk [19] in a reflexive Banach space (RBS) established an existence of fixed point for nonexpansive maps. In 2008, Suzuki [20] showed that the class of maps endowed with condition (C) is weaker than the notion of nonexpansive maps and proved some related fixed point theorems in Banach spaces. In 2017, Pant and Shukla [21] proved that the notion of generalized α-nonexpansive maps is weaker than the notion of maps endowed with condition (C). They proved some convergence theorems using Agarwal iteration [5] for these maps. Very recently, Ullah et al. [22] used M iteration for finding fixed points of generalized α-nonexpansive maps in Banach spaces. In this paper, we show under some conditions that F iteration converges better to a fixed point of generalized α -nonexpansive map as compared to the leading M iteration and hence many other iterative schemes.

Definition 2. Select a Banach space $\mathcal{J}$ such that $\mathcal{P} \subseteq \mathcal{J}$ is nonempty and $\{w_t\} \subseteq \mathcal{J}$ is bounded. We set for fix $j \in \mathcal{J}$ the following.

(a_1) asymptotic radius of the bounded sequence $\{w_t\}$ at the point j by $r(j, \{w_t\}) \coloneqq \limsup_{t\to\infty}\|j - w_t\|$;

(a_2) asymptotic radius of the bounded sequence $\{w_t\}$ with the connection of $\mathcal{P}$ by $r(\mathcal{P}, \{w_t\}) = \inf\{r(j, \{w_t\}) : j \in \mathcal{P}\}$;

(a_3) asymptotic center of the bounded sequence $\{w_t\}$ with the connection of $\mathcal{P}$ by $A(\mathcal{P}, \{w_t\}) = \{j \in \mathcal{P} : r(j, \{w_t\}) = r(\mathcal{P}, \{w_t\})\}$.

It is worth mentioning that $A(\mathcal{P}, \{w_t\})$ has a cardinality equal to one in the case of UCBS and nonempty convex in the case of weak compactness and convexity of $\mathcal{P}$ (see [23, 24]).

Definition 3 (see [25]). A Banach space $\mathcal{J}$ is called with Opial's condition in the case when every sequence $\{w_t\} \subseteq \mathcal{J}$ which is weakly convergent to $j \in \mathcal{J}$, then one has the following

$$\limsup_{t\to\infty}\|w_t - j\| < \limsup_{t\to\infty}\|w_t - j'\| \text{for each} j' \in \mathcal{P} - \{j\}. \quad (2)$$

Pant and Shukla [21] observed the following facts about generalized α-nonexpansive operators.

Proposition 4. *If $\mathcal{J}$ is a Banach space such that $\mathcal{P} \subseteq \mathcal{J}$ is closed and nonempty, then for $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ and $\alpha \in [0, 1)$, the following hold*

(i) If $\mathcal{G}$ is endowed with condition (C), then $\mathcal{G}$ is generalized α-nonexpansive

(ii) If $\mathcal{G}$ is generalized α-nonexpansive endowed with a nonempty fixed point, then $\|\mathcal{G}p' - p^*\| \le \|p' - p^*\|$ for $p' \in \mathcal{P}$ and p^* is a fixed point of $\mathcal{G}$

(iii) If $\mathcal{G}$ is generalized α-nonexpansive, then $F_{\mathcal{G}}$ is closed. Furthermore, when the underlying space $\mathcal{J}$ is strictly convex and the set $\mathcal{P}$ is convex, then the set $F_{\mathcal{G}}$ is also convex

(iv) If $\mathcal{G}$ is generalized α-nonexpansive, then for every choice of $p', p'' \in \mathcal{P}$

$$\left\|p' - \mathcal{G}p''\right\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\left\|p' - \mathcal{G}p'\right\| + \left\|p' - p''\right\|. \quad (3)$$

(v) If the underlying space $\mathcal{J}$ is with Opial condition, the operator $\mathcal{G}$ is generalized α-nonexpansive, $\{w_t\}$ is weakly convergent to l and $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$, then $l \in F_{\mathcal{G}}$

We now state an interesting property of a UCBS from [26].

Lemma 5. *Suppose $\mathcal{J}$ is any UCBS. Choose $0 < r \le \alpha_t \le s < 1$ and $\{w_t\}, \{x_t\} \subseteq \mathcal{J}$ such that $\limsup_{t\to\infty}\|w_t\| \le q$, $\limsup_{t\to\infty}\|x_t\| \le q$, and $\lim_{t\to\infty}\|\alpha_t w_t + (1 - \alpha_t)x_t\| = q$ for some $q \ge 0$. Then, consequently, $\lim_{t\to\infty}\|w_t - x_t\| = 0$.*

2. Main Results

We first provide a very basic lemma.

Lemma 6. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator satisfying with $F_{\mathcal{G}} \ne \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1), then, consequently, one has $\lim_{t\to\infty}\|w_t - p^*\|$ always exists for every taken $p^* \in F_{\mathcal{G}}$.*

Proof. We may take any $p^* \in F_{\mathcal{G}}$. Using Proposition 4(ii), we see that

$$\begin{aligned}\|u_t - p^*\| &= \|\mathcal{G}((1-\alpha_t)w_t + \alpha_t\mathcal{G}w_t) - p^*\| \le \|(1-\alpha_t)w_t \\ &\quad + \alpha_t\mathcal{G}w_t - p^*\| \le (1-\alpha_t)\|w_t - p^*\| + \alpha_t\|\mathcal{G}w_t \\ &\quad - p^*\| \le (1-\alpha_t)\|w_t - p^*\| + \alpha_t\|w_t - p^*\| \le \|w_t - p^*\|.\end{aligned} \quad (4)$$

This implies that

$$\|w_{t+1}-p^*\| = \|\mathcal{G}v_t - p^*\| \le \|v_t - p^*\| = \|\mathcal{G}u_t - p^*\| \le \|u_t - p^*\| \le \|w_t - p^*\|. \tag{5}$$

Consequently, $\|w_{t+1}-p^*\| \le \|w_t - p^*\|$, that is, $\{\|w_t - p^*\|\}$ is bounded as well as nonincreasing. This follows that $\lim_{t\to\infty}\|w_t - p^*\|$ exists for each $p^* \in F_{\mathcal{G}}$.

We now provide the necessary and sufficient requirements for the existence of fixed points for any given generalized nonexpansive mappings in a Banach space.

Theorem 7. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G}: \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator and $\{w_t\}$ is a sequence of F iterates (1). Then, $F_{\mathcal{G}} \neq \emptyset$ if and only if $\{w_t\}$ is bounded and $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$.*

Proof. Suppose that $F_{\mathcal{G}} \neq \emptyset$ and $p^* \in F_{\mathcal{G}}$. Take any $p^* \in F_{\mathcal{G}}$, and so applying Lemma 6, we have $\lim_{t\to\infty}\|w_t - p^*\|$ exists and $\{w_t\}$ is bounded. Suppose that this limit is equal to some ε, that is,

$$\lim_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{6}$$

As we have established in the proof of Lemma 6 that

$$\|u_t - p^*\| \le \|w_t - p^*\|. \tag{7}$$

This together with (6) gives that

$$\limsup_{t\to\infty}\|u_t - p^*\| \le \limsup_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{8}$$

Since p^* is in the set $F_{\mathcal{G}}$, so we may apply Proposition 4(ii) to obtain the following

$$\|\mathcal{G}w_t - p^*\| \le \|w_t - p^*\|, \Rightarrow \limsup_{t\to\infty}\|\mathcal{G}w_t - p^*\| \le \limsup_{t\to\infty}\|w_t - p^*\| = \varepsilon. \tag{9}$$

Now, if we look in the proof of Lemma 6, we can see the following

$$\|w_{t+1} - p^*\| \le \|u_t - p^*\| \Rightarrow \varepsilon = \liminf_{t\to\infty}\|w_{t+1} - p^*\| \le \liminf_{t\to\infty}\|u_t - p^*\|. \tag{10}$$

From (8) and (10), we have

$$\varepsilon = \lim_{t\to\infty}\|u_t - p^*\|. \tag{11}$$

By (11) and (1), one has

$$\begin{aligned}\varepsilon &= \lim_{t\to\infty}\|u_t - p^*\| = \lim_{t\to\infty}\|\mathcal{G}((1-\alpha_t)w_t + \alpha_t\mathcal{G}w_t) - p^*\| \le \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*) + \alpha_t(\mathcal{G}w_t - p^*)\| \\ &\le \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*)\| + \lim_{t\to\infty}\|\alpha_t(\mathcal{G}w_t - p^*)\| \\ &\le \lim_{t\to\infty}(1-\alpha_t)\|w_t - p^*\| + \lim_{t\to\infty}\alpha_t\|w_t - p^*\| = \lim_{t\to\infty}\|w_t - p^*\| \le \varepsilon.\end{aligned} \tag{12}$$

If and only if

$$\varepsilon = \lim_{t\to\infty}\|(1-\alpha_t)(w_t - p^*) + \alpha_t(\mathcal{G}w_t - p^*)\|. \tag{13}$$

One can now apply the Lemma 5, to obtain

$$\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0. \tag{14}$$

Conversely, we want to show that the set $F_{\mathcal{G}}$ is nonempty under the assumptions that $\{w_t\}$ is bounded such that $\lim_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0$. We may choose a point $p^* \in A(\mathcal{P}, \{w_t\})$. If we apply Proposition 4(iv), then one can observe the following

$$\begin{aligned} r(\mathcal{G}p^*, \{w_t\}) &= \limsup_{t\to\infty}\|w_t - \mathcal{G}p^*\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{t\to\infty}\|\mathcal{G}w_t - w_t\| \\ &\quad + \limsup_{t\to\infty}\|w_t - p^*\| = \limsup_{t\to\infty}\|w_t - p^*\| = r(p^*, \{w_t\}).\end{aligned} \tag{15}$$

We observed that $\mathcal{G}p^* \in A(\mathcal{P}, \{w_t\}$. By using the facts that this set has only element in the case of UCBS $\mathcal{J}$, one concludes $\mathcal{G}p^* = p^*$, accordingly the set $F_{\mathcal{G}}$ is nonempty.

The weak convergence of F iteration is established as follows.

Theorem 8. *Suppose $\mathcal{J}$ is any UCBS with Opial condition and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G}: \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \emptyset$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges weakly to a fixed point of $\mathcal{G}$.*

Proof. By Theorem 7, the given sequence $\{w_t\}$ is bounded. Since $\mathcal{J}$ is UCBS, $\mathcal{J}$ is RBS. Therefore, some one construct a weakly convergent sequence of $\{w_t\}$. We may assume that $\{w_{t_r}\}$ be this subsequence having weak limit $x_1 \in \mathcal{P}$. If we apply Theorem 7 on this subsequence, we obtain $\lim_{r\to\infty}\|w_{t_r} - \mathcal{G}w_{t_r}\| = 0$. Thus, by Proposition 4(v), one has $x_1 \in F_{\mathcal{G}}$. It is sufficient to show that $\{w_t\}$ converges weakly to x_1. In fact, if $\{w_t\}$ does not converge weakly to x_1. Then, there exists a subsequence $\{w_{t_s}\}$ of $\{w_t\}$ and $x_2 \in \mathcal{P}$ such that $\{w_{t_s}\}$ converges weakly to x_2 and $x_2 \neq x_1$. Again by Proposition 4(v), $x_2 \in F_{\mathcal{G}}$. By Lemma 6 together with Opial property, we have

$$\lim_{t\to\infty}\|x_n - l_1\| = \lim_{r\to\infty}\|w_{t_r} - x_1\| < \lim_{r\to\infty}\|w_{t_r} - x_2\| = \lim_{t\to\infty}\|w_t - x_2\| = \lim_{s\to\infty}\|w_{t_s} - x_2\| < \lim_{s\to\infty}\|w_{t_s} - x_1 \mid\| = \lim_{t\to\infty}\|w_t - x_1\|. \quad (16)$$

This is a contradiction. So, we have $x_1 = x_2$. Thus, $\{w_t\}$ converges weakly to $x_1 \in F_{\mathcal{G}}$.

Now we provide some strong convergence results.

Theorem 9. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and compact. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \varnothing$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. Since the domain $\mathcal{P}$ is a compact subset of $\mathcal{J}$ and $\{w_t\} \subseteq \mathcal{P}$. It follows that a subsequence $\{w_{t_r}\}$ of $\{w_t\}$ exists such that $\lim_{r\to\infty}\|w_t - p^{**}\| = 0$ for some $p^{**} \in \mathcal{P}$. In the view of Theorem 7, $\lim_{r\to\infty}\|\mathcal{P}w_{t_r} - w_{t_r}\| = 0$. Applying Proposition 4(iv), one has

$$\|w_{t_r} - \mathcal{G}p^{**}\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|w_{t_r} - \mathcal{G}w_{t_r}\| + \|w_{t_r} - p^{**}\|. \quad (17)$$

Hence, if we let $r \to \infty$, then $\mathcal{G}p^{**} = p^{**}$. The fact that p^{**} is the strong limit of $\{w_t\}$ now follows from the existence of $\lim_{t\to\infty}\|w_t - p^{**}\|$.

Theorem 10. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator with $F_{\mathcal{G}} \neq \varnothing$ and $\{w_t\}$ is a sequence of F iterates (1) and* $\liminf_{t\to\infty} d(w_t, F_{\mathcal{G}}) = 0$. *Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. By using Lemma 6, one has $\lim_{t\to\infty}\|w_t - p^*\|$ exists, for every fixed point of $\mathcal{G}$. It follows that $\lim_{t\to\infty} d(w_t, F_{\mathcal{G}})$ exists. Accordingly

$$\lim_{t\to\infty} d(w_n, F_{\mathcal{G}}) = 0. \quad (18)$$

The above limit provides us two subsequence $\{w_{t_r}\}$ and $\{p_r\}$ of $\{w_t\}$ and $F_{\mathcal{G}}$, respectively, in the following way

$$\|w_{t_r} - p_r\| \leq \frac{1}{2^r} \quad \text{for each} \quad r \geq 1. \quad (19)$$

By looking into the proof of Lemma 6, we see that $\{w_t\}$ is nonicreasing, therefore

$$\|w_{t_{r+1}} - p_r\| \leq \|w_{t_r} - p_r\| \leq \frac{1}{2^r}. \quad (20)$$

It follows that

$$\|p_{r+1} - p_r\| \leq \|p_{r+1} - w_{t_{r+1}}\| + \|w_{t_{r+1}} - p_r\| \leq \frac{1}{2^{r+1}} + \frac{1}{2^r} \leq \frac{1}{2^{r-1}} \to 0, asr \to \infty. \quad (21)$$

Consequently, we obtained that $\lim_{r\to\infty}\|p_{r+1} - p_r\| = 0$ which show that $\{p_r\}$ is Cauchy sequence in $F_{\mathcal{G}}$ and so it converges to an element p^{**}. Applying Proposition 4(iii), $F_{\mathcal{G}}$ is closed and so $p^{**} \in F_{\mathcal{G}}$. By Lemma 6, $\lim_{t\to\infty}\|w_t - p^{**}\|$ exists and hence p^{**} is the strong limit of $\{w_t\}$.

Theorem 11. *Suppose $\mathcal{J}$ is any UCBS and $\mathcal{P} \subseteq \mathcal{J}$ is convex nonempty and closed. If $\mathcal{G} : \mathcal{P} \to \mathcal{P}$ is generalized α-nonexpansive operator satisfying condition I with $F_{\mathcal{G}} \neq \varnothing$ and $\{w_t\}$ is a sequence of F iterates (1). Then, consequently, $\{w_t\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. Keeping Theorem 7 in mind, one can write

$$\liminf_{t\to\infty}\|\mathcal{G}w_t - w_t\| = 0. \quad (22)$$

From the definition of condition (I), we see that

$$\|w_t - \mathcal{G}w_t\| \geq f(d(w_t, F_{\mathcal{G}})). \quad (23)$$

Applying (22) on (23), we have

$$\liminf_{t\to\infty} f(d(w_t, F_{\mathcal{G}})) = 0. \quad (24)$$

It follows that

$$\liminf_{t\to\infty} d(w_t, F_{\mathcal{G}}) = 0. \quad (25)$$

Now applying Theorem 10, $\{w_t\}$ is strongly convergent to a fixed point of $\mathcal{G}$.

3. Example

To support the main results, we provide an example of generalized α-nonexpansive mappings, which is not endowed with condition (C). Using this example, we compare F with other iterations in the setting of generalized α-nonexpansive mappings.

Example 12. We take a set $\mathcal{P} = [7, 13]$ and set a self map on $\mathcal{G}$ by the following rule:

$$\mathcal{G}p' = \begin{cases} \dfrac{p'+7}{2} & \text{if} \quad p' < 13, \\ 7 \quad \text{if} \quad z = 13. \end{cases} \quad (26)$$

We show that $\mathcal{G}$ is generalized α-nonexpansive having $\alpha = 1/2$, but not Suzuki mapping. This example thus exceeds the class of Suzuki mappings.

Case I. When $p' = 13 = p''$, we have

$$\frac{1}{2}|p' - \mathcal{G}p''| + \frac{1}{2}|p'' - \mathcal{G}p'| + \left(1 - 2\left(\frac{1}{2}\right)\right)|p' - p''| \geq 0 = |\mathcal{G}p' - \mathcal{G}p''|. \quad (27)$$

Table 1: Numerical data generated by F, M, Picard-S, S, Ishikawa, and Mann iterative approximation schemes for the self map given in Example 12.

	F	M	Picard-S	S	Ishikawa	Mann
1	7.9	7.9	7.9	7.9	7.9	7.9
2	7.06468750	7.12937500	7.16284375	7.3256875	7.3931875	7.51750000
3	7.00464941	7.01859766	7.02946454	7.11785816	7.17177379	7.29756250
4	7.00033418	7.00267341	7.00533124	7.04264992	7.07504367	7.17109844
5	7.00002402	7.00038430	7.00096462	7.01543394	7.03278471	7.09838160
6	7.00000173	7.00005524	7.00017454	7.00558516	7.01432282	7.05656942
7	7.00000012	7.00000794	7.00003158	7.00202113	7.00625728	7.03252742
8	7.00000001	7.00000114	7.00000571	7.0007314	7.00273365	7.01870326
9	7	7.00000016	7.00000103	7.00026467	7.00119426	7.01075438
10	7	7.00000002	7.00000019	7.00009578	7.00052174	7.00618377
11	7	7	7.00000003	7.00003466	7.00022794	7.00355567
12	7	7	7.00000001	7.00001254	7.00009959	7.00204451
13	7	7	7	7.00000454	7.00004350	7.00117559
14	7	7	7	7.00000164	7.00001901	7.00067597
15	7	7	7	7.00000059	7.00000830	7.00038868
16	7	7	7	7.00000022	7.00000363	7.00022349
17	7	7	7	7.00000008	7.00000158	7.00012851
18	7	7	7	7.00000003	7.00000069	7.00007389
19	7	7	7	7.00000001	7.00000030	7.00004249
20	7	7	7	7	7.00000013	7.00002443
21	7	7	7	7	7.00000006	7.00001405
22	7	7	7	7	7.00000003	7.00000808
23	7	7	7	7	7.00000001	7.00000464
24	7	7	7	7	7	7.00000260

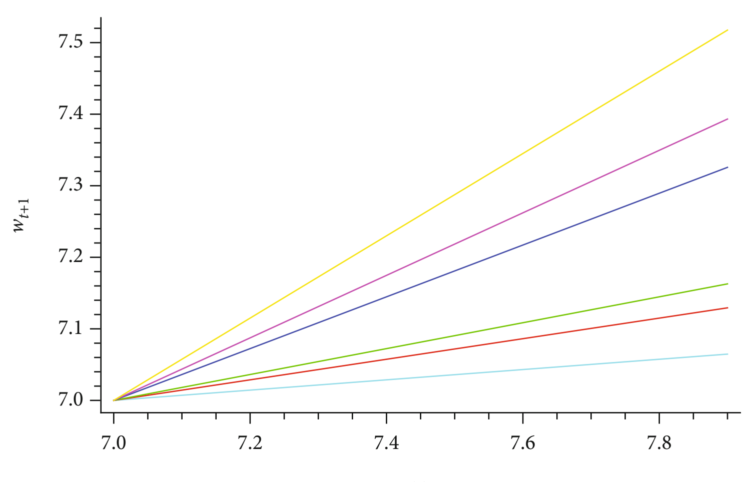

Figure 1: Convergence analysis view of F (cyan), M (red), Picard-S (green), S (blue), Ishikawa (magenta), and Mann (yellow) iteration process for the mapping given in Example 12.

Case II. Choose $p', p'' < 13$, we have

$$\begin{aligned}
&\frac{1}{2}\|p' - \mathcal{G}p''\| + \frac{1}{2}\|p'' - \mathcal{G}p'\| \\
&\quad + \left(1 - 2\left(\frac{1}{2}\right)\right)\|p' - p''\| \\
&= \frac{1}{2}\left\|p'' - \left(\frac{p'+7}{2}\right)\right\| + \frac{1}{2}\left\|p' - \left(\frac{p''+7}{2}\right)\right\| \\
&\geq \frac{1}{2}\left\|\left(p'' - \left(\frac{p'+7}{2}\right)\right) - \left(p' - \left(\frac{p''+7}{2}\right)\right)\right\| \\
&= \frac{1}{2}\left\|\frac{2p'' - p' - 7 - 2p' + p'' + 7}{2}\right\| \\
&= \frac{1}{2}\left\|\frac{3p'' - 3p'}{2}\right\| = \frac{3}{4}\|p' - p''\| \geq \frac{1}{2}\|p' - p''\| \\
&= \|\mathcal{G}p' - \mathcal{G}p''\|.
\end{aligned} \tag{28}$$

Case III. When $p' = 13$ and $p'' < 13$, we have

$$\begin{aligned}
&\frac{1}{2}\|p' - \mathcal{G}p''\| + \frac{1}{2}\|p'' - \mathcal{G}p'\| \\
&\quad + \left(1 - 2\left(\frac{1}{2}\right)\right)\|p' - p''\| \\
&= \frac{1}{2}\|p' - 7\| + \frac{1}{2}\left\|p'' - \left(\frac{p'+7}{2}\right)\right\| \geq \frac{1}{2}\|p' - 7\| \\
&= \left\|\frac{p'-7}{2}\right\| = \|\mathcal{G}p' - \mathcal{G}p''\|.
\end{aligned} \tag{29}$$

Consequently, $\|\mathcal{G}p' - \mathcal{G}p''\| \leq 1/2 \mid p' - \mathcal{G}p'' \mid + 1/2 \mid p'' - \mathcal{G}p' \mid + (1 - 2(1/2))|p' - p''|$ for every two points $p', p'' \in \mathcal{G}$. Now if one chooses $p' = 11.8$ and $p'' = 13$, we must have $|p' - p'' \mid = 1.2, |\mathcal{G}p' - \mathcal{G}p'' \mid = 2.4$ and $1/2 \mid p' - \mathcal{G}p' \mid = 1.2$. It has been observed, $1/2 \mid p' - \mathcal{G}p' \mid \leq \mid p' - p'' \mid$ and $|\mathcal{G}p' - \mathcal{G}p'' \mid > \mid p' - p'' \mid$. Thus, $\mathcal{G}$ exceeded the class of Suzuki mappings.

We now compare the effectiveness of the iterative scheme F [15] with the leading M [14] and Picard [13] and the elementry S [5], Ishikawa [3] and Mann [2] approximation scheme. We may take $\alpha_t = 0.85$ and $\beta_t = 0.65$. For the strating $w_1 = 7.9$, we can see some values in Table 1. Furthermore, Figure 1 provides information about the behavior of the leading schemes. Clearly, F iterative scheme is more effective than the other schemes in the general context of generalized α-nonexpansive maps.

Remark 13. In the view of the above discussion, we noted that the main theorems and outcome of this paper improved and extended the main results of Ullah and Arshad [14] from Suzuki mappings to generalized α-nonexpansive mappings and from the setting of M iteration to the more general setting of F iteration process. Moreover, the main results of this paper improved the results of Ali and Ali [15] from the setting of contractions to the general context of generalized α-nonexpansive mappings. We have also improved the results of Ullah et al. [22] in the sense of better rate of convergence.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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