

Research Article

Certain Subclass of m -Valent Functions Associated with a New Extended Ruscheweyh Operator Related to Conic Domains

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The main object of the present paper is to introduce certain subclass of m -valent functions associated with a new extended Ruscheweyh linear operator in the open unit disk. Also, we investigate a number of geometric properties including coefficient estimates and the Fekete–Szegő type inequalities for this subclass. Several known consequences of the main results are also pointed out.

1. Introduction

Let $\mathcal{A}(m)$ denote the class of functions of the next form:

$$f(\xi) = \xi^m + \sum_{n=m+1}^{\infty} a_n \xi^n \quad (m \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic and m -valent in the open unit disc $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$, and let $\mathcal{A}(1) = \mathcal{A}$. Also, let f, g be analytic in \mathbb{D} , and the function $f(\xi)$ is said to be subordinate to $g(\xi)$ if there exists a function $\omega(\xi)$ analytic in \mathbb{D} with $\omega(0) = 0$ and $|\omega(\xi)| < 1, \xi \in \mathbb{D}$, such that $f(\xi) = g(\omega(\xi))$. In such a case, we write $f(\xi) \prec g(\xi)$. If g is univalent function, then $f(\xi) \prec g(\xi)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ (see [1, 2] and [3]).

For functions $f(\xi)$ given by (1) and $g(\xi)$ is defined by

$$g(\xi) = \xi^m + \sum_{n=m+1}^{\infty} b_n \xi^n, \quad (2)$$

and the Hadamard product or convolution of $f(\xi)$ and $g(\xi)$ is defined by

$$(f * g)(\xi) = \xi^m + \sum_{n=m+1}^{\infty} a_n b_n \xi^n. \quad (3)$$

For $\nu \in \mathbb{C}, k \in \mathbb{R}$, and $n \in \mathbb{N}$, the Pochhammer k -symbol $(\nu)_{n,k}$ is given by (see [4])

$$(\nu)_{n,k} = \nu(\nu+k)(\nu+2k) \cdots (\nu+(n-1)k) = \prod_{i=1}^n (\nu+(i-1)k). \quad (4)$$

We define the function $\phi_m(\delta, k; \xi)$ by

$$\begin{aligned} \phi_m(\delta, k; \xi) &= \frac{\xi^m}{(1-\xi)^{\delta+mk/k}} \\ &= \xi^m + \sum_{n=m+1}^{\infty} \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} \xi^n \quad (\delta > -mk; k > 0; \xi \in \mathbb{D}). \end{aligned} \tag{5}$$

Corresponding to the function $\phi_m(\delta, k; \xi)$, we consider a linear operator $\mathcal{D}^{\delta+mk-k} : \mathcal{A}(m) \rightarrow \mathcal{A}(m)$ ($\delta > -mk, k > 0$) which is defined by means of the following Hadamard product (or convolution):

$$\mathcal{D}^{\delta+mk-k}f(\xi) = \phi_m(\delta, k; \xi) * f(\xi) = \xi^m + \sum_{n=m+1}^{\infty} \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} a_n \xi^n \quad (\xi \in \mathbb{D}). \tag{6}$$

It is easily verified from (6) that

$$k\xi \left(\mathcal{D}^{\delta+mk-k}f(\xi) \right)' = (\delta+mk)\mathcal{D}^{\delta+mk}f(\xi) - \delta\mathcal{D}^{\delta+mk-k}f(\xi) \quad (k > 0). \tag{7}$$

We note that

- (1) For $k = 1$, the operator $\mathcal{D}^{\delta+mk-k}f(\xi)$ reduced to the differential operator $\mathcal{D}^{\delta+m-1}f(\xi)$ introduced by Goel and Sohi [5] (see also [6, 7] and [8])
- (2) For $m = 1$, we obtain the k -Ruscheweyh derivative operator \mathcal{D}_k^δ ([9]), where

$$\mathcal{D}_k^\delta f(\xi) = \frac{\xi}{(1-\xi)^{\delta+k/k}} * f(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(\delta+k)_{n-1,k}}{(k)_{n-1,k}} a_n \xi^n. \tag{8}$$

- (3) For $k = m = 1$, the operator $\mathcal{D}^{\delta+mk-k}f(\xi)$ reduced to the well-familiar Ruscheweyh operator \mathcal{D}^δ ([10])
- (4) For $\delta = k - mk$, we have $\mathcal{D}^0 f(\xi) = f(\xi)$, and for $\delta = 2k - mk$, we get $\mathcal{D}^k f(\xi) = \xi^m (\xi^{1-m} f(\xi))'$

By using the linear operator $\mathcal{D}^{\delta+mk-k}f(\xi)$, we define the subclass $\beta - \mathcal{ST}_m(\delta, k, b)$ of $\mathcal{A}(m)$ as follows:

Definition 1. Let $\beta \geq 0, \delta > -mk, m \in \mathbb{N}, k > 0, b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\xi \in \mathbb{D}$. A function $f \in \mathcal{A}(m)$ is in the class $\beta - \mathcal{ST}_m(\delta, k, b)$, if it satisfies

$$\Re \left\{ 1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k}f(\xi) \right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1 \right) \right\} > \beta \left| \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k}f(\xi) \right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1 \right) \right|. \tag{9}$$

Geometrically, a function $f \in \beta - \mathcal{ST}_m(\delta, k, b)$ if and

only if

$$1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k}f(\xi) \right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1 \right), \tag{10}$$

takes all the values in the conic domain $\Omega_\beta = \psi_\beta(\mathbb{D})$, where

$$\Omega_\beta = \left\{ u + iv : u > \beta \sqrt{(u-1)^2 + v^2} \right\}, \tag{11}$$

or equivalently,

$$1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k}f(\xi) \right)'}{m\mathcal{D}^{\delta+mk-k}f(\xi)} - 1 \right) \prec \psi_\beta(\xi), \Omega_\beta = \psi_\beta(\mathbb{D}). \tag{12}$$

The boundary $\partial\Omega_\beta$ of the above set becomes the imaginary axis when $\beta = 0$, a hyperbola when $0 < \beta < 1$, a parabola when $\beta = 1$, and an ellipse when $1 < \beta < \infty$. The functions $\psi_\beta(\xi)$ are defined by

$$\psi_\beta(\xi) = \begin{cases} \frac{1+\xi}{1-\xi} & (\beta=0), \\ 1 + \frac{1}{1-\beta^2} \cos\left(\frac{2}{\pi}(\cos^{-1}\beta)i \log\left(\frac{1+\sqrt{\xi}}{1-\sqrt{\xi}}\right)\right) & (0 < \beta < 1), \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}} \right)^2 & (\beta=1), \\ 1 + \frac{1}{\beta^2-1} \sin\left(\frac{\pi}{2R(t)} \int_0^{u(\xi)/\sqrt{t}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{\beta^2}{\beta^2-1} & (1 < \beta < \infty), \end{cases} \tag{13}$$

with $u(\xi) = \xi - \sqrt{t}/1 - \sqrt{t\xi}$ ($0 < t < 1, \xi \in \mathbb{D}$), where t is chosen such that $k = \cosh(\pi R'(t)/4R(t))$, and $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ the complementary integral of $R(t)$ (see [11, 12] and [13]).

By taking specific values to the parameters β, m, δ, k , and b in the subclass $\beta - \mathcal{ST}_m(\delta, k, b)$, we obtain

- (1) $\beta - \mathcal{ST}_m(\delta, k, (1-\alpha/m) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{ST}_m^\gamma(\delta, k, \alpha)$ ($0 \leq \alpha < m; |\gamma| < \pi/2$) = $\{f \in \mathcal{A}(m) : e^{i\gamma} \xi (\mathcal{D}^{\delta+mk-k}f(\xi))' / \mathcal{D}^{\delta+mk-k}f(\xi) \prec (m-\alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + im \sin \gamma\}$ and $\beta - \mathcal{ST}_m(\delta, k, 1-\alpha/m) = \beta - \mathcal{ST}_m(\delta, k, \alpha)$ ($0 \leq \alpha < m$) = $\{f \in \mathcal{A}(m) : 1/m - \alpha(\xi (\mathcal{D}^{\delta+mk-k}f(\xi))' / \mathcal{D}^{\delta+mk-k}f(\xi) - \alpha) \prec \psi_\beta(\xi)\}$
- (2) $\beta - \mathcal{ST}_m(\delta, 1, b) = \beta - \mathcal{ST}_m(\delta, b) = \{f \in \mathcal{A}(m) : 1 + 1/b(\xi (\mathcal{D}^{\delta+m-1}f(\xi))' / m\mathcal{D}^{\delta+m-1}f(\xi) - 1) \prec \psi_\beta(\xi)\}$,
 $\beta - \mathcal{ST}_m(\delta, 1, (1-\alpha/m) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{ST}_m^\gamma(\delta, \alpha)$

$$(0 \leq \alpha < m; |\gamma| < \pi/2) = \{f \in \mathcal{A}(m): e^{i\gamma}\xi (\mathcal{D}^{\delta+m-1}f(\xi))' / \mathcal{D}^{\delta+m-1}f(\xi) < (m-\alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + im \sin \gamma\} \text{ and } \beta - \mathcal{S}\mathcal{T}_m(\delta, 1, 1 - \alpha/m) = \beta - \mathcal{S}\mathcal{T}_m(\delta, \alpha)(0 \leq \alpha < m) = \{f \in \mathcal{A}(m): 1/m - \alpha(\xi (\mathcal{D}^{\delta+m-1}f(\xi))' / \mathcal{D}^{\delta+m-1}f(\xi) - \alpha) < \psi_\beta(\xi)\}$$

(3) $\beta - \mathcal{S}\mathcal{T}_1(\delta, 1, b) = \beta - \mathcal{S}\mathcal{T}(\delta, b) = \{f \in \mathcal{A} : 1 + 1/b(\xi(\mathcal{D}^\delta f(\xi))' / \mathcal{D}^\delta f(\xi) - 1) < \psi_\beta(\xi)\},$

$$\beta - \mathcal{S}\mathcal{T}_1(\delta, 1, (1 - \alpha) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{S}\mathcal{T}^\gamma(\delta, \alpha)(0 \leq \alpha < 1; |\gamma| < \pi/2) = \{f \in \mathcal{A} : e^{i\gamma}\xi(\mathcal{D}^\delta f(\xi))' / \mathcal{D}^\delta f(\xi) < (1 - \alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + i \sin \gamma\} \text{ and } \beta - \mathcal{S}\mathcal{T}_1(\delta, 1, 1 - \alpha/m) = \beta - \mathcal{S}\mathcal{T}(\delta, \alpha)(0 \leq \alpha < 1) = \{f \in \mathcal{A} : 1/1 - \alpha(\xi(\mathcal{D}^\delta f(\xi))' / \mathcal{D}^\delta f(\xi) - \alpha) < \psi_\beta(\xi)\}$$

(4) $\beta - \mathcal{S}\mathcal{T}_1(\delta, k, b) = \beta - \mathcal{S}\mathcal{T}(\delta, k, b) = \{f \in \mathcal{A} : 1 + 1/b(\xi(\mathcal{D}_k^\delta f(\xi))' / \mathcal{D}_k^\delta f(\xi) - 1) < \psi_\beta(\xi)\},$

$$\beta - \mathcal{S}\mathcal{T}_1(\delta, k, (1 - \alpha) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{S}\mathcal{T}^\gamma(\delta, k, \alpha)(0 \leq \alpha < 1; |\gamma| < \pi/2) = \{f \in \mathcal{A} : e^{i\gamma}\xi(\mathcal{D}_k^\delta f(\xi))' / \mathcal{D}_k^\delta f(\xi) < (1 - \alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + i \sin \gamma\} \text{ and } \beta - \mathcal{S}\mathcal{T}_1(\delta, k, 1 - \alpha/m) = \beta - \mathcal{S}\mathcal{T}(\delta, k, \alpha)(0 \leq \alpha < 1) = \{f \in \mathcal{A} : 1/1 - \alpha(\xi(\mathcal{D}_k^\delta f(\xi))' / \mathcal{D}_k^\delta f(\xi) - \alpha) < \psi_\beta(\xi)\}$$

(5) $\beta - \mathcal{S}\mathcal{T}_m(k - mk, k, b) = \beta - \mathcal{S}\mathcal{T}_m(b) = \{f \in \mathcal{A}(m): 1 + 1/b(\xi f'(\xi)/mf(\xi) - 1) < \psi_\beta(\xi)\}$

$$\beta - \mathcal{S}\mathcal{T}_m(k - mk, k, (1 - \alpha/m) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{S}\mathcal{T}_m^\gamma(\alpha)(0 \leq \alpha < m; |\gamma| < \pi/2) = \{f \in \mathcal{A}(m): e^{i\gamma}\xi f'(\xi)/f(\xi) < (m - \alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + im \sin \gamma\} \text{ and (see [14])}$$

(6) $\beta - \mathcal{S}\mathcal{T}_m(k - mk, k, 1 - \alpha/m) = \beta - \mathcal{S}\mathcal{T}_m(\alpha)(0 \leq \alpha < m) = \{f \in \mathcal{A}(m): 1/m - \alpha(\xi f'(\xi)/f(\xi) - \alpha) < \psi_\beta(\xi)\};$

$$\beta - \mathcal{S}\mathcal{T}_1(0, k, b) = \beta - \mathcal{S}\mathcal{T}(b) = \{f \in \mathcal{A} : 1 + 1/b(\xi f'(\xi)/f(\xi) - 1) < \psi_\beta(\xi)\},$$

$$\beta - \mathcal{S}\mathcal{T}_1(0, k, (1 - \alpha) \cos \gamma e^{-i\gamma}) = \beta - \mathcal{S}\mathcal{T}^\gamma(\alpha)(0 \leq \alpha < 1; |\gamma| < \pi/2) = \{f \in \mathcal{A}(m): e^{i\gamma}\xi f'(\xi)/f(\xi) < (1 - \alpha) \cos \gamma \psi_\beta(\xi) + \alpha \cos \gamma + i \sin \gamma\} \text{ and (see [15]) } \beta - \mathcal{S}\mathcal{T}_1(0, k, 1 - \alpha) = \beta - \mathcal{S}\mathcal{T}(\alpha)(0 \leq \alpha < 1) = \{f \in \mathcal{A} : 1/1 - \alpha(\xi f'(\xi)/f(\xi) - \alpha) < \psi_\beta(\xi)\}$$

(7) $0 - \mathcal{S}\mathcal{T}_m(\delta, k, b) = \mathcal{S}_m(\delta, k, b) = \{f \in \mathcal{A}(m): \Re[m + 1/b(\xi(\mathcal{D}^{\delta+mk-k}f(\xi))' / \mathcal{D}^{\delta+mk-k}f(\xi) - m)] > 0\},$

$$0 - \mathcal{S}\mathcal{T}_m(\delta, k, (1 - \alpha/m) \cos \gamma e^{-i\gamma}) = \mathcal{S}_m^\gamma(\delta, k, \alpha)(0$$

$$\leq \alpha < m; |\gamma| < \pi/2) = \{f \in \mathcal{A}(m): \Re(e^{i\gamma}\xi (\mathcal{D}^{\delta+mk-k}f(\xi))' / \mathcal{D}^{\delta+mk-k}f(\xi)) > \alpha \cos \gamma\}, \text{ (see [16] and [17]) } 0 - \mathcal{S}\mathcal{T}_m(k - mk, k, b) = \mathcal{S}_m^\gamma(\alpha)(0 \leq \alpha < m; |\gamma| < \pi/2) = \{f \in \mathcal{A}(m): \Re(e^{i\gamma}\xi f'(\xi)/f(\xi)) > \alpha \cos \gamma\}, \text{ (see [18]) } 0 - \mathcal{S}\mathcal{T}_m(k - mk, k, b) = \mathcal{S}_m(b) = \{f \in \mathcal{A}(m): \Re[m + 1/b(\xi f'(\xi)/f(\xi) - m)] > 0\} \text{ and } \mathcal{S}_1(b) = \mathcal{S}(b) \text{ (see [19, 20])}$$

In order to establish our main results, we need the following lemmas.

Lemma 2 [21]. *Let $\psi_\beta(\xi)(0 \leq \beta < \infty)$ be defined by (13). If*

$$\psi_\beta(\xi) = 1 + L_1\xi + L_2\xi^2 + \dots, \tag{14}$$

then

$$L_1 = \begin{cases} \frac{2A^2}{1 - \beta^2} & (0 \leq \beta < 1), \\ \frac{8}{\pi^2} & (\beta = 1), \\ \frac{\pi^2}{4\sqrt{t}(\beta^2 - 1)R^2(t)(1+t)} & (1 < \beta < \infty), \end{cases} \tag{15}$$

$$L_2 = \begin{cases} \frac{A^2 + 2}{3}L_1 & (0 \leq \beta < 1), \\ \frac{2}{3}L_1 & (\beta = 1), \\ \frac{4R^2(t)(t^2 + 6t + 1) - \pi^2}{24\sqrt{t}R^2(t)(1+t)}L_1 & (1 \leq \beta < \infty), \end{cases} \tag{16}$$

where

$$A = \frac{2 \cos^{-1} \beta}{\pi}, \tag{17}$$

and $t \in (0, 1)$ is chosen such that $\beta = \cosh(\pi R'(t)/R(t))$, and $R(t)$ is the Legendre's complete elliptic integral of the first kind.

Lemma 3 [22]. *Let $h(\xi) = 1 + \sum_{n=1}^\infty c_n \xi^n \in \mathcal{P}$, i.e., let h be analytic in \mathbb{D} and satisfy $\Re\{h(\xi)\} > 0$ for ξ in \mathbb{D} ; then, the following sharp estimate holds*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\} \text{ for all } \nu \in \mathbb{C}. \tag{18}$$

The result is sharp for the functions given by

$$g(\xi) = \frac{1 + \xi^2}{1 - \xi^2} \text{ or } g(\xi) = \frac{1 + \xi}{1 - \xi}. \tag{19}$$

Lemma 4 [22]. If $h(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n \in \mathcal{P}$, then

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0, \\ 2 & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2 & \text{if } \nu \geq 1, \end{cases} \quad (20)$$

and when $\nu < 0$ or $\nu > 1$, the equality holds if and only if $h(\xi) = (1 + \xi)/(1 - \xi)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $h(\xi) = (1 + \xi^2)/(1 - \xi^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$h(\xi) = \left(\frac{1 + \lambda}{2}\right) \frac{1 + \xi}{1 - \xi} + \left(\frac{1 - \lambda}{2}\right) \frac{1 - \xi}{1 + \xi} \quad (0 \leq \lambda \leq 1), \quad (21)$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if g is the reciprocal of one of the functions such that equality holds in the case of $\nu = 0$.

Also, the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \left(0 \leq \nu \leq \frac{1}{2}\right), \quad (22)$$

$$|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \left(\frac{1}{2} \leq \nu \leq 1\right).$$

In this paper, we investigate a coefficient estimates and the familiar Fekete–Szegő type inequalities for the subclass $\beta - \mathcal{ST}_m(\delta, k, b)$.

2. Main Results

We will assume throughout our discussion, unless otherwise stated, that $0 \leq \beta < \infty$, $m \in \mathbb{N}$, $\delta > -mk$, $k > 0$, $b \in \mathbb{C}^*$, L_1 is given by (15), L_2 is given by (16), and $\xi \in \mathbb{D}$.

Theorem 5. Let $f \in \mathcal{A}(m)$ be given by (1). If the inequality

$$\sum_{n=m+1}^{\infty} \{(\beta + 1)(n - m) + m|b|\} \frac{(\delta + mk)_{n-m,k}}{(k)_{n-m,k}} |a_n| \leq m|b|, \quad (23)$$

holds, then the function $f \in \beta - \mathcal{ST}_m(\delta, k, b)$.

Proof. Suppose the inequality (23) holds. Also, let us assume

$$H(\xi) = 1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k} f(\xi) \right)'}{m \mathcal{D}^{\delta+mk-k} f(\xi)} - 1 \right). \quad (24)$$

We have

$$\begin{aligned} |H(\xi) - 1| &= \left| \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k} f(\xi) \right)'}{m \mathcal{D}^{\delta+mk-k} f(\xi)} - 1 \right) \right| \\ &= \frac{1}{m|b|} \left| \left(\frac{\sum_{n=m+1}^{\infty} (n - m) ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) a_n \xi^{n-m}}{1 + \sum_{n=m+1}^{\infty} ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) a_n \xi^{n-m}} \right) \right| \\ &\leq \frac{\sum_{n=m+1}^{\infty} (n - m) ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) |a_n|}{m|b| \left[1 - \sum_{n=m+1}^{\infty} ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) |a_n| \right]}. \end{aligned} \quad (25)$$

Now consider

$$\begin{aligned} \beta |H(\xi) - 1| - \Re \{H(\xi) - 1\} &\leq (\beta + 1) |H(\xi) - 1| \\ &< \frac{(\beta + 1) \sum_{n=m+1}^{\infty} ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) |a_n|}{m|b| \left[1 - \sum_{n=m+1}^{\infty} ((\delta + mk)_{n-m,k} / (k)_{n-m,k}) |a_n| \right]}. \end{aligned} \quad (26)$$

The last expression is bounded by 1 if (23) holds. This completes the proof of Theorem 5. \square

Corollary 6. If $f(\xi) \in \beta - \mathcal{ST}_m(\delta, k, b)$, then

$$|a_n| \leq \frac{m|b|(k)_{n-m,k}}{\{(\beta + 1)(n - m) + m|b|\} (\delta + mk)_{n-m,k}} \quad (n \geq m + 1). \quad (27)$$

The result is sharp for the function

$$f(\xi) = \xi^m + \frac{m|b|(k)_{n-m,k}}{\{(\beta + 1)(n - m) + m|b|\} (\delta + mk)_{n-m,k}} \xi^n \quad (n \geq m + 1). \quad (28)$$

Putting $m = 1$ in Theorem 5, we obtain the following corollary.

Corollary 7. Let $f \in \mathcal{A}$ be given by (1) with $m = 1$. If the inequality

$$\sum_{n=2}^{\infty} \{(\beta + 1)(n - 1) + |b|\} \frac{(\delta + k)_{n-1,k}}{(k)_{n-1,k}} |a_n| \leq |b|, \quad (29)$$

holds, then the function $f \in \beta - \mathcal{ST}(\delta, k, b)$.

Theorem 8. If $f \in \beta - \mathcal{ST}_m(\delta, k, b)$, then

$$|a_{m+1}| \leq \frac{m|b|L_1 k}{\delta + mk}, \quad (30)$$

and for all $n = 3, 4, 5, \dots$,

$$|a_{n+m-1}| \leq \frac{m|b|L_1(k)_{n-1,k}}{(n - 1) (\delta + mk)_{n-1,k}} \prod_{j=1}^{n-2} \left(1 + \frac{m|b|L_1}{j} \right). \quad (31)$$

Proof. Let

$$g(\xi) = 1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k} f(\xi) \right)'}{m \mathcal{D}^{\delta+mk-k} f(\xi)} - 1 \right), \quad (32)$$

where

$$g(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n, \quad (33)$$

is analytic function in \mathbb{D} , and it can be written as

$$\sum_{n=m+1}^{\infty} (n-m) \frac{(\delta+mk)_{n-m,k}}{(k)_{n-m,k}} a_n \xi^n = mb \mathcal{D}^{\delta+mk-k} f(\xi) \left(\sum_{n=1}^{\infty} c_n \xi^n \right). \quad (34)$$

Comparing the coefficients of ξ^{n+m-1} on both sides

$$\begin{aligned} (n-1) \frac{(\delta+mk)_{n-1,k}}{(k)_{n-1,k}} a_{n+m-1} \\ = mb \left[c_{n-1} + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}} c_{n-2} a_{m+1} + \dots + \frac{(\delta+mk)_{n-2,k}}{(k)_{n-2,k}} c_1 a_{n+m-2} \right]. \end{aligned} \quad (35)$$

Taking absolute on both sides and then applying the coefficient estimates $|c_n| \leq L_1$ (see [13]), we have

$$\begin{aligned} |a_{n+m-1}| \leq \frac{m|b|L_1(k)_{n-1,k}}{(n-1)(\delta+mk)_{n-1,k}} \\ \cdot \left\{ 1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}} |a_{m+1}| + \dots + \frac{(\delta+mk)_{n-2,k}}{(k)_{n-2,k}} |a_{n+m-2}| \right\}. \end{aligned} \quad (36)$$

We apply the mathematical induction on (36) so for $n = 2$

$$|a_{m+1}| \leq \frac{m|b|L_1(k)_{1,k}}{(\delta+mk)_{1,k}} = \frac{m|b|L_1 k}{\delta+mk}, \quad (37)$$

and this shows that result is true for $n = 2$. Now for $n = 3$,

$$|a_{m+2}| \leq \frac{m|b|L_1(k)_{2,k}}{2(\delta+mk)_{2,k}} \left\{ 1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}} |a_{m+1}| \right\}, \quad (38)$$

and using (37), we obtain

$$|a_{m+2}| \leq \frac{m|b|L_1(k)_{2,k}}{2(\delta+mk)_{2,k}} (1 + m|b|L_1), \quad (39)$$

which is true for $n = 3$. Let us assume that (31) is true for $n = t$, that is,

$$|a_{t+m-1}| \leq \frac{m|b|L_1(k)_{t-1,k}}{(t-1)(\delta+mk)_{t-1,k}} \prod_{j=1}^{t-2} \left(1 + \frac{m|b|L_1}{j} \right). \quad (40)$$

Consider

$$\begin{aligned} |a_{t+m}| &\leq \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}} \left\{ 1 + \frac{(\delta+mk)_{1,k}}{(k)_{1,k}} |a_{m+1}| + \dots + \frac{(\delta+mk)_{t-1,k}}{(k)_{t-1,k}} |a_{t+m-1}| \right\} \\ &\leq \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}} \left\{ 1 + m|b|L_1 + \dots + \frac{(\delta+mk)_{t-1,k}}{(k)_{t-1,k}} |a_{t+m-1}| \right\} \\ &\leq \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}} \left\{ 1 + m|b|L_1 + \frac{m|b|L_1}{2} (1 + m|b|L_1) + \dots + \frac{m|b|L_1}{t-1} \right. \\ &\quad \cdot \left. (1 + m|b|L_1) \left(1 + \frac{m|b|L_1}{2} \right) \dots \left(1 + \frac{m|b|L_1}{t-2} \right) \right\} \\ &= \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}} (1 + m|b|L_1) \left(1 + \frac{m|b|L_1}{2} \right) \dots \left(1 + \frac{m|b|L_1}{t-1} \right) \\ &= \frac{m|b|L_1(k)_{t,k}}{t(\delta+mk)_{t,k}} \prod_{j=1}^{t-1} \left(1 + \frac{m|b|L_1}{j} \right). \end{aligned} \quad (41)$$

Therefore, the result is true for $n = t + 1$. Consequently, using mathematical induction, we proved that the result holds true for all $n (n \geq 2)$. This completes the proof of Theorem 8. \square

Putting $m = 1$ in Theorem 8, we obtain the following corollary.

Corollary 9. *If $f \in \beta - \mathcal{ST}(\delta, k, b)$, then*

$$|a_2| \leq \frac{|b|L_1 k}{\delta + k}, \quad (42)$$

and for all $n = 3, 4, 5, \dots$,

$$|a_n| \leq \frac{|b|L_1(k)_{n-1,k}}{(n-1)(\delta+k)_{n-1,k}} \prod_{j=1}^{n-2} \left(1 + \frac{|b|L_1}{j} \right). \quad (43)$$

Theorem 10. *Let $f \in \beta - \mathcal{ST}_m(\delta, k, b)$. Then., $f(\mathbb{D})$ contains an open disk of radius*

$$r = \frac{\delta + mk}{(m+1)(\delta + mk) + m|b|L_1 k}. \quad (44)$$

Proof. Let $w_0 \neq 0$ be a complex number such that $f(\xi) \neq w_0$ for $\xi \in \mathbb{D}$. Then, $f_1(\xi) = w_0 f(\xi) / w_0 - f(\xi) = \xi^m + (a_{m+1} + 1/w_0) \xi^{m+1} + \dots$.

Since f_1 is univalent, so

$$\left| a_{m+1} + \frac{1}{w_0} \right| \leq m + 1. \quad (45)$$

Now using Theorem 8, we have

$$\left| \frac{1}{w_0} \right| \leq m + 1 + \frac{m|b|L_1 k}{\delta + mk}, \quad (46)$$

and hence

$$|w_0| \geq \frac{\delta + mk}{(m + 1)(\delta + mk) + m|b|L_1k}. \tag{47}$$

This completes the proof of Theorem 10. \square

Putting $m = 1$ in Theorem 10, we obtain the following corollary.

Corollary 11. *Let $f \in \beta - \mathcal{ST}(\delta, k, b)$. Then, $f(\mathbb{D})$ contains an open disk of radius*

$$r_1 = \frac{\delta + k}{2(\delta + k) + |b|L_1k}. \tag{48}$$

Theorem 12. *Let $f \in \beta - \mathcal{ST}_m(\delta, k, b)$ with the form (1). Then, for a complex number μ , we have*

$$|a_{m+2} - \mu a_{m+1}^2| \leq \frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)} \max \left\{ 1, \left| \frac{L_2}{L_1} + mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right| \right\}. \tag{49}$$

Proof. If $f \in \beta - \mathcal{ST}_m(\delta, k, b)$, then there exists a Schwarz function w , with $w(0) = 0$ and $|w(\xi)| < 1$ such that

$$1 + \frac{1}{b} \left(\frac{\xi \left(\mathcal{D}^{\delta+mk-k} f(\xi) \right)'}{m \mathcal{D}^{\delta+mk-k} f(\xi)} - 1 \right) = \psi_\beta(w(\xi)) \quad (\xi \in \mathbb{D}). \tag{50}$$

Let $h \in \mathcal{P}$ be a function defined by

$$h(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + c_1\xi + c_2\xi^2 + \dots \quad (\xi \in \mathbb{D}). \tag{51}$$

This gives

$$w(\xi) = \frac{c_1}{2}\xi + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) \xi^2 + \dots, \tag{52}$$

$$\psi_\beta(w(\xi)) = 1 + \frac{1}{2}c_1L_1\xi + \frac{1}{2} \left\{ \frac{c_1^2L_2}{2} + \left(c_2 - \frac{c_1^2}{2} \right) L_1 \right\} \xi^2 + \dots. \tag{53}$$

Using (53) in (50), we obtain

$$a_{m+1} = \frac{mbc_1L_1k}{2(\delta + mk)},$$

$$a_{m+2} = \frac{mbk^2}{2(\delta + mk)(\delta + mk + k)} \left\{ \frac{c_1^2L_2}{2} + \left(c_2 - \frac{c_1^2}{2} \right) L_1 + \frac{mbc_1^2L_1^2}{2} \right\}. \tag{54}$$

For any complex number μ , we have

$$a_{m+2} - \mu a_{m+1}^2 = \frac{mbk^2}{2(\delta + mk)(\delta + mk + k)} \left\{ \frac{c_1^2L_2}{2} + \left(c_2 - \frac{c_1^2}{2} \right) L_1 + \frac{mbc_1^2L_1^2}{2} \right\} - \mu \frac{m^2b^2c_1^2L_1^2k^2}{4(\delta + mk)^2}. \tag{55}$$

Then (55) can be written as

$$a_{m+2} - \mu a_{m+1}^2 = \frac{mbL_1k^2}{2(\delta + mk)(\delta + mk + k)} \{ c_2 - \nu c_1^2 \}, \tag{56}$$

where

$$\nu = \frac{1}{2} \left\{ 1 - \frac{L_2}{L_1} - mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right\}. \tag{57}$$

Now, taking absolute value on both sides and using Lemma 3, we obtain the required result. \square

Putting $m = 1$ in Theorem 12, we obtain the following corollary.

Corollary 13. *Let $f \in \beta - \mathcal{ST}(\delta, k, b)$ with the form (1). Then, for a complex number μ , we have*

$$|a_3 - \mu a_2^2| \leq \frac{bL_1k^2}{(\delta + k)(\delta + 2k)} \max \left\{ 1, \left| \frac{L_2}{L_1} + bL_1 \left(1 - \frac{\delta + 2k}{\delta + k} \mu \right) \right| \right\}. \tag{58}$$

Theorem 14. *Let*

$$\sigma_1 = \frac{\{mbL_1^2 + L_2 - L_1\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}, \sigma_2 = \frac{\{mbL_1^2 + L_2 + L_1\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}, \sigma_3 = \frac{\{mbL_1^2 + L_2\}(\delta + mk)}{mb(\delta + mk + k)L_1^2}. \tag{59}$$

If f given by (1) belongs to $\beta - \mathcal{ST}_m(\delta, k, b)$ with $b > 0$, then

$$|a_{m+2} - \mu a_{m+1}^2| \leq \begin{cases} \frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)} \left\{ \frac{L_2}{L_1} + mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right\} & (\mu \leq \sigma_1), \\ \frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)} & (\sigma_1 \leq \mu \leq \sigma_2), \\ -\frac{mbL_1k^2}{(\delta + mk)(\delta + mk + k)} \left\{ \frac{L_2}{L_1} + mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right\} & (\mu \geq \sigma_2). \end{cases} \tag{60}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_{m+2} - \mu a_{m+1}^2| + \frac{\delta + mk}{mb(\delta + mk + k)L_1} \left\{ 1 - \frac{L_2}{L_1} - mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right\} |a_{m+1}|^2 \leq \frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)}. \tag{61}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_{m+2} - \mu a_{m+1}^2| + \frac{\delta + mk}{mb(\delta + mk + k)L_1} \times \left\{ 1 + \frac{L_2}{L_1} + mbL_1 \left(1 - \frac{\delta + mk + k}{\delta + mk} \mu \right) \right\} |a_{m+1}|^2 \leq \frac{mbL_1 k^2}{(\delta + mk)(\delta + mk + k)}. \tag{62}$$

Proof. Applying Lemma 4 to (56) and (57), respectively, we can obtain our results asserted by Theorem 14. \square

Putting $m = 1$ in Theorem 14, we obtain

Corollary 15. *Let*

$$\sigma_4 = \frac{\{bL_1^2 + L_2 - L_1\}(\delta + k)}{b(\delta + 2k)L_1^2}, \sigma_5 = \frac{\{bL_1^2 + L_2 + L_1\}(\delta + k)}{b(\delta + 2k)L_1^2}, \sigma_6 = \frac{\{bL_1^2 + L_2\}(\delta + k)}{b(\delta + 2k)L_1^2}. \tag{63}$$

If f given by (1) belongs to $\beta - \mathcal{ST}(\delta, k, b)$ with $b > 0$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{bL_1 k^2}{(\delta + k)(\delta + 2k)} \left\{ \frac{L_2}{L_1} + bL_1 \left(1 - \frac{\delta + 2k}{\delta + k} \mu \right) \right\} & (\mu \leq \sigma_4), \\ \frac{bL_1 k^2}{(\delta + k)(\delta + 2k)} & (\sigma_4 \leq \mu \leq \sigma_5), \\ -\frac{bL_1 k^2}{(\delta + k)(\delta + 2k)} \left\{ \frac{L_2}{L_1} + bL_1 \left(1 - \frac{\delta + 2k}{\delta + k} \mu \right) \right\} & (\mu \geq \sigma_5). \end{cases} \tag{64}$$

Further, if $\sigma_4 \leq \mu \leq \sigma_6$, then

$$|a_3 - \mu a_2^2| + \frac{\delta + k}{b(\delta + 2k)L_1} \left\{ 1 - \frac{L_2}{L_1} - bL_1 \left(1 - \frac{\delta + 2k}{\delta + k} \mu \right) \right\} |a_2|^2 \leq \frac{bL_1 k^2}{(\delta + k)(\delta + 2k)}. \tag{65}$$

If $\sigma_6 \leq \mu \leq \sigma_5$, then

$$|a_3 - \mu a_2^2| + \frac{\delta + k}{b(\delta + 2k)L_1} \left\{ 1 + \frac{L_2}{L_1} + bL_1 \left(1 - \frac{\delta + 2k}{\delta + k} \mu \right) \right\} |a_2|^2 \leq \frac{bL_1 k^2}{(\delta + k)(\delta + 2k)}. \tag{66}$$

Remark 16. For different choices of the parameters β, m, δ, k , and b in the above theorems, we can obtain the corresponding results for each of the following subclasses $\beta - \mathcal{ST}_m^\gamma(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(\delta, b)$, $\beta - \mathcal{ST}_m^\gamma(\delta, \alpha)$, $\beta - \mathcal{ST}_m(\delta, \alpha)$, $\beta - \mathcal{ST}(\delta, b)$, $\beta - \mathcal{ST}^\gamma(\delta, \alpha)$, $\beta - \mathcal{ST}(\delta, \alpha)$, $\beta - \mathcal{ST}^\gamma(\delta, k, \alpha)$, $\beta - \mathcal{ST}(\delta, k, \alpha)$, $\beta - \mathcal{ST}_m(b)$, $\beta - \mathcal{ST}_m^\gamma(\alpha)$, $\beta - \mathcal{ST}_m(\alpha)$, $\beta - \mathcal{ST}(b)$, $\beta - \mathcal{ST}^\gamma(\alpha)$, $\beta - \mathcal{ST}(\alpha)$, $\mathcal{S}_m(\delta, k, b)$, $\mathcal{S}_m^\gamma(\delta, k, \alpha)$, $\mathcal{S}_m^\gamma(\alpha)$, $\mathcal{S}_m(b)$, and $\mathcal{S}(b)$ which are defined in Section 1.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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