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Research Article

Ordered Convex Metric Spaces

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The aim of this article is to introduce a new notion of ordered convex metric spaces and study some basic properties of these spaces. Several characterizations of these spaces are proven that allow making geometric interpretations of the new concepts.

1. Introduction

Menger [1] initiated the study of convexity in metric spaces which was further developed by many authors [2–4]. The terms "metrically convex" and "convex metric space" are due to [2]. Subsequently, Takahashi [5] introduced the notion of convex metric spaces and studied their geometric properties. Takahashi also proved that all normed spaces and their convex subsets are convex metric spaces and gave an example of a convex metric space which is not embedded in any normed/Banach space. Kirk [6] showed that a metric space of hyperbolic type is a convex metric space. Afterward, Shimizu and Takahashi [7] gave the concept of uniformly convex metric space, studied its properties, and constructed examples of a uniformly convex metric space. Beg [8] established some inequalities in uniformly convex complete metric spaces analogous to the parallelogram law in Hilbert spaces and their applications. Beg [9] proved that a closed convex subset of uniformly convex complete metric spaces is a Chebyshev set. Recently, Abdelhakim [10] studied convex functions on these spaces. The aim of this note is to further continue the research in this direction by introducing the concept of ordered convex metric spaces and study their structure.

We conclude with the plan of the paper. In Section 2, we recall some basic notations and definitions from the existing literature on convex metric spaces, order structure, and general topology. In Section 3, we introduce the new concept of ordered convex metric spaces and study some basic properties. Several characterizations of these spaces are also proven that allow making geometric interpretations of the new concepts Finally, Section 4 concludes with a summary statement.

2. Preliminaries

In this section, basic results about convex metric spaces and order structure are given.

Definition 1 (see [5]). Let (X, d) be a metric space and $I = [0, 1]$. A mapping $\omega : X \times X \times I \longrightarrow X$ is said to be a convex structure on X if for each $(a, b, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \omega(a, b\,; \lambda)) \leq \lambda d(u, a) + (1 - \lambda) d(u, b). \tag{1}$$

Metric space (X, d) together with the convex structure ω is called a convex metric space. A nonempty subset $K \subset X$ is said to be convex if $\omega(a, b\,; \lambda) \in K$ whenever $(a, b, \lambda) \in K \times K \times I$.

Remark 2 (see [5, 10]). The convex metric space (X, d) has the following properties:

(i) $w(a, b\,; 1) = a$, $\omega(a, b\,; 0) = b$, $\omega(a, a\,; \lambda) = a$

(ii) Open spheres $B(a, r) = \{b \in X : d(a, b) < r\}$ and closed spheres $B[a, r] = \{b \in X : d(a, b) \leq r\}$ are convex

(iii) If $\{K_\alpha : \alpha \in A\}$ is a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is convex

Any normed space and a convex subset of a normed space is a convex metric space. There are several examples in the existing literature [5, 7, 8, 10] of convex metric spaces which are not embedded in any normed space.

Definition 3 (see [11]). A binary relation $a \preccurlyeq b$ defined for some pairs a, b of elements of a set X is called an order relation in X if $\preccurlyeq$ is reflexive, transitive, and antisymmetric. A reflexive and transitive relation $\preccurlyeq$ is called a preorder.

Remark 4 (see [11]). Let $\preccurlyeq$ be a binary relation on a set X. By $a \prec b$ we mean $a \preccurlyeq b$ and $a \neq b$. Relation $\sim$ is defined as $a \sim b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$. The inverse of $\preccurlyeq$ is defined as $a \succcurlyeq b$ if $b \preccurlyeq a$. Incomparable elements a and b (i.e., $a \npreccurlyeq b$ and $a \nsucccurlyeq b$) are denoted by $a \rhd\lhd b$. Transitivity of order relation $\preccurlyeq$ implies $a \npreccurlyeq b \npreccurlyeq c \Longrightarrow a \npreccurlyeq c$ for all $a, b, c \in X$.

Definition 5 (see [11]). An ordered set is called totally ordered if it has no incomparable elements.

Proposition 6 (see Proposition 4.1 of [12]). *A topological space is disconnected if and only if it has a nonempty subset that is both open and closed.*

Proposition 7 (see [13]). *Let X be a connected topological space. If B is a connected subset of X such that $X \setminus B$ is the union of $n (n > 1)$ nonempty, pairwise disjoint open (in $X \setminus B$) sets D_i, then $B \cup D_i$ is connected for all i.*

3. Ordered Convex Metric Spaces

In this section, first, we introduce the property $(\mathscr{L})$ on a convex metric space. Next, we present some notations and definitions related to an order relation $\preccurlyeq$ on a convex metric space. Finally, we define ordered convex metric space and prove several interesting results related to ordered convex metric spaces.

Definition 8. A convex metric space X is said to have property $(\mathscr{L})$ if for all a, b in X and λ, μ, ν in I, we have

$$w(a, b; \lambda) = \omega(b, a; 1-\lambda),$$
$$\omega(\omega(a, b; \lambda), \omega(a, b; \mu); \nu) = \omega(a, b; \lambda\nu + \mu(1-\nu)). \tag{2}$$

Each normed space has property $(\mathscr{L})$, if we define $\omega(a, b; t) = ta + (1-t)b$. In Definition 8, taking $\mu = 0$ and using Remark 2, we obtain

$$\omega(\omega(a, b; \lambda), b; \nu) = \omega(a, b; \lambda\nu). \tag{3}$$

Let X be a convex metric space and $\preccurlyeq$ be an ordered relation on X. First, we define some notation for subsequent use. For any a, b, c in X and $\lambda \in I$,

$$\begin{aligned} A_{\preccurlyeq}(a,b,c) &= \{\lambda : \omega(a,b;\lambda) \preccurlyeq c\}, & A_{\succcurlyeq}(a,b,c) &= \{\lambda : \omega(a,b;\lambda \succcurlyeq c\}, \\ A_{\prec}(a,b,c) &= \{\lambda : \omega(a,b;\lambda) \prec c\}, & A_{\succ}(a,b,c) &= \{\lambda : \omega(a,b;\lambda) \succ c\}, \\ A_{\sim}(a,b,c) &= \{\lambda : \omega(a,b;\lambda) \sim c\}, & A_{\rhd\lhd}(a,b,c) &= \{\lambda : \omega(a,b;\lambda) \rhd\lhd c\}. \end{aligned} \tag{4}$$

Definition 9. (i) A relation $\preccurlyeq$ on a convex metric space X is said to be continuous if for all a, b, c in X, the sets $A_{\preccurlyeq}(a, b, c)$ and $A_{\succcurlyeq}(a, b, c)$ are closed.

(ii) A relation $\preccurlyeq$ on a convex metric space X is said to be Archimedean if for all a, b, c, d in X with $d \rhd\lhd a \prec b \rhd\lhd c$, there exists $\lambda, \mu \in (0, 1)$ such that $\omega(a, c; \lambda) \prec b$ and $a \prec \omega(b, d; \mu)$. When relation $\preccurlyeq$ is total, the space is called Archimedean.

(iii) A relation $\preccurlyeq$ on a convex metric space X is said to have betweenness property if for all a, b in X, all $c \in \{a, b\}$ and all $\lambda \in (0, 1)$,$a \preccurlyeq b$ if and only if $\omega(a, c; \lambda) \preccurlyeq \omega(b, c; \lambda)$.

Definition 10. A relation $\preccurlyeq$ on a convex metric space X, is

(i) (o)-convex if $a \preccurlyeq c$ and $b \preccurlyeq c$ implies $\omega(a, b; \lambda) \preccurlyeq c$

(ii) (o)-concave if $a \preccurlyeq c$ and $b \preccurlyeq c$ implies $\omega(a, b; \lambda) \succcurlyeq c$

(iii) (o)-linear if $a \sim b$ implies $a \sim \omega(a, b; \lambda)$

Definition 11. A convex metric space X with order relation $\preccurlyeq$ is called ordered convex metric space if $\preccurlyeq$ is continuous.

Proposition 12. *Let X be an ordered convex metric space with property $(\mathscr{L})$. Then, $\preccurlyeq$ is Archimedean $\Longleftrightarrow A_{\prec}(a, b, c)$ and $A_{\succ}(a, b, c)$ are open for any a, b, c in X.*

Proof. Let $\preccurlyeq$ be an Archimedean relation on X and $A_{\prec}(a, b, c)$ be closed. Without loss of generality, we can assume that $A_{\prec}(a, b, c)$ is nonempty. Choose λ in $A_{\prec}(a, b, c)$. Now, continuity of $\preccurlyeq$ and $\lambda \notin A_{\succcurlyeq}(a, b, c)$ imply that there exists $t > 0$ such that $N_t(\lambda) = \{\alpha : |\alpha - \lambda| < t\}$ is contained in the complement of $A_{\succcurlyeq}(a, b, c)$.

Assume there exists $\beta \in N_t(\lambda) \cap A_{\rhd\lhd}(a, b, c)$. Continuity of $\preccurlyeq$ implies that $A_{\rhd\lhd}(a, b, c)$ is an open set in I. Thus, $A_{\rhd\lhd}(a, b, c)$ is union of at most countably many mutually disjoint open intervals. Axiom of choice further implies that there exists among these intervals an open interval F such that $\beta \in F$. If $\beta \prec \lambda$, then set $\delta = \sup F$; otherwise, $\delta = \inf F$. Then, $\delta \in A_{\prec}(a, b, c)$. By Definition 8 (ii), we have

$$\omega(\omega(a, b; \delta), \omega(a, b; \beta); \gamma) = \omega(a, b; \gamma\delta + (1-\gamma)\beta) \text{ for all } \gamma. \tag{5}$$

Obviously, $(\beta, \delta) \subset F$; thus, $\gamma\beta + (1-\gamma)\delta \in A_{\rhd\lhd}(a, b, c)$, $\forall \gamma \in (0, 1)$. It contradicts that $\preccurlyeq$ is Archimedean. Hence, $A_{\prec}(a, b, c)$ is open.

Similarly, we can prove that $A_{\succ}(a, b, c)$ is open.

$\Longleftarrow$: Assume that $A_{\prec}(a, b, c)$ and $A_{\succ}(a, b, c)$ are open. Choose a, b, c in X such that $a \prec b$. Remark 2 (i) implies that $1 \in A_{\succ}(a, b, c)$. Since $A_{\succ}(a, b, c)$ is open, thus there exists $\lambda < 1$ such that $(\lambda, 1] \subset A_{\succ}(a, b, c)$. Also, $1 \in A_{\prec}(a, b, c)$ and $A_{\prec}(a, b, c)$ is open; therefore, there exists $\delta < 1$ such that $(\delta, 1] \subset A_{\prec}(a, b, c)$. Now obviously, $\omega(a, c; \lambda) \prec b$ and $a \prec \omega(b, c; \delta)$.

Next, we give Example 13 to show that we cannot drop any condition from Proposition 12. □

Example 13. Consider the convex metric space $([0,1], d)$ with d usual Euclidean distance and convex structure $\omega : [0.1] \times 0,1] \times I \longrightarrow 0,1]$ defined by $\omega(a,b;\lambda) = \lambda a + (1-\lambda)b$. Assume $\preccurlyeq$ is a reflexive relation on $[0,1]$ such that $0 \prec a$ for all $a > 0$ and no other element are comparable. Then, each of $A_{\preccurlyeq}(a,b,c)$ and $A_{\succcurlyeq}(a,b,c)$ either contains at most two elements or is equal I. Therefore, $A_{\preccurlyeq}(a,b,c)$ and $A_{\succcurlyeq}(a,b,c)$ are closed. Also $A_{\prec}(0,1,1) = \{0\}$ is not open. Moreover, $0 \prec 1$, but for all $a \in (0,1)$ and all $\lambda \in (0,1)$,$\omega(1,a;\lambda) \nprec 1$. Thus, $\preccurlyeq$ is continuous, $A_{\preccurlyeq}(a,b,c)$ and $A_{\succcurlyeq}(a,b,c)$ are not open, and $\preccurlyeq$ is not Archimedean.

Now, the following proposition is obvious.

Proposition 14. *Any totally ordered convex metric space X with property ($\mathscr{L}$) is Archimedean.*

Theorem 15. *A nontrivial continuous Archimedean order $\preccurlyeq$ on a convex metric space X with property ($\mathscr{L}$) is totally ordered.*

Proof. Let $\preccurlyeq$ be not totally ordered on the convex metric space X. Then, there exists $u, v \in X$ such that $u \rhd\lhd v$ and $a, b \in X$ with $a \prec b$. Let $a \prec u$. Then, using Remark 4 $a \prec v$ or $v \prec u$. Since $u \rhd\lhd v$, therefore $a \prec v$. Thus, $a \prec u$ and $a \prec v$. Now, we prove

$$A_{\preccurlyeq}(a,u,v) \cap A_{\preccurlyeq}(a,u,u) = A_{\prec}(a,u,v) \cap A_{\prec}(a,u,u). \quad (6)$$

Obviously, $A_{\prec}(a,u,v) \cap A_{\prec}(a,u,\mathrm{u}) \subset A_{\preccurlyeq}(a,u,v) \cap A_{\preccurlyeq}(a,u,u)$. To prove other inclusions, choose $\lambda \in A_{\preccurlyeq}(a,u,v) \cap A_{\preccurlyeq}(a,u,u)$. If $v \sim \omega(a,u;\lambda)$, then it follows from transitivity and $\omega(a,u;\lambda) \preccurlyeq u$ that $u \succcurlyeq v$, which is a contradiction to $u \rhd\lhd v$. Therefore, $v \succ \omega(a,u;\lambda)$, i.e., $\lambda \in A_{\prec}(a,u,v)$. In a similar way, if $u \sim \omega(a,u;\lambda)$, then $u \preccurlyeq v$ which contradicts $u \rhd\lhd v$. Thus, $\lambda \in A_{\prec}(a,u,u)$.

Now, $u \rhd\lhd v$ and Remark 2 (i) imply that $0 \notin A_{\prec}(a,u,v) \cap A_{\prec}(a,u,u)$ and $1 \in A_{\prec}(a,u,v) \cap A_{\prec}(a,u,u)$. Continuity of $\preccurlyeq$ further implies that $A_{\preccurlyeq}(a,u,u) \cap A_{\preccurlyeq}(a,u,v)$ is closed. Using Equality 3, we obtain that $A_{\prec}(a,u,v) \cap A_{\prec}(a,u,u)$ is a closed set. On the other hand, Proposition 12 implies that $A_{\prec}(a,u,u) \cap A_{\prec}(a,u,v)$ is an open set. Thus, we have a nonempty closed-open proper subset of I. Since I is connected, therefore it is a contradiction to Proposition 6 . Similarly, we can show a contradiction for the case $a \succ b$. Hence, $\preccurlyeq$ is a totally ordered relation. □

Corollary 16. *Let X be an ordered convex metric space with property ($\mathscr{L}$). If $\preccurlyeq$ is an Archimedean relation, then the space X is also Archimedean.*

Proposition 17. *Let X be an ordered convex metric space with property ($\mathscr{L}$); then, $\preccurlyeq$ is (o)-linear $\Longleftrightarrow A_{\sim}(a,b,c)$ is convex.*

Proof. Let $\preccurlyeq$ be (o)-linear. Choose $\lambda, \delta \in A_{\sim}(a,b,c)$ and $\beta \in I$. Define $p = \omega(a,b;\lambda)$ and $q = \omega(a,b;\delta)$. Then, $p \sim c$ and $q \sim c$. Transitivity of $\sim$ implies $p \sim q$. It follows from (o)-linearity of $\preccurlyeq$ that $p \sim \omega(p,q;\beta)$. By Definition 8 (ii), we obtain $\omega(p,q;\beta) = \omega(\omega(a,b;\lambda), \omega(a,b;\delta);\beta) = \omega(a,b;\beta\lambda + (1-\beta)\delta)$. Transitivity of $\sim$ further implies that $\omega(a,b;\beta\lambda + (1-\beta)\delta) \sim c$. Therefore, $\beta\lambda + (1-\beta)\delta \in A_{\sim}(a,b,c)$. Hence, $A_{\sim}(a,b,c)$ is convex.

$\Longleftarrow$; Assume $A_{\sim}(a,b,c)$ is convex. Choose $a, b \in X$ such that $a \sim b$ and $\lambda \in I$. From reflexivity of $\preccurlyeq$,$a \sim a$. Remark 2 implies $0, 1 \in A_{\sim}(a,b,c)$. Since by assumption $A_{\sim}(a,b,c)$ is convex. Therefore, $A_{\sim}(a,b,c) = I$. Thus, $\lambda, 1-\lambda \in A_{\sim}(a,b,c)$. Now transitivity of $\preccurlyeq$ and $a \sim b$ imply that $\omega(a,b;\lambda) \sim b$ and $\omega(a,b;1-\lambda) \sim a$. Property ($\mathscr{L}$) (see Definition 8 (i)) of X implies $\omega(a,b;1-\lambda) = \omega(b,a;\lambda)$. Hence, $\preccurlyeq$ is (o)-linear. □

Proposition 18. *Let X be an ordered convex metric space with property ($\mathscr{L}$), then $\preccurlyeq$ is (o)-convex if and only if $A_{\preccurlyeq}(a,b,c)$ is convex.*

Proof. Suppose that $\preccurlyeq$ is (o)-convex. Choose $\lambda, \delta \in A_{\preccurlyeq}(a,b,c)$ and $\beta \in I$. Define $p = \omega(a,b;\lambda)$ and $q = \omega(a,b;\delta)$. Then, $p \preccurlyeq c$ and $q \preccurlyeq c$. (o)-convexity of $\preccurlyeq$ implies that $\omega(p,q;\beta) \preccurlyeq c$. Using Definition 8, we have $\omega(p,q;\beta) = \omega(p,q;\beta\lambda + (1-\beta)\delta)$. Thus, $\beta\lambda + (1-\beta)\delta \in A_{\preccurlyeq}(a,b,c)$. Hence, $A_{\preccurlyeq}(a,\mathrm{b},c)$ is convex.

$\Longleftarrow$; Assume that $A_{\preccurlyeq}(a,b,c)$ is convex. Choose $a, b, c \in X$ such that $a \preccurlyeq c$,$b \preccurlyeq c$ and $\lambda \in I$. Remark 2 implies $0, 1 \in A_{\preccurlyeq}(a,b,c)$. As $A_{\preccurlyeq}(a,b,c)$ is convex, therefore $A_{\preccurlyeq}(a,b,c) = I$. Thus, $\omega(a,b;\lambda) \preccurlyeq c$. Hence, $\preccurlyeq$ is (o)-convex. □

Proposition 19. *Let X be an ordered convex metric space with property ($\mathscr{L}$); then, $\preccurlyeq$ is (o)-concave if and only if $A_{\pm}(a,b,c)$ is convex.*

Proof. Similar to Proposition 18. □

Theorem 20. *Let X be an Archimedean-ordered convex metric space with property ($\mathscr{L}$). The relation $\preccurlyeq$ is (o)-linear if and only if $\preccurlyeq$ is (o)-convex and (o)-concave.*

Proof. Assume $\preccurlyeq$ is (o)-linear. Proposition 17 implies $A_{\sim}(a,b,c)$ is convex, thus a connected subset of I. Obviously, $I \setminus A_{\sim}(a,b,c) = A_{\prec}(a,b,c) \cup A_{\succ}(a,b,c) \cup A_{\rhd\lhd}(a,b,c)$. Clearly, $A_{\prec}(a,b,c)$,$A_{\succ}(a,b,c)$, and $A_{\rhd\lhd}(a,b,c)$ are mutually disjoint sets. Proposition 12 and continuity of $\preccurlyeq$ imply that these all three sets are open sets. Since $A_{\preccurlyeq}(a,b,c) = A_{\prec}(a,b,c) \cup A_{\sim}(a,b,c)$. Proposition 18 and Proposition 19 further imply that (o)-concavity of $\preccurlyeq$ is equivalent to the convexity of $A_{\succcurlyeq}(a,b,c)$ and (o)-convexity of $\preccurlyeq$ is equivalent to the convexity of $A_{\preccurlyeq}(a,b,c)$. Now, Proposition 7 implies that $\preccurlyeq$ is (o)-convex and (o)-concave.

$\Longleftarrow$; Assume that $A_{\preccurlyeq}(a,b,c)$ is convex and $A_{\succcurlyeq}(a,b,c)$ is concave for all a, b, c in X. As $A_{\sim}(a,b,c) = A_{\preccurlyeq}(a,b,c) \cap A_{\pm}(a,b,c)$. Therefore, $A_{\sim}(a,b,c)$ is convex. Proposition 17 further implies that $\preccurlyeq$ is (o)-linear. □

4. Concluding Remarks

Order, convexity, and metric are three fundamental concepts in mathematics. These ideas have beautiful geometric properties with significant applications in approximation and optimization (see [14, 15]). In this work, we tried to combine these three indispensable notions of order, convexity, and metric. We introduced the new concept of ordered convex metric spaces and studied some of their properties. Several characterizations (Propositions 12, 17, 18, and 19 and Theorem 20) of these spaces are proven that allow to make geometric interpretations of the new concepts. This author's recommendation is to study other applications of ordered convex metric spaces to economics, preference modelling, control theory, functional analysis, etc.

Data Availability

No data were used to support this study.

Conflicts of Interest

Author declares that he has no conflict of interest.

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