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Research Article

Applications of Magnetohydrodynamic Couple Stress Fluid Flow between Two Parallel Plates with Three Different Kernels

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In this paper, we investigate the implementations of newly introduced nonlocal differential operators as convolution of power law, exponential decay law, and the generalized Mittag-Leffler law with fractal derivative in fluid dynamics. The new operators are referred as fractal-fractional differential operators. The governing equations for the problem are constructed with the fractal-fractional differential operators. We present the stability analysis and the error analysis.

1. Introduction

Magnetohydrodynamics (MHD) deals with the study of the motion of electrically conducting fluids in the presence of the magnetic field. MHD flow has significant importance applications between infinite parallel plates in various areas such as geophysical, astrophysical, and metallurgical processing, MHD generators, pumps, geothermal reservoirs, polymer technology, and mineral industries [1–6]. In last few decades, fractional calculus has taken much interest in many fields [7, 8]. There are many definitions for the fractional derivative operators, and among them are Caputo-Fabrizio (CF) [9] and Atangana and Baleanu (AB) [10] definitions of fractional derivatives with a nonlocal and nonsingular kernels having all the characteristics of the old definitions [7, 11–23]. Farman et al. [24] have analyzed the numerical solution of SEIR Epidemic model of measles with noninteger time fractional derivatives by using the Laplace Adomian decomposition method. Ghanbari and Djilali [25] have taken mathematical analysis of a fractional-order predator-prey model with prey social behavior and infection developed in predator population. Ghanbari and Atangana [26, 27] have given the new edge detecting techniques based on fractional derivatives with nonlocal and nonsingular kernels. Recently, another idea of differentiation has been proposed by Atanagna [28].

We organize our manuscript as follows. We present the main definitions in Section 2. We construct the problem formulation in Section 3. We present the analysis of the model with the power law kernel in Section 4. We give the analysis of the model with the exponential decay kernel in Section 5. We discuss the analysis of the model with the Mittag-Leffler kernel in Section 6. We present the error analysis in Section 7. We give the conclusion in the last section.

2. Preminaries

Definition 1. Assume that $g(\varsigma)$ is a continuous function in the (c_{11}, d_{11}) and fractal differentiable on (c_{11}, d_{11}) with order η, then the fractal-fractional derivative of g of order λ in Riemann-Liouville sense with power law kernel is introduced as [29]

$$ {}_{b}^{FFP}D_{t}^{\eta,\lambda}g(\varsigma) = \frac{1}{\Gamma(1-\eta)}\frac{d}{d\varsigma^{\lambda}}\int_{b}^{\varsigma}g(x)(\varsigma-x)^{-\eta}dx, \quad 0<\eta,\ \lambda\le 1, \tag{1} $$

where

$$ \frac{dg(x)}{dx^{\lambda}} = \lim_{\varsigma\longrightarrow x}\frac{g(\varsigma)-g(x)}{\varsigma^{\lambda}-x^{\lambda}}. \tag{2} $$

Definition 2. Assume that $g(\varsigma)$ is a continuous function in the (c_{11}, d_{11}) and fractal differentiable on (c_{11}, d_{11}) with order η, then the fractal-fractional derivative of g of order λ in Riemann-Liouville sense with the exponential decay kernel is introduced as [29]

$$ {}_{b}^{FFE}D_{t}^{\eta,\lambda}g(\varsigma) = \frac{M(\eta)}{(1-\eta)}\frac{d}{d\varsigma^{\lambda}}\int_{b}^{\varsigma}g(x)\exp\left(-\frac{\eta}{1-\eta}(\varsigma-x)\right)dx, \quad 0<\eta,\ \lambda\le 1. \tag{3} $$

Definition 3. Assume that $g(\varsigma)$ is a continuous function in the (c_{11}, d_{11}) and fractal differentiable on (c_{11}, d_{11}) with order η, then the fractal-fractional derivative of g of order λ in Riemann-Liouville sense with the generalized Mittag-Leffler kernel is introduced as [29]

$$ {}_{b}^{FFE}D_{t}^{\eta,\lambda}g(\varsigma) = \frac{AB(\eta)}{(1-\eta)}\frac{d}{d\varsigma^{\lambda}}\int_{b}^{\varsigma}g(x)E_{\eta}\left(-\frac{\eta}{1-\eta}(\varsigma-x)^{\eta}\right)dx, \quad 0<\eta,\ \lambda\le 1. \tag{4} $$

3. Problem Formulation

We consider

$$ \rho\frac{\partial u'_{11}\left(\xi'_{11},\tau'_{11}\right)}{\partial\tau'_{11}} = \mu\frac{\partial^{2}u'_{11}\left(\xi'_{11},\tau'_{11}\right)}{\partial{\xi'_{11}}^{2}} - \eta\frac{\partial^{4}u'_{11}\left(\xi'_{11},\tau'_{11}\right)}{\partial{\xi'_{11}}^{4}} - \sigma{B_0}^{2}u'_{11}\left(\xi'_{11},\tau'_{11}\right), \tag{5} $$

$$ u'_{11}\left(\xi'_{11},0\right) = 0, \text{for all } \xi'_{11}, \quad \text{(initial condition)} \tag{6} $$

$$ u'_{11}\left(0,\tau'_{11}\right) = 0, \quad u'_{11}\left(d,\tau'_{11}\right) = 0, \quad 0\le\xi'_{11}\le h, \tag{7} $$

$$ \frac{\partial^{2}u'_{11}\left(\xi'_{11},\tau'_{11}\right)}{\partial{\xi'_{11}}^{2}} = 0, \text{at } \xi'_{11}=0 \text{ and } \xi'_{11}=h \text{ for any } \tau'_{11}>0, \tag{8} $$

$$ v = \frac{u'_{11}}{U_0}, \quad t = \frac{\tau'_{11}U_0}{h}, \quad y = \frac{\xi'_{11}}{h}, \tag{9} $$

into Eqs. (5)-(8), and we obtain

$$ \frac{\partial v(y,t)}{\partial t} = \frac{1}{\text{Re}}\left(\frac{\partial^{2}v(y,t)}{\partial y^{2}} - \frac{\partial^{4}v(y,t)}{\partial y^{4}} - M_{11}v(y,t)\right), \tag{10} $$

$$ v(y,0)=0, \quad v(0,t)=0, \quad v(1,t)=1, \tag{11} $$

$$ \frac{\partial^{2}v(y,t)}{\partial y^{2}} = 0, \text{at } y=0 \text{ and } y=1 \text{ for any } t>0, \tag{12} $$

where $M_{11}=\sigma{B_0}^{2}d^{2}/\mu$ is the magnetic field parameter, and $\text{Re}=\rho U_0 d/\mu$ is the Reynold number and $h^2=\eta/\mu$. We demonstrate the geometry of the physical model in Figure 1.

4. Solution of the Problem with the Power Law Kernel

We take into consideration the Eq. (10) with fractal-fractional differential operator using Definition 1 of power law kernel as

$$ {}_{0}^{FFP}D_{t}^{\alpha_1,\beta_1}v(y,t) = \frac{1}{\text{Re}}\left(\frac{\partial^{2}v(y,t)}{\partial y^{2}} - \frac{\partial^{4}v(y,t)}{\partial y^{4}} - M_{11}v(y,t)\right), $$

$$ \begin{aligned} &\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{dt}\int_0^t v(y,\lambda)(t-\lambda)^{-\alpha_1}d\lambda \\ &\quad = \frac{\beta_1}{\text{Re}}t^{\beta_1-1}\left(\frac{\partial^{2}v(y,t)}{\partial y^{2}} - \frac{\partial^{4}v(y,t)}{\partial y^{4}} - M_{11}v(y,t)\right). \end{aligned} \tag{13} $$

The, we get

$$ v(y,\lambda) = \frac{\beta_1}{\text{Re}\,\Gamma(\alpha_1)}\int_0^t \lambda^{\beta_1-1}\left(\frac{\partial^{2}v(y,\lambda)}{\partial y^{2}} - \frac{\partial^{4}v(y,\lambda)}{\partial y^{4}} - M_{11}v(y,\lambda)\right)(t-\lambda)^{\alpha_1-1}d\lambda. \tag{14} $$

For simplicity, we take

$$ \begin{aligned} F(y,\lambda) &= \beta_1\lambda^{\beta_1-1}\left(\frac{\partial^{2}v(y,\lambda)}{\partial y^{2}} - \frac{\partial^{4}v(y,\lambda)}{\partial y^{4}} - M_{11}v(y,\lambda)\right), \\ v(y,\lambda) &= \frac{1}{\text{Re}\,\Gamma(\alpha_1)}\int_0^t F(y,\lambda)(t-\lambda)^{\alpha_1-1}d\lambda. \end{aligned} \tag{15} $$

We discretize this equation at $(y_i, t=t_{n+1})$ and get

$$ \begin{aligned} v(y_i,t_{n+1}) &= \frac{1}{\text{Re}\,\Gamma(\alpha_1)}\int_0^{t_{n+1}} F(y_i,\lambda)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda, \\ v(y_i,t_{n+1}) &= \frac{1}{\text{Re}\,\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} F(y_i,\lambda)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda. \end{aligned} \tag{16} $$

We apply the two-step Lagrange polynomial as

$$ p_j(\lambda) = \frac{\lambda-t_{j-1}}{t_j-t_{j-1}}F\left(y_i,t_j\right) - \frac{\lambda-t_j}{t_j-t_{j-1}}F\left(y_i,t_{j-1}\right). \tag{17} $$

Thus, we will get

$$\begin{aligned} v(y_i, t_{n+1}) &= \frac{1}{\operatorname{Re}\Gamma(\alpha_1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} p_j(\lambda)(t_{n+1}-\lambda)^{\alpha_1-1} d\lambda \\ &= \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} F(y_i, t_j)}{\operatorname{Re}\Gamma(\alpha_1+2)} ((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) \right. \\ &\quad \left. - (n-j)^{\alpha_1}(n-j+2+2\alpha_1)) \right] \\ &\quad - \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} F(y_i, t_{j-1})}{\operatorname{Re}\Gamma(\alpha_1+2)} \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1) \right) \right]. \end{aligned} \tag{18}$$

We have

$$F(y_i, t_j) = \beta_1 t_j^{\beta_1-1} \left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta y)^2} - \frac{v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j}{(\Delta y)^4} - M_{11} v_i^j \right). \tag{19}$$

Then, we will obtain

$$\begin{aligned} v(y_i, t_{n+1}) &= \frac{1}{\operatorname{Re}\Gamma(\alpha_1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} p_j(\lambda)(t_{n+1}-\lambda)^{\alpha_1-1} d\lambda \\ &= \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1}}{\operatorname{Re}\Gamma(\alpha_1+2)} \right. \\ &\quad \cdot \left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta y)^2} - \frac{v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j}{(\Delta y)^4} - M_{11} v_i^j \right) \\ &= \times \left. ((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) - (n-j)^{\alpha_1}(n-j+2+2\alpha_1)) \right] \\ &\quad - \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} \beta_1 t_{j-1}^{\beta_1-1}}{\operatorname{Re}\Gamma(\alpha_1+2)} \right. \\ &\quad \cdot \left(\frac{v_{i+1}^{j-1} - 2v_i^{j-1} + v_{i-1}^{j-1}}{(\Delta y)^2} - \frac{v_{i+2}^{j-1} - 4v_{i+1}^{j-1} + 6v_i^{j-1} - 4v_{i-1}^{j-1} + v_{i-2}^{j-1}}{(\Delta y)^4} - M_{11} v_i^{j-1} \right) \\ &\quad \times \left. \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1) \right) \right]. \end{aligned} \tag{20}$$

We define

$$\begin{aligned} A_{n,j}^{\alpha_1,\beta_1} &= \frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1} K_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)(\Delta y)^2}, \quad B_{n,j}^{\alpha_1,\beta_1} = \frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1} K_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)(\Delta y)^4}, \\ C_{n,j}^{\alpha_1,\beta_1} &= \frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1} M_{11} K_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)}, \\ D_{n,j-1}^{\alpha_1,\beta_1} &= \frac{h^{\alpha_1} \beta_1 t_{j-1}^{\beta_1-1} \gamma_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)(\Delta y)^2}, \quad B_{n,j-1}^{\alpha_1,\beta_1} = \frac{h^{\alpha_1} \beta_1 t_{j-1}^{\beta_1-1} \gamma_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)(\Delta y)^4}, \\ C_{n,j-1}^{\alpha_1,\beta_1} &= \frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1} M_{11} \gamma_{n,j}^{\alpha_1}}{\operatorname{Re}\Gamma(\alpha_1+2)}, \\ K_{n,j}^{\alpha_1} &= ((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) - (n-j)^{\alpha_1}(n-j+2+2\alpha_1)), \\ \gamma_{n,j}^{\alpha_1} &= \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1) \right). \end{aligned} \tag{21}$$

Then, we get

$$\begin{aligned} v_i^{n+1} &= \sum_{j=0}^{n} \left[A_{n,j}^{\alpha_1,\beta_1} \left(v_{i+1}^j - 2v_i^j + v_{i-1}^j \right) - B_{n,j}^{\alpha_1,\beta_1} \right. \\ &\quad \left. \cdot \left(v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j \right) - C_{n,j}^{\alpha_1,\beta_1} v_i^j \right] \\ &\quad - \sum_{j=0}^{n} \left[D_{n,j-1}^{\alpha_1,\beta_1} \left(v_{i+1}^{j-1} - 2v_i^{j-1} + v_{i-1}^{j-1} \right) \right. \\ &\quad \left. - E_{n,j-1}^{\alpha_1,\beta_1} \left(v_{i+2}^{j-1} - 4v_{i+1}^{j-1} + 6v_i^{j-1} - 4v_{i-1}^{j-1} + v_{i-2}^{j-1} \right) - F_{n,j-1}^{\alpha_1,\beta_1} \left(v_i^{j-1} \right) \right]. \end{aligned} \tag{22}$$

We choose $\varepsilon_i^n = \delta_n \exp(ik_m y)$. Then, we have

$$\begin{aligned} \delta_{n+1} \exp(ik_m y) &= A_{n,n}^{\alpha_1,\beta_1} [\delta_n \exp(ik_m(y+\Delta y)) - 2\delta_n \exp(ik_m y) + \delta_n \exp(ik_m(y-\Delta y))] \\ &\quad - B_{n,n}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_n \exp(ik_m(y+2\Delta y)) - 4\delta_n \exp(ik_m(y+\Delta y)) + 6\delta_n \exp(ik_m y) \\ -4\delta_n \exp(ik_m(y-\Delta y)) + \delta_n \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ &\quad - C_{n,n}^{\alpha_1,\beta_1} \delta_n \exp(ik_m y) - D_{n,n-1}^{\alpha_1,\beta_1} [\delta_{n-1} \exp(ik_m(y+\Delta y)) - 2\delta_{n-1} \exp(ik_m y) \\ &\quad + \delta_{n-1} \exp(ik_m(y-\Delta y))] \\ &\quad - E_{n,n-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{n-1} \exp(ik_m(y+2\Delta y)) - 4\delta_{n-1} \exp(ik_m(y+\Delta y)) + 6\delta_{n-1} \exp(ik_m y) \\ -4\delta_{n-1} \exp(ik_m(y-\Delta y)) + \delta_{n-1} \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ &\quad - F_{n,n-1}^{\alpha_1,\beta_1} (\delta_{n-1} \exp(ik_m y)) \\ &\quad + \sum_{j=0}^{n-1} \begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1} [\delta_j \exp(ik_m(y+\Delta y)) - 2\delta_j \exp(ik_m y) + \delta_j \exp(ik_m(y-\Delta y))] \\ -B_{n,j}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_j \exp(ik_m(y+2\Delta y)) - 4\delta_j \exp(ik_m(y+\Delta y)) + 6\delta_j \exp(ik_m y) \\ -4\delta_j \exp(ik_m(y-\Delta y)) + \delta_j \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1} \delta_n \exp(ik_m y) \end{pmatrix} \\ &\quad - \sum_{j=0}^{n-1} \begin{pmatrix} D_{n,j-1}^{\alpha_1,\beta_1} [\delta_{j-1} \exp(ik_m(y+\Delta y)) - 2\delta_{j-1} \exp(ik_m y) + \delta_{j-1} \exp(ik_m(y-\Delta y))] \\ -E_{n,j-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{j-1} \exp(ik_m(y+2\Delta y)) - 4\delta_{j-1} \exp(ik_m(y+\Delta y)) + 6\delta_{j-1} \exp(ik_m y) \\ -4\delta_{j-1} \exp(ik_m(y-\Delta y)) + \delta_{j-1} \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ -F_{n,j-1}^{\alpha_1,\beta_1} (\delta_{j-1} \exp(ik_m y)) \end{pmatrix}. \end{aligned} \tag{23}$$

After simplification, we obtain

$$\begin{aligned} \delta_{n+1} &= A_{n,n}^{\alpha_1,\beta_1} [\delta_n \exp(ik_m \Delta y) - 2\delta_n + \delta_n \exp(-ik_m \Delta y)] - B_{n,n}^{\alpha_1,\beta_1} \\ &\quad \cdot \begin{pmatrix} \delta_n \exp(2ik_m \Delta y) - 4\delta_n \exp(ik_m \Delta y) + 6\delta_n \\ -4\delta_n \exp(ik_m \Delta y) + \delta_n \exp(-2ik_m \Delta y) \end{pmatrix} \\ &\quad - C_{n,n}^{\alpha_1,\beta_1} \delta_n - D_{n,n-1}^{\alpha_1,\beta_1} [\delta_{n-1} \exp(ik_m \Delta y) - 2\delta_{n-1} + \delta_{n-1} \exp(ik_m \Delta y)] - E_{n,n-1}^{\alpha_1,\beta_1} \\ &\quad \cdot \begin{pmatrix} \delta_{n-1} \exp(2ik_m \Delta y) - 4\delta_{n-1} \exp(ik_m \Delta y) + 6\delta_{n-1} \\ -4\delta_{n-1} \exp(ik_m \Delta y) + \delta_{n-1} \exp(-2ik_m \Delta y) \end{pmatrix} - F_{n,n-1}^{\alpha_1,\beta_1} \delta_{n-1} + \sum_{j=0}^{n-1} \\ &\quad \cdot \begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1} [\delta_j \exp(ik_m \Delta y) - 2\delta_j + \delta_j \exp(-ik_m \Delta y)] \\ -B_{n,j}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_j \exp(2ik_m \Delta y) - 4\delta_j \exp(ik_m \Delta y) + 6\delta_j \\ -4\delta_j \exp(-ik_m \Delta y) + \delta_j \exp -2(ik_m \Delta y) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1} \delta_n \end{pmatrix} \\ &\quad - \sum_{j=0}^{n-1} \begin{pmatrix} D_{n,j-1}^{\alpha_1,\beta_1} [\delta_{j-1} \exp(ik_m \Delta y) - 2\delta_{j-1} + \delta_{j-1} \exp(-ik_m \Delta y)] \\ -E_{n,j-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{j-1} \exp(2ik_m \Delta y) - 4\delta_{j-1} \exp(ik_m \Delta y) + 6\delta_{j-1} \\ -4\delta_{j-1} \exp(-ik_m \Delta y) + \delta_{j-1} \exp(-2ik_m \Delta y) \end{pmatrix} \\ -F_{n,j-1}^{\alpha_1,\beta_1} \delta_{j-1} \end{pmatrix}. \end{aligned} \tag{24}$$

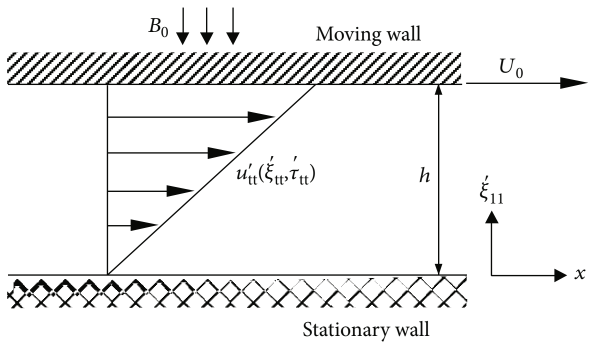

FIGURE 1: Geometry of the physical model.

We prove by induction. For $n = 0$, we obtain

$$\begin{aligned}\delta_1 &= A_{0,0}^{\alpha_1,\beta_1}[\delta_0 \exp(ik_m\Delta y) - 2\delta_0 + \delta_0 \exp(-ik_m\Delta y)] - B_{0,0}^{\alpha_1,\beta_1} \\ &\quad \cdot \begin{pmatrix} \delta_0 \exp(2ik_m\Delta y) - 4\delta_0 \exp(ik_m\Delta y) + 6\delta_n \\ -4\delta_0 \exp(ik_m\Delta y) + \delta_0 \exp(-2ik_m\Delta y) \end{pmatrix} - C_{0,0}^{\alpha_1,\beta_1}\delta_0 \\ &= \delta_0 \begin{pmatrix} A_{0,0}^{\alpha_1,\beta_1}[\exp(ik_m\Delta y) - 2 + \exp(-ik_m\Delta y)] \\ -B_{0,0}^{\alpha_1,\beta_1}\begin{pmatrix} \exp(2ik_m\Delta y) - 4\exp(ik_m\Delta y) + 6\delta_n \\ -4\exp(ik_m\Delta y) + \exp(-2ik_m\Delta y) \end{pmatrix} - C_{0,0}^{\alpha_1,\beta_1} \end{pmatrix} \\ &= \delta_0\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{0,0}^{\alpha_1,\beta_1} + 4B_{0,0}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{0,0}^{\alpha_1,\beta_1}\right).\end{aligned} \quad (25)$$

We should show $|\delta_1/\delta_0| < 1$. Therefore, we have

$$\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{0,0}^{\alpha_1,\beta_1} + 4B_{0,0}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{0,0}^{\alpha_1,\beta_1}\right| < 1.'' \quad (26)$$

Since we have for all m, we obtain

$$''\left|-4A_{0,0}^{\alpha_1,\beta_1} + 20B_{0,0}^{\alpha_1,\beta_1} - C_{0,0}^{\alpha_1,\beta_1}\right| < 1.'' \quad (27)$$

When $n > 1$, we have

$$\begin{aligned}\delta_{n+1} &= \sum_{j=0}^{n}\begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1}[\delta_j \exp(ik_m\Delta y) - 2\delta_j + \delta_j \exp(-ik_m\Delta y)] \\ -B_{n,j}^{\alpha_1,\beta_1}\begin{pmatrix} \delta_j \exp(2ik_m\Delta y) - 4\delta_j \exp(ik_m\Delta y) + 6\delta_j \\ -4\delta_j \exp(-ik_m\Delta y) + \delta_j \exp - 2(ik_m\Delta y) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1}\delta_n \end{pmatrix} \\ &\quad - \sum_{j=0}^{n-1}\begin{pmatrix} D_{n,j-1}^{\alpha_1,\beta_1}[\delta_{j-1}\exp(ik_m\Delta y) - 2\delta_{j-1} + \delta_{j-1}\exp(-ik_m\Delta y)] \\ -E_{n,j-1}^{\alpha_1,\beta_1}\begin{pmatrix} \delta_{j-1}\exp(2ik_m\Delta y) - 4\delta_{j-1}\exp(ik_m\Delta y) + 6\delta_{j-1} \\ -4\delta_{j-1}\exp(-ik_m\Delta y) + \delta_{j-1}\exp(-2ik_m\Delta y) \end{pmatrix} \\ -F_{n,j-1}^{\alpha_1,\beta_1}\delta_{j-1} \end{pmatrix}.\end{aligned} \quad (28)$$

Then, we get

$$\begin{aligned}\delta_{n+1} &= \sum_{j=0}^{n}\delta_j\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j}^{\alpha_1,\beta_1} + 4B_{n,j}^{\alpha_1,\beta_1}\right. \\ &\quad \left.\cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j}^{\alpha_1,\beta_1}\right) \\ &\quad - \sum_{j=0}^{n-1}\delta_{j-1}\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j-1}^{\alpha_1,\beta_1} + 4B_{n,j-1}^{\alpha_1,\beta_1}\right. \\ &\quad \left.\cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j-1}^{\alpha_1,\beta_1}\right).\end{aligned} \quad (29)$$

We assume that for all $n \geq 1$, $|\delta_1/\delta_0| < 1$. We want to prove that $|\delta_{n+1}/\delta_0| < 1$.. However,

$$\begin{aligned}|\delta_{n+1}| &\leq \sum_{j=0}^{n}|\delta_j|\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j}^{\alpha_1,\beta_1} + 4B_{n,j}^{\alpha_1,\beta_1}\right. \\ &\quad \cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) \\ &\quad \left. - C_{n,j}^{\alpha_1,\beta_1}\right| + \sum_{j=0}^{n-1}|\delta_{j-1}|\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j-1}^{\alpha_1,\beta_1} + 4B_{n,j-1}^{\alpha_1,\beta_1}\right. \\ &\quad \left.\cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j-1}^{\alpha_1,\beta_1}\right|.''\end{aligned} \quad (30)$$

By induction hypothesis for all $n \geq 1, |\delta_n| < |\delta_0|$, we have

$$\begin{aligned}|\delta_{n+1}| &< |\delta_0|\left(\sum_{j=0}^{n}\left|\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j}^{\alpha_1,\beta_1} + 4B_{n,j}^{\alpha_1,\beta_1}\right.\right.\right. \\ &\quad \left.\left.\cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j}^{\alpha_1,\beta_1}\right)\right| + \sum_{j=0}^{n-1}\left| \right. \\ &\quad \cdot\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j-1}^{\alpha_1,\beta_1} + 4B_{n,j-1}^{\alpha_1,\beta_1}\right. \\ &\quad \left.\left.\left.\cdot\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right|\right).\end{aligned} \quad (31)$$

This inequality is true for all m. Thus, we reach

$$\begin{aligned}|\delta_{n+1}| &< |\delta_0|\left(\sum_{j=0}^{n}\left|\left(-4A_{n,j}^{\alpha_1,\beta_1} + 20B_{n,j}^{\alpha_1,\beta_1} - C_{n,j}^{\alpha_1,\beta_1}\right)\right| + \sum_{j=0}^{n-1}\left| \right.\right. \\ &\quad \left.\left.\cdot\left(-4A_{n,j-1}^{\alpha_1,\beta_1} + 20B_{n,j-1}^{\alpha_1,\beta_1} - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right|\right).\end{aligned} \quad (32)$$

We need to show that $|\delta_{n+1}/\delta_0| < 1$. Thus, we reach

$$\left(\sum_{j=0}^{n}\left|\left(-4A_{n,j}^{\alpha_1,\beta_1} + 20B_{n,j}^{\alpha_1,\beta_1} - C_{n,j}^{\alpha_1,\beta_1}\right)\right| + \sum_{j=0}^{n-1}\left|\left(-4A_{n,j-1}^{\alpha_1,\beta_1} + 20B_{n,j-1}^{\alpha_1,\beta_1} - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right|\right) < 1. \quad (33)$$

5. Solution of the Problem with the Exponential Decay Kernel

We consider Eq. (10) with fractal-fractional differential operator using Definition 2 of exponential decay kernel as

$$
{}_0^{FFE}D_t^{\alpha_1,\beta_1} v(y,t) = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right),
$$

$$
\begin{aligned}
&\frac{M(\alpha_1)}{(1-\alpha_1)}\frac{d}{dt}\int_0^t v(y,\lambda)\exp\left(-\frac{\alpha_1}{(1-\alpha_1)}(t-\lambda)\right)d\lambda \\
&\quad = \frac{\beta_1}{\mathrm{Re}} t^{\beta_1-1}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right).
\end{aligned} \tag{34}
$$

For simplicity, we define

$$
F(y,t,v(y,t)) = \frac{\beta_1}{\mathrm{Re}} t^{\beta_1-1}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right). \tag{35}
$$

Then, we reach

$$
v(y,\lambda) = \frac{1-\alpha_1}{M(\alpha_1)} F(y,t,v(y,t)) + \frac{\alpha_1}{M(\alpha_1)}\int_0^t F(y,\lambda,v(y,\lambda))d\lambda. \tag{36}
$$

We discretize Eq. (36) at (y_i, t_{n+1}) and (y_i, t_n) as

$$
\begin{aligned}
v_i^{n+1} &= \frac{1-\alpha_1}{M(\alpha_1)} F(y_i, t_n, v_i^n) + \frac{\alpha_1}{M(\alpha_1)}\int_0^{t_{n+1}} F(y_i,\lambda,v(y_i,\lambda))d\lambda, \\
v_i^n &= \frac{1-\alpha_1}{M(\alpha_1)} F\left(y_i, t_n, v_i^{n-1}\right) + \frac{\alpha_1}{M(\alpha_1)}\int_0^{t_n} F(y_i,\lambda,v(y_i,\lambda))d\lambda.
\end{aligned} \tag{37}
$$

Then, we obtain

$$
\begin{aligned}
v_i^{n+1} &= v_i^n + \frac{1-\alpha_1}{M(\alpha_1)}\left(F(y_i,t_n,v_i^n) - F\left(y_i,t_{n-1},v_i^{n-1}\right)\right) \\
&\quad + \frac{\alpha_1}{M(\alpha_1)}\int_{t_n}^{t_{n+1}} F(y_i,\lambda,v(y_i,\lambda))d\lambda \\
&= v_i^n + \frac{1-\alpha_1}{M(\alpha_1)}\left(F(y_i,t_n,v_i^n) - F\left(y_i,t_{n-1},v_i^{n-1}\right)\right) \\
&\quad + \frac{\alpha_1}{M(\alpha_1)}\left(\frac{3h}{2}F(y_i,t_n,v_i^n) - \frac{h}{2}F\left(y_i,t_{n-1},v_i^{n-1}\right)\right),
\end{aligned} \tag{38}
$$

where

$$
\begin{aligned}
&F(y_i,t_n,v(y_i,t_n)) \\
&\quad = \frac{\beta_1}{\mathrm{Re}} t_n^{\beta_1-1}\left(\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta y)^2} - \frac{v_{i+2}^n - 4v_{i+1}^n + 6v_i^n - 4v_{i-1}^n + v_{i-2}^n}{(\Delta y)^4} - M_{11} v_i^n\right).
\end{aligned} \tag{39}
$$

Thus, we acquire

$$
\begin{aligned}
v_i^{n+1} &= v_i^n + \frac{\beta_1}{\mathrm{Re}} t_n^{\beta_1-1} \\
&\quad \cdot \left(\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta y)^2} - \frac{v_{i+2}^n - 4v_{i+1}^n + 6v_i^n - 4v_{i-1}^n + v_{i-2}^n}{(\Delta y)^4} - M_{11} v_i^n\right) \\
&\quad \cdot \left(\frac{3(1-\alpha_1) + 3h\alpha_1}{2M(\alpha_1)}\right) - \frac{\beta_1}{\mathrm{Re}} t_{n-1}^{\beta_1-1} \\
&\quad \cdot \left(\frac{v_{i+1}^{n-1} - 2v_i^{n-1} + v_{i-1}^{n-1}}{(\Delta y)^2} - \frac{v_{i+2}^{n-1} - 4v_{i+1}^{n-1} + 6v_i^{n-1} - 4v_{i-1}^{n-1} + v_{i-2}^{n-1}}{(\Delta y)^4} - M_{11} v_i^{n-1}\right) \\
&\quad \cdot \left(\frac{2(1-\alpha_1) + h\alpha_1}{2M(\alpha_1)}\right).
\end{aligned} \tag{40}
$$

For simplicity, we let

$$
\begin{aligned}
A_{n,\alpha_1,\beta_1} &= \frac{\beta_1 t_n^{\beta_1-1} N_{\alpha_1}}{\mathrm{Re}\,(\Delta y)^2}, \quad B_{n,\alpha_1,\beta_1} = \frac{\beta_1 t_n^{\beta_1-1} N_{\alpha_1}}{\mathrm{Re}\,(\Delta y)^4}, \\
C_{n,\alpha_1,\beta_1} &= \frac{\beta_1 t_j^{\beta_1-1} M_{11} N_{\alpha_1}}{\mathrm{Re}}, \\
D_{n,\alpha_1,\beta_1} &= \frac{\beta_1 t_{n-1}^{\beta_1-1} K_{\alpha_1}}{\mathrm{Re}\,(\Delta y)^2}, \quad E_{n,\alpha_1,\beta_1} = \frac{\beta_1 t_{n-1}^{\beta_1-1} K_{\alpha_1}}{\mathrm{Re}\,(\Delta y)^4}, \\
F_{n,\alpha_1,\beta_1} &= \frac{\beta_1 t_{n-1}^{\beta_1-1} M_{11} K_{\alpha_1}}{\mathrm{Re}}, \\
N_{\alpha_1} &= \frac{2(1-\alpha_1) + 3h\alpha_1}{2M(\alpha_1)}, \quad K_{\alpha_1} = \frac{2(1-\alpha_1) + h\alpha_1}{2M(\alpha_1)}.
\end{aligned} \tag{41}
$$

We choose $\varepsilon_i^n = \delta_n \exp(ik_m y)$.. Therefore, we reach

$$
\begin{aligned}
&\delta_{n+1}\exp(ik_m y) = \delta_n \exp(ik_m y) - C_{n,n}^{\alpha_1,\beta_1}\delta_n \exp(ik_m y) + A_{n,\alpha_1,\beta_1}[\delta_n \exp(ik_m(y+\Delta y)) \\
&\quad - 2\delta_n \exp(ik_m y) + \delta_n \exp(ik_m(y-\Delta y))] - B_{n,\alpha_1,\beta_1} \\
&\quad \cdot \begin{pmatrix} \delta_n \exp(ik_m(y+2\Delta y)) - 4\delta_n \exp(ik_m(y+\Delta y)) + 6\delta_n \exp(ik_m y) \\ -4\delta_n \exp(ik_m(y-\Delta y)) + \delta_n \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\
&\quad - D_{n,\alpha_1,\beta_1}[\delta_{n-1}\exp(ik_m(y+\Delta y)) - 2\delta_{n-1}\exp(ik_m y) + \delta_{n-1}\exp(ik_m(y-\Delta y))] \\
&\quad - E_{n,\alpha_1,\beta_1}\begin{pmatrix} \delta_{n-1}\exp(ik_m(y+2\Delta y)) - 4\delta_{n-1}\exp(ik_m(y+\Delta y)) + 6\delta_{n-1}\exp(ik_m y) \\ -4\delta_{n-1}\exp(ik_m(y-\Delta y)) + \delta_{n-1}\exp(ik_m(y-2\Delta y)) \end{pmatrix} \\
&\quad + F_{n,n-1}^{\alpha_1,\beta_1}(\delta_{n-1}\exp(ik_m y)).
\end{aligned} \tag{42}
$$

After simplification, we obtain

$$
\begin{aligned}
\delta_{n+1} &= \delta_n - C_{n,\alpha_1,\beta_1}\delta_n + F_{n,\alpha_1,\beta_1}\delta_{n-1} + A_{n,\alpha_1,\beta_1}[\delta_n \exp(ik_m\Delta y) \\
&\quad - 2\delta_n + \delta_n \exp(-ik_m\Delta y)] - B_{n,\alpha_1,\beta_1} \\
&\quad \cdot \begin{pmatrix} \delta_n \exp(2ik_m\Delta y) - 4\delta_n \exp(ik_m\Delta y) + 6\delta_n \\ -4\delta_n \exp(ik_m\Delta y) + \delta_n \exp(-2ik_m\Delta y) \end{pmatrix} \\
&\quad - D_{n,\alpha_1,\beta_1}[\delta_{n-1}\exp(ik_m\Delta y) - 2\delta_{n-1} + \delta_{n-1}\exp(ik_m\Delta y)] \\
&\quad + E_{n,\alpha_1,\beta_1}\begin{pmatrix} \delta_{n-1}\exp(2ik_m\Delta y) - 4\delta_{n-1}\exp(ik_m\Delta y) + 6\delta_{n-1} \\ -4\delta_{n-1}\exp(ik_m\Delta y) + \delta_{n-1}\exp(-2ik_m\Delta y) \end{pmatrix}.
\end{aligned} \tag{43}
$$

Thus, we have

$$\delta_{n+1} = \delta_n \left(1 - 4A_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4B_{n,j}^{\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{n,\alpha_1,\beta_1}\right) + \delta_{n-1} \cdot \left(4D_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) - 4E_{n,\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) + F_{n,\alpha_1,\beta_1}\right). \tag{44}$$

For $n = 0$, we get

$$\delta_1 = \delta_0 \left(1 - 4A_{0,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4B_{0,\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{0,\alpha_1,\beta_1}\right). \tag{45}$$

The $|\delta_1/\delta_0| < 1$ implies

$$\left|1 - 4A_{0,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4B_{0,\alpha_1,\beta_1} \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{0,\alpha_1,\beta_1}\right| < 1. \tag{46}$$

This is true for all m.. Thus, we get

$$\left|1 - 4A_{0,\alpha_1,\beta_1} + 20B_{0,\alpha_1,\beta_1} - C_{0,\alpha_1,\beta_1}\right| < 1. \tag{47}$$

We assume that $|\delta_n/\delta_0| < 1$.. Thus, we need to show $|\delta_{n+1}/\delta_0| < 1$.

$$|\delta_{n+1}| \le |\delta_n| \left|1 - 4A_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4B_{n,\alpha_1,\beta_1} \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{n,\alpha_1,\beta_1}\right| + |\delta_{n-1}| \left|4D_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4E_{n,\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) + F_{n,\alpha_1,\beta_1}\right| < |\delta_0| \left[\left|1 - 4A_{n,\alpha_1,\beta_1} \sin^2 \cdot \left(\frac{k_m \Delta y}{2}\right) + 4B_{n,\alpha_1,\beta_1} \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{n,\alpha_1,\beta_1}\right| + \left|4D_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) - 4E_{n,\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) + F_{n,\alpha_1,\beta_1}\right|\right]. \tag{48}$$

Thus, we obtain

$$\left|1 - 4A_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) + 4B_{n,\alpha_1,\beta_1} \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) - C_{n,\alpha_1,\beta_1}\right| + \left|4D_{n,\alpha_1,\beta_1} \sin^2\left(\frac{k_m \Delta y}{2}\right) - 4E_{n,\alpha_1,\beta_1} \cdot \left(\sin^2(k_m \Delta y) + 4\sin^2\left(\frac{k_m \Delta y}{2}\right)\right) + F_{n,\alpha_1,\beta_1}\right| < 1. \tag{49}$$

This inequality is true for all m.. Thus, we get

$$\left(\left|\left(1 - 4A_{n,\alpha_1,\beta_1} + 20B_{n,\alpha_1,\beta_1} - C_{n,\alpha_1,\beta_1}\right)\right| + \left| \cdot \left(4D_{n,\alpha_1,\beta_1} - 20E_{n,\alpha_1,\beta_1} + F_{n,\alpha_1,\beta_1}\right)\right|\right) < 1. \tag{50}$$

6. Solution of the Problem with the Generalized Mittag-Leffler Kernel

We take into consideration the Eq. (10) with fractal-fractional differential operator using Definition 3 of Mittag-Leffler kernel as

$$ {}_0^{FFM}D_t^{\alpha_1,\beta_1} v(y,t) = \frac{1}{\text{Re}}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right),$$

$$\frac{AB(\alpha_1)}{(1-\alpha_1)} \frac{d}{dt}\int_0^t v(y,\lambda) E_{\alpha_1}\left(-\frac{\alpha_1}{1-\alpha_1}(t-\lambda)^{\alpha_1}\right) d\lambda = \frac{\beta_1}{\text{Re}} t^{\beta_1 - 1}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right). \tag{51}$$

For simplicity, we define

$$F(y,t,v(y,t)) = \frac{\beta_1}{\text{Re}} t^{\beta_1 - 1}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11} v(y,t)\right). \tag{52}$$

Then, we get

$$v(y,\lambda) = \frac{1-\alpha_1}{AB(\alpha_1)} F(y,t,v(y,t)) + \frac{\alpha_1}{\Gamma(\alpha_1) AB(\alpha_1)} \int_0^t F(y,\lambda,v(y,\lambda)) d\lambda. \tag{53}$$

We discretize above Eq. (53) at (y_i, t_{n+1}) as

$$v_i^{n+1} = \frac{1-\alpha_1}{AB(\alpha_1)} F(y_i, t_{n+1}, v_i^n) + \frac{\alpha_1}{\Gamma(\alpha_1) AB(\alpha_1)} \int_0^{t_{n+1}} F(y_i, \lambda, v(y_i, \lambda))(t_{n+1}, \lambda)^{\alpha_1 - 1} d\lambda. \tag{54}$$

Then, we obtain

$$v_i^{n+1} = \frac{1-\alpha_1}{AB(\alpha_1)} F(y_i, t_{n+1}, v_i^n) + \frac{\alpha_1}{AB(\alpha_1)} \sum_{j=0}^{n} \cdot \left[\frac{h^{\alpha_1} F\left(y_i, t_j, v_i^j\right)}{\Gamma(\alpha_1+2)} \left((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) - (n-j)^{\alpha_1}(n-j+2+2\alpha_1)\right)\right] - \frac{\alpha_1}{AB(\alpha_1)} \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} F\left(y_i, t_{j-1}, v_i^{j-1}\right)}{\text{Re}\, \Gamma(\alpha_1+2)} \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1)\right)\right]. \tag{55}$$

We have

$$F(y_i, t_j, v_i^n) = \beta_1 t_n^{\beta_1 - 1}\left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta y)^2} - \frac{v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j}{(\Delta y)^4} - M_{11} v_i^j\right). \tag{56}$$

Then, we will obtain

$$\begin{aligned} v_i^{n+!} = {} & \frac{\beta_1(1-\alpha_1)}{\operatorname{Re} AB(\alpha_1)} t_j^{\beta_1-1} \\ & \cdot \left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta y)^2} - \frac{v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j}{(\Delta y)^4} - M_{11} v_i^j \right) \\ & + \sum_{j=0}^{n} \left[\frac{h^{\alpha_1} \beta_1 t_j^{\beta_1-1}}{\Gamma(\alpha_1+2)} \left(\frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta y)^2} - \frac{v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j}{(\Delta y)^4} - M_{11} v_i^j \right) \right. \\ & \left. \times \left((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) - (n-j)^{\alpha_1}(n-j+2+2\alpha_1) \right) \right] \frac{\alpha_1}{\operatorname{Re} AB(\alpha_1)} - \sum_{j=0}^{n} \\ & \cdot \left[\frac{h^{\alpha_1} \beta_1 t_{j-1}^{\beta_1-1}}{\Gamma(\alpha_1+2)} \left(\frac{v_{i+1}^{j-1} - 2v_i^{j-1} + v_{i-1}^{j-1}}{(\Delta y)^2} - \frac{v_{i+2}^{j-1} - 4v_{i+1}^{j-1} + 6v_i^{j-1} - 4v_{i-1}^{j-1} + v_{i-2}^{j-1}}{(\Delta y)^4} - M_{11} v_i^{j-1} \right) \right. \\ & \left. \times \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1) \right) \right] \frac{\alpha_1}{\operatorname{Re} AB(\alpha_1)}. \end{aligned} \tag{57}$$

For simplicity, we let

$$\begin{aligned} M_{\alpha_1} &= \frac{\beta_1 t_j^{\beta_1-1}(1-\alpha_1)}{\operatorname{Re} (\Delta y)^2 AB(\alpha_1)}, \quad K_{\alpha_1} = \frac{\beta_1 t_j^{\beta_1-1} M_{11}(1-\alpha_1)}{\operatorname{Re} AB(\alpha_1)}, \\ T_{\alpha_1} &= \frac{\beta_1 t_j^{\beta_1-1}(1-\alpha_1)}{\operatorname{Re} (\Delta y)^4 AB(\alpha_1)}, \\ A_{n,j}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_j^{\beta_1-1} K_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} (\Delta y)^2 AB(\alpha_1)}, \\ B_{n,j}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_j^{\beta_1-1} K_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} (\Delta y)^4 AB(\alpha_1)}, \\ C_{n,j}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_j^{\beta_1-1} M_{11} K_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} AB(\alpha_1)}, \\ A_{n,j-1}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_{j-1}^{\beta_1-1} \gamma_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} (\Delta y)^2 AB(\alpha_1)}, \\ B_{n,j-1}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_{j-1}^{\beta_1-1} \gamma_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} (\Delta y)^4 AB(\alpha_1)}, \\ C_{n,j-1}^{\alpha_1,\beta_1} &= \alpha_1 \frac{h^{\alpha_1}\beta_1 t_{j-1}^{\beta_1-1} M_{11} K_{n,j}^{\alpha_1}}{\Gamma(\alpha_1+2) \operatorname{Re} AB(\alpha_1)}, \\ K_{n,j}^{\alpha_1} &= \left((n+1-j)^{\alpha_1}(n-j+2+\alpha_1) - (n-j)^{\alpha_1}(n-j+2+2\alpha_1) \right), \\ \gamma_{n,j}^{\alpha_1} &= \left((n+1-j)^{\alpha_1+1} - (n-j)^{\alpha_1}(n-j+1+\alpha_1) \right). \end{aligned} \tag{58}$$

Then, we get

$$\begin{aligned} v_i^{n+1} = {} & M_{\alpha_1}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) - K_{\alpha_1} v_i^n - T_{\alpha_1} \\ & \cdot (v_{i+2}^n - 4v_{i+1}^n + 6v_i^n - 4v_{i-1}^n + v_{i-2}^n) \\ & + \sum_{j=0}^{n} \left[A_{n,j}^{\alpha_1,\beta_1} \left(v_{i+1}^j - 2v_i^j + v_{i-1}^j \right) - B_{n,j}^{\alpha_1,\beta_1} \right. \\ & \left. \cdot \left(v_{i+2}^j - 4v_{i+1}^j + 6v_i^j - 4v_{i-1}^j + v_{i-2}^j \right) - C_{n,j}^{\alpha_1,\beta_1} v_i^j \right] \\ & - \sum_{j=0}^{n} \left[A_{n,j-1}^{\alpha_1,\beta_1} \left(v_{i+1}^{j-1} - 2v_i^{j-1} + v_{i-1}^{j-1} \right) - B_{n,j-1}^{\alpha_1,\beta_1} \right. \\ & \left. \cdot \left(v_{i+2}^{j-1} - 4v_{i+1}^{j-1} + 6v_i^{j-1} - 4v_{i-1}^{j-1} + v_{i-2}^{j-1} \right) - C_{n,j-1}^{\alpha_1,\beta_1} \left(v_i^{j-1} \right) \right]. \end{aligned} \tag{59}$$

We choose $\varepsilon_i^n = \delta_n \exp(ik_m y)$.. Then, we acquire

$$\begin{aligned} \delta_{n+1} \exp(ik_m y) = {} & M_{\alpha_1}[\delta_n \exp(ik_m(y+\Delta y)) - 2\delta_n \exp(ik_m y) + \delta_n \exp(ik_m(y-\Delta y))] \\ & - K_{\alpha_1}\delta_n \exp(ik_m y) - T_{\alpha_1}(\delta_n \exp(ik_m(y+2\Delta y)) - 4\delta_n \exp(ik_m(y+\Delta y)) \\ & + 6\delta_n \exp(ik_m y) - 4\delta_n \exp(ik_m(y-\Delta y)) + \delta_n \exp(ik_m(y-2\Delta y))) \\ & + A_{n,n}^{\alpha_1,\beta_1}[\delta_n \exp(ik_m(y+\Delta y)) - 2\delta_n \exp(ik_m y) + \delta_n \exp(ik_m(y-\Delta y))] \\ & - B_{n,n}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_n \exp(ik_m(y+2\Delta y)) - 4\delta_n \exp(ik_m(y+\Delta y)) + 6\delta_n \exp(ik_m y) \\ -4\delta_n \exp(ik_m(y-\Delta y)) + \delta_n \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ & - C_{n,n}^{\alpha_1,\beta_1}\delta_n \exp(ik_m y) - A_{n,n-1}^{\alpha_1,\beta_1}[\delta_{n-1} \exp(ik_m(y+\Delta y)) - 2\delta_{n-1} \exp(ik_m y) \\ & + \delta_{n-1} \exp(ik_m(y-\Delta y))] + B_{n,n-1}^{\alpha_1,\beta_1} \\ & \cdot \begin{pmatrix} \delta_{n-1} \exp(ik_m(y+2\Delta y)) - 4\delta_{n-1} \exp(ik_m(y+\Delta y)) + 6\delta_{n-1} \exp(ik_m y) \\ -4\delta_{n-1} \exp(ik_m(y-\Delta y)) + \delta_{n-1} \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ & + C_{n,n-1}^{\alpha_1,\beta_1}(\delta_{n-1} \exp(ik_m y)) + \sum_{j=0}^{n-1} \\ & \cdot \begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1}\left[\delta_j \exp(ik_m(y+\Delta y)) - 2\delta_j \exp(ik_m y) + \delta_j \exp(ik_m(y-\Delta y))\right] \\ -B_{n,j}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_j \exp(ik_m(y+2\Delta y)) - 4\delta_j \exp(ik_m(y+\Delta y)) + 6\delta_j \exp(ik_m y) \\ -4\delta_j \exp(ik_m(y-\Delta y)) + \delta_j \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1}\delta_n \exp(ik_m y) \end{pmatrix} - \sum_{j=0}^{n-1} \\ & \cdot \begin{pmatrix} A_{n,j-1}^{\alpha_1,\beta_1}\left[\delta_{j-1} \exp(ik_m(y+\Delta y)) - 2\delta_{j-1} \exp(ik_m y) + \delta_{j-1} \exp(ik_m(y-\Delta y))\right] \\ -B_{n,j-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{j-1} \exp(ik_m(y+2\Delta y)) - 4\delta_{j-1} \exp(ik_m(y+\Delta y)) + 6\delta_{j-1} \exp(ik_m y) \\ -4\delta_{j-1} \exp(ik_m(y-\Delta y)) + \delta_{j-1} \exp(ik_m(y-2\Delta y)) \end{pmatrix} \\ -C_{n,j-1}^{\alpha_1,\beta_1}(\delta_{j-1} \exp(ik_m y)) \end{pmatrix}. \end{aligned} \tag{60}$$

Then, we get

$$\begin{aligned} \delta_{n+1} = {} & M_{\alpha_1}[\delta_n \exp(ik_m\Delta y) - 2\delta_n + \delta_n \exp(-ik_m\Delta y)] - K_{\alpha_1}\delta_n - T_{\alpha_1}(\delta_n \exp(2ik_m\Delta y) \\ & - 4\delta_n \exp(ik_m\Delta y) + 6\delta_n - 4\delta_n \exp(-ik_m\Delta y) + \delta_n \exp(-2ik_m\Delta y)) \\ & + A_{n,n}^{\alpha_1,\beta_1}[\delta_n \exp(ik_m\Delta y) - 2\delta_n + \delta_n \exp(-ik_m\Delta y)] \\ & - B_{n,n}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_n \exp(2ik_m\Delta y) - 4\delta_n \exp(ik_m\Delta y) + 6\delta_n \\ -4\delta_n \exp(ik_m\Delta y) + \delta_n \exp(-2ik_m\Delta y) \end{pmatrix} - C_{n,n}^{\alpha_1,\beta_1}\delta_n \\ & - A_{n,n-1}^{\alpha_1,\beta_1}[\delta_{n-1} \exp(ik_m\Delta y) - 2\delta_{n-1} + \delta_{n-1} \exp(ik_m\Delta y)] \\ & + B_{n,n-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{n-1} \exp(2ik_m\Delta y) - 4\delta_{n-1} \exp(ik_m\Delta y) + 6\delta_{n-1} \\ -4\delta_{n-1} \exp(ik_m\Delta y) + \delta_{n-1} \exp(-2ik_m\Delta y) \end{pmatrix} \\ & + C_{n,n-1}^{\alpha_1,\beta_1}\delta_{n-1} + \sum_{j=0}^{n-1} \begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1}\left[\delta_j \exp(ik_m\Delta y) - 2\delta_j + \delta_j \exp(-ik_m\Delta y)\right] \\ -B_{n,j}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_j \exp(2ik_m\Delta y) - 4\delta_j \exp(ik_m\Delta y) + 6\delta_j \\ -4\delta_j \exp(-ik_m\Delta y) + \delta_j \exp -2(ik_m\Delta y) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1}\delta_j \end{pmatrix} \\ & - \sum_{j=0}^{n-1} \begin{pmatrix} A_{n,j-1}^{\alpha_1,\beta_1}\left[\delta_{j-1} \exp(ik_m\Delta y) - 2\delta_{j-1} + \delta_{j-1} \exp(-ik_m\Delta y)\right] \\ -B_{n,j-1}^{\alpha_1,\beta_1} \begin{pmatrix} \delta_{j-1} \exp(2ik_m\Delta y) - 4\delta_{j-1} \exp(ik_m\Delta y) + 6\delta_{j-1} \\ -4\delta_{j-1} \exp(-ik_m\Delta y) + \delta_{j-1} \exp(-2ik_m\Delta y) \end{pmatrix} \\ -C_{n,j-1}^{\alpha_1,\beta_1}\delta_{j-1} \end{pmatrix}. \end{aligned} \tag{61}$$

We prove by induction. For $n = 0$, we have

$$\delta_1 = M_{\alpha_1}[\delta_0 \exp(ik_m\Delta y) - 2\delta_0 + \delta_0 \exp(-ik_m\Delta y)] - K_{\alpha_1}\delta_0 - T_{\alpha_1}(\delta_0 \exp(2ik_m\Delta y) - 4\delta_0 \exp(ik_m\Delta y) + 6\delta_0 - 4\delta_0 \exp(-ik_m\Delta y) + \delta_0 \exp(-2ik_m\Delta y)) + A_{0,0}^{\alpha_1,\beta_1}[\delta_0 \exp(ik_m\Delta y) - 2\delta_0 + \delta_0 \exp(-ik_m\Delta y)] - B_{0,0}^{\alpha_1,\beta_1}\begin{pmatrix} \delta_0 \exp(2ik_m\Delta y) - 4\delta_0 \exp(ik_m\Delta y) + 6\delta_n \\ -4\delta_0 \exp(ik_m\Delta y) + \delta_0 \exp(-2ik_m\Delta y) \end{pmatrix} - C_{0,0}^{\alpha_1,\beta_1}\delta_0. \tag{62}$$

Then, we get

$$\delta_1 = \delta_0 M_{\alpha_1}[\exp(ik_m\Delta y) - 2 + \exp(-ik_m\Delta y)] - K_{\alpha_1}\delta_0 - \delta_0 T_{\alpha_1}(\exp(2ik_m\Delta y) - 4\exp(ik_m\Delta y) + 6 - 4\exp(-ik_m\Delta y) + \exp(-2ik_m\Delta y)) + \delta_0 A_{0,0}^{\alpha_1,\beta_1}[\exp(ik_m\Delta y) - 2 + \exp(-ik_m\Delta y)] - \delta_0 B_{0,0}^{\alpha_1,\beta_1}\begin{pmatrix} \exp(2ik_m\Delta y) - 4\exp(ik_m\Delta y) + 6 \\ -4\exp(ik_m\Delta y) + \exp(-2ik_m\Delta y) \end{pmatrix} - C_{0,0}^{\alpha_1,\beta_1}\delta_0. \tag{63}$$

Thus, we reach

$$\delta_1 = \delta_0 M_{\alpha_1}\left[-4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right] - K_{\alpha_1}\delta_0 + 4\delta_0 T_{\alpha_1}\left(\sin^2\left(\frac{k_m\Delta y}{2}\right) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) + \delta_0\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{0,0}^{\alpha_1,\beta_1} + 4B_{0,0}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{0,0}^{\alpha_1,\beta_1}\right). \tag{64}$$

After simplification, we obtain

$$\delta_1 = \delta_0\begin{pmatrix} -4\sin^2\left(\frac{k_m\Delta y}{2}\right)\left(M_{\alpha_1} + A_{0,0}^{\alpha_1,\beta_1}\right) - K_{\alpha_1} - C_{0,0}^{\alpha_1,\beta_1} \\ +4\left(T_{\alpha_1} + B_{0,0}^{\alpha_1,\beta_1}\right)\left(\sin^2\left(\frac{k_m\Delta y}{2}\right) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) \end{pmatrix}. \tag{65}$$

We should show $|\delta_1/\delta_0| < 1.$. Therefore, we have

$$\left|\begin{matrix} -4\sin^2\left(\frac{k_m\Delta y}{2}\right)\left(M_{\alpha_1} + A_{0,0}^{\alpha_1,\beta_1}\right) - K_{\alpha_1} - C_{0,0}^{\alpha_1,\beta_1} \\ +4\left(T_{\alpha_1} + B_{0,0}^{\alpha_1,\beta_1}\right)\left(\sin^2\left(\frac{k_m\Delta y}{2}\right) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) \end{matrix}\right| < 1.'' \tag{66}$$

Since we have for all m, we obtain

$$\left|-4\left(M_{\alpha_1} + A_{0,0}^{\alpha_1,\beta_1}\right) - K_{\alpha_1} - C_{0,0}^{\alpha_1,\beta_1} + 20\left(T_{\alpha_1} + B_{0,0}^{\alpha_1,\beta_1}\right)\right| < 1.'' \tag{67}$$

When $n > 1$, we have

$$\delta_{n+1} = M_{\alpha_1}[\delta_n \exp(ik_m\Delta y) - 2\delta_n + \delta_n \exp(-ik_m\Delta y)] - K_{\alpha_1}\delta_n - T_{\alpha_1}(\delta_n \exp(2ik_m\Delta y) - 4\delta_n \exp(ik_m\Delta y) + 6\delta_n - 4\delta_n \exp(-ik_m\Delta y) + \delta_n \exp(-2ik_m\Delta y)) + \sum_{j=0}^{n}\begin{pmatrix} A_{n,j}^{\alpha_1,\beta_1}[\delta_j \exp(ik_m\Delta y) - 2\delta_j + \delta_j \exp(-ik_m\Delta y)] \\ -B_{n,j}^{\alpha_1,\beta_1}\begin{pmatrix} \delta_j \exp(2ik_m\Delta y) - 4\delta_j \exp(ik_m\Delta y) + 6\delta_j \\ -4\delta_j \exp(-ik_m\Delta y) + \delta_j \exp -2(ik_m\Delta y) \end{pmatrix} \\ -C_{n,j}^{\alpha_1,\beta_1}\delta_n \end{pmatrix} - \sum_{j=0}^{n}\begin{pmatrix} A_{n,j-1}^{\alpha_1,\beta_1}[\delta_{j-1} \exp(ik_m\Delta y) - 2\delta_{j-1} + \delta_{j-1} \exp(-ik_m\Delta y)] \\ -B_{n,j-1}^{\alpha_1,\beta_1}\begin{pmatrix} \delta_{j-1} \exp(2ik_m\Delta y) - 4\delta_{j-1} \exp(ik_m\Delta y) + 6\delta_{j-1} \\ -4\delta_{j-1} \exp(-ik_m\Delta y) + \delta_{j-1} \exp(-2ik_m\Delta y) \end{pmatrix} \\ -C_{n,j-1}^{\alpha_1,\beta_1}\delta_{j-1} \end{pmatrix}. \tag{68}$$

We suppose that for all $n \geq 1$, $|\delta_n/\delta_0| < 1.$. We want to prove that $|\delta_{n+1}/\delta_0| < 1.$. However,

$$|\delta_{n+1}| \leq |\delta_n|\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)M_{\alpha_1} + K_{\alpha_1}\right| + \sum_{j=0}^{n}|\delta_j|\left| -4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j}^{\alpha_1,\beta_1} + 4B_{n,j}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j}^{\alpha_1,\beta_1}\right| + \sum_{j=0}^{n-1}|\delta_{j-1}|\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j-1}^{\alpha_1,\beta_1} + 4B_{n,j-1}^{\alpha_1,\beta_1} \cdot \left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j-1}^{\alpha_1,\beta_1}\right|.'' \tag{69}$$

By induction hypothesis for all $n \geq 1$, $|\delta_n| < |\delta_0|$, we have

$$|\delta_{n+1}| \leq |\delta_0|\left|-4\sin^2\left(\frac{k_m\Delta y}{2}\right)M_{\alpha_1} + K_{\alpha_1}\right| + \sum_{j=0}^{n}|\delta_0|\left|\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j}^{\alpha_1,\beta_1} + 4B_{n,j}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j}^{\alpha_1,\beta_1}\right)\right| + \sum_{j=0}^{n}|\delta_0|\left|\left(-4\sin^2\left(\frac{k_m\Delta y}{2}\right)A_{n,j-1}^{\alpha_1,\beta_1} + 4B_{n,j-1}^{\alpha_1,\beta_1}\left(\sin^2(k_m\Delta y) + 4\sin^2\left(\frac{k_m\Delta y}{2}\right)\right) - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right|. \tag{70}$$

This inequality is true for all m. Thus, we reach

$$|\delta_{n+1}| \leq |\delta_0|\left|-4M_{\alpha_1} + K_{\alpha_1}\right| + \sum_{j=0}^{n}|\delta_0|\left|\left(-4A_{n,j}^{\alpha_1,\beta_1} + 20B_{n,j}^{\alpha_1,\beta_1} - C_{n,j}^{\alpha_1,\beta_1}\right)\right| + \sum_{j=0}^{n-1}|\delta_0|\left|\left(-4A_{n,j-1}^{\alpha_1,\beta_1} + 20B_{n,j-1}^{\alpha_1,\beta_1} - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right|. \tag{71}$$

We need to show that $|\delta_{n+1}/\delta_0| < 1$. Thus, we get

$$\begin{aligned} &\left|-4M_{\alpha_1} + K_{\alpha_1}\right| + \sum_{j=0}^{n}\left|\left(-4A_{n,j}^{\alpha_1,\beta_1} + 20B_{n,j}^{\alpha_1,\beta_1} - C_{n,j}^{\alpha_1,\beta_1}\right)\right| \\ &\quad + \sum_{j=0}^{n-1}\left|\left(-4A_{n,j-1}^{\alpha_1,\beta_1} + 20B_{n,j-1}^{\alpha_1,\beta_1} - C_{n,j-1}^{\alpha_1,\beta_1}\right)\right| < 1. \end{aligned} \tag{72}$$

7. Error Analysis

In this section, we will consider the error analysis.

$$ {}_{0}^{FFP}D_t^{\alpha_1,\beta_1} v(y,t) = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11}v(y,t)\right),$$

$$\begin{aligned} &\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{dt}\int_0^t v(y,t)(t-\lambda)^{-\alpha_1}d\lambda \\ &\quad = \frac{\beta_1}{\mathrm{Re}}t^{\beta_1-1}\left(\frac{\partial^2 v(y,t)}{\partial y^2} - \frac{\partial^4 v(y,t)}{\partial y^4} - M_{11}v(y,t)\right). \end{aligned} \tag{73}$$

Then, we get

$$v(y,t) = \frac{\beta_1}{\mathrm{Re}\,\Gamma(\alpha_1)}\int_0^t \lambda^{\beta_1-1}\left(\frac{\partial^2 v(y,\lambda)}{\partial y^2} - \frac{\partial^4 v(y,\lambda)}{\partial y^4} - M_{11}v(y,\lambda)\right)(t-\lambda)^{\alpha_1-1}d\lambda. \tag{74}$$

For simplicity, we take

$$F(y,\lambda) = \beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y,\lambda)}{\partial y^2} - \frac{\partial^4 v(y,\lambda)}{\partial y^4} - M_{11}v(y,\lambda)\right). \tag{75}$$

Then, we have

$$v(y,t) = \frac{1}{\mathrm{Re}\,\Gamma(\alpha_1)}\int_0^t F(y,\lambda)(t-\lambda)^{\alpha_1-1}d\lambda. \tag{76}$$

At $(y_i, t = t_{n+1})$, we get

$$\begin{aligned} v_i^{n+1} &= \frac{1}{\Gamma(\alpha_1)}\int_0^{t_{n+1}} F(y_i,\lambda)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda \\ &= \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left(P_j(\lambda) + \frac{(\lambda-t_j)(\lambda-t_{j-1})}{2!}\frac{\partial^2}{\partial\lambda^2}F(y_i,\lambda)|_{\lambda=\varepsilon_\lambda}\right)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda \\ &= \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} P_j(\lambda)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda + \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \\ &\quad\cdot\left(\frac{(\lambda-t_j)(\lambda-t_{j-1})}{2!}\frac{\partial^2}{\partial\lambda^2}F(y_i,\lambda)|_{\lambda=\varepsilon_\lambda}\right)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda \\ &= \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} P_j(\lambda)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda + R_{n,j}^{\alpha_1}, \end{aligned}$$

$$\begin{aligned} R_{n,j}^{\alpha_1} &= \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left(\frac{(\lambda-t_j)(\lambda-t_{j-1})}{2!}\frac{\partial^2}{\partial\lambda^2}\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y,\lambda)}{\partial y^2} - \frac{\partial^4 v(y,\lambda)}{\partial y^4} - M_{11}v(y,\lambda)\right)\right]\right) \\ &\quad\cdot(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda \le \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} \\ &\quad\cdot\left(\frac{|\lambda-t_j||\lambda-t_{j-1}|}{2!}\left|\frac{\partial^2}{\partial\lambda^2}\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y,\lambda)}{\partial y^2} - \frac{\partial^4 v(y,\lambda)}{\partial y^4} - M_{11}v(y,\lambda)\right)\right]\right|\right)(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda. \end{aligned} \tag{77}$$

Then, we have

$$\begin{aligned} &\frac{\partial^2}{\partial\lambda^2}\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y_i,\lambda)}{\partial y^2} - \frac{\partial^4 v(y_i,\lambda)}{\partial y^4} - M_{11}v(y_i,\lambda)\right)\right] \\ &\quad = \beta_1\left[(\beta_1-1)(\beta_1-2)\lambda^{\beta_1-3}\left(\frac{\partial^2 v(y_i,\lambda)}{\partial y^2} - \frac{\partial^4 v(y_i,\lambda)}{\partial y^4} - M_{11}v(y_i,\lambda)\right)\right. \\ &\quad + 2(\beta_1-1)\lambda^{\beta_1-2}\times\left(\frac{\partial^3 v(y_i,\lambda)}{\partial\lambda\partial y^2} - \frac{\partial^5 v(y_i,\lambda)}{\partial\lambda\partial y^4} - M_{11}\frac{\partial v(y_i,\lambda)}{\partial\lambda}\right) \\ &\quad \left.+ \lambda^{\beta_1-1}\left(\frac{\partial^4 v(y_i,\lambda)}{\partial\lambda^2\partial y^2} - \frac{\partial^6 v(y_i,\lambda)}{\partial\lambda^2\partial y^4} - M_{11}\frac{\partial^2 v(y_i,\lambda)}{\partial\lambda^2}\right)\right], \end{aligned}$$

$$\begin{aligned} \left|R_{n,j}^{\alpha_1}\right| &< \frac{1}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sup_{\tau\in[\tau_j,\tau_{j+1}]}\left|\lambda-t_j\right|\sup_{\tau\in[\tau_{j-1},\tau_{j+1}]}\left|\lambda-t_{j-1}\right| \\ &\quad\times\sup_{\tau\in[\tau_j,\tau_{j+1}]}\left|\frac{\partial^2}{\partial\lambda^2}\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y_i,\lambda)}{\partial y^2} - \frac{\partial^4 v(y_i,\lambda)}{\partial y^4} - M_{11}v(y_i,\lambda)\right)\right]\right| \\ &\quad\cdot(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda < \frac{C_1C_2}{\Gamma(\alpha_1)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\sup_{\tau\in[\tau_j,\tau_{j+1}]}\left|\frac{\partial^2}{\partial\lambda^2}\right. \\ &\quad\cdot\left.\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y_i,\lambda)}{\partial y^2} - \frac{\partial^4 v(y_i,\lambda)}{\partial y^4} - M_{11}v(y_i,\lambda)\right)\right]\right|(t_{n+1}-\lambda)^{\alpha_1-1}d\lambda. \end{aligned} \tag{78}$$

We have

$$\begin{aligned} &\sup_{\tau\in[\tau_j,\tau_{j+1}]}\left|\frac{\partial^2}{\partial\lambda^2}\left[\beta_1\lambda^{\beta_1-1}\left(\frac{\partial^2 v(y_i,\lambda)}{\partial y^2} - \frac{\partial^4 v(y_i,\lambda)}{\partial y^4} - M_{11}v(y_i,\lambda)\right)\right]\right| \\ &\quad < \beta_1\left[|(\beta_1-1)(\beta_1-2)|\lambda^{\beta_1-3}\left(\left\|\frac{\partial^2 v}{\partial y^2}\right\|_\infty + \left\|\frac{\partial^4 v}{\partial y^4}\right\|_\infty + |M_{11}|\|v\|_\infty\right)\right. \\ &\quad + 2|(\beta_1-1)|\lambda^{\beta_1-2}\times\left(\left\|\frac{\partial^3 v}{\partial\lambda\partial y^2}\right\|_\infty + \left\|\frac{\partial^5 v}{\partial\lambda\partial y^4}\right\|_\infty - M_{11}\left\|\frac{\partial v}{\partial\lambda}\right\|_\infty\right) \\ &\quad \left.+ \lambda^{\beta_1-1}\left(\left\|\frac{\partial^4 v}{\partial\lambda^2\partial y^2}\right\|_\infty + \left\|\frac{\partial^6 v}{\partial\lambda^2\partial y^4}\right\|_\infty - M_{11}\left\|\frac{\partial^2 v}{\partial\lambda^2}\right\|_\infty\right)\right]. \end{aligned} \tag{79}$$

Therefore, we acquire

$$\left|R_{n,j}^{\alpha_1}\right| < \frac{C_1 C_2}{\Gamma(\alpha_1)} \beta_1 \left[|(\beta_1 - 1)(\beta_1 - 2)| A_1 \left(\left\| \frac{\partial^2 v}{\partial y^2} \right\|_\infty + \left\| \frac{\partial^4 v}{\partial y^4} \right\|_\infty + |M_{11}| \|v\|_\infty \right) + 2|(\beta_1 - 1)| A_2 \left(\left\| \frac{\partial^3 v}{\partial \lambda \partial y^2} \right\|_\infty + \left\| \frac{\partial^5 v}{\partial \lambda \partial y^4} \right\|_\infty + M_{11} \left\| \frac{\partial v}{\partial \lambda} \right\|_\infty \right) + A_3 \left(\left\| \frac{\partial^4 v}{\partial \lambda^2 \partial y^2} \right\|_\infty + \left\| \frac{\partial^6 v}{\partial \lambda^2 \partial y^4} \right\|_\infty + M_{11} \left\| \frac{\partial^2 v}{\partial \lambda^2} \right\|_\infty \right) \right] \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \lambda)^{\alpha_1 - 1} d\lambda, \quad (80)$$

where

$$\frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1} - \lambda)^{\alpha_1 - 1} d\lambda = \frac{(\Delta t)^{\alpha_1} (n+1)^{\alpha_1}}{\Gamma(1 + \alpha_1)}. \quad (81)$$

Therefore, we obtain

$$\left|R_{n,j}^{\alpha_1}\right| < \frac{C_1 C_2}{\Gamma(\alpha_1)} \beta_1 \left[|(\beta_1 - 1)(\beta_1 - 2)| A_1 \left(\left\| \frac{\partial^2 v}{\partial y^2} \right\|_\infty + \left\| \frac{\partial^4 v}{\partial y^4} \right\|_\infty + |M_{11}| \|v\|_\infty \right) + 2|(\beta_1 - 1)| A_2 \left(\left\| \frac{\partial^3 v}{\partial \lambda \partial y^2} \right\|_\infty + \left\| \frac{\partial^5 v}{\partial \lambda \partial y^4} \right\|_\infty + M_{11} \left\| \frac{\partial v}{\partial \lambda} \right\|_\infty \right) + A_3 \left(\left\| \frac{\partial^4 v}{\partial \lambda^2 \partial y^2} \right\|_\infty + \left\| \frac{\partial^6 v}{\partial \lambda^2 \partial y^4} \right\|_\infty + M_{11} \left\| \frac{\partial^2 v}{\partial \lambda^2} \right\|_\infty \right) \right] \sum_{j=0}^{n} \frac{(\Delta t)^{\alpha_1} (n+1)^{\alpha_1}}{\Gamma(1 + \alpha_1)}. \quad (82)$$

Remark 4. Error analysis with exponential decay kernel and Mittag-Leffler kernel can be obtained likewise. Therefore, we misplaced the error analysis for them.

8. Conclusion

In this paper, we investigated the fractional MHD incompressible couple stress fluid flow between two parallel plates. We discussed the discretization and the stability analysis for three different kernels. Additionally, we discussed the error analysis of the model in details.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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