Research Article

Analytical Analysis of Fractional-Order Physical Models via a Caputo-Fabrizio Operator

Fatemah Mofarreh ¹, A. M. Zidan ^{2,3}, Muhammad Naeem ⁴, Rasool Shah ⁵,
Roman Ullah,⁶ and Kamsing Nonlaopon ⁷

¹Department of Mathematical Sciences College of Science, Princess Nourah Bint Abdulrahman University, Riyadh 11546, Saudi Arabia

²Department of Mathematics, College of Science, King Khalid University, P.O. Box: 9004, Abha 61413, Saudi Arabia

³Department of Mathematics, Faculty of Science, Al-Azhar University, Assuit 71511, Egypt

⁴Deanship of Joint First Year Umm Al-Qura University, Makkah, KSA, Saudi Arabia

⁵Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan

⁶General Requirements Department, CAS Sohar, University of Technology and Applied Sciences, Oman

⁷Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

Received 11 May 2021; Accepted 11 August 2021; Published 20 December 2021

Academic Editor: Feliz Minhos

Copyright © 2021 Fatemah Mofarreh et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper investigates a modified analytical method called the Adomian decomposition transform method for solving fractional-order heat equations with the help of the Caputo-Fabrizio operator. The Laplace transform and the Adomian decomposition method are implemented to obtain the result of the given models. The validity of the proposed method is verified by considering some numerical problems. The solution achieved has shown that the better accuracy of the suggested method. Furthermore, due to the straightforward implementation, the proposed method can solve other nonlinear fractional-order problems.

1. Introduction

The fractional generalization of differential equations has proven to be an effective and precise tool for interpreting real-world problems. Several methods have been proposed in this direction, including Ross and Miller, Podlubny, Caputo, and Kilbas et al. Many scholars, including Das and Gupta, Srivastava et al., and Singh et al., investigated the vibration equation's fractional extension using the Caputo fractional derivative. Due to the Caputo and other fractional derivatives' singular kernel, mathematical models involving these fractional derivatives do not consider the entire memory. For describing nonlocal behavioral models, fractional calculus (FC) has become a common mathematical approach. Fractional derivatives have mathematically interpreted many physical problems in recent decades; these representations have

produced excellent solutions in real-world modeling issues. Riesz, Coimbra, Hadamard, Riemann-Liouville, Grunwald-Letnikov, Weyl, Caputo, Caputo-Fabrizio, Atangana-Baleanu, among others, gave many basic definitions of fractional operators [1, 2]. A wide variety of nonlinear equations have been developed and commonly applied in numerous nonlinear physical sciences such as chemistry, biology, mathematics, and various branches of physics such as plasma physics, condensed matter physics, fluid mechanics, field theory, and nonlinear optics over the past few years. The exact result of nonlinear equations plays a crucial role in deciding the characteristics and behavior of physical processes, but as dealing with linear equations, it is impossible to find their exact results. Many effective techniques have been used to solved nonlinear FPDEs, for example, the homotopy perturbation transform technique [3, 4], the homotopy analysis transform technique

[5, 6], reduced differential transform technique [7, 8], the finite element technique [9], the finite difference technique [10], q-homotopy analysis transform method [11], and so on.

The heat equation was created in 1822 by Joseph Fourier. The analysis of Brownian motion is related to the heat equation. Robert Brown treated this motion. The interaction of particles suspended in a liquid (fluid or gas) with the liquid's fast-moving atoms causes the erratic movement of the particles. Heat refers to the complex energy of particles that are being exchanged. Faster moving atoms in the atmosphere clash with the scheme's dividers as emotional vitality is transferred from an increased environment to a cooler system, exchanging some of this energy with the system's atoms and enabling them to travel faster [12–14]. The wave model is a critical partial differential equation for explaining waves in conventional nuclear physics, such as mechanical waves. The electromagnetic radiation equation is a significant partial differential equation that occurs in areas such as acoustics, electrostatics, and quantum mechanics. Sound, light, gravity, and even matter are examples of them (in the Klein Gordon equation, relativistic quantum mechanics). Alembert and Euler discovered the one-dimensional and three-dimensional wave equations. The results of heat equations have been used of many researcher of mathematics, such as the Adomian decomposition technique [15], the optimal homotopy asymptotic technique [16], the differential transform technique [17], the variational iteration technique [18], Bernstein polynomials with the operational matrix [19], the homotopy perturbation method (HPM) [20], apply the variational iteration technique and Yang-Laplace transformation of time-fractional heat equations [21], Elzaki transformation and the differential transformation technique for nonlinear wave equation [22], and the Aboodh decomposition method [23]. The Laplace Adomian decomposition method (LADM) is a simple and efficient technique for solving linear and nonlinear FPDEs. LADM combines the Laplace transform and Adomian decomposition technique. It is observed that the proposed form, unlike RK4, does not require a predefined declaration size. In comparison to other analytical techniques, this method needs less parameters, no discretization, and no linearization [24]. Modified decomposition method is also compared to ADM to analyze the FPDE result provided in [25]. The result of Kundu Eckhaus equations is introduced in [26], using ADTM. Multistep ADTM are applied to investigate fractional partial differential equations in [27]. Laplace Adomian decomposition method is also applied for the result of fractional-order Smoke model [28]. In this article, we implemented LADM for the investigation of fractional-order heat equations. The current method has a very straightforward and simple method. Absolute error is used to measure accuracy.

2. Definitions and Preliminary Concepts

Definition 1. If $v \in H[0, T]$, $T > 0$, $\rho \in (0, 1)$, then Caputo-Fabrizio of fractional derivative is expressed as

$${}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho} [v(\mathfrak{S})] = \frac{M(\rho)}{1-\rho} \int_0^{\mathfrak{S}} v'(\zeta) K(\mathfrak{S}, \zeta) d\zeta. \quad (1)$$

$M(\rho)$ is the normalization function with $M(0) = M(1) = 1$. If $v \in H[0, T]$,

$${}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho} [v(\mathfrak{S})] = \frac{M(\rho)}{1-\rho} \int_0^{\mathfrak{S}} (v(\mathfrak{S}) - v(\zeta))(\zeta) K(\mathfrak{S}, \zeta) d\zeta. \quad (2)$$

Definition 2. The fractional Caputo-Fabrizio integral due to $\rho \in (0, 1)$ is defined by

$${}_0^{\text{CF}} I_{\mathfrak{S}}^{\rho} [v(\mathfrak{S})] = \frac{1-\rho}{M(\rho)} v(\mathfrak{S}) + \frac{\rho}{M(\rho)} \int_0^{\mathfrak{S}} v(\zeta) d\zeta, \quad \mathfrak{S} \geq 0. \quad (3)$$

Definition 3. The Laplace transformation of relation with Caputo-Fabrizio as

$$\begin{aligned} L \left[{}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho+M} [v(\mathfrak{S})] \right] &= \frac{1}{1-\rho} \left[v^{(m+\rho)}(\mathfrak{S}) \right] L \left\{ e^{-\rho \mathfrak{S}/(1-\rho)} \right\} \\ &= \frac{1}{s + \rho(1-\rho)} \left\{ s^{m+1} L(v(\mathfrak{S})) + \sum_{k=1}^m s^{m-k} g^k(0) \right\}. \end{aligned} \quad (4)$$

If $m = 0, 1$

$$L \left[{}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho} [v(\mathfrak{S})] \right] = \frac{sL\{v(\zeta)\}}{s + \rho(1-s)}, \quad (5)$$

$$L \left[{}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho+1} [v(\mathfrak{S})] \right] = \frac{s^2 L\{v(\zeta)\} + sv(0) - v'(0)}{s + \rho(1-s)}.$$

3. The General Methodology of the Present Method

In this section, the general result finds the FPDEs with the help of the Laplace Adomian decomposition method.

$${}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho} v(\zeta, \mathfrak{S}) + Gv(\zeta, \mathfrak{S}) + Nv(\zeta, \mathfrak{S}) = q(\zeta, \mathfrak{S}), \quad \zeta, \mathfrak{S} \geq 0, \quad m-1 < \rho < m, \quad (6)$$

where ${}_0^{\text{CF}} D^{\rho} = \partial^{\rho} / \partial t^{\rho}$ the Caputo operator $\rho, m \in \mathbb{N}$, where G is linear and N nonlinear term, and the source term is q .

With initial condition

$$v(\zeta, 0) = k(\zeta). \quad (7)$$

Using the Laplace transformation to Equation (6), we get

$$L \left[{}_0^{\text{CF}} D_{\mathfrak{S}}^{\rho} v(\zeta, \mathfrak{S}) \right] + L[Gv(\zeta, \mathfrak{S}) + Nv(\zeta, \mathfrak{S})] = L[q(\zeta, \mathfrak{S})], \quad (8)$$

apply the Laplace differentiation transform property, we obtain as

$$\left(\frac{sL[v(\zeta, \mathfrak{S})] - v(\zeta, 0)}{s + \rho(1-s)} \right) = L[q(\zeta, \mathfrak{S})] - L[Gv(\zeta, \mathfrak{S}) + Nv(\zeta, \mathfrak{S})], \quad (9)$$

$$L[v(\zeta, \mathfrak{F})] = \frac{k(\zeta)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L[q(\zeta, \mathfrak{F})] - \left(\frac{s + \rho(1-s)}{s} \right) L[Gv(\zeta, \mathfrak{F}) + Nv(\zeta, \mathfrak{F})]. \quad (10)$$

The LADM result $v(\zeta, \mathfrak{F})$ shows the infinite-series:

$$v(\zeta, \mathfrak{F}) = \sum_{j=0}^{\infty} v_j(\zeta, \mathfrak{F}), \quad (11)$$

the nonlinear term can be solve by Adomian polynomials

$$Nv(\zeta, \mathfrak{F}) = \sum_{j=0}^{\infty} A_j, \quad (12)$$

$$A_j = \frac{1}{j!} \left[\frac{d^j}{d\lambda^j} \left[N \sum_{j=0}^{\infty} (\lambda^j v_j) \right] \right]_{\lambda=0}, \quad j = 0, 1, 2, \dots, \quad (13)$$

putting Equation (11) and Equation (12) in Equation (10), we achieve as

$$L \left[\sum_{j=0}^{\infty} v_j(\zeta, \mathfrak{F}) \right] = \frac{k(x)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L[q(\zeta, \mathfrak{F})] - \left(\frac{s + \rho(1-s)}{s} \right) L \left[G \sum_{j=0}^{\infty} v_j(\zeta, \mathfrak{F}) + \sum_{j=0}^{\infty} A_j \right]. \quad (14)$$

Implementing of the Laplace transformation,

$$L[v_0(\zeta, \mathfrak{F})] = \frac{v(x, 0)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L[q(\zeta, \mathfrak{F})], \quad (15)$$

$$L[v_1(\zeta, \mathfrak{F})] = - \left(\frac{s + \rho(1-s)}{s} \right) L[Gv_0(\zeta, \mathfrak{F}) + A_0].$$

Usually, we can written as

$$L[v_{j+1}(\zeta, \mathfrak{F})] = - \left(\frac{s + \rho(1-s)}{s} \right) L[Gv_j(\zeta, \mathfrak{F}) + A_j], j \geq 1. \quad (16)$$

Applying inverse transform in Equation (16), we get

$$v_0(\zeta, \mathfrak{F}) = k(\zeta, \mathfrak{F}),$$

$$v_{j+1}(\zeta, \mathfrak{F}) = -L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L[Gv_j(\zeta, \mathfrak{F}) + A_j] \right]. \quad (17)$$

4. Numerical Implementation

4.1. Problem 1. Consider the one-dimensional fractional heat equation (26):

$$\frac{\partial^\rho v}{\partial \mathfrak{F}^\rho} - \frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \zeta^2} = 0, 0 < \rho \leq 1, \mathfrak{F} \geq 0, \quad (18)$$

with initial condition

$$v(\zeta, 0) = \zeta^2. \quad (19)$$

Using the Laplace Caputo-Fabrizio operator in Equation (18),

$$\left(\frac{s[L[v(\zeta, \mathfrak{F})] - v(x, 0)]}{s + \rho(1-s)} \right) = L \left[\frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \zeta^2} \right]. \quad (20)$$

Applying inverse Laplace transformation,

$$v(\zeta, \mathfrak{F}) = L^{-1} \left[\frac{v(x, 0)}{s} - \left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \zeta^2} \right\} \right]. \quad (21)$$

Implement the ADM processes, we achieve

$$v_0(\zeta, \mathfrak{F}) = L^{-1} \left[\frac{v(x, 0)}{s} \right] = L^{-1} \left[\frac{\zeta^2}{s} \right],$$

$$v_0(\zeta, \mathfrak{F}) = \zeta^2,$$

$$v_{j+1}(\zeta, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\zeta^2}{2} \frac{\partial^2 v_j}{\partial \zeta^2} \right\} \right], j = 0, 1, 2, \dots, \quad (22)$$

for $j = 0$

$$v_1(\zeta, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\zeta^2}{2} \frac{\partial^2 v_0}{\partial \zeta^2} \right\} \right],$$

$$v_1(\zeta, \mathfrak{F}) = L^{-1} \left[\zeta^2 \left(\frac{s + \rho(1-s)}{s^2} \right) \right] = \zeta^2 (1 - \rho + \rho \mathfrak{F}). \quad (23)$$

The subsequent terms are

$$v_2(\zeta, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\zeta^2}{2} \frac{\partial^2 v_1}{\partial \zeta^2} \right\} \right]$$

$$= \zeta^2 L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) \left(\frac{s + \rho(1-s)}{s^2} \right) \right],$$

$$v_2(\zeta, \mathfrak{F}) = \zeta^2 \left[1 + \rho^2 - 2\rho - 2\rho^2 t + 2\rho t + \frac{\mathfrak{F}^2 \rho^2}{2} \right],$$

$$\begin{aligned} v_3(\zeta, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\zeta^2}{2} \frac{\partial^2 v_2}{\partial \zeta^2} \right\} \right] \\ &= \zeta^2 L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) \left(\frac{(s + \rho(1-s))^2}{s^3} \right) \right], \end{aligned}$$

$$\begin{aligned} v_3(\zeta, \mathfrak{F}) &= \zeta^2 \left[3\rho(-2\rho + 1 + \rho^2)\mathfrak{F} + \frac{\rho^3 \mathfrak{F}^3}{6} - \frac{3\rho^2(\rho-1)\mathfrak{F}^2}{2} + 3\rho^2 - 3\rho + 1 - \rho^3 \right] \\ &\vdots \end{aligned} \quad (24)$$

The LADM solution for Problem 1 is

$$\begin{aligned} v(\zeta, \mathfrak{F}) &= v_0(\zeta, \mathfrak{F}) + v_1(\zeta, \mathfrak{F}) + v_2(\zeta, \mathfrak{F}) + v_3(\zeta, \mathfrak{F}) + v_4(\zeta, \mathfrak{F}) \cdots v(\zeta, \mathfrak{F}) \\ &= \zeta^2 \left(1 + (1 - \rho + \rho\mathfrak{F}) + (1 - \rho + \rho\mathfrak{F})^2 - \frac{\mathfrak{F}^2 \rho^2}{2} + \mathfrak{F}(\mathfrak{F}\rho - \rho^2 - \rho)(3 - 3\rho) \right. \\ &\quad \left. + (\rho - 1)^2(1 - \rho) + \frac{1}{6}\mathfrak{F}^3 \rho^3 \cdots \right), \end{aligned} \quad (25)$$

when $\rho = 1$, then the LADM result is

$$v(\zeta, \mathfrak{F}) = \zeta^2 \left(1 + \mathfrak{F} + \frac{\mathfrak{F}^2}{2} + \frac{\mathfrak{F}^3}{6} + \frac{\mathfrak{F}^4}{24} \cdots \right). \quad (26)$$

The close form of the Problem 1 is

$$v(\zeta, \mathfrak{F}) = \zeta^2 e^{\mathfrak{F}}. \quad (27)$$

Figure 1 shows the exact and approximate solution graphs with the help of Caputo-Fabrizio operator of example 1. In Figure 2, the first 3D graph shows the different fractional-order of ρ , and the second 2D figure shows the different fractional-order of ρ . In Table 1, values of the error, however, are proved to be proportional to the divergence of exact solutions and corresponding to the uncertainty in the approximate solutions.

4.2. Problem 2. Consider the time-fractional two-dimensional heat equation (26):

$$\frac{\partial^\rho v}{\partial \mathfrak{F}^\rho} - \frac{y^2}{2} \frac{\partial^2 v}{\partial \zeta^2} - \frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \psi^2} = 0, 0 < \rho \leq 1, \quad (28)$$

with the initial condition

$$v(\zeta, \psi, 0) = \psi^2. \quad (29)$$

Applying the Laplace transform of Equation (28), we get

$$\left(\frac{sL[v(\zeta, \psi, \mathfrak{F})] - v(\zeta, \psi, 0)}{s + \rho(1-s)} \right) = L \left[\frac{\psi^2}{2} \frac{\partial^2 v}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \psi^2} \right]. \quad (30)$$

Applying inverse Laplace transformation,

$$v(\zeta, \psi, \mathfrak{F}) = L^{-1} \left[\frac{v(\zeta, \psi, 0)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \frac{\partial^2 v}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v}{\partial \psi^2} \right\} \right]. \quad (31)$$

Applying the ADM processes, we achieve as

$$\begin{aligned} v_0(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\frac{v(\zeta, \psi, 0)}{s} \right] = L^{-1} \left[\frac{\psi^2}{s} \right], \\ v_0(\zeta, \psi, \mathfrak{F}) &= \psi^2, \\ \sum_{j=0}^{\infty} v_{j+1}(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \sum_{j=0}^{\infty} v_{xxy} + \frac{\zeta^2}{2} \sum_{j=0}^{\infty} v_{yyj} \right\} \right], \end{aligned} \quad (32)$$

for $j = 0$

$$\begin{aligned} v_1(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \frac{\partial^2 v_0}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v_0}{\partial \psi^2} \right\} \right], \\ v_1(\zeta, \psi, \mathfrak{F}) &= \zeta^2 (1 - \rho + \rho\mathfrak{F}). \end{aligned} \quad (33)$$

The subsequent terms are

$$\begin{aligned} v_2(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \frac{\partial^2 v_1}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v_1}{\partial \psi^2} \right\} \right] \\ &= \psi^2 \left(1 + \rho^2 - 2\rho - 2\rho^2 \mathfrak{F} + 2\rho\mathfrak{F} + \frac{\mathfrak{F}^2 \rho^2}{2} \right), \\ v_3(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \frac{\partial^2 v_2}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v_2}{\partial \psi^2} \right\} \right], \\ v_3(\zeta, \psi, \mathfrak{F}) &= \zeta^2 \left(3\rho(-2\rho + 1 + \rho^2)\mathfrak{F} + \frac{\rho^3 \mathfrak{F}^3}{6} - \frac{3\rho^2(\rho-1)\mathfrak{F}^2}{2} \right. \\ &\quad \left. + 3\rho^2 - 3\rho + 1 - \rho^3 \right), \\ v_4(\zeta, \psi, \mathfrak{F}) &= L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{\psi^2}{2} \frac{\partial^2 v_3}{\partial \zeta^2} + \frac{\zeta^2}{2} \frac{\partial^2 v_3}{\partial \psi^2} \right\} \right], \\ v_4(\zeta, \psi, \mathfrak{F}) &= \psi^2 \left(-4\rho(-3\rho^2 + 3\rho - 1 + \rho^3)\mathfrak{F} - \frac{2\rho^3(\rho-1)\mathfrak{F}^3}{3} \right. \\ &\quad \left. + 3\mathfrak{F}^2(\rho^4 - 2\rho^3 + \rho^2) - 4\rho^3 + 1 + \rho^4 - 4\rho + 6\rho^2 + \frac{\rho^4 \mathfrak{F}^4}{24} \right). \\ &\vdots \end{aligned} \quad (34)$$

FIGURE 1: The first graph shows the exact, and the second graph shows the solution of analytical method at $\rho = 1$ of Problem 1.

FIGURE 2: The analytical solution figures of different fractional-order of ρ of Problem 1.

The LADM solution for Problem 2 is

$$\begin{aligned}
 v(\zeta, \psi, \mathfrak{F}) &= v_0(\zeta, \psi, \mathfrak{F}) + v_1(\zeta, \psi, \mathfrak{F}) + v_2(\zeta, \psi, \mathfrak{F}) \\
 &\quad + v_3(\zeta, \psi, \mathfrak{F}) + v_4(\zeta, \psi, \mathfrak{F}) \cdots v(\zeta, \psi, \mathfrak{F}) \\
 &= \psi^2 + \zeta^2(1 - \rho + \rho\mathfrak{F}) + \psi^2 \left(1 + \rho^2 - 2\rho - 2\rho^2\mathfrak{F} + 2\rho\mathfrak{F} + \frac{\mathfrak{F}^2\rho^2}{2} \right) \\
 &\quad + \zeta^2 \left(3\rho(-2\rho + 1 + \rho^2)\mathfrak{F} + \frac{\rho^3\mathfrak{F}^3}{6} - \frac{3\rho^2(\rho-1)\mathfrak{F}^2}{2} + 3\rho^2 - 3\rho + 1 - \rho^3 \right) \\
 &\quad + \psi^2 \left(-4\rho(-3\rho^2 + 3\rho - 1 + \rho^3)\mathfrak{F} - \frac{2\rho^3(\rho-1)\mathfrak{F}^3}{3} + 3\mathfrak{F}^2(\rho^4 - 2\rho^3 + \rho^2) \right. \\
 &\quad \left. - 4\rho^3 + 1 + \rho^4 - 4\rho + 6\rho^2 + \frac{\rho^4\mathfrak{F}^4}{24} \right) + \cdots
 \end{aligned} \tag{35}$$

The closed form of the above equation is

$$v(\zeta, \psi, \mathfrak{F}) = \zeta^2 \sinh \mathfrak{F} + \psi^2 \cosh \mathfrak{F}. \tag{36}$$

Figure 3 shows the exact and approximate solution graphs with the help of Caputo-Fabrizio operator of example 2. In Figure 4, the first 3D graph shows the different fractional-order of ρ , and the second 2D figure shows the different fractional-order of ρ .

4.3. *Problem 3.* Consider the time-fractional three-dimensional heat equation (26):

$$\frac{\partial^\rho v}{\partial \mathfrak{F}^\rho} - (\zeta\psi z)^4 - \frac{1}{36} \left(\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v}{\partial \psi^2} + z^2 \frac{\partial^2 v}{\partial z^2} \right), 0 < \rho \leq 1, \mathfrak{F} \geq 0, \tag{37}$$

the initial condition is

$$v(\zeta, \psi, z, 0) = 0. \tag{38}$$

Using the Laplace transformation of Equation (37),

$$\begin{aligned}
 &\left(\frac{sL[v(\zeta, \psi, z, \mathfrak{F})] - v(\zeta, \psi, z, 0)}{s + \rho(1-s)} \right) \\
 &= L \left[(\zeta\psi z)^4 \right] + L \left[\frac{1}{36} \left(\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v}{\partial \psi^2} + z^2 \frac{\partial^2 v}{\partial z^2} \right) \right].
 \end{aligned} \tag{39}$$

FIGURE 3: The first graph shows the exact, and the second graph shows the solution of analytical method at $\rho = 1$ of Problem 2.

FIGURE 4: The analytical solution graph of different fractional-order of ρ of Problem 2.

FIGURE 5: The analytical solution graph of different fractional-order of ρ of Problem 3.

Implementing inverse Laplace transformation

$$v(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\frac{v(\zeta, \psi, z, 0)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L \left\{ (\zeta \psi z)^4 \right\} \right] + L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{1}{36} \left(\zeta^2 \frac{\partial^2 v}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v}{\partial \psi^2} + z^2 \frac{\partial^2 v}{\partial z^2} \right) \right\} \right]. \quad (40)$$

Apply the ADM processes, we achieve

$$v_0(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\frac{v(\zeta, \psi, z, 0)}{s} + \left(\frac{s + \rho(1-s)}{s} \right) L \left\{ (\zeta \psi z)^4 \right\} \right],$$

$$v_0(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left(\zeta^4 \psi^4 z^4 \left(\frac{s + \rho(1-s)}{s^2} \right) \right) = \zeta^4 \psi^4 z^4 (1 - \rho + \rho \mathfrak{F}),$$

$$\sum_{j=0}^{\infty} (\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \frac{1}{36} \left(\zeta^2 \sum_{j=0}^{\infty} v_{xj} + \psi^2 \sum_{j=0}^{\infty} v_{yj} + z^2 \sum_{j=0}^{\infty} v_{zj} \right) \right\} \right], \quad (41)$$

FIGURE 6: The first graph shows the exact, and the second graph shows the solution of analytical method at $\rho = 1$ of Problem 3.

FIGURE 7: The analytical solution graph of different fractional-order of ρ of Problem 3.

for $j = 0$

$$v_1(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \zeta^2 \frac{\partial^2 v_0}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v_0}{\partial \psi^2} + z^2 \frac{\partial^2 v_0}{\partial z^2} \right\} \right],$$

$$v_1(\zeta, \psi, z, \mathfrak{F}) = \zeta^4 \psi^4 z^4 \left(1 + \rho^2 - 2\rho - 2\rho^2 \mathfrak{F} + 2\rho \mathfrak{F} + \frac{\mathfrak{F}^2 \rho^2}{2} \right).$$

(42)

The subsequent terms are

$$v_2(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \zeta^2 \frac{\partial^2 v_1}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v_1}{\partial \psi^2} + z^2 \frac{\partial^2 v_1}{\partial z^2} \right\} \right],$$

$$v_2(\zeta, \psi, z, \mathfrak{F}) = \zeta^4 \psi^4 z^4 \left[3\rho(-2\rho + 1 + \rho^2) \mathfrak{F} + \frac{\rho^3 \mathfrak{F}^3}{6} - \frac{3\rho^2(\rho - 1) \mathfrak{F}^2}{2} + 3\rho^2 - 3\rho + 1 - \rho^3 \right],$$

$$v_3(\zeta, \psi, z, \mathfrak{F}) = L^{-1} \left[\left(\frac{s + \rho(1-s)}{s} \right) L \left\{ \zeta^2 \frac{\partial^2 v_2}{\partial \zeta^2} + \psi^2 \frac{\partial^2 v_2}{\partial \psi^2} + z^2 \frac{\partial^2 v_2}{\partial z^2} \right\} \right],$$

$$v_3(\zeta, \psi, z, \mathfrak{F}) = \zeta^4 \psi^4 z^4 \left(-4\rho(-3\rho^2 + 3\rho - 1 + \rho^3) \mathfrak{F} - \frac{2\rho^3(\rho - 1) \mathfrak{F}^3}{3} + 3\mathfrak{F}^2(\rho^4 - 2\rho^3 + \rho^2) - 4\rho^3 + 1 + \rho^4 - 4\rho + 6\rho^2 + \frac{\rho^4 \mathfrak{F}^4}{24} \right).$$

(43)

The LADM solution for Problem 3 is

$$v(\zeta, \psi, z, \mathfrak{F}) = v_0(\zeta, \psi, z, \mathfrak{F}) + v_1(\zeta, \psi, z, \mathfrak{F}) + v_2(\zeta, \psi, z, \mathfrak{F}) + v_3(\zeta, \psi, z, \mathfrak{F}) + \dots$$

$$= \zeta^4 \psi^4 z^4 \left\{ (1 - \rho + \rho \mathfrak{F}) + \left(1 + \rho^2 - 2\rho - 2\rho^2 \mathfrak{F} + 2\rho \mathfrak{F} + \frac{\mathfrak{F}^2 \rho^2}{2} \right) + \left(3\rho(-2\rho + 1 + \rho^2) \mathfrak{F} + \frac{\rho^3 \mathfrak{F}^3}{6} - \frac{3\rho^2(\rho - 1) \mathfrak{F}^2}{2} + 3\rho^2 - 3\rho + 1 - \rho^3 \right) + \left(-4\rho(-3\rho^2 + 3\rho - 1 + \rho^3) \mathfrak{F} - \frac{2\rho^3(\rho - 1) \mathfrak{F}^3}{3} + 3\mathfrak{F}^2(\rho^4 - 2\rho^3 + \rho^2) - 4\rho^3 + 1 + \rho^4 - 4\rho + 6\rho^2 + \frac{\rho^4 \mathfrak{F}^4}{24} \right) + \dots \right\},$$

(44)

TABLE 1: The solutions of example 1 table at different fractional-order of ρ .

$\mathfrak{F}$	ξ	Absolute error ($\rho = 0.4$)	Absolute error ($\rho = 0.6$)	Absolute error ($\rho = 0.8$)	Absolute error ($\rho = 1$)
0.1	1	$5.1298036000 \times 10^{-03}$	$8.5968820000 \times 10^{-04}$	$1.1526960000 \times 10^{-05}$	$1.4693000000 \times 10^{-10}$
	2	$1.0533254900 \times 10^{-02}$	$1.7251291000 \times 10^{-03}$	$2.3064850000 \times 10^{-05}$	$2.1224000000 \times 10^{-09}$
	3	$1.5936706200 \times 10^{-02}$	$2.5905700000 \times 10^{-03}$	$3.4602750000 \times 10^{-05}$	$2.7754000000 \times 10^{-09}$
	4	$2.1340157400 \times 10^{-02}$	$3.4560108000 \times 10^{-03}$	$4.6140640000 \times 10^{-05}$	$3.4285000000 \times 10^{-09}$
	5	$2.6743608600 \times 10^{-02}$	$4.3214516000 \times 10^{-03}$	$5.7678540000 \times 10^{-05}$	$4.0815000000 \times 10^{-09}$
0.2	1	$6.5526269000 \times 10^{-03}$	$1.2845443000 \times 10^{-04}$	$1.9524030000 \times 10^{-05}$	$8.9043500000 \times 10^{-08}$
	2	$1.3585147000 \times 10^{-02}$	$2.5823037000 \times 10^{-03}$	$3.9081030000 \times 10^{-05}$	$1.2243480000 \times 10^{-08}$
	3	$2.0617667100 \times 10^{-03}$	$3.8800631000 \times 10^{-03}$	$5.8638020000 \times 10^{-05}$	$1.5582620000 \times 10^{-08}$
	4	$2.7650187200 \times 10^{-02}$	$5.1778225000 \times 10^{-03}$	$7.8195020000 \times 10^{-05}$	$1.8921750000 \times 10^{-08}$
	5	$3.4682707200 \times 10^{-02}$	$6.4755819000 \times 10^{-03}$	$9.7752020000 \times 10^{-05}$	$2.2260880000 \times 10^{-08}$
0.3	1	$7.5239217000 \times 10^{-03}$	$1.6203247000 \times 10^{-04}$	$2.6503270000 \times 10^{-05}$	$9.9570730000 \times 10^{-08}$
	2	$1.5715901300 \times 10^{-02}$	$3.2621458000 \times 10^{-03}$	$5.3069340000 \times 10^{-05}$	$1.2801951000 \times 10^{-08}$
	3	$2.3907880800 \times 10^{-02}$	$4.9039669000 \times 10^{-03}$	$7.9635420000 \times 10^{-05}$	$1.5646829000 \times 10^{-08}$
	4	$3.2099860300 \times 10^{-02}$	$6.5457880000 \times 10^{-03}$	$1.0620149000 \times 10^{-05}$	$1.8491707000 \times 10^{-08}$
	5	$4.0291839800 \times 10^{-02}$	$8.1876092000 \times 10^{-03}$	$1.3276756000 \times 10^{-05}$	$2.1336585000 \times 10^{-08}$
0.4	1	$8.2762123000 \times 10^{-03}$	$1.9075950000 \times 10^{-03}$	$3.2874570000 \times 10^{-05}$	$5.7825882000 \times 10^{-09}$
	2	$1.7398405800 \times 10^{-02}$	$3.8455493000 \times 10^{-03}$	$6.5848290000 \times 10^{-05}$	$6.7463529000 \times 10^{-08}$
	3	$2.6520599300 \times 10^{-02}$	$5.7835036000 \times 10^{-03}$	$9.8822010000 \times 10^{-05}$	$7.7101176000 \times 10^{-08}$
	4	$3.5642792800 \times 10^{-02}$	$7.7214580000 \times 10^{-03}$	$1.3179574000 \times 10^{-05}$	$8.6738823000 \times 10^{-08}$
	5	$4.4764986200 \times 10^{-02}$	$9.6594122000 \times 10^{-03}$	$1.6476946000 \times 10^{-05}$	$9.6376470000 \times 10^{-08}$
0.5	1	$8.8947364000 \times 10^{-03}$	$2.1627817000 \times 10^{-03}$	$3.8817930000 \times 10^{-04}$	$2.3900000000 \times 10^{-08}$
	2	$1.8806520800 \times 10^{-02}$	$4.3652454000 \times 10^{-03}$	$7.7777130000 \times 10^{-04}$	$3.5300000000 \times 10^{-08}$
	3	$2.8718305200 \times 10^{-02}$	$6.5677091000 \times 10^{-03}$	$1.1673633000 \times 10^{-05}$	$4.6600000000 \times 10^{-08}$
	4	$3.8630089800 \times 10^{-02}$	$8.7701729000 \times 10^{-03}$	$1.5569554000 \times 10^{-04}$	$5.8000000000 \times 10^{-08}$
	5	$4.8541874400 \times 10^{-02}$	$1.0972636700 \times 10^{-03}$	$1.9465475000 \times 10^{-04}$	$6.9300000000 \times 10^{-08}$

when $\rho = 1$, then the LADM solution is

$$v(\zeta, \psi, z, \mathfrak{F}) = \zeta^4 \psi^4 z^4 \left(\mathfrak{F} + \frac{\mathfrak{F}^2}{2!} + \frac{\mathfrak{F}^3}{3!} + \frac{\mathfrak{F}^4}{4!} \dots \right). \quad (45)$$

This is the close form solution of the above Problem 3

$$v(\zeta, \psi, z, \mathfrak{F}) = \left(e^{\mathfrak{F}} - 1 \right) \zeta^4 \psi^4 z^4. \quad (46)$$

Figure 5 shows the exact and approximate solution graphs with the help of Caputo-Fabrizio operator of example 3. In Figure 6, the 3D graph shows the different fractional-order of ρ . In Figure 7, 2D figures show the different fractional-order of ρ with respect to ζ and $\mathfrak{F}$.

5. Conclusions

In this paper, the fractional-order approximate results of heat equations are calculated via LADM with the Caputo-

Fabrizio operator. The solutions have shown the highest agreement with the actual results for the examples. This method results for some problems have revealed the accuracy of the suggested technique. It is observed that the achieved analytical series solutions for the first few terms are very precise and converge very rapidly to the solutions of real physical problems. In the future, LADM can be applied to investigate the analytical result of other linear and nonlinear fractional partial differential equations, which are frequently implemented in engineering and science.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

One of the coauthors (A. M. Zidan) extends his appreciation to the Deanship of Scientific Research at King Khalid University, Abha 61413, Saudi Arabia, for funding this work through a research group program under grant number R.G.P.-2/142/42. This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program to support publication in the top journal (Grant no. 42-FTTJ-42).

References

- [1] J. A. T. M. J. Sabatier, O. P. Agrawal, and J. T. Machado, *Advances in Fractional Calculus*, Springer, Dordrecht, 2007.
- [2] A. Atangana and D. Baleanu, "Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer," *Journal of Engineering Mechanics*, vol. 143, no. 5, p. D4016005, 2017.
- [3] M. Madani, M. Fathizadeh, Y. Khan, and A. Yildirim, "On the coupling of the homotopy perturbation method and Laplace transformation," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1937-1945, 2011.
- [4] Y. Khan and Q. Wu, "Homotopy perturbation transform method for nonlinear equations using He's polynomials," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 1963-1967, 2011.
- [5] V. F. Morales-Delgado, J. F. Gomez-Aguilar, H. Ypez-Martinez, D. Baleanu, R. F. Escobar-Jimenez, and V. H. Olivares-Peregrino, "Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular," *Advances in Difference Equations*, vol. 2016, 2016.
- [6] Y. Li, B. T. Nohara, and S. Liao, "Series solutions of coupled Van der Pol equation by means of homotopy analysis method," *Journal of Mathematical Physics*, vol. 51, no. 6, article 063517, 2010.
- [7] Y. Keskin and G. Oturanc, "Reduced differential transform method for partial differential equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 6, pp. 741-750, 2009.
- [8] P. K. Gupta, "Approximate analytical solutions of fractional Benney-Lin equation by reduced differential transform method and the homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2829-2842, 2011.
- [9] K. H. Huebner, D. L. Dewhirst, D. E. Smith, and T. G. Byrom, *The Finite Element Method for Engineers*, John Wiley & Sons, 2001.
- [10] G. D. Smith, G. D. Smith, and G. D. S. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, Oxford University Press, 1985.
- [11] R. Shah, H. Khan, and D. Baleanu, "Fractional Whitham-Broer-Kaup equations within modified analytical approaches," *Axioms*, vol. 8, no. 4, p. 125, 2019.
- [12] A. M. Lopes and J. T. Machado, "Entropy analysis of soccer dynamics," *Entropy*, vol. 21, no. 2, p. 187, 2019.
- [13] A. Bejan, "Second-Law Analysis in Heat Transfer and Thermal Design," in *Advances in Heat Transfer*, vol. 15, pp. 1-58, Elsevier, 1982.
- [14] A. Bejan, "A study of entropy generation in fundamental convective heat transfer," *Journal of Heat Transfer*, vol. 101, no. 4, pp. 718-725, 1979.
- [15] A. H. Bokhari, G. Mohammad, M. T. Mustafa, and F. D. Zaman, "Adomian decomposition method for a nonlinear heat equation with temperature dependent thermal properties," *Mathematical Problems in Engineering*, vol. 2009, Article ID 926086, 12 pages, 2009.
- [16] S. Sarwar, S. Alkhalaf, S. Iqbal, and M. A. Zahid, "A note on optimal homotopy asymptotic method for the solutions of fractional order heat- and wave-like partial differential equations," *Computers & Mathematics with Applications*, vol. 70, no. 5, pp. 942-953, 2015.
- [17] A. Secer, "Approximate analytic solution of fractional heat-like and wave-like equations with variable coefficients using the differential transforms method," *Advances in Difference Equations*, vol. 2012, no. 1, 2012.
- [18] D.-H. Shou and J.-H. He, "Beyond Adomian method: the variational iteration method for solving heat-like and wave-like equations with variable coefficients," *Physics Letters A*, vol. 372, no. 3, pp. 233-237, 2008.
- [19] D. Rostamy and K. Karimi, "Bernstein polynomials for solving fractional heat- and wave-like equations," *Fractional Calculus and Applied Analysis*, vol. 15, no. 4, pp. 556-571, 2012.
- [20] L. Jin, "Homotopy perturbation method for solving partial differential equations with variable coefficients," *International Journal of Contemporary Mathematical Sciences*, vol. 3, pp. 1395-1407, 2008.
- [21] C.-F. Liu, S.-S. Kong, and S.-J. Yuan, "Reconstructive schemes for variational iteration method within Yang-Laplace transform with application to fractal heat conduction problem," *Thermal Science*, vol. 17, no. 3, pp. 715-721, 2013.
- [22] M. Naeem, O. F. Azhar, A. M. Zidan, K. Nonlaopon, and R. Shah, "Mixed of Elzaki transform and projected differential transform method for a nonlinear wave-like equations with variable coefficients," *Journal of Function Spaces*, 2021.
- [23] R. I. Nuruddeen and A. M. Nass, "Numerical Analysis of Fractional-Order Parabolic Equations via Elzaki Transform," *Stochastic Modelling and Applications*, vol. 21, pp. 23-30, 2017.
- [24] H. Jafari, C. M. Khalique, and M. Nazari, "Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1799-1805, 2011.
- [25] M. Z. Mohamed, "Comparison between the Laplace decomposition method and Adomian decomposition in time-space fractional nonlinear fractional differential equations," *Applied Mathematics*, vol. 9, no. 4, pp. 448-458, 2018.
- [26] O. G. Gaxiola, "The Laplace-Adomian decomposition method applied to the Kundu-Eckhaus equation," *International Journal of Mathematics And its Applications*, vol. 5, no. 1-A, pp. 1-12, 2017.
- [27] M. Al-Zurigat, "Solving nonlinear fractional differential equation using a multi-step Laplace Adomian decomposition method," *Annals of the University of Craiova-Mathematics and Computer Science Series*, vol. 39, no. 2, pp. 200-210, 2012.
- [28] F. Haq, K. Shah, G. ur Rahman, and M. Shahzad, "Numerical solution of fractional order smoking model via Laplace Adomian decomposition method," *Alexandria Engineering Journal*, vol. 57, no. 2, pp. 1061-1069, 2018.