Research Article

Some Properties of Solutions to Multiterm Fractional Boundary Value Problems with p -Laplacian Operator

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In this paper, we study some properties of positive solutions to a class of multipoint boundary value problems for nonlinear multiterm fractional differential equations with p -Laplacian operator. Using the Banach contraction mapping principle, the existence, the uniqueness, the positivity, and the continuous dependency on m -point boundary conditions of the solutions to the given problem are investigated. Also, two examples are presented to demonstrate our main results.

1. Introduction

This paper deals with a class of multipoint boundary value problems for nonlinear multiterm fractional differential equations with p -Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}x(t))) = f(t, x(t), D_{0+}^{\gamma}x(t)), & 0 < t < 1, \\ D_{0+}^{\gamma}x(0) = 0, D_{0+}^{\delta}x(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\delta}x(\eta_i), \\ D_{0+}^{\alpha}x(0) = 0, \varphi_p(D_{0+}^{\alpha}x(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0+}^{\alpha}x(\eta_i)), \end{cases} \quad (1)$$

where $1 < \alpha, \beta \leq 2, 0 < \gamma, \delta < 1, 0 < \xi_i, \eta_i, \zeta_i < 1, i = 1, 2, \dots, m-2$ and $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}, D_{0+}^{\delta}$ are the standard Riemann-Liouville derivatives. And it is assumed that the nonlinear source term f satisfies $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty))$, and φ_p is defined as $\varphi_p(s) = |s|^{p-2}s, p > 1$.

Fractional differential equations have developed into a powerful tool to mathematically model and solve many real-world problems that arises in various fields such as physics,

chemistry, biology, and mechanics (see [1, 2] and references therein). Especially, since the turbulent flow was one of the fundamental problems in the field of fluid dynamics, Leibenson [3] and Esteban and Vazquez [4] proposed a mathematical model for one-dimensional, polytropic, and turbulent flow of a gas in a porous medium as

$$u_t = \frac{\partial}{\partial x} \left(\frac{\partial u^m}{\partial x} \left| \frac{\partial u^m}{\partial x} \right|^{p-1} \right), \quad (2)$$

where $u(t, x)$ was a scaled density at every point x and time t . According to them, if $p = 1$, the equation refers to the laminar filtration, in which case, it is often known as the fast diffusion equation if $0 < m < 1$ and the linear heat equation if $m = 1$; otherwise, the porous media equation. On the other hand, the case $m = 1, p \neq 1$ has been widely studied in non-Newtonian fluid dynamics (see [5] and the references therein). By some substitutions and tricks, this nonlinear problem was reduced into the following p -Laplacian equation to investigate the existence and properties of solutions to it [6, 7]:

$$(\varphi_p(u'(t)))' = f(t, u(t), u'(t)). \quad (3)$$

Due to the significance of equation (3), many works have been studied to establish a lot of valuable existence and multiplicity results for various classes of higher-order and generalized p -Laplacian differential equations with integer-order derivatives [8–10]. For example, Guo et al. [8] dealt with the existence of at least three positive solutions to the quasilinear second-order differential equation

$$\left(\varphi_p\left(u'(t)\right)\right)' + a(t)f(t, u(t)) = 0, 0 < t < 1, \quad (4)$$

subject to one of the following m -point boundary conditions:

$$\varphi_p\left(u'(0)\right) = \sum_{i=1}^{m-2} a_i \varphi_p\left(u'(\xi_i)\right), u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i) \quad (5)$$

or

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \varphi_p\left(u'(1)\right) = \sum_{i=1}^{m-2} b_i \varphi_p\left(u'(\xi_i)\right), \quad (6)$$

by using the five functional fixed point theorem. Generalizing the results of the above studies for integer-order differential equations, many researchers have obtained great outcomes of the existence of solutions to single-term fractional differential equations with p -Laplacian operator (see [11–15]). From the representative results, Lv [11] established the existence and multiplicity of positive solutions to m -point boundary value problems of p -Laplacian fractional differential equations with a parameter:

$$\begin{cases} D_{0+}^{\beta}\left(\varphi_p\left(D_{0+}^{\alpha}u(t)\right)\right) + \varphi_p(\lambda)f(t, u(t)) = 0, 0 < t < 1, 1 < \alpha \leq 2, 0 < \beta \leq 1, \\ u(0) = 0, D_{0+}^{\gamma}u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma}u(\eta_i), D_{0+}^{\alpha}u(0) = 0, 0 < \gamma \leq 1, \end{cases} \quad (7)$$

by using the theory of the fixed-point index in a cone and the monotone iterative technique. In addition, Li and Qi [12] improved the works of Guo et al. [8] to show the existence of positive solutions for the following multipoint boundary value problems of nonlinear fractional differential equations with p -Laplacian operator:

$$\begin{cases} {}^C D_{0+}^{\beta}\left(\varphi_p\left({}^C D_{0+}^{\alpha}u(t)\right)\right) + f(t, u(t)) = 0, 0 < t < 1, n-1 < \alpha \leq n, l-1 < \beta \leq l, \\ \varphi_p\left({}^C D_{0+}^{\alpha}u(0)\right)^{(i)} = 0, i = 1, 2, \dots, l-1, \\ \varphi_p\left({}^C D_{0+}^{\alpha}u(1)\right) = \sum_{i=1}^{m-2} b_i \left[\varphi_p\left({}^C D_{0+}^{\alpha}u(\xi_i)\right)\right], \\ u^{(i)}(0) = 0, i = 1, 2, \dots, n-1, \\ u(0) = \sum_{i=1}^{m-2} a_i u^{(j)}(\xi_i), \end{cases} \quad (8)$$

where $l > 1, n \geq 2, j \in (0, \alpha)$ are integers and ${}^C D_{0+}^{\alpha}, {}^C D_{0+}^{\beta}$ are the Caputo fractional derivatives. It is noted that the nota-

tions D_{0+}^{α} and ${}^C D_{0+}^{\alpha}$ appearing throughout the paper refer to the Riemann-Liouville and Caputo fractional derivatives, respectively, unless otherwise stated.

However, since the differential equation studied by Leibenson [3] is a multiterm one, it is important to extend the research results for single-term fractional differential equations to the case of multiterm. To do this, the existence of solutions to nonlinear multiterm fractional differential equations with a p -Laplacian operator have been studied by employing useful techniques in a nonlinear functional analysis such as the Banach contraction mapping principle, the Schauder fixed-point theorem, and the Krasnoselskii fixed-point theorem (see [16–20]). In particular, it is of great significance to study the case where α, β, γ are in $(0, 1)$ or $(1, 2)$, when considering the following fractional differential equation:

$$D^{\beta}\left(\varphi_p\left(D^{\alpha}u\right)\right)(t) = f(t, u(t), D^{\gamma}u(t)), \quad (9)$$

generalized equation (3) because of the inclusion of first-order derivatives outside and inside of φ_p in it. Here, D^{α}, D^{β} , and D^{γ} refer to fractional derivatives in several senses such as the Riemann-Liouville and the Caputo. Therefore, by using the coincidence degree theory, Chen et al. [16] studied two-point boundary value problems of fractional differential equations with p -Laplacian operator:

$$\begin{cases} {}^C D_{0+}^{\beta}\left(\varphi_p\left({}^C D_{0+}^{\alpha}x(t)\right)\right) = f\left(t, x(t), {}^C D_{0+}^{\alpha}x(t)\right), t \in [0, 1], 0 < \alpha, \beta \leq 1, \\ {}^C D_{0+}^{\alpha}x(0) = {}^C D_{0+}^{\alpha}x(1) = 0, \end{cases} \quad (10)$$

to obtain new results on the existence of solutions to them. Besides, according to Liu et al. [17], the following four-point boundary value problems of p -Laplacian fractional differential equations with mixed fractional derivatives:

$$\begin{cases} D_{0+}^{\alpha}\left(\varphi_p\left({}^C D_{0+}^{\beta}u(t)\right)\right) = f\left(t, u(t), {}^C D_{0+}^{\beta}u(t)\right), t \in (0, 1), 1 < \alpha, \beta \leq 2, \\ {}^C D_{0+}^{\beta}u(0) = u'(0) = 0, \\ u(1) = r_1 u(\eta), {}^C D_{0+}^{\beta}u(1) = r_2 {}^C D_{0+}^{\beta}u(\xi), \end{cases} \quad (11)$$

were considered to establish a lower and upper solution method which was used to prove the existence of positive solutions to them. After analyzing the results for nonlinear multiterm fractional differential equations with p -Laplacian operator, we have found that most of these previous works only dealt with some kinds of p -Laplacian equations that had certain rules for the fractional derivatives in nonlinear source terms and were limited to the cases of two-point, three-point, and four-point boundary conditions.

On the other hand, Dishlieva [21] and Li et al. [22] investigated the continuous dependency of solutions to their differential equations on initial conditions, barrier curves, and source terms in their studies but not on boundary conditions.

This also implies that as far as we know, no works concerned with this issue for fractional differential equations with p -Laplacian operator can be done.

Motivated by the analysis mentioned above, this paper is aimed at studying some properties of positive solutions to m -point boundary value problems of nonlinear multi-term fractional differential equations with p -Laplacian operator including the existence, the uniqueness, and the continuous dependency on boundary conditions.

This study is organized as follows. In Section 2, we give some necessary definitions and preliminary results which will be used to prove our main results. In Section 3, we prove the existence and uniqueness of positive solutions to p -Laplacian fractional boundary value problem (1) and continuous dependence of them on the perturbations with respect to the coefficients in m -point boundary conditions. Finally, in Section 4, we illustrate our results by giving two examples.

Throughout the whole paper, it is also supposed that

$$3 < \alpha + \beta \leq 4, \gamma < \delta, \alpha - \delta - 1 > 0, \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\delta-1} < 1, \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} < 1. \quad (12)$$

2. Preliminaries

For the sake of convenience of the readers, some necessary definitions and lemmas will be presented here.

The Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbf{R}$ are defined as

$$\begin{aligned} (I_{0+}^\alpha f)(t) &:= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (D_{0+}^\alpha f)(t) \\ &:= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \end{aligned} \quad (13)$$

where $n = [\alpha] + 1$, provided that the right-hand sides are pointwise defined on $(0, \infty)$ (see [23]).

Lemma 1 (see [24]). Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then,

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \quad (14)$$

for some $C_i \in \mathbf{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α .

Since $1 < \alpha \leq 2$, $0 < \gamma < \delta < 1$, $\alpha - \delta > 1$, it is obvious that $1 < \alpha - \gamma < 2$. And putting $A := 1 - \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1}$ and $B := 1 - \sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1}$, the following lemmas hold.

Lemma 2 (see [11]). Let $y \in C[0, 1]$. Then, the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha-\gamma} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = 0, D_{0+}^{\delta-\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\delta-\gamma} u(\eta_i), \end{cases} \quad (15)$$

has a unique solution which is given by

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad (16)$$

where $G(t, s) = G_1(t, s) + G_2(t, s)$,
in which

$$\begin{aligned} G_1(t, s) &= \frac{1}{\Gamma(\alpha-\gamma)} \begin{cases} t^{\alpha-\gamma-1} (1-s)^{\alpha-\delta-1} - (t-s)^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\gamma-1} (1-s)^{\alpha-\delta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ G_2(t, s) &= \frac{t^{\alpha-\gamma-1}}{A\Gamma(\alpha-\gamma)} \left[\sum_{i=1}^{m-2} \xi_i \left[\eta_i^{\alpha-\delta-1} (1-s)^{\alpha-\delta-1} - (\eta_i-s)^{\alpha-\delta-1} \right] \right. \\ &\quad \left. + \sum_{s \leq \eta_i} \xi_i \eta_i^{\alpha-\delta-1} (1-s)^{\alpha-\delta-1} \right], \quad t, s \in [0, 1]. \end{aligned} \quad (17)$$

Lemma 3 (see [11]). If $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\delta-1} < 1$, then the function $G(t, s)$ in Lemma 2 satisfies the following conditions:

- (i) $G(t, s) > 0$, for any $t, s \in (0, 1)$
- (ii) $G(t, s) \leq G_*(s, s)$, for any $t, s \in [0, 1]$

where

$$G_*(s, s) = \frac{1}{A\Gamma(\alpha-\gamma)} (1-s)^{\alpha-\delta-1}. \quad (18)$$

Lemma 4 (see [25]). Let $z \in C[0, 1]$. Then, the fractional differential equation

$$\begin{cases} D_{0+}^\beta v(t) + z(t) = 0, \\ v(0) = 0, v(1) = \sum_{i=1}^{m-2} \zeta_i v(\eta_i), \end{cases} \quad (19)$$

has a unique solution which is given by

$$v(t) = \int_0^1 H(t, s) z(s) ds, \quad (20)$$

where $H(t, s) = H_1(t, s) + H_2(t, s)$, in which

$$\begin{aligned} H_1(t, s) &= \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1, \end{cases} \\ H_2(t, s) &= \frac{t^{\beta-1}}{B\Gamma(\beta)} \left[\sum_{s < \eta_i} \zeta_i \left[\eta_i^{\beta-1}(1-s)^{\beta-1} - (\eta_i - s)^{\beta-1} \right] \right. \\ &\quad \left. + \sum_{s \geq \eta_i} \zeta_i \eta_i^{\beta-1}(1-s)^{\beta-1} \right], \quad t, s \in [0, 1]. \end{aligned} \quad (21)$$

Lemma 5 (see [25]). *If $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} < 1$, then the function $H(t, s)$ in Lemma 4 satisfies the following conditions:*

- (i) $H(t, s) > 0$, for any $t, s \in (0, 1)$
- (ii) $H(t, s) \leq H_*(s, s)$, for any $t, s \in [0, 1]$

where

$$H_*(s, s) = \frac{1}{B\Gamma(\beta)} (1-s)^{\beta-1}. \quad (22)$$

The following properties of $\varphi_p(\cdot)$ which will be used later can be found in [26]:

- (i) If $1 < p < 2$, $xy > 0$, and $|x|, |y| \geq m > 0$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)m^{p-2}|x-y| \quad (23)$$

- (ii) If $p > 2$, $|x|, |y| \leq M$, then

$$|\varphi_p(x) - \varphi_p(y)| \leq (p-1)M^{p-2}|x-y| \quad (24)$$

3. Main Results

3.1. Existence and Uniqueness of Positive Solutions to Problem (1)

Definition 6. A function $x \in \{u \mid u \in C[0, 1], D_{0+}^\alpha u \in C[0, 1], D_{0+}^\beta(\varphi_p(D_{0+}^\alpha u)) \in C[0, 1]\}$ is called a solution of problem (1) if it satisfies the fractional differential equation and the boundary conditions of (1).

Lemma 7. *If the function x is a solution of problem (1), then $u(t) := D_{0+}^\gamma x(t)$ is a solution of the integral equation*

$$u(t) = \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds, \quad (25)$$

in $C[0, 1]$ and conversely, if $u \in C[0, 1]$ is a solution of the integral equation (25), then $x(t) = I_{0+}^\gamma u(t)$ is a solution of problem (1), where q is a number such that $(1/p) + (1/q) = 1$.

Proof. Let x be a solution of the problem (1). Then, since $x \in C[0, 1]$, $D_{0+}^\alpha x \in C[0, 1]$, by using Lemma 1, it can be easily seen that

$$x(t) = I_{0+}^\alpha D_{0+}^\alpha x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \quad (26)$$

The upper continuities of the functions $x(t)$, $I_{0+}^\alpha D_{0+}^\alpha x(t)$ at $t = 0$ provide us that $c_2 = 0$. Applying D_{0+}^γ on both sides, we have

$$\begin{aligned} D_{0+}^\gamma x(t) &= D_{0+}^\gamma I_{0+}^\alpha D_{0+}^\alpha x(t) + D_{0+}^\gamma c_1 t^{\alpha-1} \\ &= I_{0+}^{\alpha-\gamma} D_{0+}^\alpha x(t) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} c_1 t^{\alpha-\gamma-1}. \end{aligned} \quad (27)$$

Combining with $D_{0+}^\alpha x \in C[0, 1]$, $\alpha - \gamma - 1 > 0$, we can see that $D_{0+}^\gamma x \in C[0, 1]$. Since $x \in C[0, 1]$, $D_{0+}^\gamma x \in C[0, 1]$, Lemma 1 yields that

$$x(t) = I_{0+}^\gamma D_{0+}^\gamma x(t) + c_3 t^{\gamma-1}. \quad (28)$$

Therefore, it follows from the upper continuities of the functions $x(t)$, $I_{0+}^\gamma D_{0+}^\gamma x(t)$ at $t = 0$ that $c_3 = 0$. This implies that by putting $u(t) := D_{0+}^\gamma x(t)$, it holds that $x(t) = I_{0+}^\gamma u(t)$. Thus, we can know that u is a solution of the following problem:

$$\begin{cases} D_{0+}^\beta \left(\varphi_p(D_{0+}^{\alpha-\gamma} u(t)) \right) = f(t, I_{0+}^\gamma u(t), u(t)), & 0 < t < 1, \\ u(0) = 0, D_{0+}^{\delta-\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\delta-\gamma} u(\eta_i), \\ D_{0+}^{\alpha-\gamma} u(0) = 0, \varphi_p(D_{0+}^{\alpha-\gamma} u(1)) = \sum_{i=1}^{m-2} \zeta_i \varphi_p(D_{0+}^{\alpha-\gamma} u(\eta_i)). \end{cases} \quad (29)$$

Put $w(t) := D_{0+}^{\alpha-\gamma} u(t)$. Lemma 2 shows that

$$u(t) = - \int_0^1 G(t, s) w(s) ds. \quad (30)$$

Also, by applying Lemma 4, the notation $v(t) = \varphi_p(w(t))$ gives us that

$$v(t) = - \int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds. \quad (31)$$

Since $\varphi_p^{-1} = \varphi_q$, by equations (30) and (31), we obtain

$$u(t) = \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds. \quad (32)$$

Conversely, let the function $u \in C[0, 1]$ be a solution of integral equation (25). From this, by putting $x(t) := I_{0+}^{\gamma} u(t)$, it can be also obtained that $x \in C[0, 1]$. Denote as follows:

$$z(t) := f(t, I_{0+}^{\gamma} u(t), u(t)), v(t) := \varphi_q \left(\int_0^1 H(t, s) z(s) ds \right). \quad (33)$$

From the continuities of the functions u, f , and H , it is obvious that $z \in C[0, 1]$, $v \in C[0, 1]$. The definition of the function $G(t, s)$ implies that

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) v(s) ds \\ &= -I_{0+}^{\alpha-\gamma} v(t) - \frac{\Gamma(\alpha-\delta)}{A\Gamma(\alpha-\gamma)} \\ &\quad \cdot \left[\sum_{i=1}^{m-2} \xi_i I_{0+}^{\alpha-\delta} v(t) \Big|_{t=\eta_i} - I_{0+}^{\alpha-\delta} v(t) \Big|_{t=1} \right] t^{\alpha-\gamma-1}, \end{aligned} \quad (34)$$

so, we have

$$D_{0+}^{\alpha} x(t) = D_{0+}^{\alpha-\gamma} u(t) = v(t). \quad (35)$$

This means that $D_{0+}^{\alpha} x \in C[0, 1]$. Also, combining the definition of the function $H(t, s)$ with the relation

$$\begin{aligned} D_{0+}^{\beta} \left(\varphi_p(D_{0+}^{\alpha} x(t)) \right) &= D_{0+}^{\beta} \left(\varphi_p(v(t)) \right) \\ &= D_{0+}^{\beta} \left(\int_0^1 H(t, s) z(s) ds \right), \end{aligned} \quad (36)$$

we can get

$$\begin{aligned} &\int_0^1 H(t, s) z(s) ds \\ &= -I_{0+}^{\beta} z(t) - \frac{1}{B} \left[\sum_{i=1}^{m-2} \zeta_i I_{0+}^{\beta} z(t) \Big|_{t=\eta_i} - I_{0+}^{\beta} z(t) \Big|_{t=1} \right] t^{\beta-1}. \end{aligned} \quad (37)$$

This yields that

$$D_{0+}^{\beta} \left(\varphi_p(D_{0+}^{\alpha} x(t)) \right) = z(t), \quad (38)$$

and therefore, it holds that $D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha} x)) \in C[0, 1]$. Since equations (15) and (19) have unique solutions by Lemmas 2 and 4, u , the solution of integral equation (25), satisfies the fractional differential equation and the boundary conditions of the problem (29). To sum up, it can be proved that x is a solution of problem (1).

Let E be the Banach space $C[0, 1]$ equipped with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and put $P := \{u \in E \mid u(t) \geq 0, t \in [0, 1]\}$.

Since $\forall u \in P, \forall t \in [0, 1], I_{0+}^{\gamma} u(t) = (1/\Gamma(\gamma)) \int_0^t (t-s)^{\gamma-1} u(s) ds$

≥ 0 , it holds that $I_{0+}^{\gamma} u \in P$. Therefore, define an operator T as

$$Tu(t) := \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^{\gamma} u(\tau), u(\tau)) d\tau \right) ds, u \in P, \quad (39)$$

then, it satisfies $T(P) \subset P$. This implies that integral equation (25) has a solution in P if and only if operator T has a fixed point. It is also obvious that the fixed point of operator T is a solution of integral equation (25) in P .

Put $\bar{M} := [(\alpha-\delta)A\Gamma(\alpha-\gamma)]^{p-1} B\Gamma(\beta)$. The following hypotheses will be used throughout the paper.

(H1) There exist $a, b, c \in L[0, 1]$ such that

$$M_a := \int_0^1 a(t) dt > 0, M_b := \int_0^1 b(t) dt > 0, M_c := \int_0^1 c(t) dt > 0,$$

$$\frac{M_b}{\Gamma(\gamma+1)^{p-1}} + M_c < \bar{M},$$

$$\forall (t, x, y) \in [0, 1] \times [0, R_1] \times [0, R_2],$$

$$f(t, x, y) < a(t) + b(t)x^{p-1} + c(t)y^{p-1},$$

(40)

where $R_2 = (M_a/(\bar{M} - (M_b/(\Gamma(\gamma+1)^{p-1})) - M_c))^{q-1}$ and $R_1 = R_2/(\Gamma(\gamma+1))$

(H2) There exist $L_1, L_2 > 0$ such that for any $t \in [0, 1]$ and any $(x_1, y_1), (x_2, y_2) \in [0, R_1] \times [0, R_2]$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1 |x_1 - x_2| + L_2 |y_1 - y_2| \quad (41)$$

(H3) There exist $\bar{m}, \mu > 0$ such that for any $(t, x, y) \in [0, 1] \times [0, R_1] \times [0, R_2]$,

$$f(t, x, y) \geq \bar{m} t^{\mu-1} \quad (42)$$

Lemma 8. Assume that the hypothesis (H1) is satisfied. Then, it holds that

$$T(B_{R_2}) \subset B_{R_2}, \quad (43)$$

where $B_{R_2} = \{u \in P \mid \|u\| \leq R_2\}$.

Proof. Since $T(B_{R_2}) \subset P$, it is sufficient to prove that $\forall u \in B_{R_2}, \|Tu\| \leq R_2$. It can be easily evaluated that for any $t \in [0, 1]$,

$$\begin{aligned} I_{0+}^{\gamma} u(t) &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds \leq \frac{R_2}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} ds \\ &= \frac{R_2}{\Gamma(\gamma+1)} \cdot t^{\gamma} \leq R_1. \end{aligned} \quad (44)$$

By using Lemma 5 and hypothesis (H1), we have

$$\begin{aligned}
 |Tu(t)| &= \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \\
 &\leq \int_0^1 G(t,s) \varphi_q \left(\max_{s \in [0,1]} H_*(s,s) \int_0^1 (a(\tau) + b(\tau)(I_{0+}^\gamma u(\tau))^{p-1} \right. \\
 &\quad \left. + c(\tau)u(\tau)^{p-1}) d\tau \right) ds \\
 &\leq \int_0^1 G(t,s) \varphi_q \left(\frac{1}{B\Gamma(\beta)} \left(M_a + \left(\frac{M_b}{\Gamma(\gamma+1)^{p-1}} + M_c \right) R_2^{p-1} \right) \right) ds.
 \end{aligned} \tag{45}$$

Applying the properties of the function $G(t,s)$ indicated in Lemma 3, the following holds:

$$\begin{aligned}
 |Tu(t)| &\leq \left(\frac{1}{B\Gamma(\beta)} \left(M_a + \left(\frac{M_b}{\Gamma(\gamma+1)^{p-1}} + M_c \right) R_2^{p-1} \right) \right)^{q-1} \int_0^1 G_*(s,s) ds \\
 &= \frac{\left(M_a + \left(\frac{M_b}{\Gamma(\gamma+1)^{p-1}} + M_c \right) R_2^{p-1} \right)^{q-1}}{B^{q-1} \Gamma(\beta)^{q-1} A(\alpha-\delta) \Gamma(\alpha-\gamma)}.
 \end{aligned} \tag{46}$$

The relation $(1/p) + (1/q) = 1$ shows that $p-1$ is an inverse power of $q-1$. So, it can be rewritten as

$$|Tu(t)| \leq \left(\left(M_a + \left(\frac{M_b}{\Gamma(\gamma+1)^{p-1}} + M_c \right) R_2^{p-1} \right) / \bar{M} \right)^{q-1}. \tag{47}$$

Since it is satisfied for R_2 that

$$M_a = \left(\bar{M} - \frac{M_b}{\Gamma(\gamma+1)^{p-1}} - M_c \right) R_2^{p-1}, \tag{48}$$

we can get

$$|Tu(t)| \leq R_2. \tag{49}$$

This concludes the lemma.

Lemma 9. *If the hypothesis (H3) holds, then for any $t \in [0, 1]$ and any $u \in B_{R_2}$,*

$$\int_0^1 H(t,s) f(s, I_{0+}^\gamma u(s), u(s)) ds \geq K_0 t^{\beta-1}, \tag{50}$$

where $K_0 = ((\bar{m}\Gamma(\mu))/B\Gamma(\beta+\mu)) \sum_{i=1}^{m-2} \zeta_i (\eta_i^{\beta-1} - \eta_i^{\beta-1+\mu})$.

Proof. It follows from Lemma 4 and hypothesis (H3) that

$$\begin{aligned}
 &\int_0^1 H(t,s) f(s, I_{0+}^\gamma u(s), u(s)) ds \\
 &\geq \bar{m} \left(\int_0^1 H_1(t,s) s^{\mu-1} ds + \int_0^1 H_2(t,s) s^{\mu-1} ds \right).
 \end{aligned} \tag{51}$$

By simple calculation, the integration terms in the right side of inequality (51) can be estimated as

$$\begin{aligned}
 \int_0^1 H_1(t,s) s^{\mu-1} ds &= \frac{1}{\Gamma(\beta)} \left[t^{\beta-1} \int_0^1 (1-s)^{\beta-1} s^{\mu-1} ds \right. \\
 &\quad \left. - \int_0^t (t-s)^{\beta-1} s^{\mu-1} ds \right] \\
 &= \frac{1}{\Gamma(\beta)} \left[t^{\beta-1} B(\mu, \beta) - t^{\beta-1+\mu} B(\mu, \beta) \right] \\
 &= \frac{\Gamma(\mu)}{\Gamma(\beta+\mu)} \left[t^{\beta-1} - t^{\beta-1+\mu} \right] \geq 0, \\
 \int_0^1 H_2(t,s) s^{\mu-1} ds &= \frac{t^{\beta-1}}{B\Gamma(\beta)} \sum_{i=1}^{m-2} \zeta_i \left[\eta_i^{\beta-1} \int_0^1 (1-s)^{\beta-1} s^{\mu-1} ds \right. \\
 &\quad \left. - \int_0^{\eta_i} (\eta_i-s)^{\beta-1} s^{\mu-1} ds \right] \\
 &= \frac{t^{\beta-1}}{B\Gamma(\beta)} \sum_{i=1}^{m-2} \zeta_i \left[\eta_i^{\beta-1} B(\mu, \beta) - \eta_i^{\beta-1+\mu} B(\mu, \beta) \right] \\
 &= \left[\frac{\Gamma(\mu)}{B\Gamma(\beta+\mu)} \sum_{i=1}^{m-2} \zeta_i (\eta_i^{\beta-1} - \eta_i^{\beta-1+\mu}) \right] t^{\beta-1}.
 \end{aligned} \tag{52}$$

These estimations provide us the conclusion (50).

Denote $K_1 := (1/\Gamma(\beta+1)) [1 + (1/B) \sum_{i=1}^{m-2} \zeta_i (\eta_i^{\beta-1} - \eta_i^{\beta})]$. Then, it can be found in [25] that for any $t \in [0, 1]$,

$$\int_0^1 H(t,s) ds \leq K_1 t^{\beta-1}. \tag{53}$$

Theorem 10. *Suppose that $p > 2$ and hypotheses (H1)-(H3) are satisfied. If*

$$\frac{(q-1)K_1}{A(\alpha-\delta)\Gamma(\alpha-\gamma)K_0^{2-q}} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) < 1, \tag{54}$$

then, integral equation (25) has a unique solution in B_{R_2} .

Proof. To prove this theorem, operators T_0, T_1 are defined as

$$\begin{aligned}
 T_0 u(t) &:= \varphi_q \left(\int_0^1 H(t,s) f(s, I_{0+}^\gamma u(s), u(s)) ds \right), \quad T_1 u(t) \\
 &:= \int_0^1 G(t,s) u(s) ds.
 \end{aligned} \tag{55}$$

Obviously, we can see that for any $u \in B_{R_2}$, $Tu(t) = (T_1 \circ T_0 u)(t)$, and $Tu(0) = 0$. Because $p > 2$, it is satisfied that

$1 < q < 2$. So, by employing Lemma 8, Lemma 9, and property (23), we can obtain that for any $u, v \in B_{R_2}$ and any $t \in (0, 1]$,

$$\begin{aligned}
& |T_0 u(t) - T_0 v(t)| \\
&= \left| \varphi_q \left(\int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds \right) \right. \\
&\quad \left. - \varphi_q \left(\int_0^1 H(t, s) f(s, I_{0+}^\gamma v(s), v(s)) ds \right) \right| \\
&\leq (q-1) \left(K_0 t^{\beta-1} \right)^{q-2} \left| \int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds \right. \\
&\quad \left. - \int_0^1 H(t, s) f(s, I_{0+}^\gamma v(s), v(s)) ds \right| \\
&\leq (q-1) \left(K_0 t^{\beta-1} \right)^{q-2} \int_0^1 H(t, s) (L_1 |I_{0+}^\gamma u(s) - I_{0+}^\gamma v(s)| \\
&\quad + L_2 |u(s) - v(s)|) ds \\
&\leq (q-1) \left(K_0 t^{\beta-1} \right)^{q-2} \int_0^1 H(t, s) \left(\frac{L_1}{\Gamma(\gamma+1)} \|u - v\| + L_2 \|u - v\| \right) ds \\
&\leq (q-1) \left(K_0 t^{\beta-1} \right)^{q-2} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\| \int_0^1 H(t, s) ds.
\end{aligned} \tag{56}$$

Substituting inequality (53) into (56), we have

$$\begin{aligned}
|T_0 u(t) - T_0 v(t)| &\leq (q-1) K_0^{q-2} K_1 \\
&\quad \cdot \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\| t^{(\beta-1)(q-1)}.
\end{aligned} \tag{57}$$

Thus, it holds that

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \left| \int_0^1 G(t, s) (T_0 u)(s) ds - \int_0^1 G(t, s) (T_0 v)(s) ds \right| \\
&\leq (q-1) K_0^{q-2} K_1 \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\| \\
&\quad \cdot \int_0^1 G(t, s) s^{(\beta-1)(q-1)} ds.
\end{aligned} \tag{58}$$

Since $(\beta-1)(q-1) > 0$, by Lemma 3, we can get

$$\begin{aligned}
\int_0^1 G(t, s) s^{(\beta-1)(q-1)} ds &\leq \int_0^1 G(t, s) ds \leq \int_0^1 G_*(s, s) ds \\
&= \frac{1}{A(\alpha-\delta)\Gamma(\alpha-\gamma)}.
\end{aligned} \tag{59}$$

This yields that

$$\|Tu - Tv\| \leq \frac{(q-1)K_1}{A(\alpha-\delta)\Gamma(\alpha-\gamma)K_0^{2-q}} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\|. \tag{60}$$

Combining inequalities (54) and (60) with Lemma 8, the operator $T : B_{R_2} \rightarrow B_{R_2}$ is a contraction mapping and the Banach contraction mapping principle provides us that it

has a unique fixed point in B_{R_2} . So, integral equation (25) has a unique solution in B_{R_2} .

Lemma 11. Assume that the hypothesis (H1) holds. Then, for any $u \in B_{R_2}$ and any $t \in [0, 1]$, it is satisfied that

$$\int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds \leq M_0, \tag{61}$$

where

$$M_0 = \frac{M_a + ((M_b I(\Gamma(\gamma+1)^{p-1})) + M_c) R_2^{p-1}}{B\Gamma(\beta)}. \tag{62}$$

Proof. In a similar way to the proof of Lemma 8, it can be easily proved that

$$\begin{aligned}
& \int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds \\
&\leq \max_{s \in [0,1]} H_*(s, s) \int_0^1 f(s, I_{0+}^\gamma u(s), u(s)) ds \\
&\leq \frac{1}{B\Gamma(\beta)} \int_0^1 (a(s) + b(s) (I_{0+}^\gamma u(s))^{p-1} + c(s) u(s)^{p-1}) ds \\
&\leq \frac{1}{B\Gamma(\beta)} \left(\int_0^1 a(s) ds + R_1^{p-1} \int_0^1 b(s) ds + R_2^{p-1} \int_0^1 c(s) ds \right) = M_0.
\end{aligned} \tag{63}$$

Theorem 12. Suppose that $1 < p < 2$ and hypotheses (H1) and (H2) hold. If

$$\frac{(q-1)R_2^{2-p}}{\beta M} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) < 1, \tag{64}$$

then, integral equation (25) has a unique solution in B_{R_2} .

Proof. In the case $1 < p < 2$, due to $(1/p) + (1/q) = 1$, it holds that $q > 2$. Similar to Theorem 10, defining operators T_0, T_1 with Lemma 8, Lemma 11, and property (24) give us that for any $u, v \in B_{R_2}$ and any $t \in (0, 1]$,

$$\begin{aligned}
& |T_0 u(t) - T_0 v(t)| \\
&= \left| \varphi_q \left(\int_0^1 H(t, s) f(s, I_{0+}^\gamma u(s), u(s)) ds \right) \right. \\
&\quad \left. - \varphi_q \left(\int_0^1 H(t, s) f(s, I_{0+}^\gamma v(s), v(s)) ds \right) \right| \\
&\leq (q-1) M_0^{q-2} \int_0^1 H(t, s) |f(s, I_{0+}^\gamma u(s), u(s)) - f(s, I_{0+}^\gamma v(s), v(s))| ds \\
&\leq (q-1) M_0^{q-2} \int_0^1 H(t, s) (L_1 |I_{0+}^\gamma u(s) - I_{0+}^\gamma v(s)| + L_2 |u(s) - v(s)|) ds \\
&\leq (q-1) M_0^{q-2} \int_0^1 H(t, s) \left(\frac{L_1}{\Gamma(\gamma+1)} \|u - v\| + L_2 \|u - v\| \right) ds \\
&\leq (q-1) M_0^{q-2} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\| \int_0^1 H_*(s, s) ds \\
&= \frac{(q-1)M_0^{q-2}}{B\Gamma(\beta+1)} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\|.
\end{aligned} \tag{65}$$

So, we have

$$\begin{aligned}
 |Tu(t) - Tv(t)| &= \left| \int_0^1 G(t, s)(T_0 u)(s)ds - \int_0^1 G(t, s)(T_0 v)(s)ds \right| \\
 &\leq \int_0^1 G(t, s)|T_0 u(s) - T_0 v(s)|ds \\
 &\leq \frac{(q-1)M_0^{q-2}}{B\Gamma(\beta+1)} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\| \\
 &\quad \cdot \int_0^1 G(t, s)ds \\
 &\leq \frac{(q-1)M_0^{q-2}}{(\alpha-\delta)AB\Gamma(\alpha-\gamma)\Gamma(\beta+1)} \\
 &\quad \cdot \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \|u - v\|.
 \end{aligned} \tag{66}$$

Obviously, for M_0 , the following holds:

$$\frac{M_0^{q-2}}{(\alpha-\delta)AB\Gamma(\alpha-\gamma)\Gamma(\beta+1)} = \frac{1}{\beta \bar{M}^{q-1}} \cdot (\bar{M} R_2^{p-1})^{q-2} = \frac{R_2^{2-p}}{\beta \bar{M}}. \tag{67}$$

Therefore, by employing inequality (64), equality (67), and Lemma 8, the operator $T : B_{R_2} \rightarrow B_{R_2}$ is a contraction mapping. It follows from the Banach contraction mapping principle that the operator T has a unique fixed point in B_{R_2} . This completes the proof.

Remark 13. Theorems 10 and 12 show the existence and uniqueness of solutions to integral equation (25) in the cases $p > 2$ and $1 < p < 2$, respectively. The Banach contraction mapping principle, employed in the proofs of these theorems, guarantees that the iterative sequences can be constructed to converge to the exact solution of integral equation (25). In other words, the iterative sequence $\{u_n\}_{n=0}^\infty$ initialized at any point $u_0 \in B_{R_2}$ and defined by

$$u_{n+1} = Tu_n, \quad n = 0, 1, 2, \dots, \tag{68}$$

converges to the exact solution of integral equation (25), u , in B_{R_2} . This provides the possibility to obtain the approximate solution u_n of integral equation (25) by using Theorems 10 and 12.

Theorem 14. *Problem (1) has a unique solution in Ω if and only if the solution to integral equation (25) uniquely exists in B_{R_2} , where $\Omega = I_{0+}^\gamma(B_{R_2})$. In particular, if hypothesis (H3) is satisfied, then the solution to problem (1) is positive.*

Proof. By Lemma 7, for x , a solution to problem (1), and u satisfying integral equation (25), the following relation holds:

$$x(t) = I_{0+}^\gamma u(t), \quad t \in [0, 1]. \tag{69}$$

This proves the first part of the theorem.

For the second part, it is necessary that if hypothesis (H3) holds, then

$$\forall t \in (0, 1], x(t) > 0. \tag{70}$$

To prove this, we must see if the following holds:

$$\forall t \in (0, 1], u(t) > 0 \tag{71}$$

Since $u \in B_{R_2}$, using Lemma 9, it can be easily obtained that

$$\begin{aligned}
 u(t) &\geq K_0^{q-1} \int_0^1 G(t, s)s^{(\beta-1)(q-1)}ds \\
 &= K_0^{q-1} \left[\int_0^1 G_1(t, s)s^{(\beta-1)(q-1)}ds + \int_0^1 G_2(t, s)s^{(\beta-1)(q-1)}ds \right].
 \end{aligned} \tag{72}$$

Considering that $G_1(t, s) \geq 0$ and $(\beta-1)(q-1) > 0$, we can get

$$\forall t \in [0, 1], \int_0^1 G_1(t, s)s^{(\beta-1)(q-1)}ds \geq 0. \tag{73}$$

From the definition of function $G_2(t, s)$, we have

$$\begin{aligned}
 &\int_0^1 G_2(t, s)s^{(\beta-1)(q-1)}ds \\
 &= \frac{t^{\alpha-\gamma-1}}{A\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-2} \xi_i \left[\eta_i^{\alpha-\delta-1} \int_0^1 (1-s)^{\alpha-\delta-1} s^{(\beta-1)(q-1)}ds \right. \\
 &\quad \left. - \int_0^{\eta_i} (\eta_i - s)^{\alpha-\delta-1} s^{(\beta-1)(q-1)}ds \right] \\
 &= \left[\frac{B(\alpha-\delta, (\beta-1)(q-1)+1)}{A\Gamma(\alpha-\gamma)} \right. \\
 &\quad \left. \cdot \sum_{i=1}^{m-2} \xi_i \left(\eta_i^{\alpha-\delta-1} - \eta_i^{\alpha-\delta+(\beta-1)(q-1)} \right) \right] t^{\alpha-\gamma-1}.
 \end{aligned} \tag{74}$$

Denote as follows:

$$\begin{aligned}
 K_2 &:= \frac{B(\alpha-\delta, (\beta-1)(q-1)+1)}{A\Gamma(\alpha-\gamma)} \\
 &\quad \cdot \sum_{i=1}^{m-2} \xi_i \left(\eta_i^{\alpha-\delta-1} - \eta_i^{\alpha-\delta+(\beta-1)(q-1)} \right).
 \end{aligned} \tag{75}$$

Since $K_0, K_2 > 0$ and $u(t) \geq K_0^{q-1} K_2 t^{\alpha-\gamma-1}$, it is obvious that (71) holds. Therefore, we can see that

$$x(t) = I_{0+}^\gamma u(t) \geq I_{0+}^\gamma \left(K_0^{q-1} K_2 t^{\alpha-\gamma-1} \right) = K_0^{q-1} K_2 \cdot \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)} t^{\alpha-1}. \quad (76)$$

The proof is completed.

Remark 15. Theorems 10, 12, and 14 show new results that differ from some of the references listed in Section 1. One of these is the result of the existence, uniqueness, and positivity of solutions to p -Laplacian fractional differential equations with multipoint boundary conditions where the nonlinear source terms contain a more generalized form of fractional derivatives than those considered in [16, 17, 20]. The other is the possibility of constructing an iterative sequence that converges to a unique solution of a given problem, which is different from [16, 18, 19]. Combining Theorem 14 with Theorems 10 and 12, the iterative sequence $\{x_n\}_{n=0}^\infty$ given by

$$x_n = I_{0+}^\gamma u_n, \quad n = 0, 1, 2, \dots, \quad (77)$$

converges to the exact solution of problem (1), x , in Ω , where $\{u_n\}_{n=0}^\infty$ is constructed by (68). This gives an opportunity to get the approximate solution, x_n , for problem (1), although there might be some difficulties in its numerical computation.

3.2. Continuous Dependence of a Solution on Boundary Conditions. In this subsection, the continuous dependence of a unique positive solution x to problem (1) on boundary conditions will be established. To do this, denote a unique solution to the problem

$$\begin{cases} D_{0+}^\beta \left(\varphi_p(D_{0+}^\alpha x(t)) \right) = f(t, x(t), D_{0+}^\gamma x(t)), & 0 < t < 1, \\ D_{0+}^\gamma x(0) = 0, D_{0+}^\delta x(1) = \sum_{i=1}^{m-2} \tilde{\xi}_i D_{0+}^\delta x(\eta_i), \\ D_{0+}^\alpha x(0) = 0, \varphi_p(D_{0+}^\alpha x(1)) = \sum_{i=1}^{m-2} \tilde{\zeta}_i \varphi_p(D_{0+}^\alpha x(\eta_i)), \end{cases} \quad (78)$$

by $\tilde{x}$.

As can be seen above, the positive solution x to problem (1) is given by

$$x(t) = I_{0+}^\gamma u(t), \quad t \in [0, 1], \quad (79)$$

where u is a solution of integral equation (25). Also, the positive solution $\tilde{x}$ to problem (78) is expressed by

$$\tilde{x}(t) = I_{0+}^\gamma \tilde{u}(t), \quad t \in [0, 1], \quad (80)$$

where $\tilde{u}$ is a solution of the integral equation

$$\tilde{u}(t) = \int_0^1 \tilde{G}(t, s) \phi_q \left(\int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right) ds, \quad (81)$$

in which

$$\begin{aligned} \tilde{A} &= 1 - \sum_{i=1}^{m-2} \tilde{\xi}_i \eta_i^{\alpha-\delta-1}, \quad \tilde{B} = 1 - \sum_{i=1}^{m-2} \tilde{\zeta}_i \eta_i^{\beta-1}, \\ \tilde{G}_2(t, s) &= \frac{t^{\alpha-\gamma-1}}{\tilde{A}\Gamma(\alpha-\gamma)} \left[\sum_{s < \eta_i} \tilde{\xi}_i \left[\eta_i^{\alpha-\delta-1} (1-s)^{\alpha-\delta-1} - (\eta_i-s)^{\alpha-\delta-1} \right] \right. \\ &\quad \left. + \sum_{s \geq \eta_i} \tilde{\xi}_i \eta_i^{\alpha-\delta-1} (1-s)^{\alpha-\delta-1} \right], \\ \tilde{H}_2(t, s) &= \frac{t^{\beta-1}}{\tilde{B}\Gamma(\beta)} \left[\sum_{s < \eta_i} \tilde{\zeta}_i \left[\eta_i^{\beta-1} (1-s)^{\beta-1} - (\eta_i-s)^{\beta-1} \right] \right. \\ &\quad \left. + \sum_{s \geq \eta_i} \tilde{\zeta}_i \eta_i^{\beta-1} (1-s)^{\beta-1} \right], \\ \tilde{G}(t, s) &= G_1(t, s) + \tilde{G}_2(t, s), \quad \tilde{H}(t, s) = H_1(t, s) + \tilde{H}_2(t, s), \\ \tilde{M} &= \left[(\alpha-\delta) \tilde{A}\Gamma(\alpha-\gamma) \right]^{p-1} \tilde{B}\Gamma(\beta), \\ \tilde{R}_2 &= \left(M_a / \left(\tilde{M} - \frac{M_b}{\Gamma(\gamma+1)^{p-1}} - M_c \right) \right)^{q-1}. \end{aligned} \quad (82)$$

Then, the continuous dependence of a unique positive solution to the problem (1) on boundary conditions can be formulated by

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{x} = x, \quad (83)$$

where

$$\begin{aligned} \lambda_\xi &:= \max \left\{ |\tilde{\xi}_1 - \xi_1|, |\tilde{\xi}_2 - \xi_2|, \dots, |\tilde{\xi}_{m-2} - \xi_{m-2}| \right\}, \\ \lambda_\zeta &:= \max \left\{ |\tilde{\zeta}_1 - \zeta_1|, |\tilde{\zeta}_2 - \zeta_2|, \dots, |\tilde{\zeta}_{m-2} - \zeta_{m-2}| \right\}. \end{aligned} \quad (84)$$

Considering relations (79) and (80), it is sufficient to prove that

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{u} = u. \quad (85)$$

The solutions of integral equations (25) and (81), u and $\tilde{u}$, uniquely exist in B_{R_2} , $B_{\tilde{R}_2}$, respectively. Since A, B ,

$\bar{M}$, and R_2 are continuous at $\xi_1, \xi_2, \dots, \xi_{m-2}, \zeta_1, \zeta_2, \dots, \zeta_{m-2}$, it holds that

$$\begin{aligned} \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{A} &= A, \\ \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{B} &= B, \\ \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{\bar{M}} &= \bar{M}, \\ \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{R}_2 &= R_2. \end{aligned} \quad (86)$$

Define a function $\bar{G}_2(t, s)$ as follows:

$$\begin{aligned} \bar{G}_2(t, s) &:= \frac{1}{\Gamma(\alpha - \gamma)} \begin{cases} t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-1} - (t-s)^{\alpha-\delta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (87)$$

Then, it can be easily seen that

$$G_2(t, s) = \frac{1}{A} \sum_{i=1}^{m-2} \xi_i \bar{G}_2(\eta_i, s) \cdot t^{\alpha-\gamma-1}, \quad \tilde{G}_2(t, s) = \frac{1}{\bar{A}} \sum_{i=1}^{m-2} \tilde{\xi}_i \bar{G}_2(\eta_i, s) \cdot t^{\alpha-\gamma-1}. \quad (88)$$

And it can be found in [27] that

$$H_2(t, s) = \frac{1}{B} \sum_{i=1}^{m-2} \zeta_i H_1(\eta_i, s) \cdot t^{\beta-1}, \quad \tilde{H}_2(t, s) = \frac{1}{\bar{B}} \sum_{i=1}^{m-2} \tilde{\zeta}_i H_1(\eta_i, s) \cdot t^{\beta-1}. \quad (89)$$

For a fixed t , the functions $\bar{G}_2(t, s)$ and $H_1(t, s)$ have maximum values at $s = t$, i.e.,

$$\begin{aligned} \max_{s \in [0,1]} \bar{G}_2(t, s) &= \frac{t^{\alpha-\delta-1}(1-t)^{\alpha-\delta-1}}{\Gamma(\alpha - \gamma)}, \\ \max_{s \in [0,1]} H_1(t, s) &= \frac{t^{\beta-1}(1-t)^{\beta-1}}{\Gamma(\beta)}. \end{aligned} \quad (90)$$

Lemma 16. Let $y \in C[0, 1]$. Then, the following holds:

$$\begin{aligned} \forall t \in [0, 1], |\tilde{v}(t) - v(t)| &\leq \frac{\|y\|}{\Gamma(\alpha - \gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\xi}_i}{\bar{A}} - \frac{\xi_i}{A} \right| \\ &\quad \cdot \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right) \cdot t^{\alpha-\gamma-1}, \end{aligned} \quad (91)$$

where

$$v(t) = \int_0^t G(t, s) y(s) ds, \quad \tilde{v}(t) = \int_0^t \tilde{G}(t, s) y(s) ds. \quad (92)$$

Proof. Using relation (88), we can see that for any $t \in [0, 1]$,

$$\begin{aligned} |\tilde{v}(t) - v(t)| &\leq \int_0^1 |\tilde{G}(t, s) - G(t, s)| y(s) ds \\ &\leq \|y\| \int_0^1 \left| \frac{1}{\bar{A}} \sum_{i=1}^{m-2} \tilde{\xi}_i \bar{G}_2(\eta_i, s) - \frac{1}{A} \sum_{i=1}^{m-2} \xi_i \bar{G}_2(\eta_i, s) \right| t^{\alpha-\gamma-1} ds \\ &\leq \|y\| \sum_{i=1}^{m-2} \left| \frac{\tilde{\xi}_i}{\bar{A}} - \frac{\xi_i}{A} \right| \int_0^1 \bar{G}_2(\eta_i, s) ds \cdot t^{\alpha-\gamma-1}. \end{aligned} \quad (93)$$

By applying (90), we obtain

$$|\tilde{v}(t) - v(t)| \leq \frac{\|y\|}{\Gamma(\alpha - \gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\xi}_i}{\bar{A}} - \frac{\xi_i}{A} \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right) \cdot t^{\alpha-\gamma-1}. \quad (94)$$

Lemma 17. Let $z \in C[0, 1]$. Then, it holds that

$$\begin{aligned} \forall t \in [0, 1], |\tilde{w}(t) - w(t)| &\leq \frac{\|z\|}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\bar{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \cdot t^{\beta-1}, \end{aligned} \quad (95)$$

where

$$w(t) = \int_0^t H(t, s) z(s) ds, \quad \tilde{w}(t) = \int_0^t \tilde{H}(t, s) z(s) ds. \quad (96)$$

Proof. The proof is similar to Lemma 16, so it is omitted.

Theorem 18. Suppose that all the assumptions of Theorem 10 are satisfied. Then, it holds that

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{u} = u. \quad (97)$$

Proof. It can be obtained that for any $t \in [0, 1]$,

$$\begin{aligned} |\tilde{u}(t) - u(t)| &\leq \left| \int_0^1 \tilde{G}(t, s) \varphi_q \left(\int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right) ds \right. \\ &\quad \left. - \int_0^1 \tilde{G}(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \right| \\ &\quad + \left| \int_0^1 \tilde{G}(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \right. \\ &\quad \left. - \int_0^1 G(t, s) \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \right| \end{aligned}$$

$$\begin{aligned}
& \leq \int_0^1 \tilde{G}(t, s) \left| \varphi_q \left(\int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right) \right. \\
& \quad \left. - \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) \right| ds \\
& \quad + \left| \int_0^1 (\tilde{G}(t, s) - G(t, s)) \varphi_q \right. \\
& \quad \left. \cdot \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \right|. \quad (98)
\end{aligned}$$

Lemma 9 indicates that for $u \in B_{R_2}$, $\tilde{u} \in B_{\tilde{R}_2}$,

$$\begin{aligned}
& \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \\
& \geq K_0 s^{\beta-1}, \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \geq \tilde{K}_0 s^{\beta-1}, \quad (99)
\end{aligned}$$

where $\tilde{K}_0 = (\tilde{m} \sim \Gamma(\tilde{\mu})) / (\tilde{B} \Gamma(\beta + \tilde{\mu})) \sum_{i=1}^{m-2} \tilde{\zeta}_i (\eta_i^{\beta-1} - \eta_i^{\beta-1+\tilde{\mu}})$. And by Lemma 11, we know that

$$\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \leq M_0. \quad (100)$$

Considering Lemma 17 and inequality (53), it is easy to see that

$$\begin{aligned}
& \left| \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right. \\
& \quad \left. - \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\
& \leq \left| \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right.
\end{aligned}$$

$$\begin{aligned}
& \left. - \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\
& \quad + \left| \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right. \\
& \quad \left. - \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\
& \leq \int_0^1 \tilde{H}(s, \tau) |f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) - f(\tau, I_{0+}^\gamma u(\tau), u(\tau))| d\tau \\
& \quad + \left| \int_0^1 (\tilde{H}(s, \tau) - H(s, \tau)) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\
& \leq \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \|\tilde{u} - u\| \int_0^1 \tilde{H}(s, \tau) d\tau \\
& \quad + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \cdot s^{\beta-1} \\
& \leq \left[\tilde{K}_1 \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \|\tilde{u} - u\| \right. \\
& \quad \left. + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \right] \cdot s^{\beta-1}, \quad (101)
\end{aligned}$$

where

$$\tilde{K}_1 = \frac{1}{\Gamma(\beta+1)} \left[1 + \frac{1}{\tilde{B}} \sum_{i=1}^{m-2} \tilde{\zeta}_i \left(\eta_i^{\beta-1} - \eta_i^\beta \right) \right], \quad (102)$$

and $C = \max_{\substack{t \in [0,1] \\ x \in [0,R_1] \\ y \in [0,R_2]}} |f(t, x, y)|$.

Therefore, combining Lemma 16 with $1 < q < 2$, it follows that

$$\begin{aligned}
|\tilde{u}(t) - u(t)| & \leq \frac{(q-1)}{\min \{K_0^{2-q}, \tilde{K}_0^{2-q}\}} \left[\tilde{K}_1 \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \|\tilde{u} - u\| + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \right] \\
& \quad \cdot \int_0^1 \tilde{G}_*(s, s) s^{(\beta-1)(q-1)} ds + \frac{M_0}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{A}} - \frac{\zeta_i}{A} \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right) \\
& \leq \frac{(q-1) \left[\tilde{K}_1 \left(\left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} \right) + \max \{L_2, \tilde{L}_2\} \right) \|\tilde{u} - u\| + (C/\Gamma(\beta)) \sum_{i=1}^{m-2} \left| \left(\frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right) \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \right]}{\tilde{A}(\alpha-\delta) \Gamma(\alpha-\gamma) \min \{K_0^{2-q}, \tilde{K}_0^{2-q}\}} \\
& \quad + \frac{M_0}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{A}} - \frac{\zeta_i}{A} \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right). \quad (103)
\end{aligned}$$

Since

$$\frac{(q-1)K_1}{A(\alpha-\delta)\Gamma(\alpha-\gamma)K_0^{2-q}} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) < 1, \quad (104)$$

and A, B, K_0, K_1, L_1, L_2 are continuous at $\xi_1, \xi_2, \dots, \xi_{m-2}, \zeta_1, \zeta_2, \dots, \zeta_{m-2}$, we can see that for sufficient small λ_ξ and λ_ζ ,

$$\begin{aligned} \Lambda_1 := & \frac{(q-1)\tilde{K}_1}{\tilde{A}(\alpha-\delta)\Gamma(\alpha-\gamma) \min \{K_0^{2-q}, \tilde{K}_0^{2-q}\}} \\ & \cdot \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) < 1. \end{aligned} \quad (105)$$

So, putting

$$\Lambda_2 := \frac{(q-1)C}{\tilde{A}(\alpha-\delta)\Gamma(\beta)\Gamma(\alpha-\gamma) \min \{K_0^{2-q}, \tilde{K}_0^{2-q}\}}, \quad (106)$$

$$\Lambda_3 := \frac{M_0}{\Gamma(\alpha-\gamma)},$$

we have

$$\|\tilde{u} - u\| \leq \frac{\Lambda_2 \sum_{i=1}^{m-2} \left| \left(\frac{\tilde{\zeta}_i}{\tilde{B}} \right) - \left(\frac{\zeta_i}{B} \right) \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) + \Lambda_3 \sum_{i=1}^{m-2} \left| \left(\frac{\tilde{\xi}_i}{\tilde{A}} \right) - \left(\frac{\xi_i}{A} \right) \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right)}{1 - \Lambda_1}. \quad (107)$$

Finally, the fact that

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \frac{\tilde{\xi}_i}{\tilde{A}} = \frac{\xi_i}{A}, \quad \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \frac{\tilde{\zeta}_i}{\tilde{B}} = \frac{\zeta_i}{B}, \quad i = 1, 2, \dots, m-2 \quad (108)$$

proves the conclusion of this theorem.

Theorem 19. Suppose that all the assumptions of Theorem 12 hold. Then, it is satisfied that

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{u} = u. \quad (109)$$

Proof. Similar to Theorem 18, we have that for any $t \in [0, 1]$,

$$\begin{aligned} |\tilde{u}(t) - u(t)| \leq & \int_0^1 \tilde{G}(t, s) \left| \varphi_q \left(\int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau \right) \right. \\ & \left. - \varphi_q \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) \right| ds \\ & + \left| \int_0^1 (\tilde{G}(t, s) - G(t, s)) \varphi_q \right. \\ & \cdot \left. \left(\int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right) ds \right|. \end{aligned} \quad (110)$$

By Lemma 11, we know that for $u \in B_{R_2}$, $\tilde{u} \in B_{\tilde{R}_2}$,

$$\begin{aligned} \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau & \leq M_0, \\ \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau & \leq \tilde{M}_0, \end{aligned} \quad (111)$$

where

$$\tilde{M}_0 = \frac{M_a + ((M_b / (\Gamma(\gamma+1)^{p-1})) + M_c) \tilde{R}_2^{p-1}}{\tilde{B}\Gamma(\beta)}. \quad (112)$$

From Lemma 17 and the inequality (53), we obtain

$$\begin{aligned} & \left| \int_0^1 \tilde{H}(s, \tau) f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) d\tau - \int_0^1 H(s, \tau) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\ & \leq \int_0^1 \tilde{H}(s, \tau) |f(\tau, I_{0+}^\gamma \tilde{u}(\tau), \tilde{u}(\tau)) - f(\tau, I_{0+}^\gamma u(\tau), u(\tau))| d\tau \\ & \quad + \left| \int_0^1 (\tilde{H}(s, \tau) - H(s, \tau)) f(\tau, I_{0+}^\gamma u(\tau), u(\tau)) d\tau \right| \\ & \leq \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \|\tilde{u} - u\| \\ & \quad \cdot \int_0^1 \tilde{H}_*(s, s) d\tau + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \cdot s^{\beta-1} \\ & \leq \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \frac{\|\tilde{u} - u\|}{\tilde{B}\Gamma(\beta+1)} \\ & \quad + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\zeta}_i}{\tilde{B}} - \frac{\zeta_i}{B} \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right), \end{aligned} \quad (113)$$

where

$$C = \max_{\substack{t \in [0,1] \\ x \in [0, R_1] \\ y \in [0, R_2]}} |f(t, x, y)|. \quad (114)$$

Since $q > 2$, using Lemma 16, we can get

$$\begin{aligned}
 |\tilde{u}(t) - u(t)| &\leq \left[\left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) \frac{\|\tilde{u} - u\|}{\tilde{B}\Gamma(\beta+1)} + \frac{C}{\Gamma(\beta)} \sum_{i=1}^{m-2} \left| \left(\tilde{\zeta}_i/\tilde{B} \right) - (\zeta_i/B) \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \right] \\
 &\quad \cdot (q-1) \max \left\{ M_0^{q-2}, \tilde{M}_0^{q-2} \right\} \int_0^1 \tilde{G}_*(s, s) ds + \frac{M_0}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\xi}_i}{\tilde{A}} - \frac{\xi_i}{A} \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right) \\
 &\leq \frac{\left[\left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) (\|\tilde{u} - u\| / (\tilde{B}\Gamma(\beta+1))) + (C/\Gamma(\beta)) \sum_{i=1}^{m-2} \left| \tilde{\zeta}_i/\tilde{B} - \zeta_i/B \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) \right]}{\tilde{A}(\alpha-\delta)\Gamma(\alpha-\gamma)} \\
 &\quad \cdot (q-1) \max \left\{ M_0^{q-2}, \tilde{M}_0^{q-2} \right\} + \frac{M_0}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-2} \left| \frac{\tilde{\xi}_i}{\tilde{A}} - \frac{\xi_i}{A} \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right).
 \end{aligned} \tag{115}$$

From the condition

$$\begin{aligned}
 &\frac{(q-1)M_0^{q-2}}{(\alpha-\delta)AB\Gamma(\alpha-\gamma)\Gamma(\beta+1)} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) \\
 &= \frac{(q-1)R_2^{2-p}}{\beta\tilde{M}} \left(\frac{L_1}{\Gamma(\gamma+1)} + L_2 \right) < 1,
 \end{aligned} \tag{116}$$

it holds that for sufficiently small λ_ξ and λ_ζ ,

$$\begin{aligned}
 \Lambda_4 &:= \frac{(q-1) \max \left\{ M_0^{q-2}, \tilde{M}_0^{q-2} \right\}}{\tilde{A}(\alpha-\delta)\Gamma(\alpha-\gamma)\tilde{B}\Gamma(\beta+1)} \\
 &\quad \cdot \left(\frac{\max \{L_1, \tilde{L}_1\}}{\Gamma(\gamma+1)} + \max \{L_2, \tilde{L}_2\} \right) < 1.
 \end{aligned} \tag{117}$$

Denote as follows:

$$\Lambda_5 := \frac{(q-1) \max \left\{ M_0^{q-2}, \tilde{M}_0^{q-2} \right\} C}{\tilde{A}(\alpha-\delta)\Gamma(\beta)\Gamma(\alpha-\gamma)}, \quad \Lambda_6 := \frac{M_0}{\Gamma(\alpha-\gamma)}. \tag{118}$$

This gives us that

$$\|\tilde{u} - u\| \leq \frac{\Lambda_5 \sum_{i=1}^{m-2} \left| \left(\tilde{\zeta}_i/\tilde{B} \right) - (\zeta_i/B) \right| \left(\eta_i^{\beta-1} (1 - \eta_i)^{\beta-1} \right) + \Lambda_6 \sum_{i=1}^{m-2} \left| \left(\tilde{\xi}_i/\tilde{A} \right) - (\xi_i/A) \right| \left(\eta_i^{\alpha-\delta-1} (1 - \eta_i)^{\alpha-\delta-1} \right)}{1 - \Lambda_4}. \tag{119}$$

Since

$$\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \frac{\tilde{\xi}_i}{\tilde{A}} = \frac{\xi_i}{A}, \quad \lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \frac{\tilde{\zeta}_i}{\tilde{B}} = \frac{\zeta_i}{B}, \quad i = 1, 2, \dots, m-2, \tag{120}$$

the proof is completed.

Remark 20. It follows from Theorems 18 and 20 that $\lim_{\substack{\lambda_\xi \rightarrow 0 \\ \lambda_\zeta \rightarrow 0}} \tilde{x} = x$. Therefore, it follows that the unique positive solution to problem (1) is continuously dependent on the perturbations with respect to the coefficients in m -point boundary conditions. This is a new result different from the

achievements of Dishlieva [21] and Li et al. [22] who studied the continuous dependency of solutions on initial conditions, barrier curves, and source terms in differential equations but not on boundary conditions.

4. Examples

Example 21. Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{1.5}(\varphi_3(D_{0+}^{1.75}x(t))) = 10\sqrt{t} + 0.2x(t) + 0.1[D_{0+}^{0.2}x(t)]^2, & 0 < t < 1, \\ D_{0+}^{0.2}x(0) = 0, D_{0+}^{0.4}x(1) = 0.3D_{0+}^{0.4}x(0.3) + 0.2D_{0+}^{0.4}x(0.7), \\ D_{0+}^{1.75}x(0) = 0, \varphi_3(D_{0+}^{1.75}x(1)) = 0.2\varphi_3(D_{0+}^{1.75}x(0.3)) + 0.5\varphi_3(D_{0+}^{1.75}x(0.7)). \end{cases} \tag{121}$$

Problem (121) can be regarded as the boundary value problem (1) where $f(t, x, y) = 10\sqrt{t} + 0.2x + 0.1y^2$, $p = 3$, $\alpha = 1.75$, $\beta = 1.5$, $\gamma = 0.2$, $\delta = 0.4$, $\eta_1 = 0.3$, $\eta_2 = 0.7$, $\xi_1 = 0.3$, $\xi_2 = 0.2$, $\zeta_1 = 0.2$, and $\zeta_2 = 0.5$. Then, we have

$$\begin{aligned} q &= 1.5 < 2, \\ A &= 0.626632\dots, \\ B &= 0.472125\dots, \\ \bar{M} &= 0.236575\dots. \end{aligned} \quad (122)$$

Since $x \leq 4 + (x^2/16)$, denoting $a(t) = 10.8$, $b(t) = 0.0125$, and $c(t) = 0.1$ yields that

$$\begin{aligned} M_a &= 10.8, \\ M_b &= 0.0125, \\ M_c &= 0.1, \\ f(t, x, y) &< a(t) + b(t)x^{p-1} + c(t)y^{p-1}. \end{aligned} \quad (123)$$

Putting $\bar{m} = 10$, $\mu = 1.5$, we find that

$$f(t, x, y) \geq \bar{m}t^{\mu-1}. \quad (124)$$

R_2, R_1 can be calculated as

$$\begin{aligned} R_2 &= 9.41849\dots, \\ R_1 &= \frac{R_2}{\Gamma(\gamma+1)} = 10.2579\dots. \end{aligned} \quad (125)$$

Also, put $L_1 = 0.2$, $L_2 = 0.1 \cdot 2$, and $R_2 = 1.8837\dots$. Then, we can see that for any $t \in [0, 1]$ and any $(x_1, y_1), (x_2, y_2) \in [0, R_1] \times [0, R_2]$,

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_1, y_2)| &\leq 0.2|x_1 - x_2| + 0.1|y_1^2 - y_2^2| \\ &\leq L_1|x_1 - x_2| + L_2|y_1 - y_2|. \end{aligned} \quad (126)$$

A simple calculation provides that

$$\begin{aligned} K_0 &= 2.48598\dots, \\ K_1 &= 1.07439\dots, \\ K &= 0.952215\dots. \end{aligned} \quad (127)$$

By Theorems 10 and 14, $K < 1$ shows that fractional boundary value problem (121) has a unique positive solution in $I_{0+}^\gamma(B_{R_2})$. Besides, Theorem 18 shows that the unique positive solution to the problem (121) is continuously dependent on its boundary conditions.

Example 22. Consider the boundary value problem

$$\begin{cases} D_{0+}^{1.5}(\varphi_{1.5}(D_{0+}^{1.75}x(t))) = 0.1t + 0.1[x(t)]^2 + 0.1D_{0+}^{0.2}x(t), 0 < t < 1, \\ D_{0+}^{0.2}x(0) = 0, D_{0+}^{0.4}x(1) = 0.3D_{0+}^{0.4}x(0.3) + 0.2D_{0+}^{0.4}x(0.7), \\ D_{0+}^{1.75}x(0) = 0, \varphi_{1.5}(D_{0+}^{1.75}x(1)) = 0.2\varphi_{1.5}(D_{0+}^{1.75}x(0.3)) + 0.5\varphi_{1.5}(D_{0+}^{1.75}x(0.7)). \end{cases} \quad (128)$$

Problem (128) can be regarded as the boundary value problem (1) where $f(t, x, y) = 0.1t + 0.1x^2 + 0.1y$, $p = 1.5$, $\alpha = 1.75$, $\beta = 1.5$, $\gamma = 0.2$, $\delta = 0.4$, $\eta_1 = 0.3$, $\eta_2 = 0.7$, $\xi_1 = 0.3$, $\xi_2 = 0.2$, $\zeta_1 = 0.2$, and $\zeta_2 = 0.5$. Then, we can get

$$\begin{aligned} q &= 3 > 2, \\ A &= 0.626632\dots, \\ B &= 0.472125\dots, \\ \bar{M} &= 0.362822\dots. \end{aligned} \quad (129)$$

Denote $a(t) = 0.1$, $b(t) = 0.1$, $c(t) = 0.1$ and we have

$$\begin{aligned} M_a &= 0.1, \\ M_b &= 0.1, \\ M_c &= 0.1, \\ R_2 &= 0.398248\dots, \\ R_1 &= 0.433742\dots. \end{aligned} \quad (130)$$

Also, it holds that for any $(t, x, y) \in [0, 1] \times [0, R_1] \times [0, R_2]$,

$$f(t, x, y) = 0.1t + 0.1x^2 + 0.1y < 0.1 + 0.1\sqrt{x} + 0.1\sqrt{y}. \quad (131)$$

Since $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq 0.1|x_1^2 - x_2^2| + 0.1|y_1 - y_2| \leq 0.1 \cdot 2R_1|x_1 - x_2| + 0.1|y_1 - y_2|$ for any $t \in [0, 1]$ and any $(x_1, y_1), (x_2, y_2) \in [0, R_1] \times [0, R_2]$, we can put

$$\begin{aligned} L_1 &= 0.0867483\dots, \\ L_2 &= 0.1. \end{aligned} \quad (132)$$

By simple calculation, we have

$$K = 0.45102\dots. \quad (133)$$

Since $K < 1$, using Theorems 12 and 14, boundary value problem (128) has a unique positive solution in $I_{0+}^\gamma(B_{R_2})$. Also, by employing Theorem 20, the continuous dependence of the unique positive solution to problem (128) on its boundary conditions is proved.

Comparing the problems (121) and (128) considered in Examples 22 and 23, respectively, there exists two main

differences between them. One is the value of p . If $p = 2$, it is obvious that $\varphi_p(s) = s$. As can be seen in the inequalities (23) and (24), some properties of $\varphi_p(\cdot)$ depend on whether the value of p is greater than 2 or not. So, Examples 22 and 23 demonstrated the existence, the uniqueness, the positivity, and the continuous dependency on four-point boundary conditions of the solutions to the problems (121) and (128) in the cases of $p > 2$ and $1 < p < 2$ by using Theorems 10 and 18 and Theorems 12 and 20, respectively. The other difference lies in the nonlinear source term $f(t, x, y)$. The source term in problem (121) was nonlinear to the fractional derivative of the unknown function, whereas, in problem (128), the nonlinearity appeared for the unknown function itself. However, similar techniques were applied to these two nonlinear source terms in the problems (121) and (128) to illustrate our main results.

5. Conclusion

In this paper, we have proved the existence, the uniqueness, the positivity, and the continuous dependency on multipoint boundary conditions of solutions to m -point boundary value problems for nonlinear multiterm fractional differential equations with p -Laplacian operator by employing the Banach contraction mapping principle. For the application of this study, two examples have been illustrated.

Data Availability

No data were used to support this study.

Conflicts of Interest

There is no competing interest among the authors regarding the publication of the article.

Authors' Contributions

All authors carried out the proof and conceived of the study. All authors read and approved the final manuscript.

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