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Research Article

Existence of Positive Solutions for Second-Order Third-Point Semipositive BVP

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In this paper, we study the existence of positive solutions for the following nonlinear second-order third-point semi-positive BVP. We derive an explicit interval of positive parameters, which for any l, μ in this interval, the existence of positive solutions to the boundary value problem is guaranteed under the condition that $a(t,x), b(t,x)$ are all superlinear (sublinear), or one is superlinear, the other is sublinear.

1. Introduction

In the applied mathematical field, three-point BVP can describe many phenomena. Moshinsky [1] introduced the vibrations of a guy wire with a uniform cross-section and composed of N parts of different densities using a multipoint BVP. Timoshenko [2] also revealed that the theory of elastic stability can be used by the method of a three-point BVP. Il'in and Moviseev [3] were the first to study this aspect. Since then, more general nonlinear BVP have been studied by several authors [4–25].

In their paper [7], Ma and Wang obtained the existence of positive solutions for a three-point BVP by Krasnoselskii's fixed theorem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, & 0 \le t \le 1, \\ u(0) = 0, & u(1) = \alpha u(\eta), \end{cases} \tag{1}$$

where α is a positive constant, $0 < \eta < 1$, $a(t) \in C([0,1], \mathbf{R}^+)$, $b(t) \in C([0,1], \mathbf{R}^-)$, $f \in C(\mathbf{R}^+, \mathbf{R}^+)$,$h \in C([0,1], \mathbf{R}^+)$ and there exists $x_0 \in (0,+\infty)$ such that $h(x_0) > 0$.

In our paper, we study the existence of positive solutions of second-order third-point semipositive BVP:

$$\begin{cases} (Lx)(t) + \lambda a(t,x) + \mu b(t,x), & 0 \le t \le 1, \\ x(0) = 0, & x(1) = \alpha x(\xi), \end{cases} \tag{2}$$

where $(Lu)(t) = u''(t) + f(t)u'(t) + g(t)u(t)$, λ, μ are positive parameters, $0 < \xi < 1$, $f(t) \in C[0,1]$, and $g(t) \in C([0,1], (-\infty,0))$. And our paper also allows that $a(t,x), b(t,x)$ are both semipositive and lower unbounded.

Our main tool is the following fixed point index theory.

Theorem 1 [4]. *We suppose that $K \subset E$ is a cone in E, in which E is a real Banach space, the open bounded set Ω_1, Ω_2 is in E, $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \longrightarrow K$. Suppose operator T can be completely continuous and satisfies one of the following conditions:*

(i) $\|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_1; \quad \|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_2$

(ii) $\|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_1; \quad \|Tx\| \le \|x\|, \quad \forall x \in K \cap \partial\Omega_2$

Then, operator T has at least one fixed point x^* in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Theorem 2 [4]. *We suppose that $P \subset E$ is a cone in E, in which E is a real Banach space, the open bounded set $\Omega_1, \Omega_2, \Omega_3$ is in E, $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, $\bar{\Omega}_2 \subset \Omega_3$, and $T : P \cap \Omega_3 \longrightarrow P$. Suppose operator A is completely continuous and satisfies the following conditions:*

$$\begin{aligned} &\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \\ &\|Tx\| \ge \|x\|, \quad Ax \ne x, \forall x \in P \cap \partial\Omega_2, \\ &\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_3. \end{aligned} \tag{3}$$

Then, operator T has at least two fixed points x^* and x^{**} in $P \cap (\bar{\Omega}_3/\Omega_1)$, and $x^* \in P \cap (\Omega_2/\Omega_1)$ and $x^{**} \in P \cap (\bar{\Omega}_3/\bar{\Omega}_2)$.

2. Preliminaries and Lemmas

We set a Banach space $E = C([0,1],(-\infty,+\infty))$ with norm $\|x\| = \max_{t\in I}|x(t)|$. We know of the following lemmas from Ref. [6].

Lemma 3. *Setting$\xi_1(t)$as the positive solution of the equation, we have:*

$$\begin{cases} (L\xi_1)(t) = 0, & 0 \le t \le 1, \\ \xi_1(0) = 0, & \xi_1(1) = 1. \end{cases} \tag{4}$$

Then, $\xi_1(t) \in [0,1]$ is strictly increasing on $[0,1]$, and $\xi_1{'}(0) > 0$.

Lemma 4. *Setting$\xi_2(t)$as the positive solution of the equation, we have:*

$$\begin{cases} (L\xi_2)(t) = 0, & 0 \le t \le 1, \\ \xi_2(0) = 1, & \xi_2(1) = 0. \end{cases} \tag{5}$$

Then, $\xi_2(t) \in [0,1]$ is strictly decreasing on $[0,1]$.

From Lemma 3 and Lemma 4, we know that $0 < \xi_1(t) < 1$, $0 < \xi_2(t) < 1$. In the rest of our paper, the following condition is used:

(C1) $0 < \alpha\xi_1(\eta) < 1$, where$\xi_1(t)$is given by Lemma 3

Throughout this paper, we shall use the following notation:

$$G(t,s) = \frac{1}{\zeta}\begin{cases} \xi_1(t)\xi_2(s), & 0 \le t \le s \le 1, \\ \xi_1(s)\xi_2(t), & 0 \le s \le t \le 1, \end{cases} \tag{6}$$

where $\zeta = \xi_1'(0)\xi_2(0)$.

Obviously, from Ref. [6], we can be assured that when (C1) holds, the BVP

$$\begin{cases} (Lx)(t) + y(t) = 0, & 0 \le t \le 1, \\ x(0) = 0, & x(1) = \alpha u(\xi), \end{cases} \tag{7}$$

is equivalent to the following integral equation:

$$x(t) = \int_0^1 G(t,s)e(s)y(s)ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)y(s)ds, \tag{8}$$

where $e(t) = \exp\left(\int_0^t f(s)ds\right)$.

Set $z(t) = \min\left((\xi_1(t)/\|\xi_1\|), (\xi_2(t)/\|\xi_2\|)\right)$. From (6), for $t \in [0,1]$, we know that

$$z(t)G(s,s) \le G(t,s) \le G(s,s). \tag{9}$$

We present some other lemmas that are important to our main results.

Lemma 5 [7]. *Assume that for any $y \in C([0,1],(0,+\infty))$, $x(t)$ is the solution of the following BVP:*

$$\begin{cases} (Lx)(t) + y(t) = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\xi). \end{cases} \tag{10}$$

Then, we have

$$x(t) \ge z(t)\|x\|, t \in [0,1]. \tag{11}$$

Lemma 6. *Assume that $\bar{w}$ is a solution of the following BVP:*

$$\begin{cases} (Lx)(t) = -B(t), & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\xi), \end{cases} \tag{12}$$

where $B \in C(0,1)$, $M > 0$. Then, there exists constant $M > 0$ and satisfies

$$\bar{w}(t) \le M\|B\|z(t), \quad t \in [0,1]. \tag{13}$$

Proof. For $t \in [0,1]$, we can have

$$\bar{w}(t) = \int_0^1 G(t,s)e(s)B(s)ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)B(s)ds. \tag{14}$$

Obviously, for $t \in [0,1]$, we have

$$\begin{aligned}
&\int_0^1 G(t,s)e(s)B(s)ds \\
&= \frac{1}{\zeta}\left[\int_0^t \xi_1(s)\xi_2(t)e(s)B(s)ds + \int_t^1 \xi_1(t)\xi_2(s)e(s)M(s)ds\right] \\
&\le \frac{p(1)}{\zeta}\left[\xi_1(t)\xi_2(t)\int_0^t B(s)ds + \xi_1(t)\xi_2(t)\int_t^1 B(s)ds\right] \\
&= \frac{e(1)\|\xi_1\|\|\xi_2\|}{\zeta}\left[\frac{\xi_1(t)}{\|\xi_1\|}\frac{\xi_2(t)}{\|\xi_2\|}\int_0^t M(s)ds\right. \\
&\quad \left.+ \frac{\xi_1(t)}{\|\xi_1\|}\frac{\xi_2(t)}{\|\xi_2\|}\int_t^1 B(s)ds\right] \\
&\le \frac{e(1)\|\xi_1\|\|\xi_2\|}{\zeta} z(t)\int_0^1 B(s)ds \le M_1 z(t)\|B\|,
\end{aligned} \tag{15}$$

where $M_1 = (e(1)\|\xi_1\|\|\xi_2\|)/\zeta$.

By the same method, we can know that

$$\frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)B(s)ds \le M_2 z(t)\|B\|, \tag{16}$$

where $M_2 = (\alpha e(1)\|\xi_1\|\|\xi_2\|)/(1-\alpha\xi_1(\xi))$.

So, by choosing constant $M \ge M_1 + M_2$, we have

$$\bar{w}(t) \le M\|B\|z(t), \quad 0 \le t \le 1. \tag{17}$$

□

Lemma 7 [7]. *Let $0 \le \overline{\lim}_{x\infty}(b(t,x)/x) \le L_2$, $t \in [0,1]$. Define the following function:*

$$G(\tau) = \max_{0\le t\le 1, 0\le x\le \tau} b(t,\tau). \tag{18}$$

Then

(i) G is a nondecreasing function for τ

(ii) $0 \le \overline{\lim}_{\rho\infty}(G(\tau)/\tau) \le K_2$

For g assumptions:

(C2) $a(t,x), b(t,x) \in C, ([0,1]\times[0,+\infty)R)$

From (C2), there exists a function$B(t) \in C[0,1]$, $B(t) > 0$, which satisfies

$$\begin{aligned}
&a(t,x) \ge -B(t), \\
&b(t,x) \ge -B(t), \\
&\qquad \forall t \in (0,1), x \ge 0,
\end{aligned} \tag{19}$$

where $M\|B\| < 1$. M is given by Lemma 6.

(C3) $B_1 \le a_\infty^- \le \infty, B_2 \le b_\infty^- \le \infty$

(C4) $0 \le a_\infty^+ \le b_1, 0 \le b_\infty^+ \le b_2$

(C5) $K_1 \le a_\infty^- \le \infty, 0 \le b_\infty^+ \le K_2$,

where

$$\begin{aligned}
&\min(B_1,B_2) \ge 2\left((\lambda+\mu)\min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds\right)^{-1}, \\
&b_1 + b_2 \le \left((\lambda+\mu)p(1)\left[\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\right)^{-1}, \\
&K_1 \ge 2\left(\lambda \min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds\right)^{-1}, \\
&a_\infty^- = \underline{\lim}_{x\longrightarrow\infty}\frac{a(t,x)}{x}, \\
&a_\infty^+ = \overline{\lim}_{x\longrightarrow\infty}\frac{a(t,x)}{x}, \\
&b_\infty^- = \underline{\lim}_{x\longrightarrow\infty}\frac{b(t,x)}{x}, \\
&b_\infty^+ = \overline{\lim}_{x\longrightarrow\infty}\frac{b(t,x)}{x}.
\end{aligned} \tag{20}$$

Let $\varepsilon = \min_{0\le t\le 1} z(t)$, and

$$\begin{aligned}
&\bar{H}(t,x) = \begin{cases} H(t,x), & x \ge 0, \\ F(t,0), & x < 0, \end{cases} \\
&\bar{Y}(t,x) = \begin{matrix} Y(t,x), & x \ge 0, \\ G(t,0), & x < 0, \end{matrix}
\end{aligned} \tag{21}$$

where $H(t,x) = a(t,x) + B(t)$, $Y(t,x) = b(t,x) + B(t)$.

For any $l > 0$, we set

$$\begin{aligned}
H_l &= \max_{0\le t\le 1, 0\le x\le l} \bar{H}(t,x), \\
Y_l &= \max_{0\le t\le 1, 0\le x\le l} \bar{Y}(t,x).
\end{aligned} \tag{22}$$

From Lemma 6, letting $w(t) = \bar{w}(t)$, then $x(t)$ is the positive solution of problem (2) if and only if $\tilde{x}(t) = x(t) + w(t)$ is the solution of the following problem:

$$\begin{cases} (Lx)(t) + \lambda\bar{H}(t,x-w) + \mu\bar{Y}(t,x-w) = 0, \\ x(0) = 0, \quad x(1) = \alpha u(\xi), \end{cases} \tag{23}$$

and $\tilde{x}(t) > w(t)$, $0 < t < 1$; here, $\bar{H}$, $\bar{Y}$ is given by (21).

Defining the cone P in E, we have

$$P = \{x \in E : x(t) \ge \|x\|q(t), \quad t \in [0,1]\}. \tag{24}$$

Obviously, problem (18) is equivalent to

$$x(t)=\int_0^1 Y(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds. \tag{25}$$

Defining the operator $T:E\longrightarrow E$, we have

$$(Tx)(t)=\int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s) \cdot \left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds. \tag{26}$$

Obviously $T(P)\subset P$ and T is completely continuous.

3. Our Main Three Results

Theorem 8. *Suppose condition (C1), condition (C2), and condition (C3) hold. Then, for the small number λ,μ, problem (2) has at least one positive solution.*

Proof. Firstly, we choose sufficiently small λ,μ which satisfies the following:

$$\lambda+\mu<\left([H_1+Y_1]p(1)\left[\int_0^1 G(s,s)ds+\frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\right)^{-1}. \tag{27}$$

Letting $\Omega_1=\{x\in E:\|x\|<1\}$, for any $x\in P\cap\partial\Omega_1$, $t\in[0,1]$, by the definition of operator T, we have

$$\begin{aligned}(Tx)(t)&\le\int_0^1 G(s,s)[\lambda H_1+\mu Y_1]ds+\frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\cdot\int_0^1 G(\xi,s)e(s)[\lambda H_1+\mu Y_1]ds\\ &\le(\lambda+\mu)[H_1+Y_1]e(1)\cdot\left[\int_0^1 G(s,s)ds+\frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]<1=\|x\|.\end{aligned} \tag{28}$$

Thus, we have

$$\|Tx\|\le\|x\|,\quad \forall x\in P\cap\partial\Omega_1. \tag{29}$$

Secondly, by (C3), we know that there exists constant $l_1>0$ which satisfies

$$a(t,x)\ge B_1x,\, b(t,x)\ge B_2x, \forall x\ge l_1,\, t\in[0,1]. \tag{30}$$

Letting $r=\max\{2M\|B\|,(2l_1/\varepsilon),2\}$, then $r>1$. Set $\Omega_2=\{x\in E:\|x\|<r\}$, for any $x\in P\cap\partial\Omega_2$, $t\in[0,1]$, we have

$$x(t)-w(t)\ge x(t)-M\|B\|z(t)\ge x(t)-\frac{M\|B\|_1}{r}x(t)\ge\frac{1}{2}x(t). \tag{31}$$

Therefore, we have $x(t)-w(t)\ge(1/2)x(t)\ge(\|x\|/2)z(t)\ge(\varepsilon r/2)\ge l_1$.

Thus, by the definition of $\bar{H}$, $\bar{Y}$ and (30), we can have

$$\begin{aligned}&\lambda\bar{H}(s,x(s)-w(s))+\mu\bar{Y}(s,x(s)-w(s))\\ &\quad\ge B_1\lambda(x(s)-w(s))+B_2\mu(x(s)-w(s))\\ &\quad\ge\min(B_1,B_2)(\lambda+\mu)(x(s)-w(s)).\end{aligned} \tag{32}$$

We have

$$\begin{aligned}(Tx)(t)&\ge\int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds\\ &\ge\min_{0\le t\le1}\int_0^1 G(t,s)\min(B_1,B_2)(\lambda+\mu)(x(s)-w(s))ds\\ &\ge\frac{1}{2}(\lambda+\mu)\min(B_1,B_2)\min_{0\le t\le1}\int_0^1 G(t,s)x(s)ds.\end{aligned} \tag{33}$$

Then, by Lemma 5, we have

$$\|(Tx)(t)\|\ge\frac{1}{2}(\lambda+\mu)\min(B_1,B_2)\min_{0\le t\le1}\int_0^1 G(t,s)e(s)z(s)ds\|x(t)\|. \tag{34}$$

Therefore, by the definition of B_1,B_2, we have

$$\|Tx\|\ge\|x\|,\quad\forall x\in K\cap\partial\Omega_2. \tag{35}$$

Then, by (29), (35) and Theorem 1, operator T has at last one fixed point $\tilde{x}(t)\in P\cap(\bar{\Omega}_2/\Omega_1)$, i.e., $\tilde{x}(t)$ is the solution of problem (2), and it is easy to know $\|\tilde{x}\|\ge1$.

Finally, by (C2) and Lemma 3, we have

$$\tilde{x}(t)\ge\|\tilde{x}\|z(t)\ge z(t)>M\|B\|z(t)\ge\bar{w}(t)=w(t). \tag{36}$$

Thus, $x=\tilde{x}-w$ is the positive solution of problem (2). ☐

Theorem 9. *We suppose that condition (C1), (C2), and (C4) hold, and the following condition also holds:*

(C6) There exist constant $D>0$, $\rho>0$, and we have

$$\begin{aligned}&a(t,x)\ge\rho,\\ &b(t,x)\ge\rho,\\ &\quad x\in[D,\infty),\, t\in[0,1].\end{aligned} \tag{37}$$

Then, for the small number λ, μ, problem (2) has at least one positive solution.

Proof. Firstly, let $r = \max\{2M\|B\|, (2D/\varepsilon), 2\}$, and

$$1 = 2r\left(\min_{0\le t\le 1}\int_0^1 Y(t,s)e(s)(\lambda+\mu)\rho ds\right)^{-1}. \tag{38}$$

Set $\Omega_1 = \{x \in E : \|x\| < r\}$, for any $x \in P \cap \partial\Omega_1$, $s \in [0,1]$, we have

$$x(s) - w(s) \ge x(s) - M\|B\|z(s) \ge x(s) - \frac{M\|B\|}{r}x(s) \ge \frac{1}{2}x(s). \tag{39}$$

Thus, $x(s) - w(s) \ge (1/2)x(s) \ge (\|x\|/2)z(s) \ge (\varepsilon r/2) \ge D$. Therefore, by (C6) and the definition of operator T, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \\ &\quad + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s) \\ &\quad \cdot \left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds.\end{aligned} \tag{40}$$

For $B(t) > 0$, $t \in (0,1)$, we have

$$\begin{aligned}(Tx)(t) &\ge \int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \\ &\ge \int_0^1 G(t,s)e(s)[\lambda(\rho+B(s)) + \mu(\rho+B(s))]ds \\ &\ge \frac{1}{2}\min_{0\le t\le 1}\int_0^1 G(t,s)e(s)(\lambda+\mu)\rho ds = r = \|x\|.\end{aligned} \tag{41}$$

We can know that by the above discussion, we have

$$\|Tx\| \ge \|x\|, \forall x \in P \cap \partial\Omega_1. \tag{42}$$

Secondly, by (C4), we can have

$$\begin{gathered} 0 \le \lim_{x\longrightarrow\infty}\frac{\bar{H}(s,x-w)}{u} \le b_1, \\ 0 \le \lim_{x\longrightarrow\infty}\frac{\bar{Y}(s,x-w)}{u} \le b_2, \\ s \in [0,1]. \end{gathered} \tag{43}$$

Then, there exists constant $l_2 > 0$ which satisfies

$$\begin{gathered} \bar{H}(s,x-w) \le b_1 x, \\ \bar{Y}(s,x-w) \le b_2 x, \\ \forall x \ge l_2, s \in [0,1]. \end{gathered} \tag{44}$$

Letting $R = \max\{2l_2, 2r\}$, then $r < R$. Set $\Omega_2 = \{x \in E : \|x\| < R\}$, for any $x \in P \cap \partial\Omega_2$, $t \in [0,1]$, we have

$$\begin{aligned}(Tx)(t) &\le \int_0^1 G(s,s)[\lambda b_1 x(s) + \mu b_2 x(s)]ds \\ &\quad + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)[\lambda b_1 x(s) + \mu b_2 x(s)]ds.\end{aligned} \tag{45}$$

Thus, we have

$$\begin{aligned}(Tx)(t) &\le (\lambda+\mu)[b_1+b_2]e(1) \\ &\quad \cdot \left[\int_0^1 G(s,s)x(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)x(s)ds\right].\end{aligned} \tag{46}$$

So, we have

$$\begin{aligned}\|(Tx)(t)\| &\le (\lambda+\mu)[b_1+b_2]e(1) \\ &\quad \cdot \left[\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\|x\|.\end{aligned} \tag{47}$$

Then, we can have by the definition of b_1, b_2

$$\|Tx\| \le \|x\|, \forall x \in K \cap \partial\Omega_2. \tag{48}$$

□

Then, similar to the proof of heorem 8, we have that result of heorem 9 by Theorem 1.

Theorem 10. *Suppose condition (C1), condition (C2), and condition (C5) hold. Then, for sufficiently small λ, μ, problem (2) has at least two positive solutions.*

Proof. Firstly, by Lemma 7, there exists constant $\tau > 0$ which satisfies

$$G(\tau) \le K_2\tau. \tag{49}$$

Therefore, setting $\Omega_1 = \{x \in E : \|x\| < \tau\}$, for any $x \in P \cap \partial\Omega_1$, $t \in [0,1]$, by the above discussion, for the quite small λ, μ, we have

$$[\lambda H_\tau + \mu Y(\tau)]e(1)\left(\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right) \le \tau. \tag{50}$$

We have

$$\begin{aligned}(Tx)(t) &\le [\lambda H_\tau + \mu Y(\tau)] \\ &\quad \cdot \left[\int_0^1 G(t,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)ds\right] \\ &\le [\lambda H_\tau + \mu Y(\tau)]e(1) \\ &\quad \cdot \left(\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right) \\ &\le \tau = \|x\|.\end{aligned} \tag{51}$$

Then, we have

$$\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1. \tag{52}$$

Secondly, by (C5), there exists a constant $l_3 > 1$, which satisfies

$$a(t,x) \ge K_1 x, \quad \forall x \ge l_3. \tag{53}$$

Letting $r = \max\{2M\|B\|, (2l_3/\varepsilon), 2\tau\}$, and $\Omega_2 = \{x \in E : \|x\| < r\}$, for any $x \in P \cap \partial\Omega_2$, $t \in [0,1]$, we have

$$x(t) - w(t) \ge x(t) - M\|B\|z(t) \ge x(t) - \frac{M\|B\|}{r}x(t) \ge \frac{1}{2}x(t). \tag{54}$$

Then, $x(t) - w(t) \ge (1/2)x(t) \ge (\|x\|/2)z(t) \ge (r\varepsilon/2) \ge l_3$.

Therefore, by the definitions of $\bar{H}$, $\bar{Y}$ and the above discussion, we have

$$\begin{aligned}(Tx)(t) &\ge \int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \\ &\ge \int_0^1 G(t,s)e(s)\lambda K_1(x-w)ds \\ &\ge \frac{K_1}{2}\lambda \min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds \ge r = \|x\|.\end{aligned} \tag{55}$$

Thus, we have

$$\|Tx\| \ge \|x\|, \forall x \in P \cap \partial\Omega_2. \tag{56}$$

Finally, letting

$$\begin{aligned}R = \max\Bigg\{&[\lambda H_R + \mu Y_R]\left(\int_0^1 G(s,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\right. \\ &\left.\cdot \int_0^1 G(\xi,s)e(s)ds\right], 2r\Bigg\},\end{aligned} \tag{57}$$

then, $\tau < r < R$. Set $\Omega_3 = \{x \in E : \|x\| < R\}$, for any $x \in P \cap \partial\Omega_3$, $t \in [0,1]$, by the definition of operator T, we have

$$\begin{aligned}(Tx)(t) &\le [\lambda H_R + \mu Y_R] \\ &\quad \cdot \left(\int_0^1 G(s,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)ds\right].\end{aligned} \tag{58}$$

Thus, we have

$$\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_3. \tag{59}$$

Then, similar to the proof of heorem 8, we have the result of heorem 10 by Theorem 2. □

Remark 11. The results of these three theorems in our paper also hold under the condition in which nonlinear $a(t,x)$, $b(t,x)$ are both lower semicontinuous.

Remark 12. We can obtain the results of Theorem 10 if we replace condition (C5) with (C6)$K_1 \le b_\infty^- \le \infty$, $0 \le a_\infty^+ \le K_2$.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript, and all authors typed, read, and approved the final manuscript.

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