

Research Article

Final Value Problem for Parabolic Equation with Fractional Laplacian and Kirchhoff's Term

Nguyen Hoang Luc ¹, Devendra Kumar ², Le Dinh Long ¹ and Ho Thi Kim Van ¹

¹Division of Applied Mathematics, Thu Dau Mot University, Binh Duong Province, Vietnam

²Department of Mathematics, University of Rajasthan, Jaipur, India

Correspondence should be addressed to Ho Thi Kim Van; hothikimvan@tdmu.edu.vn

Received 17 April 2021; Accepted 15 June 2021; Published 3 July 2021

Academic Editor: Santosh Kumar

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In this paper, we study a diffusion equation of the Kirchhoff type with a conformable fractional derivative. The global existence and uniqueness of mild solutions are established. Some regularity results for the mild solution are also derived. The main tools for analysis in this paper are the Banach fixed point theory and Sobolev embeddings. In addition, to investigate the regularity, we also further study the nonwell-posed and give the regularized methods to get the correct approximate solution. With reasonable and appropriate input conditions, we can prove that the error between the regularized solution and the search solution is towards zero when δ tends to zero.

1. Introduction

The aim of this study is to investigate the final value for the space fractional diffusion equation

$$\begin{cases} \frac{{}^{\mathcal{C}}\partial^\alpha}{\partial t^\alpha} v(x, t) + (\|\nabla v\|_{L^2}) (-\Delta)^\beta v(x, t) = F(x, t), & x \in \Omega, t \in (0, T), \\ v(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ v(x, T) = f(x), & x \in \Omega, \end{cases} \quad (1)$$

where the symbol $\frac{{}^{\mathcal{C}}\partial^\alpha}{\partial t^\alpha} v(t)$ is called the conformable derivative which is defined clearly in Section 2. Here, $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with the smooth boundary $\partial\Omega$, and $T > 0$ is a given positive number. The function F represents the external forces or the advection term of a diffusion phenomenon, etc., and the function f is the final datum which will be specified later.

The applications of the conformable derivative are interested in various models such as the harmonic oscillator, the damped oscillator, and the forced oscillator (see, e.g., [1]), electrical circuits (see, e.g., [2]), chaotic systems in dynamics

(see, e.g., [3]), and quantum mechanics (see, e.g., [4]). From the paper, see, e.g., [5], we must confirm that the study of the ODE problem with the conformable derivative is very different from the study of the PDE problem with a conformable derivative. Results and research methods of the well-posedness for the ODE and PDE model are not the same and are completely different. The following two remarks confirm what we have just pointed out.

Remark 1. Let us first discuss conformable ODEs. Let v be the functions whose domain of its value is \mathbb{R} . If $\alpha = 1$, $\frac{{}^{\mathcal{C}}\partial^\alpha}{\partial t^\alpha}$ becomes the classical derivative. If $0 < \alpha < 1$, by the paper of [6], we know that the relation between the conformable derivative and the classical derivative by the following lemma.

Lemma 2. If $v : [0, T] \rightarrow \mathbb{R}$, then a conformable derivative of order α at $s > 0$ of v exists if and only if it is differentiable at s , and the following equality is true:

$$\frac{{}^{\mathcal{C}}\partial^\alpha v(s)}{\partial s^\alpha} = s^{1-\alpha} \frac{\partial v(s)}{\partial s}. \quad (2)$$

Remark 3. In the following, we mention the PDEs with conformable derivative where D is a Sobolev space, such as $L^2(\Omega)$, $W^{r,p}(\Omega)$, and $D(A^r)$. When we study the PDE model, we often do with a multivariable function $v : (0, T) \rightarrow D$, where D is a Sobolev space. This means that, for each t , $v(t)$ can take on values in many classes of spaces with $D_1 \hookrightarrow D \hookrightarrow D_2 \dots$. Some illustrated examples given in [5] say that (2) may be not true on Sobolev spaces.

Let us mention some recent works on diffusion equations with a conformable derivative, for example, [2, 5, 7–16]. Some interesting papers on fractional diffusion equations can be found in [17–24] and the references therein.

When $\alpha = 1$, the main equation of Problem (1) appears in many population dynamics. By the work of Chipot and Lovat [25], we know that the diffusion coefficient B is dependent on the entire population in the domain instead of local density; that is, the moves are guided by considering the global state of the vehicle. The function u is a descriptive population density (e.g., bacteria) spread. According to article [26], we find that model (1) is a type of Kirchhoff equation, arising in vibration theory; see, for example, [27].

- (i) This paper is the first study on the final value problem for a diffusion equation with a Kirchhoff-type equation and conformable derivative. Since our models are nonlinear, in order to establish the existence and uniqueness of solutions, we have to use the Banach contracting mapping theorem combined with some techniques to evaluate inequality and some Sobolev embeddings. One of the most difficult points is finding the appropriate functional spaces for the solution
- (ii) The second result is to investigating the regularized solution for our problem. We show the ill-posedness of the problem and give Fourier regularization. The most difficult thing that we have to overcome is finding the appropriate space, to prove that the regularized solution converges with the exact solution

It can be said that our article is one of the first results, giving a general and comprehensive picture, considering both the frequency and the inaccuracy of Kirchhoff's diffusion equation with fractional time and space derivative. Using complex and interoperable assessment techniques, we find the right keys and tools to achieve both of our goals.

This paper is organized as follows. In Section 3, we present the existence of the backward Problem (1) with the simple case $F = 0$. In the appropriate terms of the terminal data f , we show that the mild solution of (1) in the case $\beta < 1$ converges with the mild solution of the same problem in the case $\beta = 1$ when $\beta \rightarrow 1^-$. Finally, in Section 4, we consider a backward problem with an inhomogeneous source term. The first part of this section discusses the existence of a mild solution under the appropriate conditions of the source function F . Furthermore, we also give an example, which shows that the problem is not stable, and then look for the approximate solution. Using the Fourier truncation method, we involve the regularized solution. Convergence error between

the regularized solution and the correct solution has also been established, with some suitable conditions of input value data.

2. Preliminaries

2.1. Conformable Derivative Model. Let the function $v : [0, \infty) \rightarrow D$, where D is a Banach space.

If for each $t > 0$, the limitation

$$\mathcal{C}\partial^\alpha v(t) := \lim_{\varepsilon \rightarrow 0} \frac{v(t + \varepsilon t^{1-\alpha}) - v(t)}{\varepsilon} \text{ in } D, \quad (3)$$

finitely exists, then it is called the conformable derivative of order $\alpha \in (0, 1]$ of v . We can refer the reader to [6, 8, 14, 28, 29].

We introduce fractional powers of \mathcal{A} as follows:

$$D(\mathcal{A}^\nu) = \left\{ g \in L^2(\Omega) : \sum_{j=1}^{\infty} |\langle g, w_j \rangle|^2 \lambda_j^{2\nu} < \infty \right\}. \quad (4)$$

The space $D(\mathcal{A}^\nu)$ is a Banach space in the following with the corresponding norm:

$$\|g\|_{D(\mathcal{A}^\nu)} := \left(\sum_{j=1}^{\infty} |\langle g, w_j \rangle|^2 \lambda_j^{2\nu} \right)^{1/2}, \quad g \in D(\mathcal{A}^\nu). \quad (5)$$

The information for negative fractional power $\mathcal{A}^{-\nu}$ can be provided by [30]. For any $\theta > 0$, we introduce the following Hölder continuous space of exponent θ

$$\begin{aligned} C^\theta([0, T]; \mathcal{B}) \\ = \left\{ v \in C([0, T]; \mathcal{B}) : \sup_{0 \leq s < t \leq T} \frac{\|v(\cdot, t) - v(\cdot, s)\|_{\mathcal{B}}}{|t - s|^\theta} < \infty \right\}, \end{aligned} \quad (6)$$

corresponding to the following norm:

$$\|v\|_{C^\theta([0, T]; \mathcal{B})} = \sup_{0 \leq s < t \leq T} \frac{\|v(\cdot, t) - v(\cdot, s)\|_{\mathcal{B}}}{|t - s|^\theta}. \quad (7)$$

For any $0 < \theta < 1$, let us introduce the following space:

$$\begin{aligned} \mathcal{C}^\theta((0, T]; \mathcal{B}) \\ = \left\{ v \in C((0, T]; L^2(\Omega)) : \sup_{0 < t \leq T} t^\theta \|v(\cdot, t)\|_{\mathcal{B}} < \infty \right\}, \end{aligned} \quad (8)$$

corresponding to the norm $\|v\|_{\mathcal{C}^\theta((0, T]; \mathcal{B})} := \sup_{0 < t \leq T} t^\theta \|v(\cdot, t)\|_{\mathcal{B}}$.

Let us define the space as follows:

$$\mathbb{X}_{\beta,\alpha}(\Omega) = \left\{ g \in L^2(\Omega), \sum_{j=1}^{\infty} \lambda_j^{2+2\beta} \cdot \exp\left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha}\right) \langle g, w_j \rangle^2 < \infty \right\}. \quad (9)$$

3. Backward Problem for Homogeneous Case

In this section, we consider the final value problem for the homogeneous equation with a space fractional derivative as follows:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} v(x, t) + B(\|\nabla v\|_{L^2})(-\Delta)^\beta v(x, t) = 0, & x \in \Omega, t \in (0, T), \\ v(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ v(x, T) = f(x), & x \in \Omega, \end{cases} \quad (10)$$

where $0 < M_0 \leq B(\xi) \leq M_1$ and $\xi \in [0, T]$. The following theorem states the existence and uniqueness of the solution of Problem (10).

Theorem 4. *Let $f \in \mathbb{X}_{\beta,\alpha}(\Omega)$. Then, Problem (10) has a unique mild solution $v \in C([0, T]; H^1(\Omega))$ which satisfies that*

$$v(x, t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (11)$$

Furthermore, this solution is not stable in the L^2 norm.

Proof. We express a mild solution of (10) by Fourier series as follows:

$$v(x, t) = \sum_{j=1}^{\infty} \langle v(\cdot, t), w_j \rangle w_j(x). \quad (12)$$

It follows from Problem (10) and the equality $\langle (-\Delta)^\beta v(\cdot, t), w_j \rangle = \lambda_j^\beta \langle v(\cdot, t), w_j \rangle$ that

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} \langle v(\cdot, t), w_j \rangle + \lambda_j^\beta B(\|\nabla v\|_{L^2}) \langle v(\cdot, t), w_j \rangle = 0, & t \in (0, T), \\ \langle u(\cdot, 0), w_j \rangle = \langle u_0, w_j \rangle. \end{cases} \quad (13)$$

Note that this formula

$$\frac{\partial^\alpha}{\partial t^\alpha} \langle u(\cdot, t), w_j \rangle = t^{1-\alpha} \frac{\partial}{\partial t} \langle u(\cdot, t), w_j \rangle, \quad (14)$$

is correct; we get that

$$\frac{\partial}{\partial t} \langle v(\cdot, t), w_j \rangle + \lambda_j^\beta t^{\alpha-1} B(\|\nabla v\|_{L^2}) \langle v(\cdot, t), w_j \rangle = 0. \quad (15)$$

Multiply both sides of equation (15) by the quantity $\exp\left(\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right)$, we reach the following assertion:

$$\frac{\partial}{\partial t} \left(\langle v(\cdot, t), w_j \rangle \exp\left(\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \right) = 0, \quad (16)$$

where we have used the fact that

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\exp\left(\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \right) \\ &= \exp\left(\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) t^{\alpha-1} \lambda_j^\beta B(\|\nabla v(\cdot, t)\|_{L^2}). \end{aligned} \quad (17)$$

Integrating the two sides of the latter equation 0 to t , we obtain the following confirmation:

$$\langle v(\cdot, t), w_j \rangle \exp\left(\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) = \langle v(\cdot, 0), w_j \rangle. \quad (18)$$

It yields that

$$\langle v(\cdot, t), w_j \rangle = \exp\left(-\int_0^{t^\alpha/\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \langle v(\cdot, 0), w_j \rangle. \quad (19)$$

Therefore, we find that

$$v(x, t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (20)$$

For $v \in L^\infty(0, T; H^1(\Omega))$, we consider the following function:

$$\mathcal{Q}(v)(x, t) = \sum_{j=1}^{\infty} \exp\left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v(\cdot, s)\|_{L^2}) ds\right) \langle f, w_j \rangle w_j(x). \quad (21)$$

We shall prove by induction if $w_1, w_2 \in L^\infty(0, T; H^1(\Omega))$, then

$$\begin{aligned} \left\| \mathcal{Q}^k(w_1) - \mathcal{Q}^k(w_2) \right\|_{L^2(\Omega)} &\leq \left(\frac{\left(\|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2((T^\alpha - t^\alpha)/\alpha) \right)^{1/2}}{k!} \right) \\ &\cdot \|w_1 - w_2\|_{L^\infty(0,T;L^2(\Omega))} \quad \forall q \leq 1. \end{aligned} \quad (22)$$

For $m = 1$, using the inequality $|e^a - e^b| \leq |a - b| \max(e^a, e^b)$ for any $a, b \in \mathbb{R}$, we have

$$\begin{aligned} &\left\| \mathcal{Q}(w_1) - \mathcal{Q}(w_2) \right\|_{H^1(\Omega)}^2 \\ &= \sum_{j=1}^{\infty} \lambda_j^{2+2\beta} \left[\exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_1(\cdot, s)\|_{L^2}) ds \right) \right. \\ &\quad \left. - \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_2(\cdot, s)\|_{L^2}) ds \right) \right]^2 \langle f, w_j \rangle^2 \\ &\leq \sum_{j=1}^{\infty} \lambda_j^{2-2\beta} \left| \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_1(\cdot, s)\|_{L^2}) ds \right. \\ &\quad \left. - \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_2(\cdot, s)\|_{L^2}) ds \right|^2 \\ &\quad \cdot \exp \left(\frac{2T^\alpha - 2t^\alpha}{\alpha} M_1 \lambda_j \right) \langle f, w_j \rangle^2 \\ &\leq K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \|\nabla(w_1 - w_2)(\cdot, s)\|_{L^2(\Omega)}^2 ds \\ &\leq K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \frac{T^\alpha - t^\alpha}{\alpha} \|w_1 - w_2\|_{L^\infty(0,T;H^1(\Omega))}^2. \end{aligned} \quad (23)$$

Assume that (22) holds for $m = k$. We show that (22) holds for $m = k + 1$. Indeed, we have

$$\begin{aligned} &\left\| \mathcal{Q}^{k+1}(w_1) - \mathcal{Q}^{k+1}(w_2) \right\|_{H^1(\Omega)}^2 \\ &= \sum_{j=1}^{\infty} \lambda_j^2 \left[\exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla \mathcal{Q}^k(w_1)(\cdot, s)\|_{L^2}) ds \right) \right. \\ &\quad \left. - \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla \mathcal{Q}^k(w_2)(\cdot, s)\|_{L^2}) ds \right) \right]^2 \langle f, w_j \rangle^2 \\ &\leq \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2 \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \left\| \mathcal{Q}^k(w_1)(\cdot, s) - \mathcal{Q}^k(w_2)(\cdot, s) \right\|_{H^1(\Omega)}^2 ds \\ &\leq \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2 \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \frac{\left(\|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2((T^\alpha - t^\alpha)/\alpha) \right)^k}{k!} ds \\ &\leq \frac{\left(\|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2((T^\alpha - t^\alpha)/\alpha) \right)^{k+1}}{(k+1)!} \|w_1 - w_2\|_{L^\infty(0,T;H^1(\Omega))}^2. \end{aligned} \quad (24)$$

By the theory of the induction principle, (22) holds for all $w_1, w_2 \in L^\infty(0, T; H^1(\Omega))$. Since the fact that

$$\lim_{k \rightarrow +\infty} \left(\frac{\left(\|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 K_b^2((T^\alpha - t^\alpha)/\alpha) \right)^k}{k!} \right)^{1/2} = 0, \quad (25)$$

there exists a positive integer number k_0 such that \mathcal{Q}^{k_0} is a contraction. It follows that the equation $\mathcal{Q}^{k_0} v = v$ has a unique solution $v \in L^\infty(0, T; H^1(\Omega))$. It is easy to see that v is also a fixed point of \mathcal{Q} . \square

Theorem 5. Assume that $f \in X\beta_{+\gamma,\alpha}(\Omega)$ for any $\gamma > \beta$. Let us choose v_0 such that

$$2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 v_0^{-2\delta}}{\alpha^{1-\delta} (1-\delta)} < 1. \quad (26)$$

Let any $0 < \varepsilon < \gamma - \beta$. Then, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} &\left\| v_{\alpha,\beta} - w_\alpha \right\|_{L_{v_0}^\infty(0,T;H^1(\Omega))}^2 \\ &\leq \frac{C_\varepsilon (1-\beta)^\varepsilon \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}}{\sqrt{1 - 2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)} \left(T^{\alpha(1-\delta)} |C_\delta|^2 v_0^{-2\delta} / \alpha^{1-\delta} (1-\delta) \right)}}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} L_{v_0}^\infty(0, T; H^1(\Omega)) &= \left\{ v \in L^\infty(0, T; H^1(\Omega)), \|v\|_{L_{v_0}^\infty(0,T;H^1(\Omega))} \right. \\ &\quad \left. = \sup_{0 \leq t \leq T} e^{v_0(t-T)} \|v(t)\|_{H^1(\Omega)} < \infty \right\}. \end{aligned} \quad (28)$$

Proof. Let $v_{\alpha,\beta}$ be the solution of Problem (11). Let w_α be the solution to Problem (11) with $\beta = 1$. Then, we get

$$\begin{aligned} v_{\alpha,\beta}(x, t) &= \sum_{j=1}^{\infty} \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v_{\alpha,\beta}(\cdot, s)\|_{L^2}) ds \right) \\ &\quad \cdot \langle f, w_j \rangle w_j(x), \\ w_\alpha(x, t) &= \sum_{j=1}^{\infty} \exp \left(\lambda_j \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \\ &\quad \cdot \langle f, w_j \rangle w_j(x). \end{aligned} \quad (29)$$

We have

$$\begin{aligned}
& v_{\alpha,\beta}(x, t) - w_\alpha(x, t) \\
&= \sum_{j=1}^{\infty} \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v_{\alpha,\beta}(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x) \\
&\quad - \sum_{j=1}^{\infty} \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x) \\
&\quad + \sum_{j=1}^{\infty} \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x) \\
&\quad - \sum_{j=1}^{\infty} \exp \left(\lambda_j \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x) \\
&= \mathcal{D}_1 + \mathcal{D}_2,
\end{aligned} \tag{30}$$

whereby

$$\begin{aligned}
\mathcal{D}_1 &= \sum_{j=1}^{\infty} \exp \left(\lambda_j^\beta \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x), \\
\mathcal{D}_2 &= \sum_{j=1}^{\infty} \exp \left(\lambda_j \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right) \langle f, w_j \rangle w_j(x).
\end{aligned} \tag{31}$$

The term \mathcal{D}_1 is bounded by

$$\begin{aligned}
\|\mathcal{D}_1\|_{H^1(\Omega)}^2 &\leq \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \\
&\quad \cdot \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla v_{\alpha,\beta}(\cdot, s)\|_{L^2}) ds \right. \\
&\quad \left. - \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right)^2 \langle f, w_j \rangle^2 \\
&\leq K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \|\nabla(v_{\alpha,\beta} - w_\alpha)(\cdot, s)\|_{L^2(\Omega)}^2 ds.
\end{aligned} \tag{32}$$

The term \mathcal{D}_2 is estimated as follows:

$$\begin{aligned}
\|\mathcal{D}_2\|_{H^1(\Omega)}^2 &\leq \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) (\lambda_j^\beta - \lambda_j)^2 \\
&\quad \cdot \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} B(\|\nabla w_\alpha(\cdot, s)\|_{L^2}) ds \right)^2 \langle f, w_j \rangle^2 \\
&\leq M_1^2 \left(\frac{T^\alpha - t^\alpha}{\alpha} \right)^2 \sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \\
&\quad \cdot (\lambda_j^\beta - \lambda_j)^2 \langle f, w_j \rangle^2.
\end{aligned} \tag{33}$$

Consider the following subset:

$$A_1 = \{j \in \mathbb{N}, \lambda_j \leq 1\}, \quad A_2 = \{j \in \mathbb{N}, \lambda_j > 1\}. \tag{34}$$

If $j \in A_1$, then using the inequality $1 - e^{-z} \leq C_\varepsilon z^\varepsilon$, we get

$$\lambda_j^\beta - \lambda_j = \lambda_j^\beta (1 - \lambda_j^{1-\beta}) = \lambda_j^\beta C_\varepsilon (1 - \beta)^\varepsilon \lambda_j^{-\varepsilon} = C_\varepsilon (1 - \beta)^\varepsilon \lambda_j^{\beta-\varepsilon}, \tag{35}$$

which allows us to obtain

$$\begin{aligned}
&\sum_{j \in A_1} \lambda_j^{2\beta} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) (\lambda_j^\beta - \lambda_j)^2 \langle f, w_j \rangle^2 \\
&\leq |C_\varepsilon|^2 (1 - \beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2-2\varepsilon} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle f, w_j \rangle^2.
\end{aligned} \tag{36}$$

If $j \in A_2$, then using the inequality $1 - e^{-z} \leq C_\varepsilon z^\varepsilon$, we find

$$|\lambda_j^\beta - \lambda_j| = \lambda_j (1 - \lambda_j^{\beta-1}) = \lambda_j C_\varepsilon (1 - \beta)^\varepsilon \lambda_j^\varepsilon = C_\varepsilon (1 - \beta)^\varepsilon \lambda_j^{\beta+\varepsilon}. \tag{37}$$

Hence, we obtain

$$\begin{aligned}
&\sum_{j \in A_2} \lambda_j^{2\beta+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) (\lambda_j^\beta - \lambda_j)^2 \langle f, w_j \rangle^2 \\
&\leq |C_\varepsilon|^2 (1 - \beta)^{2\varepsilon} \sum_{j \in A_2} \lambda_j^{4\beta+2+2\varepsilon} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle f, w_j \rangle^2.
\end{aligned} \tag{38}$$

Combining (36) and (38), we find that

$$\begin{aligned}
&\lambda_j^{2\beta} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) (\lambda_j^\beta - \lambda_j)^2 \langle f, w_j \rangle^2 \\
&\leq C |C_\varepsilon|^2 (1 - \beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2\varepsilon+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle f, w_j \rangle^2.
\end{aligned} \tag{39}$$

Let us choose $0 < \varepsilon < \gamma - \beta$. Then, we follow from (33) and the latter equality that

$$\begin{aligned}
\|\mathcal{D}_2\|^2 &\leq CM_1^2 \left(\frac{T^\alpha - t^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{4\beta+2\varepsilon+2} \\
&\quad \cdot \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle f, w_j \rangle^2 \\
&\leq CM_1^2 \left(\frac{T^\alpha - t^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \sum_{j \in A_1} \lambda_j^{2\beta+2\gamma_2} \\
&\quad \cdot \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle f, w_j \rangle^2 \\
&= CM_1^2 \left(\frac{T^\alpha - t^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}^2.
\end{aligned} \tag{40}$$

This above inequality together with (32) and (3) yields that

$$\begin{aligned}
e^{2\nu(t-T)} \|v_{\alpha,\beta}(\cdot, t) - w_\alpha(\cdot, t)\|_{H^1(\Omega)}^2 &\leq 2e^{2\nu(t-T)} \|\mathcal{D}_1\|_{H^1(\Omega)}^2 + 2e^{2\nu(t-T)} \|\mathcal{D}_2\|_{H^1(\Omega)}^2 \\
&\leq 2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 e^{2\nu(t-T)} \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \|v_{\alpha,\beta}(\cdot, s) - w_\alpha(\cdot, s)\|_{H^1(\Omega)}^2 ds \\
&\quad + 2CM_1^2 e^{2\nu(t-T)} \left(\frac{T^\alpha - t^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}^2.
\end{aligned} \tag{41}$$

It is easy to get that

$$\begin{aligned}
e^{2\nu(t-T)} \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} \|v_{\alpha,\beta}(\cdot, s) - w_\alpha(\cdot, s)\|_{H^1(\Omega)}^2 ds &= \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)} e^{2\nu(s-T)} \|v_{\alpha,\beta}(\cdot, s) - w_\alpha(\cdot, s)\|_{H^1(\Omega)}^2 ds \\
&\leq \left(\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)} ds \right) \|v_{\alpha,\beta} - w_\alpha\|_{L_v^\infty(0,T;H^1(\Omega))}^2.
\end{aligned} \tag{42}$$

It follows from the inequality

$$e^{-z} \leq C_\delta z^{-\delta}, \quad 0 < \delta < 1, \tag{43}$$

we get

$$\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)} ds \leq |C_\delta|^2 \nu^{-2\delta} \int_{t^\alpha/\alpha}^{T^\alpha/\alpha} (s-t)^{-\delta} ds. \tag{44}$$

The inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$ leads to

$$\begin{aligned}
\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} (s-t)^{-\delta} ds &= \frac{(T^\alpha/\alpha - t)^{1-\delta} - (t^\alpha/\alpha - t)^{1-\delta}}{1-\delta} \\
&\leq \frac{(T^\alpha/\alpha - t^\alpha/\alpha)^{1-\delta}}{1-\delta} \leq \frac{T^{\alpha(1-\delta)}}{\alpha^{1-\delta}(1-\delta)},
\end{aligned} \tag{45}$$

which allows us to get immediately that

$$\int_{t^\alpha/\alpha}^{T^\alpha/\alpha} e^{2\nu(t-s)} ds \leq \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}. \tag{46}$$

It follows from (41) and (42) that for any $t \in [0, T]$

$$\begin{aligned}
e^{2\nu(t-T)} \|v_{\alpha,\beta}(\cdot, t) - w_\alpha(\cdot, t)\|_{H^1(\Omega)}^2 &\leq 2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \|v_{\alpha,\beta} - w_\alpha\|_{L_v^\infty(0,T;H^1(\Omega))}^2 \\
&\quad + 2CM_1^2 \left(\frac{T^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}^2.
\end{aligned} \tag{47}$$

Since the right-hand side of (47) is independent of t , we deduce that

$$\begin{aligned}
\|v_{\alpha,\beta} - w_\alpha\|_{L_v^\infty(0,T;H^1(\Omega))}^2 &\leq 2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \nu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \|v_{\alpha,\beta} - w_\alpha\|_{L_v^\infty(0,T;H^1(\Omega))}^2 \\
&\quad + 2CM_1^2 \left(\frac{T^\alpha}{\alpha} \right)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}^2.
\end{aligned} \tag{48}$$

Then, we find that

$$\begin{aligned}
\|v_{\alpha,\beta} - w_\alpha\|_{L_v^\infty(0,T;H^1(\Omega))}^2 &\leq \frac{2CM_1^2 (T^\alpha/\alpha)^2 |C_\varepsilon|^2 (1-\beta)^{2\varepsilon} \|f\|_{\mathbb{X}_{\beta+\gamma,\alpha}(\Omega)}^2}{1 - 2K_b^2 \|f\|_{\mathbb{X}_{\beta,\alpha}(\Omega)}^2 \left(T^{\alpha(1-\delta)} |C_\delta|^2 \nu_0^{-2\delta} / \alpha^{1-\delta} (1-\delta) \right)}.
\end{aligned} \tag{49}$$

□

4. Backward Problem for Inhomogeneous Case

In this section, we consider the final value problem for homogeneous equation as follows:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} u(x, t) + B(\|\nabla u\|_{L^2}) (-\Delta)^\beta u(x, t) = F(x, t), & x \in \Omega, t \in (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, t \in (0, T), \\ u(x, T) = 0, & x \in \Omega, \end{cases} \tag{50}$$

where F is defined later.

4.1. Existence and Uniqueness of the Mild Solution. In this subsection, we state the existence and uniqueness of the mild solution. In order to give the main results, we require the condition F which belongs to the space $L^{2p}(0, T; X\beta, \alpha(\Omega))$.

Theorem 6. Let $0 < \beta \leq 1$ and F be the source function that belongs to $L^{2p}(0, T; X\beta, \alpha(\Omega))$ for any $1 < p < 1/(1 - \alpha)$. Let B be the functions which satisfy $M_0 \leq B(z) \leq M_1$, $z \in [0, T]$ and

$$|B(z_1) - B(z_2)| \leq K_b |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}. \quad (51)$$

Then, Problem (50) has a unique mild solution $u \in L_{\mu_0}^\infty(0, T; H^1(\Omega))$, where μ_0 is small enough. The function u satisfies that

$$\begin{aligned} u(x, t) = & - \sum_{j=1}^{\infty} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla u(\cdot, r)\|_{L^2}) dr \right) \right. \\ & \left. \cdot \langle F(\cdot, s), w_j \rangle ds \right) w_j. \end{aligned} \quad (52)$$

Furthermore, this solution is not stable in the L^2 norm.

Proof. By a simple calculation, we get the following equality:

$$\begin{aligned} \langle v(\cdot, t), w_j \rangle = & \exp \left(- \int_0^{t^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds \right) \langle u(\cdot, 0), w_j \rangle \\ & + \int_0^t s^{\alpha-1} \exp \left(- \int_{s^{\alpha/\alpha}}^{t^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \\ & \cdot \langle F(\cdot, s), w_j \rangle ds. \end{aligned} \quad (53)$$

By letting $t = T$ and noting that $v(x, T) = 0$, we find that

$$\begin{aligned} & \exp \left(- \int_0^{T^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, s)\|_{L^2}) ds \right) \langle u(\cdot, 0), w_j \rangle \\ & + \int_0^T s^{\alpha-1} \exp \left(- \int_{s^{\alpha/\alpha}}^{T^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \\ & \cdot \langle F(\cdot, s), w_j \rangle ds = 0. \end{aligned} \quad (54)$$

Therefore, we obtain

$$\begin{aligned} \langle u(\cdot, 0), w_j \rangle = & - \int_0^T s^{\alpha-1} \exp \left(\int_0^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \\ & \cdot \langle F(\cdot, s), w_j \rangle ds. \end{aligned} \quad (55)$$

Combining (53) and (55), we deduce that

$$\begin{aligned} \langle v(\cdot, t), w_j \rangle = & \int_0^t s^{\alpha-1} \exp \left(- \int_{s^{\alpha/\alpha}}^{t^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \\ & \cdot \langle F(\cdot, s), w_j \rangle ds - \int_0^T s^{\alpha-1} \exp \\ & \cdot \left(- \int_{s^{\alpha/\alpha}}^{t^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \langle F(\cdot, s), w_j \rangle ds \\ = & - \int_t^T s^{\alpha-1} \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \\ & \cdot \langle F(\cdot, s), w_j \rangle ds. \end{aligned} \quad (56)$$

Let us denote by $L_\mu^\infty(0, T; \mathbb{V})$ the functional subspace of $L^\infty(0, T; \mathbb{V})$ corresponding to the norm

$$\begin{aligned} \|g\|_{L_\mu^\infty(0, T; \mathbb{V})} = & \max_{0 \leq t \leq T} \|\exp(\mu(t-T))g(\cdot, t)\|_{\mathbb{V}}, \\ & \forall g \in L_\mu^\infty(0, T; \mathbb{V}), \end{aligned} \quad (57)$$

where

$$\mathbb{V} = \left\{ \mu \in \mathbb{R}, g \in L^2(\Omega), \sum_{j=1}^{\infty} \exp(\mu(t-T)) \langle g, w_j \rangle^2 < \infty \right\}. \quad (58)$$

Set the following function:

$$\begin{aligned} \mathcal{P}v(t) = & - \sum_{j=1}^{\infty} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \right. \\ & \left. \cdot \langle F(\cdot, s), w_j \rangle ds \right) w_j, \end{aligned} \quad (59)$$

and we let

$$\mathcal{M}(s, t, j, w) = \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla w(\cdot, r)\|_{L^2}) dr \right). \quad (60)$$

So, using Parseval's equality, we get that

$$\begin{aligned} & \|\mathcal{P}w_1 - \mathcal{P}w_2\|_{H^1(\Omega)}^2 \\ = & \sum_{j=1}^{\infty} \lambda_j \left(\int_t^T s^{\alpha-1} (\mathcal{M}(s, t, j, w_1) \right. \\ & \left. - \mathcal{M}(s, t, j, w_2)) \langle F(\cdot, s), w_j \rangle ds \right)^2 \\ \leq & \sum_{j=1}^{\infty} \lambda_j \left(\int_t^T s^{\alpha-1} ds \right) \left(\int_0^t s^{\alpha-1} (\mathcal{M}(s, t, j, w_1) \right. \\ & \left. - \mathcal{M}(s, t, j, w_2))^2 \langle F(\cdot, s), w_j \rangle^2 ds \right). \end{aligned} \quad (61)$$

Using the inequality $|e^a - e^b| \leq |a - b| \max(e^a, e^b)$, we continue to treat the term $\mathcal{M}(s, t, j, w_1) - \mathcal{M}(s, t, j, w_2)$ as follows:

$$\begin{aligned} & |\mathcal{M}(s, t, j, w_1) - \mathcal{M}(s, t, j, w_2)| \\ &= \left| \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla w_1(\cdot, r)\|_{L^2}) dr \right) \right. \\ &\quad \left. - \exp \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \lambda_j^\beta B(\|\nabla w_2(\cdot, r)\|_{L^2}) dr \right) \right| \\ &\leq \exp \left(\frac{T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) K_b \lambda_j^\beta \int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \|u(\cdot, r) - v(\cdot, r)\|_{H^1} dr. \end{aligned} \quad (62)$$

Therefore, applying the Hölder inequality, we get

$$\begin{aligned} & |\mathcal{M}(s, t, j, v_1) - \mathcal{M}(s, t, j, v_2)|^2 \\ &\leq \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) |K_b|^2 \lambda_j^{2\beta} \frac{T^{2\alpha}}{\alpha^2} \\ &\quad \cdot \int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \|v_1(\cdot, r) - v_2(\cdot, r)\|_{H^1(\Omega)}^2 ds. \end{aligned} \quad (63)$$

Inserting (61) and (63) yields the following inequality:

$$\begin{aligned} & \exp(2\mu(t - T)) \|\mathcal{P}v_1 - \mathcal{P}v_2\|_{H^1(\Omega)}^2 \\ &\leq |K_b|^2 \frac{T^{3\alpha}}{\alpha^3} \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} e^{2\mu(t-s)} e^{2\mu(s-T)} \|v_1(\cdot, s) - v_2(\cdot, s)\|_{L^2} ds \right) \\ &\quad \times \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} s^{\alpha-1} \left(\sum_{j=1}^{\infty} \lambda_j^{2\beta+2} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \langle F(\cdot, s), w_j \rangle^2 \right) ds \right) \\ &\leq \frac{K_b^2 T^{3\alpha}}{3\alpha} \left(\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \exp(2\mu(t-s)) ds \right) \\ &\quad \times \left(\max_{0 \leq s \leq T} \exp(2\mu(s-T)) \|v_1(\cdot, s) - v_2(\cdot, s)\|_{H^1(\Omega)}^2 \right) \\ &\quad \cdot \left(\int_0^t s^{\alpha-1} \|F(\cdot, s)\|_{\mathbb{X}_{\beta, \alpha}(\Omega)}^2 ds \right). \end{aligned} \quad (64)$$

Take any $\delta \in (0, 1)$. By a similar explanation as (46), we find that

$$\int_{t^{\alpha/\alpha}}^{s^{\alpha/\alpha}} \exp(2\mu(t-s)) ds \leq \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)}. \quad (65)$$

By applying the Hölder inequality, we also obtain that

$$\begin{aligned} & \int_0^t s^{\alpha-1} \|F(\cdot, s)\|_{\mathbb{X}_{\beta, \alpha}(\Omega)}^2 ds \\ &\leq \left(\int_0^t s^{(\alpha-1)p^*} ds \right)^{1/p^*} \left(\int_0^t \|F(\cdot, s)\|_{\mathbb{X}_{\beta, \alpha}(\Omega)}^{2p} ds \right)^{1/p} \\ &\leq \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))}^2. \end{aligned} \quad (66)$$

From some observations as above, we deduce that

$$\begin{aligned} & \exp(2\mu(t - T)) \|\mathcal{P}v_1 - \mathcal{P}v_2\|_{H^1(\Omega)}^2 \\ &\leq \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \\ &\quad \cdot \|F\|_{L^{2p}(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))}^2 \|v_1 - v_2\|_{L_\mu^\infty(0, T; H^1(\Omega))}^2. \end{aligned} \quad (67)$$

Since the right-hand side of the latter estimate is independent of t , we find that

$$\begin{aligned} & \|\mathcal{P}v_1 - \mathcal{P}v_2\|_{L_\mu^\infty(0, T; H^1(\Omega))}^2 \\ &\leq \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \\ &\quad \cdot \|F\|_{L^{2p}(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))}^2 \|v_1 - v_2\|_{L_\mu^\infty(0, T; H^1(\Omega))}^2. \end{aligned} \quad (68)$$

Let us choose μ_0 such that

$$\mu_0^{2\delta} > \frac{K_b^2 T^{3\alpha}}{3\alpha} \frac{T^{\alpha(1-\delta)} |C_\delta|^2}{\alpha^{1-\delta}(1-\delta)} \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))}^2. \quad (69)$$

Then, we can conclude that \mathcal{P} is a contraction mapping in the space $L_{\mu_0}^\infty(0, T; H^1(\Omega))$. Next, we continue to show that if $v \in L_{\mu_0}^\infty(0, T; H^1(\Omega))$, then $\mathcal{P}v \in L_{\mu_0}^\infty(0, T; H^1(\Omega))$. If $v_1 = 0$, then

$$\mathcal{P}v_1(t) = \sum_{j=1}^{\infty} \left(\int_t^T s^{\alpha-1} \langle F(\cdot, s), w_j \rangle ds \right) w_j. \quad (70)$$

Hence, from Parseval's equality, we find that

$$\begin{aligned} \|\mathcal{P}v_1\|_{H^1(\Omega)}^2 &= \sum_{j=1}^{\infty} \lambda_j^2 \left(\int_t^T s^{\alpha-1} \langle F(\cdot, s), w_j \rangle ds \right)^2 \\ &\leq \left(\int_t^T s^{\alpha-1} ds \right) \left(\sum_{j=1}^{\infty} \int_t^T s^{\alpha-1} \lambda_j^2 \langle F(\cdot, s), w_j \rangle^2 ds \right) \\ &\leq \frac{T^\alpha}{\alpha} \int_t^T s^{\alpha-1} \|F(\cdot, s)\|_{H^1(\Omega)}^2 ds \\ &\leq \frac{T^\alpha}{\alpha} \left(\int_0^t s^{(\alpha-1)p^*} ds \right)^{1/p^*} \left(\int_0^t \|F(\cdot, s)\|_{H^1(\Omega)}^{2p} ds \right)^{1/p} \\ &\leq \frac{p-1}{\alpha p-1} T^{(\alpha p-1)/(p-1)} \|F\|_{L^{2p}(0, T; H^1(\Omega))}^2. \end{aligned} \quad (71)$$

This says that $\mathcal{P}v_1$ belongs to the space $L_{\mu_0}^\infty(0, T; H^1(\Omega))$. Using (68), we arrive at the confirmation that $\mathcal{P}v$ belongs to $L_{\mu_0}^\infty(0, T; H^1(\Omega))$ if $v \in L_{\mu_0}^\infty(0, T; H^1(\Omega))$. For any $m \in \mathbb{N}$, let u_m be the function that satisfies the following integral equation:

$$u_m(x, t) = - \sum_{j=1}^{\infty} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha}^{s^\alpha} \lambda_j^\beta B(\|\nabla u_m(\cdot, r)\|_{L^2}) dr \right) \cdot \langle F(\cdot, s), w_j \rangle ds \right) w_j(x). \quad (72)$$

Let us assume that

$$F_m(x, t) = \frac{1}{\lambda_m} \sum_{j=1}^{\infty} w_j(x). \quad (73)$$

It is not difficult to verify that $F_m \in L^\infty(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))$, so we get that $F_m \in L^{2p}(0, T; \mathbb{X}_{\beta, \alpha}(\Omega))$.

Using Theorem 6, we conclude that equation (72) has a unique solution $u_m \in L^\infty(0, T; H^1(\Omega))$. By the fact that $B(z) \geq M_0 \forall z \in \mathbb{R}$, we obtain the following estimate:

$$\begin{aligned} \|u_m(\cdot, t)\|_{L^2(\Omega)}^2 &= \frac{1}{\lambda_m^2} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha}^{s^\alpha} \lambda_j^\beta B(\|\nabla u_m(\cdot, r)\|_{L^2}) dr \right) \right)^2 \\ &\geq \frac{\left(\int_t^T s^{\alpha-1} \exp \left(M_0 \lambda_j^\beta ((s^\alpha - t^\alpha)/\alpha) \right) ds \right)^2}{\lambda_m^2}. \end{aligned} \quad (74)$$

The estimate is true for all $t \in [0, T]$, so it is easy to see that

$$\begin{aligned} \|u_m\|_{L^\infty(0, T; L^2(\Omega))} &\geq \frac{\int_0^T s^{\alpha-1} \exp \left(M_0 \lambda_j^\beta (s^\alpha/\alpha) \right) ds}{\lambda_m} \\ &\geq \frac{\exp \left(\lambda_m^\beta M_0 (T^\alpha/\alpha) \right)}{M_0 \lambda_m^{\beta+1}}. \end{aligned} \quad (75)$$

When m tends to $+\infty$, we can check that $\|f_m\|_{L^2(\Omega)} = 1/\lambda_m$ go to zero when $m \rightarrow +\infty$ and

$$\begin{aligned} \lim_{m \rightarrow +\infty} \|u_m\|_{C([0, T]; L^2(\Omega))} &= \lim_{m \rightarrow +\infty} \frac{\exp \left(\lambda_m^\beta M_0 (T^\alpha/\alpha) \right)}{M_0 \lambda_m^{\beta+1}} = +\infty. \end{aligned} \quad (76)$$

This shows that Problem (50) is ill-posed in the sense of Hadamard in the L^2 -norm. \square

4.2. Fourier Truncation Method. In this section, we will provide a regularized solution and solve the problem by the Fourier truncation method as follows:

$$u^{N, \delta}(x, t) = - \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha}^{s^\alpha} \lambda_j^\beta B(\|\nabla u^{N, \delta}(\cdot, r)\|_{L^2}) dr \right) \cdot \langle F^\delta(\cdot, s), w_j \rangle ds \right) w_j(x). \quad (77)$$

Here, $N = N(\delta)$ goes to infinity as δ tends to zero which is called a parameter regularization. The function F is disturbed by the observed data $F^\delta \in L^\infty(0, T; L^2(\Omega))$ provided by

$$\|F^\delta - F\|_{L^\infty(0, T; L^2(\Omega))} \leq \delta. \quad (78)$$

The main results of this subsection are given by the theorem below.

Theorem 7. Let $\nu > 0$ such that F belongs to the space $L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))$. Let F^δ be as above. Let us assume that Problem (50) has a unique mild solution $u \in L^\infty(0, T; D(\mathcal{A}^{\nu+\theta}))$ for $\theta > 0$. Let us choose N such that

$$\lim_{\delta \rightarrow 0} \lambda_N^\nu \exp \left(\frac{T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \delta = 0, \quad \lim_{\delta \rightarrow 0} \lambda_N = +\infty. \quad (79)$$

Here $\nu \geq 1/2$. Then, there exists a positive $\bar{\mu}$ large enough such that Problem has a unique solution $v^{N, \delta} \in L_{\bar{\mu}}^\infty(0, T; D(\mathcal{A}^\nu))$. Moreover, we have the following estimate:

$$\begin{aligned} \|v^{N, \delta} - u\|_{L_{\bar{\mu}}^\infty(0, T; D(\mathcal{A}^\nu))}^2 &\leq \frac{6T^{2\alpha}}{\alpha^2} \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \delta^2 \\ &\quad + 6\lambda_N^{-2\theta} \|u\|_{L^\infty(0, T; D(\mathcal{A}^{\nu+\theta}))}^2. \end{aligned} \quad (80)$$

Remark 8. Since $\lambda_N \sim N^{2/d}$, we can choose a natural number N such that

$$\lambda_N = \left(\frac{\alpha(1-b) \log(1/b)}{T^\alpha M_1} \right)^{1/\beta}. \quad (81)$$

Proof. Part 1: prove that the nonlinear integral equation (77) has a unique mild solution.

Let any $v \in C([0, T]; H^1(\Omega))$, we denote by the following function

$$\begin{aligned} \mathcal{G}(v)(x, t) &= - \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha}^{s^\alpha} \lambda_j^\beta B(\|\nabla v(\cdot, r)\|_{L^2}) dr \right) \cdot \langle F^\delta(\cdot, s), w_j \rangle ds \right) w_j(x). \end{aligned} \quad (82)$$

By applying Parseval's equality, we follow from (82) that

$$\begin{aligned}
& \|\mathcal{G}(v_1)(\cdot, t) - \mathcal{G}(v_2)(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \\
&= \sum_{j=1}^N \lambda_j^{2v} \left(\int_t^T s^{\alpha-1} (\mathcal{M}(s, t, j, v_1) \right. \\
&\quad \left. - \mathcal{M}(s, t, j, v_2)) \langle F(\cdot, s), w_j \rangle ds \right)^2 \\
&\leq \frac{T^\alpha}{\alpha} \lambda_N^{2v} \sum_{j=1}^N \left(\int_0^t s^{\alpha-1} (\mathcal{M}(s, t, j, w_1) \right. \\
&\quad \left. - \mathcal{M}(s, t, j, w_2)) \langle F^\delta(\cdot, s), w_j \rangle ds \right)^2.
\end{aligned} \tag{83}$$

If $1 \leq j \leq N$, then we have in view of (63) that

$$\begin{aligned}
& |\mathcal{M}(s, t, j, v_1) - \mathcal{M}(s, t, j, v_2)|^2 \\
&\leq \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) |K_b|^2 \lambda_j^{2\beta} \frac{T^{2\alpha}}{\alpha^2} \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \|v_1(\cdot, r) - v_2(\cdot, r)\|_{H^1(\Omega)}^2 ds \\
&\leq \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta} K_b^2 \frac{T^{2\alpha}}{\alpha^2} \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \|v_1(\cdot, r) - v_2(\cdot, r)\|_{H^1(\Omega)}^2 ds.
\end{aligned} \tag{84}$$

The above two observations (65) lead to

$$\begin{aligned}
& \exp(2\mu(t-T)) \|\mathcal{G}(v_1)(\cdot, t) - \mathcal{G}(v_2)(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \\
&\leq |K_b|^2 \frac{T^{3\alpha}}{\alpha^3} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2v} \\
&\quad \cdot \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} e^{2\mu(t-s)} e^{2\mu(s-T)} \|v_1(\cdot, s) - v_2(\cdot, s)\|_{H^1(\Omega)}^2 ds \right) \\
&\quad \cdot \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} s^{\alpha-1} \left(\sum_{j=1}^\infty \langle F^\delta(\cdot, s), w_j \rangle^2 \right) ds \right) \\
&\leq |K_b|^2 \frac{T^{3\alpha}}{\alpha^3} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2v} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \\
&\quad \cdot \left(\max_{0 \leq s \leq T} \exp(2\mu(s-T)) \|v_1(\cdot, s) - v_2(\cdot, s)\|_{H^1(\Omega)}^2 \right) \\
&\quad \cdot \left(\int_0^t s^{\alpha-1} \|F^\delta(\cdot, s)\|_{L^2(\Omega)}^2 ds \right) \\
&\leq |K_b|^2 \frac{T^{4\alpha}}{\alpha^4} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2v} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \\
&\quad \cdot \|F^\delta\|_{L^\infty(0, T; L^2(\Omega))}^2 \|v_1 - v_2\|_{L^\infty(0, T; H^1(\Omega))}^2.
\end{aligned} \tag{85}$$

Because the right-hand side of the latter estimate is independent of t and noting the Sobolev embedding $D(\mathcal{A}^v) \hookrightarrow H^1(\Omega)$, we arrive at

$$\begin{aligned}
& \|\mathcal{G}(v_1) - \mathcal{G}(v_2)\|_{L_\mu^\infty(0, T; D(\mathcal{A}^v))}^2 \\
&\leq |K_b|^2 \frac{T^{4\alpha}}{\alpha^4} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2v} \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta}(1-\delta)} \\
&\quad \cdot \|F^\delta\|_{L^\infty(0, T; L^2(\Omega))}^2 \|v_1 - v_2\|_{L_\mu^\infty(0, T; D(\mathcal{A}^v))}^2.
\end{aligned} \tag{86}$$

Let us choose μ_1 such that

$$\begin{aligned}
& |K_b|^2 \frac{T^{4\alpha}}{\alpha^4} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2v} \frac{T^{\alpha(1-\delta)} |C_\delta|^2}{\alpha^{1-\delta}(1-\delta)} \\
&\quad \cdot \|F^\delta\|_{L^\infty(0, T; L^2(\Omega))}^2 < \mu_1^{2\delta}.
\end{aligned} \tag{87}$$

It is easy to see that \mathcal{G} is a contracting mapping on the space $L_{\mu_1}^\infty(0, T; D(\mathcal{A}^v))$. Therefore, we can conclude that there exists a uniqueness solution $v^{N, \delta}$ for Problem (77).

Next, we continue to give the upper bound of the term $\|v^{N, \delta}(\cdot, t) - u(\cdot, t)\|_{D(\mathcal{A}^v)}^2$. First, we have

$$\begin{aligned}
& v^{N, \delta}(x, t) - u(x, t) \\
&= - \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B \left(\|\nabla v^{N, \delta}(\cdot, r)\|_{L^2} \right) dr \right) \right. \\
&\quad \cdot \left. \langle F^\delta(\cdot, s) - F(\cdot, s), w_j \rangle ds \right) w_j(x) \\
&\quad + \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \left(\mathcal{M}(s, t, j, v^{N, \delta}) \right. \right. \\
&\quad \left. \left. - \mathcal{M}(s, t, j, u) \right) \langle F(\cdot, s), w_j \rangle ds \right) w_j(x) \\
&\quad + \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B \left(\|\nabla u(\cdot, r)\|_{L^2} \right) dr \right) \right. \\
&\quad \cdot \left. \langle F(\cdot, s), w_j \rangle ds \right) w_j.
\end{aligned} \tag{88}$$

The above equality and Parseval's equality allow us to get that

$$\begin{aligned}
& \|v^{N, \delta}(\cdot, t) - u(\cdot, t)\|_{D(\mathcal{A}^v)}^2 \\
&\leq \underbrace{3 \sum_{j=1}^N \lambda_j^{2v} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B \left(\|\nabla v^{N, \delta}(\cdot, r)\|_{L^2} \right) dr \right) \langle F^\delta(\cdot, s) - F(\cdot, s), w_j \rangle ds \right)^2}_{J_1} \\
&\quad + \underbrace{3 \sum_{j=1}^N \lambda_j^{2v} \left(\int_t^T s^{\alpha-1} \left(\mathcal{M}(s, t, j, v^{N, \delta}) - \mathcal{M}(s, t, j, u) \right) \langle F(\cdot, s), w_j \rangle ds \right)^2}_{J_2} \\
&\quad + \underbrace{3 \sum_{j=N+1}^\infty \lambda_j^{2v} \left(\int_t^T s^{\alpha-1} \exp \left(\int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B \left(\|\nabla u(\cdot, r)\|_{L^2} \right) dr \right) \langle F(\cdot, s), w_j \rangle ds \right)^2}_{J_3}.
\end{aligned} \tag{89}$$

Since the condition $\mathcal{B}(z) \leq M_1 \forall z \in \mathbb{R}$ and applying Hölder inequality, the quantity J_1 is bounded by

$$\begin{aligned} J_1 &\leq 3\lambda_N^{2\nu} \left(\int_t^T s^{\alpha-1} \right) \sum_{j=1}^N \left(\int_t^T s^{\alpha-1} \exp \left(2 \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \lambda_j^\beta B \left(\left\| \nabla v^{N,\delta}(\cdot, r) \right\|_{L^2} \right) dr \right) \right. \\ &\quad \cdot \left. \left\langle F^\delta(\cdot, s) - F(\cdot, s), w_j \right\rangle^2 ds \right) \\ &\leq \frac{3T^\alpha}{\alpha} \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \left(\int_0^t s^{\alpha-1} \left\| F^\delta(\cdot, s) - F \right\|_{L^2(\Omega)}^2 ds \right) \\ &\leq \frac{3T^\alpha}{\alpha} \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \left(\int_t^T s^{\alpha-1} ds \right) \left\| F^\delta - F \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ &\leq \frac{3T^{2\alpha}}{\alpha^2} \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \delta^2, \end{aligned} \quad (90)$$

where we have used the fact that $\|F^\delta - F\|_{L^\infty(0, T; L^2(\Omega))} \leq \delta$.

The quantity J_2 is estimated as follows:

$$\begin{aligned} &\exp(2\mu(t-T))J_2 \\ &\leq 3 \exp(2\mu(t-T)) \left(\int_t^T s^{\alpha-1} ds \right) \\ &\quad \cdot \left[\sum_{j=1}^N \int_t^T s^{\alpha-1} \lambda_j^{2\nu} \left(\mathcal{M}(s, t, j, v^{N,\delta}) \right. \right. \\ &\quad \left. \left. - \mathcal{M}(s, t, j, u) \right)^2 \left\langle F(\cdot, s), w_j \right\rangle^2 ds \right]. \end{aligned} \quad (91)$$

We have in view of (63) that

$$\begin{aligned} &\left| \mathcal{M}(s, t, j, v^{N,\delta}) - \mathcal{M}(s, t, j, u) \right|^2 \\ &\leq \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \lambda_j^{2\beta} \frac{K_b^2 T^{2\alpha}}{\alpha^2} \\ &\quad \cdot \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} \left\| v^{N,\delta}(\cdot, r) - u(\cdot, r) \right\|_{D(\mathcal{A}^\nu)}^2 dr. \end{aligned} \quad (92)$$

This leads to the following estimate:

$$\begin{aligned} &\exp(2\mu(t-T))J_2 \\ &\leq \frac{3K_b^2 T^{3\alpha}}{\alpha^3} \int_{t^\alpha/\alpha}^{s^\alpha/\alpha} e^{2\mu(t-s)} \\ &\quad \cdot e^{2\mu(s-T)} \left\| v^{N,\delta}(\cdot, r) - u(\cdot, r) \right\|_{D(\mathcal{A}^\nu)}^2 \\ &\quad \cdot dr \int_0^T s^{\alpha-1} \left(\sum_{j=1}^\infty \lambda_j^{2\beta+2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_j^\beta}{\alpha} \right) \left\langle F(\cdot, s), w_j^2 \right\rangle ds \right) \\ &\leq \frac{3K_b^2 T^{4\alpha}}{\alpha^4} \|F\|_{L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta} (1-\delta)} \\ &\quad \cdot \left\| v^{N,\delta} - u \right\|_{L_\mu^\infty(0, T; D(\mathcal{A}^\nu))}^2. \end{aligned} \quad (93)$$

The term J_3 is estimated as follows:

$$\begin{aligned} J_3 &= 3 \sum_{j=N+1}^\infty \lambda_j^{2\nu} \left\langle u(\cdot, t), w_j \right\rangle^2 \\ &= 3 \sum_{j=N+1}^\infty \lambda_j^{-2\theta} \lambda_j^{2\nu+2\theta} \left\langle u(\cdot, t), w_j \right\rangle^2 \\ &\leq 3\lambda_N^{-2\theta} \|u\|_{L^\infty(0, T; D(\mathcal{A}^{\nu+\theta}))}^2. \end{aligned} \quad (94)$$

Combining (89), (90), (93), and (94), we find that

$$\begin{aligned} &\exp(2\mu(t-T)) \left\| v^{N,\delta}(\cdot, t) - u(\cdot, t) \right\|_{D(\mathcal{A}^\nu)}^2 \\ &\leq \exp(2\mu(t-T)) (J_1 + J_2 + J_3) \\ &\leq \frac{3T^{2\alpha}}{\alpha^2} \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \delta^2 + \frac{3K_b^2 T^{4\alpha}}{\alpha^4} \\ &\quad \cdot \|F\|_{L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta} (1-\delta)} \\ &\quad \cdot \left\| v^{N,\delta} - u \right\|_{L_\mu^\infty(0, T; D(\mathcal{A}^\nu))}^2 + 3\lambda_N^{-2\theta} \|u\|_{L^\infty(0, T; D(\mathcal{A}^{\nu+\theta}))}^2. \end{aligned} \quad (95)$$

We choose $\bar{\mu}$ such that both the following inequalities are satisfied:

$$\begin{aligned} &\frac{3K_b^2 T^{4\alpha}}{\alpha^4} \|F\|_{L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2}{\alpha^{1-\delta} (1-\delta)} \leq \frac{1}{2} \bar{\mu}^{2\delta}, \\ &|K_b|^2 \frac{T^{4\alpha}}{\alpha^4} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \lambda_N^{2\beta+2\nu} \frac{T^{\alpha(1-\delta)} |C_\delta|^2}{\alpha^{1-\delta} (1-\delta)} \\ &\quad \cdot \|F^\delta\|_{L^\infty(0, T; L^2(\Omega))}^2 < \bar{\mu}^{2\delta}. \end{aligned} \quad (96)$$

Some observations above give us the following confirmation:

$$\begin{aligned} &\left\| v^{N,\delta} - u \right\|_{L_\mu^\infty(0, T; D(\mathcal{A}^\nu))}^2 \\ &\leq \frac{(3T^{2\alpha}/\alpha^2) \lambda_N^{2\nu} \exp \left(\frac{2T^\alpha M_1 \lambda_N^\beta}{\alpha} \right) \delta^2 + 3\lambda_N^{-2\theta} \|u\|_{L^\infty(0, T; D(\mathcal{A}^{\nu+\theta}))}^2}{1 - (3K_b^2 T^{4\alpha}/\alpha^4) \|F\|_{L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))}^2 \left(\frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta} (1-\delta)} \right)}. \end{aligned} \quad (97)$$

Since the fact that

$$1 - \frac{3K_b^2 T^{4\alpha}}{\alpha^4} \|F\|_{L^\infty(0, T; \mathbb{X}_{\beta+\nu, \alpha}(\Omega))}^2 \frac{T^{\alpha(1-\delta)} |C_\delta|^2 \mu^{-2\delta}}{\alpha^{1-\delta} (1-\delta)} \geq \frac{1}{2}, \quad (98)$$

We easily obtain the desired result (80). \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to the work. The four authors read and approved the final manuscript.

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